On Frequency Sensitivity and Mode Orthogonality of Flexible Robotic Manipulators

Fei-Yue Wang, *Fellow, IEEE*, and Yanqing Gao

***Abstract*—This paper presents sensitivity analysis of vibration frequencies of flexible manipulators with respect to variations of systems parameters such as rotational inertia of hub, and mass, moment, and side of tip load. Both Euler-Bernoulli and Timoshenko dynamical models of flexible manipulators are discussed. By using variational method, sensitivity indices are obtained with explicit expressions for measuring the sensitivity of frequencies. Based on variational formulations, a novel method for deriving the orthogonal relations among vibration modal shape functions of flexible manipulators is introduced. With this method, the orthogonal relations can be derived easily without invoking the tedious process of differentiation and integration by part, as commonly used in their derivation.**

***Index Terms*—Flexible manipulators, Euler-Bernoulli dynamical model, Timoshenko dynamical model, variational formulation, vibration frequency, frequency sensitivity, sensitivity analysis, mode orthogonality.**

I. INTRODUCTION

FLEXIBLE manipulators are considered as effective robotic devices for high performance and low cost, especially for less energy consumptions[1−5]. However, due to their structural deformation, extra difficulty and complexity must be dealt with in modeling, analysis, and control of flexible robotic manipulators[6−9]. One of basic problems in flexible manipulators is the determination and analysis of fundamental frequencies, a key factor that affects the system stability and motion speed. Mode orthogonality is another minor issue in analysis and control of flexible manipulators, since mode shape functions are often used in shape modulation and control synthesis[2−5, 9, 10]. Here, we summarize some basic results in frequency sensitivity and mode orthogonality of flexible manipulator systems.

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II. BASIC EQUATIONS

Two types of dynamical models for one-link flexible manipulators are considered in this paper: Euler-Bernoulli model and Timoshenko model. The first model considers only the effect of rotatory inertia on vibration, while the second one takes the effect of both rotatory inertia and shear deformation into account. The detail derivation of the two models can be found in [11], here we just list their dimensionless forms as follows.

Euler-Bernoulli Model:

$$z'''' + \delta m^2 z'' - m^2 z = 0, \tag{1}$$

with boundary conditions,

$$\begin{aligned} &z(0) = 0,\\ &z''(0) + \eta m^2 z'(0) = 0, \end{aligned} \tag{2}$$

$$\begin{aligned} &z''(1) - m^2[\kappa z'(1) + \zeta\mu z(1)] = 0,\\ &z'''(1) + m^2[(\delta + \mu\zeta) z'(1) + \mu z(1)] = 0. \end{aligned} \tag{3}$$

Timoshenko Model:

$$\begin{aligned} &\alpha'' - \sigma(\alpha - z') + \delta m^2 \alpha = 0,\\ &\sigma(\alpha - z')' - m^2 z = 0, \end{aligned} \tag{4}$$

with boundary conditions,

$$\begin{aligned} &z(0) = 0,\\ &\alpha'(0) + \eta m^2 \alpha(0) = 0, \end{aligned} \tag{5}$$

$$\begin{aligned} &\alpha'(1) - m^2[\kappa\alpha(1) + \zeta\mu z(1)] = 0,\\ &\sigma[\alpha(1) - z'(1)] + \mu m^2[z(1) + \zeta\alpha(1)] = 0. \end{aligned} \tag{6}$$

Where a prime indicates differentiation with respect to the dimensionless coordinate ξ, $m = c\omega$, and ω is the frequency of vibration. Other dimensionless variables and parameters are defined as,

$$\xi = \frac{x}{L},\ z = \frac{v}{L},\ \delta = \frac{S}{L^2},\ \sigma = \frac{CL^2}{D},\ c^2 = \frac{\rho L^4}{D}, \tag{7}$$

$$\mu = \frac{M_p}{\rho L},\ \eta = \frac{I_H}{\rho L^3},\ \kappa = \frac{J_p}{\rho L^3},\ \zeta = \frac{a_c}{L}, \tag{8}$$

in which x is the coordinate along the longitudinal axis of the link, L the link length, v and α the total deflection and rotation of the link. S and C are link parameters characterizing the effect of rotatory inertia and shear deformation, and D the bending rigidity. ρ is the mass density per unit length of the link, and I_H the rotational inertia of the hub. Finally, M_p, a_c, and J_p are the mass, the distance of mass center from the free end of link, and the moment of tip load.

Note that in the dynamical model presented in [12], the size of tip load was not considered (i.e., $\zeta = 0$).

III. Vibrational Formulations

In order to discuss the sensitivity of frequency with respect to the variation of system parameters, an explicit expression for frequency is desired. For this purpose, we transfer equation (1)-(6) into their equivalent variational forms.

One can show easily that equation (1)-(3) are equivalent to the following variational equation,

$$m^2 = st_z \frac{\int_0^1 z''^2 \mathrm{d}\xi}{\int_0^1 (z^2+\delta z'^2)\mathrm{d}\xi + \eta z'(0)^2 + \mu z(1)^2 + 2\zeta\mu z(1)z'(1) + \kappa z'(1)^2} \tag{9}$$

where z is only subject to the *geometric* boundary condition $z(0) = 0$, and st_z stands for the stationary value with respect to z. In other words, the solution of equations (1)-(3) must take m^2 achieve its stationary value, or minimum value in the case of fundamental frequency, and *vice versa*.

Similarly, the corresponding vibrational equation for equations (4)-(6) can be found as,

$$m^2 = st_{\alpha,z} \frac{\int_0^1 [\alpha'^2 + \sigma(\alpha - z')^2]\mathrm{d}\xi}{\int_0^1 (z^2+\delta\alpha^2)\mathrm{d}\xi + \eta\alpha(0)^2 + \mu z(1)^2 + 2\zeta\mu z(1)\alpha(1) + \kappa\alpha(1)^2} \tag{10}$$

again, only z is subject to $z(0) = 0$.

IV. Sensitivity Analysis

A variation in hub inertia, or more likely, in tip load will induce a corresponding change in the frequencies of flexible arms. It is extremely important to know how the frequencies changes as systems parameters vary.

From equations (9) and (10), variation of m^2 due to small changes of η, μ, ξ, and κ can expressed as,

$$\frac{\delta m^2}{m^2} = -S_\eta \frac{\delta\eta}{\eta} - S_\mu \frac{\delta\mu}{\mu} - S_{\mu\xi} \frac{\delta(\mu\xi)}{\mu\xi} - S_\kappa \frac{\delta\kappa}{\kappa}, \tag{11}$$

where, for the Euler-Bernoulli model,

$$\begin{aligned} S_\eta &= \frac{\eta z'(0)^2}{\Delta_S}, \quad S_\mu = \frac{\mu z(1)^2}{\Delta_S}, \\ S_{\mu\xi} &= \frac{2\zeta\mu z(1)z'(1)}{\Delta_S}, \quad S_\kappa = \frac{\kappa z'(1)^2}{\Delta_S}, \end{aligned} \tag{12}$$

$$\begin{aligned} \Delta_S = \int_0^1 (z^2 + \delta z'^2)\mathrm{d}\xi + \eta z'(0)^2 + \mu z(1)^2 \\ + 2\zeta\mu z(1)z'(1) + \kappa z'(1)^2, \end{aligned} \tag{13}$$

while for Timoshenko model,

$$\begin{aligned} S_\eta &= \frac{\eta\alpha(0)^2}{\Delta_S}, \quad S_\mu = \frac{\mu z(1)^2}{\Delta_S}, \\ S_{\mu\xi} &= \frac{2\zeta\mu z(1)\alpha(1)}{\Delta_S}, \quad S_\kappa = \frac{\kappa\alpha(1)^2}{\Delta_S}, \end{aligned} \tag{14}$$

$$\begin{aligned} \Delta_S = \int_0^1 (z^2 + \delta\alpha^2)\mathrm{d}\xi + \eta\alpha(0)^2 + \mu z(1)^2 \\ + 2\zeta\mu z(1)\alpha(1) + \kappa\alpha(1)^2. \end{aligned} \tag{15}$$

Note that although $\delta\eta$, $\delta\mu$, $\delta\xi$, and $\delta\eta$ cause the corresponding variations δz and $\delta\alpha$ in z and α, respectively, these variations will not affect the values of m^2, since m^2 obtains its stationary value at z and α. This is why we do not need to consider the variations of z and α in the above sensitivity analysis.

It should be pointed out that sensitivity indices can also be formally defined as follows,

$$\begin{aligned} S_\chi &= -\frac{\%\text{ change in } m^2}{\%\text{ change in } \chi} = -\frac{\delta m^2/m^2}{\delta\chi/\chi} = -\frac{\chi}{m^2}\frac{\delta m^2}{\delta\chi}, \\ \chi &= \eta, \mu, \mu\zeta, \kappa. \end{aligned} \tag{16}$$

Clearly, $0 \leq S_\chi \leq 1$, therefore, vibration frequencies always decreases as system parameters increases their values.

V. Orthogonality Derivation

Consider two different vibration frequencies m_1 and m_2, an their corresponding modal shape functions (z_1, α_1) and (z_2, α_2) (or z_1 and z_2 only, for Euler-Bernoulli model). It is well known that some orthogonal relations exist between (z_1, α_1) and (z_2, α_2). Usually, those orthogonal relations are derived directly from dynamic equations (1)-(6) by integrations and manipulation, and in general the process of derivations is quite tedious[13, 14]. Here we will show that these orthogonal relations can be obtained from the variational formulations (9) and (10) in a much simpler way.

For Euler-Bernoulli model, let,

$$z(\xi) = c_1 z_1(\xi) + c_2 z_1(\xi) \tag{17}$$

where c_1 and c_2 are two arbitrary constants. Obviously, $z(0) = 0$, so it satisfies the geometric boundary condition. Substitute (17) into (9), one finds that m^2 can be expressed as a function of c_1 and c_2,

$$m^2 = \frac{N_{11}c_1^2 + 2N_{12}c_1c_2 + N_{22}c_2^2}{M_{11}c_1^2 + 2M_{12}c_1c_2 + M_{22}c_2^2} \tag{18}$$

where

$$\begin{aligned} N_{ij} &= \int_0^1 z_i'' z_j'' \mathrm{d}\xi, \\ M_{ij} &= \int_0^1 (z_i z_j + \delta z_i' z_j')\mathrm{d}\xi + \eta z_i'(0) z_j'(0) + \mu z_i(1) z_j(1) \\ &\quad + \zeta\mu[z_i(1)z_j'(1) + z_j(1)z_i'(1)] + \kappa z_i'(1)z_j'(1), \\ &\quad i, j = 1, 2. \end{aligned}$$

Since m^2 achieves its stationary values at both $(c_1, c_2) = (1, 0)$ and $(c_1, c_2) = (0, 1)$ (recall that z_1 and z_2 are vibration modal shape functions), $\partial m^2/\partial c$ must be zero at these two points. Hence,

$$N_{11} - m_1^2 M_{11} = 0, \quad N_{12} - m_1^2 M_{12} = 0,$$

$$N_{12} - m_2^2 M_{12} = 0, \quad N_{22} - m_2^2 M_{22} = 0,$$

which leads to

$$N_{12} = 0, \quad M_{12} = 0, \quad m_1^2 = \frac{N_{11}}{M_{11}}, \quad m_2^2 = \frac{N_{22}}{M_{22}}.$$

While the last two equations are expected, the first two gives the following orthogonal relations,

$$\int_0^1 z_i'' z_j'' \mathrm{d}\xi = 0, \quad i \neq j \tag{19}$$

$$\int_0^1 (z_i z_j + \delta z_i' z_j')\mathrm{d}\xi + \eta z_i'(0) z_j'(0) + \mu z_i(1) z_j(1) + \zeta\mu[z_i(1)z_j'(1) + z_i'(1)z_j(1)] + \kappa z_i'(1) z_j'(1) = 0, \quad i \neq j \tag{20}$$

whose special forms have been found in [3, 4].

For Timoshenko model, similarly, let,

$$z(\xi) = c_1 z_1(\xi) + c_2 x_2(\xi), \tag{21}$$

$$\alpha(\xi) = c_3 \alpha_1(\xi) + c_4 \alpha_2(\xi), \tag{22}$$

where $c = (c_1, c_2, c_3, c_4)^{\mathrm{T}}$ is an arbitrary constant vector, From (10), m^2 now can be expressed in terms of c as,

$$m^2 = \frac{c^{\mathrm{T}} N c}{c^{\mathrm{T}} M c} \tag{23}$$

where

$$N = \begin{pmatrix} N_{11} & N_{12} & -N_{13} & -N_{14} \\ N_{12} & N_{22} & -N_{23} & -N_{24} \\ -N_{13} & -N_{23} & N_{33} & N_{34} \\ -N_{14} & -N_{24} & N_{34} & N_{44} \end{pmatrix},$$

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{12} & M_{22} & M_{23} & M_{24} \\ M_{13} & M_{23} & M_{33} & M_{34} \\ M_{14} & M_{24} & M_{34} & M_{44} \end{pmatrix}, \tag{24}$$

$$N_{i(j+2)} = \int_0^1 \sigma z_i' \alpha_j \mathrm{d}\xi,$$

$$N_{ij} = \int_0^1 \sigma z_i' z_j' \mathrm{d}\xi,$$

$$N_{(i+2)(j+2)} = \int_0^1 (\alpha_i' \alpha_j' + \sigma \alpha_i \alpha_j)\mathrm{d}\xi, \tag{25}$$

$$M_{ij} = \int_0^1 z_i z_j \mathrm{d}\xi + \mu z_i(1) z_j(1),$$

$$M_{i(j+2)} = \zeta\mu z_i(1)\alpha_j(1), \tag{26}$$

$$M_{(i+2)(j+2)} = \int_0^1 \delta\alpha_i\alpha_j \mathrm{d}\xi + \eta\alpha_i(0)\alpha_j(0) + \kappa\alpha_i(1)\alpha_j(1), \tag{27}$$

$$1 \leq i,\ j \leq 2$$

By the same argument, $\partial m^2/\partial c$ must be zero at the two points $c = (1, 0, 1, 0)^{\mathrm{T}}$ and $c = (0, 1, 0, 1)^{\mathrm{T}}$. Since,

$$\frac{\partial m^2}{\partial c} = 0 \implies (N - m^2 M)c = 0,$$

which leads to,

$$\begin{aligned} N_{11} - m_1^2 M_{11} - (N_{13} + m_1^2 M_{13}) = 0, \\ N_{12} - m_1^2 M_{12} - (N_{23} + m_1^2 M_{23}) = 0, \\ N_{33} - m_1^2 M_{33} - (N_{13} + m_1^2 M_{13}) = 0, \\ N_{34} - m_1^2 M_{34} - (N_{14} + m_1^2 M_{14}) = 0, \\ N_{12} - m_2^2 M_{12} - (N_{14} + m_2^2 M_{14}) = 0, \\ N_{22} - m_2^2 M_{22} - (N_{24} + m_2^2 M_{24}) = 0, \\ N_{34} - m_2^2 M_{34} - (N_{23} + m_2^2 M_{23}) = 0, \\ N_{44} - m_2^2 M_{44} - (N_{24} + m_2^2 M_{24}) = 0. \end{aligned}$$

Combine these equations appropriately, one can show that,

$$N_{12} + N_{34} - N_{23} - N_{14} - m_i^2(M_{12} + M_{34} + M_{23} + M_{14}) = 0, \quad i = 1, 2. \tag{28}$$

Since $m_1 \neq m_2$, it must have,

$$N_{12} + N_{34} = N_{23} + N_{14},$$
$$M_{12} + M_{34} + M_{23} + M_{14} = 0. \tag{29}$$

It follows that,

$$\int_0^1 [\alpha_i' \alpha_j' + \sigma(\alpha_i - z_i')(\alpha_j - z_j')]\mathrm{d}\xi = 0, \tag{30}$$

$$\int_0^1 (z_i z_j + \delta\alpha_i\alpha_j)\mathrm{d}\xi + \eta\alpha_i(0)\alpha_j(0) + \mu z_i(1) z_j(1) + \zeta\mu[z_i(1)\alpha_j(1) + z_j(1)\alpha_i(1)] + \kappa\alpha_i(1)\alpha_j(1) = 0, \tag{31}$$

for $i \neq j$, $1 \leq i,\ j \leq 2$.

Additional relations can also be derived. For example, we have

$$\begin{aligned} \frac{N_{ii} - N_{i(i+2)}}{M_{ii} + M_{i(i+2)}} &= \frac{N_{(i+2)(i+2)} - N_{i(i+2)}}{N_{(i+2)(i+2)} + M_{i(i+2)}} \\ &= \frac{N_{ij} - N_{j(i+2)}}{M_{ij} - M_{j(i+2)}} \\ &= \frac{N_{(i+2)(i+2)} - N_{i(j+2)}}{M_{(i+2)(i+2)} - N_{i(j+2)}} \end{aligned} \tag{32}$$

for $i \neq j$, $1 \leq i,\ j \leq 2$.

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