Letter

Multi-Timescale Distributed Approach to Generalized-Nash-Equilibrium Seeking in Noncooperative Nonconvex Games

Banghua Huang, Yang Liu, *Senior Member, IEEE*, Kit Ian Kou, and Weihua Gui

Dear Editor,

The distributed generalized-Nash-equilibrium (GNE) seeking in noncooperative games with nonconvexity is the topic of this letter. Inspired by the sequential quadratic programming (SQP) method, a multi-timescale multi-agent system (MAS) is developed, and its convergence to a critical point of the game is proven. To illustrate the qualities and efficacy of the theoretical findings, a numerical example is elaborated.

With the rapid development of MAS theories and distributed optimization, distributed Nash-equilibrium (NE) seeking has become a meaningful research topic [1]–[7]. In a paradigm of noncooperative games, cost function and action set of an individual player depend on its own action, and together with the other player's actions. Besides, NEs are important solutions among the strategies of a noncooperative game. At an NE, if other players keep their current strategies, then no player can improve its benefits by changing its strategies. In the game modeled in the practical networks (e.g., [8]), each player is capable to receive partial decision information (the player's own information and its neighbors' information) only. Hence, it is essential for each player to estimate the strategies, in a distributed manner, adopted by all the other players. To this end, many distributed NE seeking methods have been proposed and captured attention from many areas [1], [9]–[15]. Specifically, the landmark leader-following consensus-based distributed NE seeking strategy was first proposed in [1]. Then, numerous effective methods for seeking NEs have emerged in the past few years. For example, a distributed method for seeking generalized NE based on gradient projection is proposed in [9], a distributed continuous-time penalty method is proposed for seeking GNE in [10], a distributed method is proposed for seeking NEs with Markovian switching topologies [11], a distributed method is proposed for seeking GNE with nonsmooth cost functions in [12], a distributed NE seeking method with actuator constraints is proposed in [13], a distributed NE seeking method subject to quantitative communications is proposed in [14], a distributed NE seeking method with bounded disturbances is proposed in [15]. In the studies aforementioned, the cost function and action sets in the games are assumed to be convex. However, the distributed NE seeking methods for the games with nonconvex cost functions and constraints are not studied.

The SQP method is one of the effective approaches for constrained nonlinear or nonconvex optimization [16]. Driven by the idea of SQP, a two-timescale neural network is developed for nonconvex optimization [17]. To echo the nonconvex optimization solver based on multi-timescale neural networks, we develop a multi-timescale MAS for seeking the critical point of the nonconvex game. The main contributions are summarized as follows: 1) Based on the idea of SQP, we develop a multi-timescale MAS for GNE-seeking of nonconvex games. 2) Compared with the existing distributed NE seeking approach for the convex games in [1], [9]–[15], the proposed MAS with proper timescales is proven to be convergent to a critical point of a nonconvex game.

Notations: $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{R}^n$ denotes the set of all n-dim real vectors. $\otimes$ denotes the Kronecker product operator. $\times$ denotes the Cartesian product operator. $\|\cdot\|$ denotes the 2-norm; For a group of vectors $t_1,\dots,t_N$, $\text{col}[t_1,\dots,t_N]=(t_1^T,\dots,t_N^T)^T$. $(\cdot)^+=\max\{0,\cdot\}$ is the popular ReLU function. For a vector $u\in\mathbb{R}^N$ and $\alpha>0$, $\mathcal{B}(u,\alpha)=\{y\in\mathbb{R}^N \mid \|u-y\|\le\alpha\}$. I_N indicates an N-dim identity matrix. For a matrix M, $\delta_{\min}(M)$ denotes the minimal eigenvalue of M.

Graph theory fundamentals: Let $\mathcal{G}=(\mathcal{V},\mathcal{E},A)$ be an undirected graph with edges set $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ and noeds set $\mathcal{V}=\{1,\dots,N\}$; $A=(a_{ij})$ denotes the adjacency matrix of $\mathcal{G}$. If $(i,j)\in\mathcal{E}$, then $a_{ij}>0$, which indicates that i and j can communicate with each other. $\mathcal{N}_i=\{j\in\mathcal{V}\mid a_{ij}>0\}$ denotes the neighbor index set of node i. $D=\text{diag}((D_1,\dots,D_N)^T)$ where $D_i=\sum_{j=1}^N a_{ij}$. $L=D-A$ denotes the Laplacian matrix of $\mathcal{G}$. Graph $\mathcal{G}$ is called undirected if $A=A^T$, and an undirected graph $\mathcal{G}$ is called connected if there is a path bewteen any two distinct nodes. $C=\text{diag}(a_{11},\dots,a_{1N},\dots,a_{N1},\dots,a_{NN})$.

Problem statement: Consider the following game with N players:

$$\min_{x_i\in X_i} f_i(x_i,\mathbf{x}_{-i})=f_i(\mathbf{x}) \tag{1}$$

where $f_i(x_i,\mathbf{x}_{-i})$ is the cost function of player i, $\mathbf{x}_{-i}=\text{col}[x_1,\dots,x_{i-1},x_{i+1},\dots,x_N]$ represents the vector of all player actions except for player i and $\mathbf{x}=\text{col}[x_1,\dots,x_N]$. Besides, $X_i\triangleq\{\mathbf{x}\in\mathbb{R}^N|g_i(\mathbf{x})\le 0, h_i(\mathbf{x})=0\}$ and $X=X_1\times\cdots\times X_N$.

Let $\mathbf{x}^*=\text{col}[x_1^*,\dots,x_N^*]$ and $\mathbf{x}_{-i}^*=\text{col}[x_1^*,\dots,x_{i-1}^*,x_{i+1}^*,\dots,x_N^*]$. Point $\mathbf{x}^*$ is called a Local GNE of game (1) if $f_i(x_i^*,\mathbf{x}_{-i}^*)\le f_i(x_i,\mathbf{x}_{-i}^*)$ holds for $\mathbf{x}\in\mathcal{N}(\mathbf{x}^*)\subset X$ where $\mathcal{N}(x^*)$ is a neighborhood of $\mathbf{x}^*$.

Several necessary assumptions are provided.

Assumption 1: For $i\in\mathcal{V}, X_i$ is a nonempty; $f_i(x_i,\mathbf{x}_{-i})$, $g_i(\mathbf{x})$ and $h_i(\mathbf{x})$ are smooth.

Assumption 2: The graph $\mathcal{G}$ is connected.

Assumption 3: There exists at least one GNE in game (1).

Let $\nabla f(\mathbf{x})=\text{col}[\nabla_{x_1}f_1(\mathbf{x}_1),\dots,\nabla_{x_N}f_N(\mathbf{x}_N)]$, $g(\mathbf{x})=\text{col}[g_1(\mathbf{x}),\dots,g_N(\mathbf{x})]$ and $h(\mathbf{x})=\text{col}[h_1(\mathbf{x}),\dots,h_N(\mathbf{x})]$.

Definition 1: A point $(\mathbf{x},\lambda,\mu)$ is called a critical point of game (1) if it satisfies the following equations:

$$\begin{aligned}&\nabla f(\mathbf{x})+\nabla g(\mathbf{x})^T\lambda+\nabla h(\mathbf{x})^T\mu=\mathbf{0}\\&g(\mathbf{x})^T\lambda=0,\ \lambda\ge\mathbf{0},\ g(\mathbf{x})\le\mathbf{0}\\&h(\mathbf{x})=\mathbf{0}.\end{aligned} \tag{2}$$

where λ and μ are Lagrange multipliers.

Equations (2) is said be a Karush-Kuhn-Tucker (KKT) conditions in optimization theory. Under several assumptions, e.g., second-order sufficiency conditions [18], a point satisfying KKT conditions (2) is a local solution. Therefore, the seeking of critical points contributes to the seeking of local GNEs.

The SQP method is an effective method for constrained nonconvex optimization, and it has two stages [16]: At the first stage, a quadratic programming subproblem of game (1) is formulated as follows:

Corresponding author: Yang Liu.

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B. Huang is with the School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, China (e-mail: huangbanghua@zjnu.edu.cn).

Y. Liu is with the Key Laboratory of Intelligent Education Technology and Application of Zhejiang Province, Jinhua 321004, and also with the School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, China (e-mail: liuyang@zjnu.edu.cn).

K. Kou is with the Faculty of Science and Technology, University of Macau, Macau, China (e-mail: kikou@umac.mo).

W. Gui is with the School of Automation, Central South University, Changsha 410083, China (e-mail: gwh@csu.edu.cn).

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$$\min_{\mathbf{z}_k} \frac{1}{2}\mathbf{z}_k^T M\mathbf{z}_k + \nabla f(\mathbf{x}_k)^T \mathbf{z}_k$$
$$\text{s.t.} \quad \nabla h(\mathbf{x}_k)^T \mathbf{z}_k + h(\mathbf{x}_k) = 0$$
$$\nabla g(\mathbf{x}_k)^T \mathbf{z}_k + g(\mathbf{x}_k) \le 0 \tag{3}$$

where $\mathbf{x}_k$ and $\mathbf{z}_k$ is the decision vector and the directional vector at the k-th iteration, and M is a given positive definite matrix. At the second stage, $\mathbf{x}_k$ is updated by the following iteration rule:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha z_k^* \tag{4}$$

where α is the step size and z_k^* is the optimal solution of subproblem (3).

Remark 1: In the idea of SQP, we do not need to solve nonconvex constrained game (1) directly, instead, we use rule (4) to update the decision variable and we solve optimization subproblem (3) at each iteration of (4). Note that subproblem (3) at each iteration is a convex optimization problem with respect to $\mathbf{z}$, which avoids using the nonconvexity of cost functions.

Main results: Inspired by [1] and [17], a continuous-time generalization of a SQP approach in the form of multi-timescale MAS-based distributed optimization is proposed. Let y_{ij} be player i's estimate on x_j, $\mathbf{y}_i = \text{col}[y_{i1},\ldots,y_{iN}]$ and $\mathbf{y} = \text{col}[\mathbf{y}_1,\ldots,\mathbf{y}_N]$. Then, a multi-timescale MAS for seeking the NE of the game (1) is described

$$\begin{aligned}
\epsilon_1 \frac{d\mathbf{x}}{dt} &= \mathbf{z}\\
\epsilon_2 \frac{d\mathbf{z}}{dt} &= -M\mathbf{z} - \nabla F(\mathbf{y}) - \nabla G(\mathbf{y})^T\lambda - \nabla H(\mathbf{y})^T\mu\\
\epsilon_2 \frac{d\lambda}{dt} &= -\lambda + (\lambda + G(\mathbf{y}) + \nabla G(\mathbf{y})^T\mathbf{z})^+\\
\epsilon_2 \frac{d\mu}{dt} &= H(\mathbf{y}) + \nabla H(\mathbf{y})^T\mathbf{z}\\
\epsilon_3 \frac{d\mathbf{y}}{dt} &= -(L\otimes I_N + C)(\mathbf{y} - 1_N\otimes\mathbf{x})
\end{aligned} \tag{5}$$

where ϵ_1, ϵ_2 and ϵ_3 are positive time constants. In addition, $\nabla F(\mathbf{y}) = \text{col}[\nabla_{x_1} f_1(\mathbf{y}_1),\ldots,\nabla_{x_N} f_N(\mathbf{y}_N)]$, $G(\mathbf{y}) = \text{col}[g_1(\mathbf{y}),\ldots,g_N(\mathbf{y})]$, $\lambda = \text{col}[\lambda_1,..,\lambda_N]$, $H(\mathbf{y}) = \text{col}[h_1(\mathbf{y}),\ldots,h_N(\mathbf{y})]$, and $\mu = \text{col}[\mu_1,..,\mu_N]$.

In multi-timescale MAS (5), the $\mathbf{x}$-layer (i.e., $\epsilon_1 d\mathbf{x}/dt = \mathbf{z}$) is a continuous counterpart of the second stage of SQP (i.e., update rule (4)), and its timescale is ϵ_1. The $\mathbf{z}$-layer, λ-layer, and μ-layer are designed for the first stage of SQP (i.e., solving subproblem (3)), and their timescale is ϵ_2. The $\mathbf{y}$-layer is an estimate layer with its timescale being ϵ_3, and it is leveraged to estimate the other player's states by an individual player [1]. M is a given positive definite matrix and it is a diagonal block matrix shown as $M = \text{blkdiag}(M_1,\ldots,M_N)$ with M_i being a positive definite matrix. Player i or agent i just need to determine its individual M_i, which is in a distributed manner.

Let $\bar{\mathbf{p}} = \text{col}[\bar{\mathbf{x}},\bar{\mathbf{z}},\bar{\lambda},\bar{\mu},\bar{\mathbf{y}}]$ be an equilibrium of MAS (5). Now, we prove the equivalence between the critical point of game (1) and the equilibrium $\bar{\mathbf{p}}$.

Theorem 1: Under Assumptions 1−3, $\bar{\mathbf{p}}$ is a critical point of game (1).

Proof: According to the definition of equilibria, $\bar{\mathbf{p}}$ satisfies the following equations:

$$\begin{aligned}
\mathbf{0} &= \bar{\mathbf{z}}\\
\mathbf{0} &= -M\bar{\mathbf{z}} - \nabla F(\bar{\mathbf{y}}) - \nabla G(\bar{\mathbf{y}})^T\bar{\lambda} - \nabla H(\bar{\mathbf{y}})^T\bar{\mu}\\
\mathbf{0} &= -\bar{\lambda} + (\bar{\lambda} + G(\bar{\mathbf{y}}) + \nabla G(\bar{\mathbf{y}})^T\bar{\mathbf{z}})^+\\
\mathbf{0} &= H(\bar{\mathbf{y}}) + \nabla H(\bar{\mathbf{y}})^T\bar{\mu}\\
\mathbf{0} &= -(L\otimes I_N + C)(\bar{\mathbf{y}} - 1_N\otimes\bar{\mathbf{x}}).
\end{aligned}$$

Since the graph is undirected and connected, the equation $\mathbf{0} = -(L\otimes I_N + C)(\bar{\mathbf{y}} - 1_N\otimes\bar{\mathbf{x}})$ yields $\bar{\mathbf{y}} = 1_N\otimes\bar{\mathbf{x}}$. According to $\mathbf{0} = \bar{\mathbf{z}}$, we obtain

$$\begin{cases}
\mathbf{0} = -\nabla f(\bar{\mathbf{x}}) - \nabla g(\bar{\mathbf{x}})^T\lambda - \nabla h(\bar{\mathbf{x}})^T\mu\\
\mathbf{0} = -\bar{\lambda} + (\bar{\lambda} + g(\bar{\mathbf{x}}))^+\\
\mathbf{0} = h(\bar{\mathbf{x}}).
\end{cases} \tag{6}$$

According to the variational theory, $\mathbf{0} = -\bar{\lambda} + (\bar{\lambda} + g(\bar{\mathbf{x}}))^+$ leads to $\bar{\lambda} \ge \mathbf{0}$, $g(\bar{\mathbf{x}}) \le \mathbf{0}$ and $g(\bar{\mathbf{x}})^T\lambda = 0$. Bringing in (6), we obtain

$$\begin{aligned}
&\nabla f(\bar{\mathbf{x}}) + \nabla g(\bar{\mathbf{x}})^T\lambda + \nabla h(\bar{\mathbf{x}})^T\mu = \mathbf{0}\\
&g(\bar{\mathbf{x}})^T\lambda = 0, \quad \bar{\lambda} \ge \mathbf{0}, \quad g(\bar{\mathbf{x}}) \le \mathbf{0}\\
&h(\bar{\mathbf{x}}) = \mathbf{0}.
\end{aligned}$$

Therefore, $\bar{\mathbf{p}}$ is a critical point of game (1). ■

Now, we prove the convergence of MAS (5) with proper timescales to its equilibrium. Let $\mathbf{u} = \text{col}[\mathbf{x},\mathbf{z},\lambda,\mu]$. First, a function is defined as follows:

$$V_1(\mathbf{u}) = -K_1(\mathbf{u})^T K_2(\mathbf{u}) - \frac{1}{2}\|K_2(\mathbf{u})\|^2 + \frac{1}{2}\left\|\mathbf{w} - \bar{\mathbf{w}}\right\|^2$$

where

$$K_1(\mathbf{u}) = \begin{pmatrix} M\mathbf{z} + \nabla f(\mathbf{x}) + \nabla g(\mathbf{x})^T\lambda + \nabla h(\mathbf{x})^T\mu\\ -g(\mathbf{x}) - \nabla g(\mathbf{x})^T\mathbf{z}\\ -h(\mathbf{x}) - \nabla h(\mathbf{x})^T\mathbf{z} \end{pmatrix}$$

and

$$K_2(\mathbf{u}) = \begin{pmatrix} -M\mathbf{z} - \nabla f(\mathbf{x}) - \nabla g(\mathbf{x})^T\lambda - \nabla h(\mathbf{x})^T\mu\\ -\lambda + (\lambda + g(\mathbf{x}) + \nabla g(\mathbf{x})^T\mathbf{z})^+\\ h(\mathbf{x}) + \nabla h(\mathbf{x})^T\mathbf{z} \end{pmatrix}$$

with $\mathbf{w} = \text{col}[\mathbf{z},\lambda,\mu]$.

According to Assumption 1, $K_2(\mathbf{u})$ is locally Lipschitz continuous with respect to $\mathbf{x}$. Then, according to the proof of Theorem 1 in [17], for any $\mathbf{u} \in \mathcal{B}(\bar{\mathbf{u}},\varepsilon)$ with any positive ε, there exists l_1 and l_2 such that $\|\mathbf{x} - \bar{\mathbf{x}}\| \le l_1\left\|\mathbf{z} - \bar{\mathbf{z}}\right\|$ and $\|\nabla_{\mathbf{x}} V_1(\mathbf{u})\| \le l_2\left\|\mathbf{z} - \bar{\mathbf{z}}\right\|$.

Theorem 2: Under Assumptions 1−3, if $\epsilon_2/\epsilon_1 \le (4\delta_{\min}(M))/(4(l_1 + l_2) + N)$ and $\epsilon_3/\epsilon_1 \le 4/N$, then MAS (5) is convergent to its equilibrium.

Proof: Accroding to proof of Theorem 1 in [1], there exists a domain $D = \{\mathbf{p} - \bar{\mathbf{p}} \mid \|\mathbf{p} - \bar{\mathbf{p}}\| \le r\}$ for some positive constant r. According to Theorem 1 and inequality in [19], we obtain

$$K_1(\mathbf{u})^T K_2(\mathbf{u}) \le -\|K_2(\mathbf{u})\|$$

and

$$\begin{aligned}
\epsilon_2 \nabla_{\mathbf{w}} V_1(\mathbf{u}) \times \frac{d\mathbf{w}}{dt} &\le -\frac{1}{2}(\mathbf{w} - \bar{\mathbf{w}})^T(\nabla_{\mathbf{w}} K_1(\boldsymbol{u}) + \nabla_{\mathbf{w}} K_1^T(\boldsymbol{u}))(\mathbf{w} - \bar{\mathbf{w}})\\
&\le -\delta_{\min}(M)\left\|\mathbf{z} - \bar{\mathbf{z}}\right\|^2.
\end{aligned}$$

$\nabla_{\mathbf{w}} K_1(\boldsymbol{u})$ is derived as follows:

$$\nabla_{\mathbf{w}} K_1(\boldsymbol{u}) = \begin{pmatrix} M & \nabla \boldsymbol{g}(\boldsymbol{x}) & \nabla \boldsymbol{h}^T(\boldsymbol{x})\\ -\nabla \boldsymbol{g}^T(\boldsymbol{x}) & \boldsymbol{O}_1 & \boldsymbol{O}_2\\ -\nabla \boldsymbol{h}^T(\boldsymbol{x}) & \boldsymbol{O}_3 & \boldsymbol{O}_4 \end{pmatrix}.$$

Based on Theorem 1 in [17] and Theorem 1 in [19], we derived

$$\begin{aligned}
\frac{dV_1(\mathbf{u})}{dt} &= \nabla_{\mathbf{x}} V_1(\mathbf{u}) \times \frac{d\mathbf{x}}{dt} + \nabla_{\mathbf{w}} V_1(\mathbf{u}) \times \frac{d\mathbf{w}}{dt}\\
&\le \frac{1}{\epsilon_1}\left(l_2\left\|\mathbf{z} - \bar{\mathbf{z}}\right\|^2\right) - \frac{1}{\epsilon_2}\left(\delta_{\min}(M)\left\|\mathbf{z} - \bar{\mathbf{z}}\right\|^2\right).
\end{aligned} \tag{7}$$

Consider the following Lyapunov function:

$$V(\mathbf{p}) = V_1(\mathbf{u}) + \frac{1}{2}V_2(\mathbf{x},\mathbf{y}) + \frac{1}{2}\left\|\mathbf{x} - \bar{\mathbf{x}}\right\|^2$$

where $V_2(\mathbf{x},\mathbf{y}) = (\mathbf{y} - 1_N\otimes\mathbf{x})^T(L\otimes I_N + C)(\mathbf{y} - 1_N\otimes\mathbf{x})$.

Combining (7), we derive the derivative of V

$$\begin{aligned}
\dot{V} \le\ & \frac{l_2}{\epsilon_1}\left\|\mathbf{z} - \bar{\mathbf{z}}\right\|^2 - \frac{1}{\epsilon_2}\left(\delta_{\min}(M)\left\|\mathbf{z} - \bar{\mathbf{z}}\right\|^2\right)\\
&+ \frac{1}{\epsilon_1}(\mathbf{x} - \bar{\mathbf{x}})^T(\mathbf{z} - \bar{\mathbf{z}}) - \frac{1}{\epsilon_3}\|(L\otimes I_N + C)(\mathbf{y} - 1_N\otimes\mathbf{x})\|^2\\
&+ \frac{\sqrt{N}}{\sqrt{\epsilon_1}}\|(L\otimes I_N + C)(\mathbf{y} - 1_N\otimes\mathbf{x})\|\|\mathbf{z} - \bar{\mathbf{z}}\|\\
\le\ & \left(\frac{l_2 + l_1}{\epsilon_1} - \frac{\delta_{\min}(M)}{\epsilon_2} + \frac{\sqrt{N}}{2\sqrt{\epsilon_1}}\right)\|\mathbf{z} - \bar{\mathbf{z}}\|^2\\
&+ \left(\frac{\sqrt{N}}{2\sqrt{\epsilon_1}} - \frac{1}{\epsilon_3}\right)\|(L\otimes I_N + C)(\mathbf{y} - 1_N\otimes\mathbf{x})\|^2 \le 0.
\end{aligned}$$

Based on the invariance principle, we have $\mathbf{z} = \bar{\mathbf{z}}$ and $\mathbf{y} = 1_N\otimes\mathbf{x}$.

Note that $\|\mathbf{x}-\bar{\mathbf{x}}\| \le l_1 \|\mathbf{z}-\bar{\mathbf{z}}\|$, then $\mathbf{x}=\bar{\mathbf{x}}$. Thus, MAS (5) is convergent to its equilibrium. ■

Remark 2: From Theorems 1 and 2, note that MAS (5) is convergent to the critical point, even if one of the cost functions or one of the constraints is nonconvex, which is different from the methods in [1], [9]–[14].

A numerical example: This section provides a numerical example to illustrate the efficiency of MAS (5).

The player's cost functions are $f_1(\mathbf{x}) = -x_1^3 + 3x_1x_2$, $f_2(\mathbf{x}) = -(-2x_1 + 4x_2 + \frac{1}{2}x_4 + x_5)^2 + 48x_2$, $f_3(\mathbf{x}) = -(x_1 + 4x_3 - x_4 - x_5)^2$, $f_4(\mathbf{x}) = -(2x_1 + 4x_3 + 8x_4 - x_5)^2$, and $f_5(\mathbf{x}) = -(x_1 + 4x_3 + 8x_4 + 17x_5)^2$ for players 1 – 5, respectively [1]. Besides, the game is subject to the following constraints: $g_1(\mathbf{x}) = x_1^3 - 8$, $g_2(\mathbf{x}) = x_1x_2^2 - 10$, $g_3(\mathbf{x}) = x_3^3x_5 - 5$, $g_4(\mathbf{x}) = x_4^3x_5$, $g_5(\mathbf{x}) = x_1x_5^5 - 2$, $h_1(\mathbf{x}) = x_1^2 - 1.5x_1x_2$, and $h_4(\mathbf{x}) = 12x_4^3x_5 + 1/216$. The NE is $\mathbf{x}^* = [3/2, 9/4, -19/48, -1/6, 1/12]^T$. Let $\mathbf{x}(0) = [1, -2, 1, 1, 1]^T$. Let $\epsilon_1 = 1$, $\epsilon_2 = 1/500$, $\epsilon_3 = 1/800$, and $M = 10I_N$. Besides, the players are linked via the topology shown in Fig. 1.

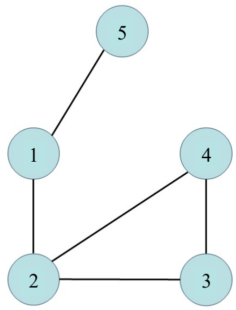
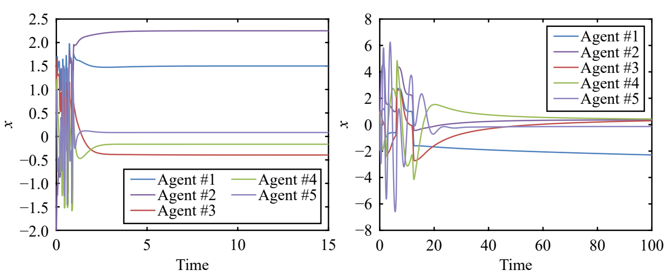


Fig. 1. The communication network topology among players.

Fig. 2(a) implies that MAS (5) is convergent to $\mathbf{x}^*$, which illustrates the validity of Theorems 1 and 2. In contrast, Fig. 2(b) shows that the MAS in [9] is not convergent to NE $\mathbf{x}^*$.



(a) MAS (5) (b) MAS (6) in [9]

Fig. 2. The transient states of MAS (5) and MAS (6) in [9].

Conclusion: In this letter, we suggest a distributed GNE-seeking strategy for nonconvex games. To echo the SQP optimization method, we develop a multi-timescale MAS for constrained games in the presence of nonconvexity. With a proper timescale setting, we prove the convergence of the MAS to one of the game's critical points. To demonstrate the viability of the suggested multi-timescale MAS, we also offer an example.

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