Letter

A Finite-Time Convergent Analysis of Continuous Action Iterated Dilemma

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Dear Editor,

In this letter, a finite-time convergent analysis of continuous action iterated dilemma (CAID) is proposed. In traditional evolutionary game theory, the strategy of the player is binary (cooperation or defection), which limits the number of strategies a player can choose from. Meanwhile, there are no effective methods to analyze the convergence and its convergence time in previous works. To solve these problems, we make several innovations in this letter. Firstly, CAID is proposed by enriching the players' strategies as continuous, which means the player can choose an intermediate state between cooperation and defection. And discount rate is considered to imitate that players cannot learn accurately based on strategic differences. Then, to analyze the convergence of CAID, the Lyapunov function is designed. Furthermore, to analyze the convergence time of CAID, a finite-time convergent analysis based on the Lyapunov function is introduced. In this case, simulation results show the effectiveness of our analysis.

With the rapid development of network science, the evolutionary game theory has been applied in economics, artificial intelligence, and multi-agent systems successfully [1]–[4]. Zhang *et al*. [5] investigate the emergence of oscillatory behavior in evolutionary games that are played using reinforcement learning, providing insights into the evolution of collective behavior. In [6], the evolutionary game theory is utilized to model the evolution process of attacking strategies employed by malicious users, taking into account the dynamics and diversity of these strategies. The driving force mechanism of information is constructed using evolutionary game theory by Xiao *et al*. [7] to investigate the factors influencing user behavior during the rumor spreading process. Notably, the strategy of the player is binary or limited in these works, which makes it hard to be consistent with reality. In real-world games, players' strategies are not limited to full cooperation or full defection. Thus, CAID with continuous strategies of players is proposed in this letter, which means the players can be in an intermediate state of full cooperation and full defection.

Convergence and convergence time are important qualities of evolutionary game theory [8]. In [9], the delayed networked evolutionary games model is proposed. Meanwhile, the convergence and evolutionarily stable profiles are analyzed. Mai *et al*. [10] design a centralized evolutionary game-based pool selection algorithm to analyze the colony behaviors of devices. The convergence in a regular network was analyzed using the Jacobian matrix by Ranjbar-Sahraei *et al*. [11]. As we can see in these works, the Jacobin matrix is usually introduced to prove the convergence of the evolutionary game, which has a high correlation with the connecting relationship of players. When the relationship between players is very complex, this method cannot be applied. At the same time, it should be noted that the analysis of convergence time has always been hard in the evolutionary game, so proposing a new method to analyze the convergence and convergence time is practical and essential.

As an important research topic in the field of control theory, finite-time convergence has attracted extensive attention worldwide [12]–[16]. Wang and Xiao [17] study the bidirectional interaction case and the unidirectional interaction case of multi-agent systems through finite-time stability. The focus of the authors in [18] is on the design of a novel disturbance rejection control scheme for a flexible Timoshenko manipulator subject to extraneous disturbances that exhibits finite-time convergence. In [19], the authors present a method for designing an adaptive fast finite-time stabilizer by utilizing an analytical strategy and carefully selecting appropriate Lyapunov functions. Notably, these studies are concentrated in the field of control and are rarely used in evolutionary game theory. These effective studies can provide us with methods to analyze the convergence and convergence time of the evolutionary game. Based on this, we propose a finite-time analysis of CAID based on the Lyapunov function.

In this letter, a finite-time convergent analysis of CAID is proposed. The contributions can be summarized as: 1) To enrich the strategies of players of the evolutionary game, CAID is introduced with continuous strategies. And discount rate is considered to imitate that players cannot learn accurately based on strategic differences. 2) The convergence of CAID is analyzed by the Lyapunov function. 3) The convergence time of CAID is analyzed by the proposed finite-time convergence method based on the Lyapunov function. Finally, the proposed method is demonstrated through simulation examples using the continuous action iterated prisoner's dilemma (CAIPD) and continuous action iterated snowdrift dilemma (CAISD), showing its effectiveness.

Problem statement: Suppose there are N players, and the connecting relationship is described by the adjacency matrix $W \in \mathbb{R}^{N\times N}$. If player i and player j is connecting, $w_{ij}=1$; $w_{ij}=0$, otherwise. The fully connected network is used in the letter, so it can be obtained $w_{ij}=w_{ji}=1$ for any pair of players (i,j). Compared to the binary strategy in traditional evolutionary game theory, the CAID is proposed in this letter, in which the strategy of player i is continuous $x_i \in [0,1]$. Notably, $x_i=1$ represents the full cooperation and $x_i=0$ represents the full defection. The payoff matrix of CAID between player i and player j can be defined as

$$\begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix} \tag{1}$$

where a_0, a_1, a_2, a_3 represents the payoff.

According to Darwin's principle of survival of the fittest, the strategy fitness between player i and player j can be described as

$$\begin{aligned} F(x_i,x_j) &= a_0x_ix_j + a_1(1-x_i)x_j + a_2x_i(1-x_j) \\ &\quad + a_3(1-x_i)(1-x_j) \\ &= (a_0-a_1-a_2+a_3)x_ix_j + (a_1-a_3)x_j \\ &\quad + (a_2-a_3)x_i + a_3. \end{aligned} \tag{2}$$

Then, the difference ΔF_{ji} is calculated by

$$\Delta F_{ji} = F(x_j,x_i) - F(x_i,x_j). \tag{3}$$

Drawing inspiration from the imitation dynamics, players adopt the strategy of one of their neighbors with a certain probability, resulting in a dynamic that can be formulated as follows:

$$x_i(k+1) = (1-p_{ij})x_i(k) + p_{ij}x_j(k) \tag{4}$$

where k is the iterated number. $p_{ij} = \epsilon \mathrm{sign}(\beta|\Delta F_{ji}|)$ with $\mathrm{sign}(\beta|\Delta F_{ji}|) = 1/(1+\exp(-\beta|\Delta F_{ji}|))$ and $\beta>0$, $\epsilon>0$ are constants.

Based on $\Delta x_i(k) = x_i(k+1)-x_i(k)$, the dynamic model of strategy adaptation in two-players CAID is

$$\dot{x}_i(t) = p_{ij}(x_j(t)-x_i(t)). \tag{5}$$

Considering the connecting relationship of N players, the strategy fitness of player i is

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$$
\begin{aligned}
F(x_i) &= \sum_{j=1}^{N} F(x_i, x_j) = \sum_{j=1}^{N} [a_0 x_i x_j + a_1(1-x_i)x_j \\
&\quad + a_2 x_i(1-x_j) + a_3(1-x_i)(1-x_j)] \\
&= \sum_{j=1}^{N} [(a_0 - a_1 - a_2 + a_3)x_i x_j + (a_1 - a_3)x_j \\
&\quad + (a_2 - a_3)x_i + a_3].
\end{aligned} \tag{6}
$$

Thus, one can conclude the difference ΔF_{ji} between the fitness of player i and j as follows:

$$\Delta F_{ji} = F(x_j) - F(x_i). \tag{7}$$

Similar to (5), the dynamic model of strategy adaptation in N-players CAID is

$$\dot{x}_i(t) = \frac{1}{N}[\sum_{j=1}^{N} p_{ij}(x_j - x_i)]. \tag{8}$$

However, in the dynamic model (8), players can learn accurately based on strategic differences, which is not in line with actual games. Thus, we propose a new CAID dynamics model (9) in which player learning exists with a discount rate $0 < \alpha < 1$

$$\dot{x}_i(t) = \frac{1}{N}[\sum_{j=1}^{N} p_{ij}\text{sign}(x_j - x_i)|x_j - x_i|^{\alpha}] \tag{9}$$

where

$$\text{sign}(r) = \begin{cases} 1, & r > 0 \\ 0, & r = 0 \\ -1, & r < 0. \end{cases} \tag{10}$$

Main results: To analyze the convergence of (9), some lemmas are introduced firstly.

Lemma 1 [17]: Suppose that function $V(t)$ is differentiable such that

$$\dot{V} \le -KV(t)^q \tag{11}$$

where $K > 0$ and $0 < q < 1$. Then, $V(t)$ will reach zero at finite time t^* as

$$t^* = \frac{V(0)^{1-q}}{K(1-q)} \tag{12}$$

and $V(t) = 0$ for all $t \ge t^*$.

Lemma 2 [20]: For a connected graph G that is undirected, the following well-known property holds:

$$\min_{x \neq 0, 1^T x = 0} \frac{x^T L x}{||x||^2} = \lambda_2(L) \tag{13}$$

where L is the Laplacian of graph G and $\lambda_2(L)$ is the second smallest eigenvalue of L.

Lemma 3 [17]: Let $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n \ge 0$ and let $0 < p < 1$. Then,

$$(\sum_{i=1}^{n} \Upsilon_i)^p \le \sum_{i=1}^{n} \Upsilon_i^p \le n^{1-p}(\sum_{i=1}^{n} \Upsilon_i)^p. \tag{14}$$

Theorem 1: If the connecting relationship of players is fully connected, then the dynamic model (9) is finite-time convergent.

Proof: Set

$$\gamma = \frac{1}{N}\sum_{i=1}^{N} x_i(t) \tag{15}$$

and it can be concluded that γ is invariant because $\sum_{i=1}^{N} \dot{x}_i(t) = 0$ according to (9). Define the error as $e_i = x_i(t) - \gamma$. we can get

$$e_i - e_j = x_i - \gamma - x_j + \gamma = x_i - x_j.$$

Meanwhile, as for γ is invariant, we can get $\dot{e}_i = \dot{x}_i - \dot{\gamma} = \dot{x}_i$.

Take the Lyapunov function as

$$V = \frac{1}{2}\sum_{i=1}^{N} e_i^2. \tag{16}$$

Then, it can be obtained that

$$
\begin{aligned}
\dot{V} &= \sum_{i=1}^{N} e_i \dot{e}_i = \frac{1}{2N}\sum_{i,j=1}^{N} (p_{ij} e_i \text{sign}(e_j - e_i)|e_j - e_i|^{\alpha} \\
&\quad + p_{ji} e_j \text{sign}(e_i - e_j)|e_j - e_i|^{\alpha}) \\
&= \frac{1}{2N}\sum_{i,j=1}^{N} p_{ij}\text{sign}(e_j - e_i)(e_i - e_j)|e_j - e_i|^{\alpha} \\
&= -\frac{1}{2N}\sum_{i,j=1}^{N} p_{ij}|e_j - e_i|^{\alpha+1} = -\frac{1}{2N}\sum_{i,j=1}^{N} ((p_{ij})^{\frac{2}{\alpha+1}}|e_j - e_i|^2)^{\frac{\alpha+1}{2}}
\end{aligned} \tag{17}
$$

where $p_{ij} = p_{ji} > 0$.

Suppose $\dot{V} \neq 0$, based on Lemma 3, we can get

$$
\begin{aligned}
\dot{V} &\le -\frac{1}{2N}(\sum_{i,j=1}^{N} (p_{ij})^{\frac{2}{\alpha+1}}|e_j - e_i|^2)^{\frac{\alpha+1}{2}} \\
&= -\frac{1}{2N}\Big(\frac{\sum_{i,j=1}^{N} (p_{ij})^{\frac{2}{\alpha+1}}|e_j - e_i|^2}{V} V\Big)^{\frac{\alpha+1}{2}}.
\end{aligned} \tag{18}
$$

The last equation follows from that $\sum_{i,j=1}^{N} (p_{ij})^{\frac{2}{\alpha+1}}|e_j - e_i|^2 \neq 0$. In fact, if $\sum_{i,j=1}^{N} (p_{ij})^{\frac{2}{\alpha+1}}|e_j - e_i|^2 = 0$, then by the connectivity of graph, $e_i = e_j$ for all players, namely, $\mathbf{e} = [e_1, e_2, \ldots, e_N]^T \in \text{span}(1)$. Because $\mathbf{1}^T\mathbf{e} = \sum_{i=1}^{N} x_i(t) - N\gamma = \sum_{i=1}^{N} x_i(t) - \sum_{i=1}^{N} x_i(t) = 0$, $\mathbf{e} = 0$, and thus $V = 0$, which contradicts our assumption. Therefore, $\sum_{i,j=1}^{N} (p_{ij})^{\frac{2}{\alpha+1}}|e_j - e_i|^2 \neq 0$.

$$\sum_{i,j=1}^{N} (p_{ij})^{\frac{2}{\alpha+1}}|e_j - e_i|^2 = \sum_{i,j=1}^{N} (p_{ij})^{\frac{2}{\alpha+1}}(e_j - e_i)^2 = 2\mathbf{e}^T L((p_{ij})^{\frac{2}{\alpha+1}})\mathbf{e} \tag{19}$$

where $l_{ij} = -(p_{ij})^{\frac{2}{\alpha+1}}$ if $j \neq i$ and $l_{ij} = \sum_{k=1,k\neq i}^{N} (p_{ik})^{\frac{2}{\alpha+1}}$ if $j = i$ in $L((p_{ij})^{\frac{2}{\alpha+1}}) = [l_{ij}] \in \mathbb{R}^{N\times N}$. As for $\mathbf{e} \perp 1$, we can get (20) based on Lemma 2,

$$
\begin{aligned}
\frac{\sum_{i,j=1}^{N} (p_{ij})^{\frac{2}{\alpha+1}}|e_j - e_i|^2}{V} &= \frac{2\mathbf{e}^T L((p_{ij})^{\frac{2}{\alpha+1}})\mathbf{e}}{\frac{1}{2}\mathbf{e}^T\mathbf{e}} \\
&\ge 4\lambda_2(L((p_{ij})^{\frac{2}{\alpha+1}})) > 0.
\end{aligned} \tag{20}
$$

Thus,

$$
\begin{aligned}
\dot{V} &\le -\frac{1}{2N}(4\lambda_2(L((p_{ij})^{\frac{2}{\alpha+1}})))^{\frac{\alpha+1}{2}} V^{\frac{\alpha+1}{2}} \\
&\le -\frac{1}{2N}\min(4\lambda_2(L((p_{ij})^{\frac{2}{\alpha+1}})))^{\frac{\alpha+1}{2}} V^{\frac{\alpha+1}{2}}
\end{aligned} \tag{21}
$$

where $0 < \frac{\alpha+1}{2} < 1$.

Then, based on the Lemma 1, the system can realize finite-time convergence. ■

Simulation examples: The prisoner's dilemma is a classic example of an evolutionary game that illustrates how individual rationality can conflict with group rationality. CAIPD is introduced here as an example. The payoff matrix can be described as

$$\begin{bmatrix} m-k & -k \\ m & 0 \end{bmatrix} \tag{22}$$

where m denotes the benefit gained by the individual and k denotes the price paid by the cooperator, and the parameters satisfy $m > k$.

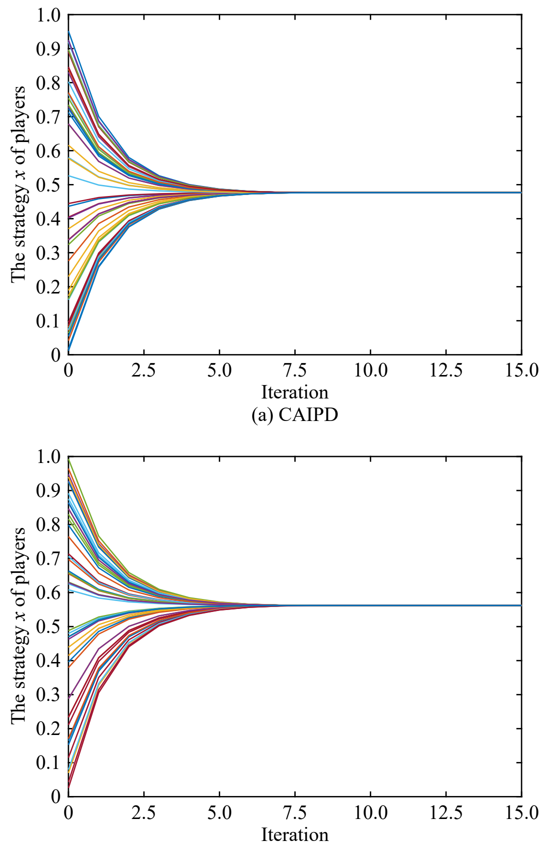
Snowdrift dilemma is another classic evolutionary game. We also introduce CAISD here. The payoff matrix is

$$\begin{bmatrix} m-\frac{k}{2} & m-k \\ m & 0 \end{bmatrix} \tag{23}$$

where $m > k$.

Set $m = 5$, $k = 1$, $\beta = 1$, $\epsilon = 0.5$ and $\alpha = 0.5$, the simulation results in Fig. 1 show the convergence of the CAIPD and CAISD in the full connected network.

Conclusion: This letter has proposed a finite-time analysis of



(a) CAIPD

(b) CAISD

Fig. 1. The convergence simulation results of CAIPD and CAISD.

CAID, which provide a method to analyze the convergence and convergence time of CAID. Firstly, the CAID with continuous strategies has been designed to enrich the binary or limited strategies in traditional evolutionary game theory. And discount rate is considered to imitate that players cannot learn accurately based on strategic differences. Then, the finite-time analysis based on the Lyapunov function has been proposed to avoid the influence caused by the complex connecting relationship of players. Furthermore, CAIPD and CAISD have been introduced as examples to demonstrate the effectiveness of the proposed method.

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