Letter

Stabilization With Prescribed Instant via Lyapunov Method

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Dear Editor,

This letter presents a prescribed-instant stabilization approach to high-order integrator systems by the Lyapunov method. Under the presented controller, the settling time of controlled systems is independent of the initial conditions and equals the prescribed time instant. With this method, the prescribed-instant stabilization method can be easily proved and extended. To be more specific, two differential inequalities of Lyapunov functions are presented to clamp/constrain the settling time to the prescribed time instant from both the left and right sides. This thought serves as an example to present a general framework to verify the designed stabilization property. Actually, the prescribed-time stability (PSTS) [1] can not prescribe the exact settling time. It can only prescribe the upper bound of the settling time and is different with this work. The detailed argumentation will be presented after a brief review of the existing important research.

Traditional asymptotic stability ensures system states converge to equilibrium as time goes to infinity. It implies that system states never reach an equilibrium within any specific time period. Finite-time stability (FNTS) guarantees that the system states convergence happens in a finite time but mostly depending on system parameters and initial conditions [2]. Moreover, when some states are unavailable, a finite time differentiator or observer can be used. Then, the correct information can be estimated after a finite time [3]. This makes it easier to get the closed-loop system stability. However, this finite time increases as the initial values of system states increase and there are no uniform bounds of the settling time. To solve this problem, some new algorithms should be considered.

The fixed-time stability (FXTS) guarantees the settling time be bounded by a constant, which is determined by the controller parameters [4]. However, its settling time is usually far smaller than the fixed time. The predefined-time stability (PDTS) gives an accurate supremum of the settling time. This supremum is usually an explicit parameter of the controller [5]. The FXTS and PDTS restrict the settling time by using the power of system states. The PSTS introduces the time t into the controller and ensures the system states converge to zero in a prescribed time T_p, where T_p is also an explicit parameter of the controller [6]. Some simulation results of PSTS even show the settling time equal to the prescribed time T_p [7]. This means that prescribed-time control may have the ability to achieve the so-called prescribed-instant stability (PSIS). However, their corresponding theoretical analysis cannot explain this ability, except for some first-order systems. A detailed analysis can be seen in Remarks 1 and 2.

Up to now, only a few research with strict proof showed that the settling time can be arbitrarily manipulated. For example, the work in [8] showed this property by presenting a detailed analysis of the infinitesimal order of the system states. Although its PSIS analysis is presented only for the first-order state, the full-state PSIS is easy to come out with. In [9], reduction to absurdity was utilized to get the exact settling time. The PSIS was established with strict proof for the first time. However, these methods of proof are circumscribed and can hardly be generalized. For more complex systems with mismatched disturbance, state constraint, or input saturation, it is difficult to analyze and obtain the PSIS by the method given in [8], [9]. However, most of these problems have been well studied in existing control theory with the Lyapunov method. We believe that the PSIS will be developed quickly as long as it is analyzed under the framework of the Lyapunov method. Further applications such as consensus control of multi-agent systems may provide amazing performance [10], [11].

This letter considers n-order integrator systems, of which the settling time under the presented controller is exactly the prescribed time instant. A corresponding proof based on the Lyapunov method is utilized to provide a general framework to ensure the exact settling time. Moreover, this framework can also help to decrease potential conservativeness in the settling time existed in the traditional PSTS.

Problem statement: Consider the following system:

$$\dot{x} = g(x,u)$$

where $g\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth continuous function of x and u, $x \in \mathbb{R}^n$ denotes the states, and $u \in \mathbb{R}$ is the control variable.

Consider the control variable as $u(x,\eta)$, where $\eta \in \mathbb{R}^m$ denote a vector of explicit controller parameters. We can obtain the closed-loop system in (1) with the initial value $x(t_0) = x_0$. We default the initial time $t_0 = 0$. Consider the closed-loop form of the system $x = g(x,u)$ as follows:

$$\dot{x} = f(x,\eta) := g_1(x,u(x,\eta)). \quad (1)$$

Definition 1 [4], [5]: The origin of the system (1) is said to be

1) FXTS if the settling time $T(x_0)$ satisfies $T(x_0) \le T_f(\eta)$, $\forall x_0 \in \mathbb{R}^n$, where $T_f(\eta)$ is positive and depends on the controller parameters $\eta = [\eta_1,\ldots,\eta_m] \in \mathbb{R}^m$, or

2) PDTS if the settling time $T(x_0)$ satisfies $\sup_{\forall x_0 \in \mathbb{R}^n} T(x_0) = T_p$, where $T_p > 0$ is one of the controller parameters, i.e., $\eta = [\eta_1 = T_p,\eta_2,\ldots,\eta_m] \in \mathbb{R}^m$.

Consider the control variable as $u(x,t,\eta)$.

$$\dot{x} = f(x,t,\eta) := g_2(x,u(x,t,\eta)). \quad (2)$$

Definition 2 [9]: Consider the positive constant T_p is one of the controller parameters η. The origin of the system (2) is said to be

1) PSTS if there exist parameters $\eta \in \mathbb{R}^m$ such that $T(x_0) \le T_p$, $\forall x_0 \in \mathbb{R}^n$, or

2) PSIS if there exist parameters $\eta \in \mathbb{R}^m$ such that $T(x_0) = T_p$, $\forall x_0 \in \mathbb{R}^n$.

Consider the following linear system:

$$\begin{cases} \dot{x}_i = x_{i+1}, \ \ i = 1,\ldots,n-1 \\ \dot{x}_n = u. \end{cases} \quad (3)$$

Since the PSIS has already been defined and proved in [9], the main contribution of this letter is presenting a Lyapunov method to verify that the controller can ensure the system in (3) is PSIS.

The expressions of Theorem 2 in [1] and Theorem 1 in [6] may mislead the readers to think that the PSTS also ensures $T(x_0) = T_p$. To be strictly clear, it is urgent to emphasize the following fact.

Remark 1: The proof in [1] is one of the main thought of proof to obtain PSTS. The key step is to obtain the following inequality:

$$\frac{dV(t)}{dt} \le -2k\mu(t)V(t), \quad k > 0, \quad t \in [0,T_p) \quad (4)$$

where V is a Lyapunov function of the controlled system and

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$$\mu(t)=\frac{T_p^{m+n}}{(T_p-t)^{m+n}},\quad t\in[0,T_p). \tag{5}$$

If the formula in (4) is an equality, there is no doubt that the settling time equals the prescribed time T_p. However, for a high-order system, it is difficult to obtain the equality form in (4). As a result, $V(t)$ is reset to zero in the prescribed time no longer than T_p irrespectively of the initial value $V(t_0)\in\mathbb{R}_{\geq 0}$ (Section 2 in [12]). We have $T(x_0)\leq T_p$.

Remark 2: Another thought of proof of PSTS in some research such as [6] is based on some time scale transformation from $t\in[0,T_p)$ to $\tau\in[0,+\infty)$. The mostly-used transformation is

$$\begin{cases}\tau=-\ln(\dfrac{T_p-t}{T_p})\\ t=T_p(1-e^{-\tau}).\end{cases} \tag{6}$$

Since $\frac{d\tau}{dt}=\frac{1}{T_p-t}$, we have two functions equivalent to each other

$$\begin{cases}\dfrac{dV(t)}{dt}=-\dfrac{1}{T_p-t}V(t), & t\in[0,T_p)\\ \dfrac{dV(t(\tau))}{d\tau}=\dfrac{dV(t)}{dt}\dfrac{dt}{d\tau}=-V(t(\tau)), & \tau\in[0,+\infty).\end{cases} \tag{7}$$

If one can prove that the Lyapunov function $V(t(\tau))$ converges exponentially, or $V(t(\tau))\to 0$ as $\tau\to+\infty$, the conclusion is definitely obtained that $V(t)\to 0$ just as $t\to T_p$. However, Lemma 2 in [6] only shows $\frac{dV(t(\tau))}{d\tau}\leq -V(t(\tau))$. There stands a chance that $\frac{dV(t(\tau))}{d\tau}\leq -V^{1.5}(t(\tau))-V^{0.5}(t(\tau))$. As a result, $V(t(\tau))=0,\forall\tau\geq\pi$. Since $V(t(\tau))$ converges to zero before τ is $+\infty$, $V(t)$ converges to zero before $t=T_p$, i.e., $T(x_0)\leq T_p$.

We have clearly presented that the proofs of the PSTS are not sufficient to obtain $T(x_0)=T_p$, although some simulations of PSTS have shown the property of $T(x_0)=T_p$.

Remark 3: The work in [13] defined the properties of PSTS (PSIS) as the so called free-will weak (strong) arbitrary time stability. It recognized that only one differential inequality (as (4) in Remark 1) from the Lyapunov function could not obtain PSIS directly. However, the free-will strong arbitrary time stability, which is consistent with the presented PSIS, can be established as long as

$$\frac{dV}{dt}=-\frac{k(1-e^{-V})}{(T_p-t)},\quad k>1,\quad t\in[0,T_p). \tag{8}$$

Although the proof of free-will strong arbitrary time stability for high-order systems remains open, yet this proof can be completed once the following inequalities are considered:

$$\begin{cases}\dot V_1\leq\dfrac{-k_1(1-e^{-V_1})}{(T_p-t)}, & k_1>1,\quad t\in[0,T_p)\\ \dot V_2\geq\dfrac{-k_2(1-e^{-V_2})}{(T_p-t)}, & k_2>k_1,\quad t\in[0,T_p).\end{cases}$$

Specifically, one can obtain the PSIS by limiting the settling time from both the left and right sides of Lyapunov function derivative formula. This is also the main thought of this letter.

To realize the above-mentioned thought, one can find a differential function whose solution converges to zero just at the prescribed instant T_p. Definition 5 of [9] presented a series of such functions named reference convergence differential functions (RCDFs).

Example 1: Let us see some typical RCDFs (ψ) given in [9]

$$\begin{cases}\dot v_1=-\psi_{v_1}=-\dfrac{\eta(v_1^2+1)\arctan(v_1)}{T_p-t}, & v_1(t)=\tan(T_p-t)^{\eta}\\ \dot v_2=-\psi_{v_2}=-\dfrac{\eta v_2}{T_p-t}, & v_2(t)=(T_p-t)^{\eta}\\ \dot v_3=-\psi_{v_3}=-\dfrac{\eta(1-e^{-|v_3|})}{T_p-t}\mathrm{sign}(v_3), & v_3(t)=\ln(1+(T_p-t)^{\eta}).\end{cases}$$

It is noted that ψ_v is also a function of t, T_p, and η. We denote it as $\psi(v,t,T_p,\eta)$, which can be written as $\frac{\eta\zeta(v)}{T_p-t}$. $\zeta(v)$ is a continuous increasing odd function of v. Moreover, $\zeta(v)=O(v)$ (infinitesimal of the same order), and

$$\lim_{t\to T_p}\psi(v,t,T_p,\eta)=\lim_{t\to T_p}\frac{v(t)-0}{t-T_p}\sim(T_p-t)^{\eta-1},\ \forall\eta>1.$$

In addition, the work in [9] also provide other RCDFs may help to promote the proof of PSTS in existing research to obtain PSIS.

Main results:

Controller design: The controller is designed with backstepping method and is presented as a recursive form. The desired value of x_1 is $x_{1,d}=c$, where c is a constant. Replacing z_i by z_{i-1}, the recurrence relation ($i\geq 2$) presented as follows is a little different from that in [9]:

$$x_{i+1,d}=\dot x_{i,d}-z_{i-1}-\psi_i \tag{9}$$

where $z_i=x_i-x_{i,d}$ and $x_{2,d}=-\psi_1$. It is noted that $\psi_i(z_i,t,T_p,\eta_i)$ belongs to the same RCDF. To prevent the singularity problem of the control signal at $t=T_p$, η_i in ψ_i is designed to satisfy $\eta_i>n+1-i$, $i=1,2,\dots,n$. The control of system (3) is

$$u=\begin{cases}x_{n+1,d}, & 0\leq t<T_p\\ 0, & T_p\leq t.\end{cases} \tag{10}$$

According to (9) and (10), when $t\in[0,T_p)$, we have

$$\begin{cases}\dot z_1=\dot x_1=x_2=z_2+x_{2,d}=z_2-\psi_1\\ \dot z_i=\dot x_i-\dot x_{i,d}=x_{i+1}-x_{i+1,d}-z_{i-1}-\psi_i\\ \quad=z_{i+1}-z_{i-1}-\psi_i\\ \dot z_n=\dot x_n-\dot x_{n,d}=u-\dot x_{n,d}=x_{n+1,d}-\dot x_{n,d}\\ \quad=\dot x_{n,d}-z_{n-1}-\psi_n-\dot x_{n,d}=-z_{n-1}-\psi_n.\end{cases} \tag{11}$$

In the following, we will present the PSIS of the transformed system (11), and then obtain the PSIS of the system (3).

Prescribed-instant stability:

Lemma 1: Suppose the function $h(x)$: $\mathbb{R}\to\mathbb{R}$ is concave $\forall x\in\mathbb{I}\in\mathbb{R}$ and positive constants $\lambda_1,\lambda_2,\dots,\lambda_n$ satisfy $\sum_{i=1}^n\lambda_i=1$. Then,

$$\sum_{i=1}^n\lambda_i h(x_i)\leq h(\sum_{i=1}^n\lambda_i x_i),\quad\forall x_i\in\mathbb{I}. \tag{12}$$

Especially, if $\lambda_1=\lambda_2=\cdots=\lambda_n=\frac{1}{n}$, we have

$$\frac{\sum_{i=1}^n h(x_i)}{n}\leq h(\frac{\sum_{i=1}^n x_i}{n}),\ \forall x_i\in\mathbb{I}. \tag{13}$$

This lemma is the so-called Jenson inequality.

Theorem 1: The origin of the system (3) under the controller (10) is PSIS with prescribed time instant T_p. The control signal $u(t)$ converges to zero at $t=T_p$, and $x(t)=u(t)=0$, $\forall t\geq T_p$.

Proof: As long as the controller is designed as (10), the dynamics of z_i is given by (11). Choose the Lyapunov function as $V_n=\sum_{i=1}^n z_i^2$. According to (11),

$$\dot V_n=\sum_{i=1}^n 2z_i\dot z_i=\sum_{i=1}^n\frac{-2\eta_i|z_i|\zeta(|z_i|)}{T_p-t},\quad t\in[0,T_p). \tag{14}$$

Denote the vector $\boldsymbol{z}=[z_1,z_2,\dots,z_n]\in\mathbb{R}^n$. According to the properties given in Example 1, $\zeta(|z|)$ is an increasing positive function, so $-|z|\zeta(|z|)$ is concave. The time derivative of V_n satisfies

$$\begin{aligned}\dot V_n&\leq\frac{2\min(\eta_1,\eta_2,\dots,\eta_n)}{T_p-t}\sum_{i=1}^n[-|z_i|\zeta(|z_i|)]\\ &\leq-\frac{2n\min(\eta_1,\eta_2,\dots,\eta_n)}{T_p-t}\frac{\|z\|_1}{n}\zeta(\frac{\|z\|_1}{n})\\ &\leq-\frac{2n\min(\eta_1,\eta_2,\dots,\eta_n)}{T_p-t}\frac{\sqrt{V_n}}{n}\zeta(\frac{\sqrt{V_n}}{n})\end{aligned} \tag{15}$$

where the second inequality used Lemma 1, and the third inequality is because $\|z\|_1\geq\|z\|_2=\sqrt{V_n}$. Let $a_1=\frac{\sqrt{V_n}}{n}$, we have

$$\dot a_1=-\frac{\dot V_n}{2n\sqrt{V_n}}\leq-\frac{\min(\eta_1,\eta_2,\dots,\eta_n)\zeta(a_1)}{n(T_p-t)},\ (V_n\neq 0,\forall z_i\neq 0). \tag{16}$$

This means a_1 converges to and reaches zero before or at $t=T_p$, as well as V_n. One can obtain that z_i converges to and reaches zero before or at $t=T_p$.

Another inequality of V_n is given by

$$\dot{V}_n \geq -\frac{2(\eta_1+\eta_2+\cdots+\eta_n)\|z\|_\infty\zeta(\|z\|_\infty)}{T_p-t} \geq -\frac{2(\eta_1+\eta_2+\cdots+\eta_n)\sqrt{V_n}\zeta(\sqrt{V_n})}{T_p-t}. \quad (17)$$

Let $a_2 = \sqrt{V_n}$ of which the derivative is

$$\dot{a}_2 = -\frac{\dot{V}_n}{2\sqrt{V_n}} \geq -\frac{(\eta_1+\eta_2+\cdots+\eta_n)\zeta(a_2)}{T_p-t}, \ (V_n \neq 0, \forall z_i \neq 0). \quad (18)$$

This means a_2 does not reach zero before $t = T_p$, as well as V_n. Hence, V_n converges to and reaches zero at $t = T_p$. Meanwhile, z_1 to z_n converge to and reaches zero.

Since $z_i \to 0$ as $t \to T_p$, by combining systems (3), (9) and (10), one can obtain the following relationship as $t \to T_p$:

$$\begin{cases} x_2 = x_{2,d} = -\psi_1 = u_1 \\ x_3 = x_{3,d} = -\dot{\psi}_1 - z_1 - \psi_2 = u_2 \\ x_4 = x_{4,d} = -\ddot{\psi}_1 - \dot{z}_1 - \dot{\psi}_2 - z_2 - \psi_3 = u_3 \\ \quad \vdots \\ x_{n+1} = x_{n+1,d} = \dot{x}_{n,d} - z_{n-1} - \psi_n = u_n. \end{cases} \quad (19)$$

Each equation in (19) contains ψ_i, z_i, and their derivatives, and ψ_1 is derivatives the most times. For example, u_n contains $\psi_1^{(n-1)}$ and $\psi_i^{(n-i)}$. As mentioned in Example 1, $\psi \sim (T-t)^{\eta-1}$, $\psi^{(n-1)} \sim (T_p-t)^{\eta-n}$ as $t \to T_p$. As long as the parameters are selected as $\eta_i > n+1-i$, the derivative of each ψ_i will tend to zero as $t \to T_p$. Hence, the controller (10) will tends to zero as $t \to T_p$, and so do the states of the system (3). Because $u(t) = 0$, $\forall t \geq T_p$, we have $x(t) = 0$, $\forall t \geq T_p$. Therefore, the system (3) is PSIS with the prescribed time instant $t = T_p$. ∎

Numerical example: Consider a simple pendulum system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 + \frac{1}{ml^2}T \end{cases}$$

where x_1 denotes the angle, and T is the torque. Moreover, $l = 0.5$ m, $m = 0.1$ kg, g $= 9.81$ m/s^2, and $k = 0.01$. We set the initial values as $x_1(0) = 0.09$ rad and $x_2(0) = 0.1$ rad/s. The control objective is making $x_1 = x_{1,d} = 0.15$ rad at $T_p = 0.5$ s, i.e., $z_1 = x_1 - 0.15$. Let $T = ml^2(\frac{g}{l}\sin x_1 + \frac{k}{m}x_2 + u)$. Then, $\dot{x}_1 = x_2, \dot{x}_2 = u$.

Choose the RCDFs as

$$\begin{cases} \psi_1 = \dfrac{\eta_1(1-e^{-|z_1|})}{T_p-t}\mathrm{sign}(z_1), \ \eta_1 = 3 \\ \psi_2 = \dfrac{\eta_2(1-e^{-|z_2|})}{T_p-t}\mathrm{sign}(z_2), \ \eta_2 = 2. \end{cases}$$

According to (9) and (10), the specific controller is

$$u = \begin{cases} -z_1 - \dfrac{\eta_1(1-e^{-|z_1|})}{(T_p-t)^2}\mathrm{sign}(z_1) \\ -\dfrac{\eta_1 e^{-|z_1|}}{T_p-t}\mathrm{sign}(z_1) - \dfrac{\eta_2(1-e^{-|z_2|})}{T_p-t}\mathrm{sign}(z_2), & 0 \leq t < T_p \\ 0, & T_p \leq t. \end{cases}$$

As presented in Fig. 1, each state is stabilized to the desired value at $t = T_p = 0.5$ s. One characteristic of the PSIS is presented in Fig. 1(b), i.e., u strikes zero at $t = T_p = 0.5$ s.

Conclusion: This letter provides a proof framework based on the Lyapunov method to ensure that the real convergence time of a high-order integrator system equals the prescribed time instant. Therefore, the settling time is irrelevant to the initial conditions and can be any physically feasible assigned time instant. A simulation with a simple pendulum system has verified this approach. Extending the proposed method to get PSIS for the systems with matched and mismatched disturbance is a consequential topic in the future.

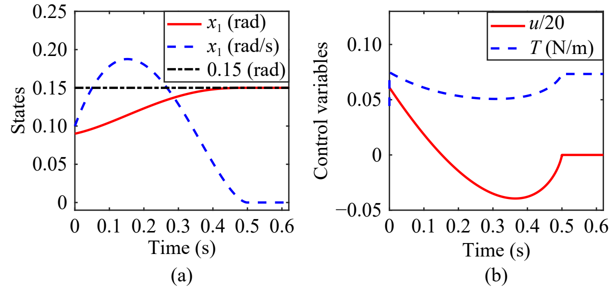

Fig. 1. Trajectories of (a) x_1 (angle), and x_2 (angular velocity); (b) u (equivalent control), and T (torque applied to the pendulum).

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