Letter

Fixed-Time and Predefined-Time Stability of Impulsive Systems

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Dear Editor,

This letter establishes several criteria for fixed-time stability and predefined-time stability of impulsive systems. First, sufficient conditions for fixed-time stability of impulsive systems are presented to treat the destabilizing impulses and hybrid impulses involving multiple jump maps by fixed-time control without linear feedback regulation. It determines the robustness of nonlinear systems against impulsive disturbance which has destabilizing and hybrid effect to dynamics. Then, the predefined-time stability of impulsive systems is ensured under the requirement that the hybrid impulses have stabilizing accumulative effect. Finally, the validity of theoretical results is verified by the allocation of mobile agents on a segment via fixed-time control with impulsive regulation.

Stability analysis is the effect way to discover not only the dynamics of nonlinear systems but, increasingly, the insightful mechanism of many practical systems in engineering, physics, and social sciences. Consequently, various stability properties have been carried out for the analysis of nonlinear systems. Exponential stability and asymptotic stability describe that the states of nonlinear systems converge to the equilibrium point as time goes to infinity and can be achieved by conventional feedback controller. Different from exponential stability and asymptotic stability, finite-time stability determines the states to reach the equilibrium point in certain settling time and shows great advantages of convergence rate and robustness against disturbance [1]. However, the settling time of finite-time stable systems depends on the initial value and probably increases as the norm of initial value grows. Then, the fixed-time stability (FXS) was developed to estimate the upper bound of settling time which is independent of initial value. For instance, several necessary and sufficient conditions for FXS of continuous systems were presented in terms of Lyapunov functions in [2]. Then, the estimation of settling time of fixed-time stable systems was enhanced to derive a new nonconservative upper bound and a new predefined-time convergent algorithm in [3]. The settling time of predefined-time stable systems is set a priori as a parameter of the system, which is more desirable in practical systems [4]. Therefore, the predefined-time stability (PTS) of nonlinear systems has derived plenty of valuable works (see [5]–[7] and references therein). But, the PTS of impulsive systems is rarely considered in existing works.

Impulsive system is a kind of hybrid systems which can be classified into two categories: impulsive systems with stabilizing impulses and destabilizing impulses. The stabilizing impulses contribute to the stability of systems and act as the controller in the stabilization of systems. For instance, the impulses were shown to accelerate the convergence rate of finite-time stable systems in [8]. In [9], the fixed-time stabilization of impulsive systems was achieved by fixed-time controller under the requirement that the impulses have stabilizing or inactive accumulative effect. On the other hand, the destabilizing impulses model the sudden changes at certain time and are usually regarded as disturbance in practical systems. In [10], the impulsive systems with destabilizing impulsive were stabilized by finite-time controller with the restricted range of admissible initial values, that is local finite-time stability. Then, FXS of impulsive systems with destabilizing impulses was achieved by fixed-time controller with linear feedback term which regulates destabilizing impulses in [11]. However, the fixed-time controller have both fractional power and power larger than one, so it ought to be powerful enough to regulate the destabilizing impulses. But how to determine the FXS and PTS of impulsive systems without the linear feedback regulation remains unsolved due to the difficulty of bridging the non-Lipschitz continuous dynamics and the destabilizing effect or hybrid effect caused by impulses, which is the primary concern of this letter.

Thus motivated, this letter focuses on the FXS and PTS of impulsive systems. The novelty lies in three aspects: 1) New FXS result for impulsive systems with destabilizing impulses is established under a generalized dwell-time condition without the linear feedback regulation; 2) The FXS results are extended to the case of hybrid impulses with multiple jump maps to show the robustness against impulsive disturbance; 3) Sufficient conditions for PTS are given to ensure the stability of impulsive systems with hybrid impulses. It shows that the PTS of nonlinear systems can be ascertained by fixed-time control with impulsive regulation, which is also verified by the predefined-time allocation of mobile agents.

Problem formulation: Consider the following impulsive system:

$$\begin{cases}\dot{x}(t)=f(x(t)), & t\in\mathbb{R}_{\geq 0}\setminus\mathcal{T}\\ x(t)=g_k(x(t^-)), & t\in\mathcal{T}\\ x(0)=x_0\in\mathbb{R}^n\end{cases}\tag{1}$$

where $x(t)=[x_1(t),x_2(t),\ldots,x_n(t)]^T$, $f:\mathbb{R}^n\to\mathbb{R}^n$, $g_k:\mathbb{R}^n\to\mathbb{R}^n$, $k\in\mathbb{N}_+$, and $\mathcal{T}=\{t_k\}_{k\in\mathbb{N}_+}$ ($\{t_k\}$ for short). Assume that the impulsive strength function satisfies $g_k(0)=0$, $k\in\mathbb{N}_+$, and the impulse time sequence $\{t_k\}$ satisfies $0=t_0<t_1<t_2<\cdots<t_k<\cdots$ and $t_k\to+\infty$ as $k\to+\infty$ to prevent the occurrence of accumulation points. In addition, we assume that f satisfies suitable conditions such that system (1) admits unique right-continuous solution in forward time. $N(t,s)$ denotes the number of impulsive times in $(s,t]$. Here, we define several sets of impulse time sequences for later use. For $\tau_a>0$ and $N_0\geq 0$, $\mathcal{F}_-[\tau_a,N_0]$ and $\mathcal{F}_+[\tau_a,N_0]$ denote the classes of impulse sequences $\{t_k\}$ satisfying

$$N(t,s)\leq\frac{t-s}{\tau_a}+N_0,\quad N(t,s)\geq\frac{t-s}{\tau_a}-N_0$$

where τ_a and N_0 are the average impulsive interval and the elasticity number. For $\xi>0$, $\mathcal{F}_\xi$ denotes the class of impulse sequences $\{t_k\}$ satisfying that there exists a function $h\in\mathcal{L}$ with $h(0)>1$ such that

$$\xi N(t,s)-(t-s)\leq\ln h(t-s)\tag{2}$$

where $\mathcal{L}$ denotes the class of continuous functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ which strictly decrease to 0 as $t\to+\infty$. Taking $h(t)=e^{\mu-\lambda t}$ with $\mu=\xi N_0$ and $\lambda=1-\xi/\tau_a>0$, the dwell-time condition (2) degenerates into the classical average impulsive interval condition. If we take $h(t)=(t+1)e^{\mu-\lambda t}$ for $t>\frac{1}{\lambda}$ and $h(t)=(1+\frac{1}{\lambda})e^{\mu-\lambda t}$ for $0<t\leq\frac{1}{\lambda}$, the dwell-time condition (2) leads to $N(t,s)\leq\frac{1}{\xi}\ln(t-s+1)+\frac{t-s}{\tau_a}+N_0$ for large $t-s$. It is considerably weaker than the classical average impulsive interval condition. $\mathcal{P}$ denotes the class of continuous functions $\alpha:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ which satisfy $\alpha(0)=0$ and $\alpha(x)>0$ for $x>0$.

Definition 1: The origin of system (1) is said to be

• Finite-time stable, if it is Lyapunov stable and for any $x_0\in\mathbb{R}^n$ there exists $0\leq T<\infty$ such that $x(t,x_0)=0$ for $t\geq T$. $T(x_0)=\inf\{T\geq 0: x(t,x_0)=0,\ t\geq T\}$ is called the settling-time function of system (1);

• Fixed-time stable over the class $\mathcal{F}$, if it is finite-time stable and $\sup_{x_0\in\mathbb{R}^n}T(x_0)<+\infty$ for any $\{t_k\}\in\mathcal{F}$;

• Predefined-time stable over the class $\mathcal{F}$, if it is fixed-time stable

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and for any predefined $T_c > 0$, $\sup_{x_0\in\mathbb{R}^n} T(x_0) \le T_c$ for any $\{t_k\} \in \mathcal{F}$.

Definition 2: Function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be a Lyapunov function for system (1) with $\varphi, \phi_k \in \mathcal{P}$, $k \in \mathbb{N}_+$, if it is locally Lipschitz continuous, radially unbounded, and satisfies that

1) $V(g_k(x)) \le \phi_k(V(x))$, for all $x \in \mathbb{R}^n$, $k \in \mathbb{N}_+$;

2) $D^+V[x(t)] \le -\varphi(V(x(t)))$, where $D^+V[x(t)]$ is the upper right-hand Dini derivative along system (1) [8].

Main results: In this section, several criteria for FXS and PTS of impulsive systems with destabilizing impulses and hybrid impulses are established as follows.

Theorem 1: Suppose that V is a Lyapunov function for system (1) with φ, $\phi_k \in \mathcal{P}$, $k \in \mathbb{N}_+$. If there exists a positive constant ξ such that

$$\Upsilon = \int_0^{\sup_{x\in\mathbb{R}^n} V(x)} \frac{ds}{\varphi(s)} < +\infty, \quad \int_a^{\phi_k(a)} \frac{ds}{\varphi(s)} \le \xi, \quad \forall a > 0$$

then system (1) is fixed-time stable over the class $\mathcal{F}_\xi$. Moreover, the settling time is bounded by $h^{-1}(e^{-\Upsilon})$.

Proof: According to Theorem 48 of [12], system (1) is Lyapunov stable. Since $V : \mathbb{R}^n \to \mathbb{R}_+$ is radially unbounded, there exist $\mathcal{K}_\infty$-functions ω_1, ω_2 such that $\omega_1(|x|) \le V(x) \le \omega_2(|x|)$. If there exists $T \ge 0$ such that $V(x(t)) = 0$, $x(t) = 0$ for $t \ge T$ due to the fact that $x(t) \equiv 0$ is the equilibrium point. Without loss of generality, assume that $x_0 \ne 0$. Since V is a Lyapunov function, it yields that

$$\begin{cases} D^+y(t) \le -\varphi(y(t)), & t \in \mathbb{R}_{\ge 0} \setminus \mathcal{T} \quad (3)\\ y(t) \le \phi_k(y(t^-)), & k \in \mathcal{T} \quad (4)\end{cases}$$

where $y(t) = V(x(t))$. For $t \in [t_k, t_{k+1})$, it follows from (3) that:

$$F_\epsilon(y(t)) - F_\epsilon(y(t_k)) \le -(t - t_k)$$

where $F_\epsilon(y) = \int_\epsilon^y \frac{ds}{\varphi(s)}$ and ϵ is a positive constant. Combining (4) and the property of impulse sequence in class $\mathcal{F}_\xi$, it obtains that

$$\begin{aligned} F_\epsilon(y(t)) &\le F_\epsilon(y(t_k)) - F_\epsilon(y(t_k^-)) + F_\epsilon(y(t_k^-)) - (t - t_k)\\ &= \int_\epsilon^{y(t_k)} \frac{ds}{\varphi(s)} - \int_\epsilon^{y(t_k^-)} \frac{ds}{\varphi(s)} + F_\epsilon(y(t_k^-)) - (t - t_k)\\ &= \int_{y(t_k^-)}^{y(t_k)} \frac{ds}{\varphi(s)} + F_\epsilon(y(t_k^-)) - (t - t_k)\\ &= \int_{y(t_k^-)}^{V(g_k(x(t_k^-)))} \frac{ds}{\varphi(s)} + F_\epsilon(y(t_k^-)) - (t - t_k)\\ &\le \int_{y(t_k^-)}^{\phi_k(V(x(t_k^-)))} \frac{ds}{\varphi(s)} + F_\epsilon(y(t_k^-)) - (t - t_k)\\ &= \int_{y(t_k^-)}^{\phi_k(y(t_k^-))} \frac{ds}{\varphi(s)} + F_\epsilon(y(t_k^-)) - (t - t_k)\\ &\le \xi + F_\epsilon(y(t_k^-)) - (t - t_k)\\ &\le \xi + F_\epsilon(y(t_{k-1})) - F(y(t_{k-1}^-))\\ &\quad + F_\epsilon(y(t_{k-1}^-)) - (t - t_{k-1})\\ &\le 2\xi + F_\epsilon(y(t_{k-1}^-)) - (t - t_{k-1}) \le \cdots\\ &\le F_\epsilon(V(x_0)) + \xi N(t,0) - t\\ &\le F_\epsilon(V(x_0)) + \ln h(t)\end{aligned}$$

for $t \in [t_k, t_{k+1})$. Therefore, for $t \ge 0$,

$$V(x(t)) \le \max\{0, F_\epsilon^{-1}(F_\epsilon(V(x_0)) + \ln h(t))\}.$$

Since $\varphi \in \mathcal{P}$, F_ϵ and F_ϵ^{-1} is continuous and strictly increasing. Combining $h \in \mathcal{L}$, $F_\epsilon^{-1}(F_\epsilon(V(x_0)) + \ln h(t))$ is strictly decreasing to $-\infty$ as $t \to +\infty$. Therefore, there exists $T(x_0) = h^{-1}[-\exp(\int_\epsilon^{V(x_0)} \frac{ds}{\varphi(s)})]$ such that $x(t) = 0$ for $t \ge T(x_0)$ and

$$\sup_{x_0\in\mathbb{R}^n} T(x_0) \le \sup_{x_0\in\mathbb{R}^n} h^{-1}\left[-\exp\left(\int_0^{V(x_0)} \frac{ds}{\varphi(s)}\right)\right] = h^{-1}\left(e^{-\Upsilon}\right)$$

which indicates the FXS of system (1) over the class $\mathcal{F}_\xi$. ■

Remark 1: In [2], the FXS of autonomous systems was established by Lyapunov function with condition as the first inequality of Theorem 1. Then, the settling time estimation of fixed-time stable systems was enhanced in [3]. In this work, the second inequality of Theorem 1 measures the jump maps of impulses in systems and bridges the non-Lipschitz continuous dynamics and impulsive effect. It indicates that the fixed-time stable systems is robust against impulsive disturbance. In [9] and [11], FXS of impulsive systems was developed, but either the impulses were measured to have stabilizing effect or the destabilizing impulses were regulated by linear feedback control together with fixed-time control. In contrast, Theorem 1 can handle both stabilizing impulses and destabilizing impulses which can be regulated without linear feedback as the following corollary.

Corollary 1: Suppose that there exists a Lyapunov function V for system (1) with $\varphi(s) = (ps^\alpha + qs^\beta)^\delta$ and $\phi_k(s) = e^d s$, $k \in \mathbb{N}_+$, where $p,q,\alpha,\beta,\delta,d > 0$, and $0 < \alpha\delta < 1 < \beta\delta$. If there exists a positive constant τ_a such that $d/\tau_a < \min\{p^\delta, q^\delta\}$, then system (1) is fixed-time stable over the class $\mathcal{F}_-[\tau_a, N_0]$. Moreover, the settling time is bounded by $\frac{\Upsilon+\xi N_0}{1-\xi/\tau_a}$, where $\xi = d/\min\{p^\delta, q^\delta\}$, $\Upsilon = \frac{\Gamma(m_\alpha)\Gamma(m_\beta)}{p^\delta\Gamma(\delta)(\beta-\alpha)}(\frac{p}{q})^{m_\alpha}$, $m_\alpha = \frac{1-\alpha\delta}{\beta-\alpha}$, and $m_\beta = \frac{\beta\delta-1}{\beta-\alpha}$.

Proof: The proof is straightforward based on Theorem 1 and the settling time estimation in [3] so as to be omitted. ■

Remark 2: It should be pointed out that the existing results about FXS of nonlinear systems without impulses such as [13] and [14] can be extended to the impulsive case by the second inequality of Theorem 1. In addition, the criteria for FXS of impulsive systems are presented in terms of Lyapunov function. By choosing appropriate Lyapunov function, the sufficient conditions of nonlinear functions f and g for FXS of systems (1) can be obtained. For example, if we choose the Lyapunov function $V(x) = x^T x$ which is frequently used in recent literature such as [9] and [13], the FXS conditions are given by $x^T f(x) \le \varphi(x^T x)$ and $g_k^T(x)g_k(x) \le \phi_k(x^T x)$ where φ and ϕ_k are defined in Corollary 1. It shows the applicability of Theorem 1.

Next, sufficient conditions for FXS and PTS of impulsive systems are presented where the impulses are measured to have hybrid effect to dynamics. For τ_a, $\tau_a' > 0$ and N_0, $N_0' \ge 0$, $\mathcal{F}[\tau_a, N_0, \tau_a', N_0']$ denotes the class of impulse sequences $\mathcal{T}$ satisfying $\mathcal{T}_D \cup \mathcal{T}_S = \mathcal{T}$, $\mathcal{T}_D \cap \mathcal{T}_S = \emptyset$, $\mathcal{T}_D \in \mathcal{F}_-[\tau_a, N_0]$, and $\mathcal{T}_S \in \mathcal{F}_+[\tau_a', N_0']$, where $\mathcal{T}_D$ and $\mathcal{T}_S$ denote the sets composed of impulse times of stabilizing impulses and destabilizing impulses, respectively.

Theorem 2: Suppose that there exist a Lyapunov function V for system (1) with $\varphi(s) = (ps^\alpha + qs^\beta)^\delta$ and $\phi_k(s) = e^{d_k} s$, where p,q, $\alpha,\beta,\delta > 0$, $0 < \alpha\delta < 1 < \beta\delta$, $d_k \le d$ for $d_k > 1$ and $d_k \le -d'$ for $d_k < 1$, $d,d' > 0$, $k \in \mathbb{N}_+$. If there exist positive constants τ_a, τ_a' such that $\frac{d}{\tau_a \min\{p^\delta,q^\delta\}} - \frac{d'}{\tau_a' \max\{p^\delta,q^\delta\}} < 1$, then system (1) is fixed-time stable over the class $\mathcal{F}[\tau_a, N_0, \tau_a', N_0']$.

Proof: The proof is similar to that of Theorem 1 and thus is given by sketching the outline focusing on the different parts. Since $\varphi(s) = (ps^\alpha + qs^\beta)^\delta$ and $\phi_k(s) = e^{d_k} s$, it yields that

$$\int_a^{\phi_k(a)} \frac{ds}{\varphi(s)} \le \xi_k, \quad \forall a > 0, \quad k \in \mathbb{N}_+$$

where $\xi_k = \frac{d}{\min\{p^\delta,q^\delta\}}$ for $t_k \in \mathcal{T}_D$ and $\xi_k = -\frac{d'}{\max\{p^\delta,q^\delta\}}$ for $t_k \in \mathcal{T}_S$. Then, it follows from (3) and (4) that for $t \in [t_k, t_{k+1})$:

$$\begin{aligned} F_\epsilon(y(t)) &\le F_\epsilon(V(x_0)) + \sum_{i=1}^k \xi_i - t\\ &= \int_\epsilon^{V(x_0)} \frac{ds}{\varphi(s)} + \frac{dN_D(t,0)}{\min\{p^\delta,q^\delta\}} - \frac{d'N_S(t,0)}{\max\{p^\delta,q^\delta\}} - t\\ &\le \int_0^{\sup_{x\in\mathbb{R}^n} V(x)} \frac{ds}{\varphi(s)} + \frac{dN_0}{\min\{p^\delta,q^\delta\}} + \frac{d'N_0'}{\max\{p^\delta,q^\delta\}}\\ &\quad + \Big(\frac{d}{\tau_a \min\{p^\delta,q^\delta\}} - \frac{d'}{\tau_a' \max\{p^\delta,q^\delta\}} - 1\Big)t\end{aligned}$$

where $N_D(t,s)$ and $N_S(t,s)$ denote the numbers of impulsive times of destabilizing and stabilizing impulses in interval $(s,t]$. As $t \to +\infty$, $V(x(t)) \to -\infty$ based on the continuity and monotonicity of F_ϵ and F_ϵ^{-1}. Thus, system (1) is fixed-time stable over the class $\mathcal{F}[\tau_a, N_0, \tau_a', N_0']$ and subsequently the settling time is estimated by $T(x_0) \le \frac{\Upsilon+\Lambda}{c}$ where $\Upsilon = \frac{\Gamma(m_\alpha)\Gamma(m_\beta)}{p^\delta\Gamma(\delta)(\beta-\alpha)}(\frac{p}{q})^{m_\alpha}$, $m_\alpha = \frac{1-\alpha\delta}{\beta-\alpha}$, $m_\beta = \frac{\beta\delta-1}{\beta-\alpha}$, $\Lambda = \frac{dN_0}{\min\{p^\delta,q^\delta\}} + \frac{d'N_0'}{\max\{p^\delta,q^\delta\}}$, and $c = 1 + \frac{d'}{\tau_a' \max\{p^\delta,q^\delta\}} - \frac{d}{\tau_a \min\{p^\delta,q^\delta\}}$. ■

Theorem 3: Let $T_c > 0$ be any predefined constant and suppose that there exist a Lyapunov function V for system (1) as defined in Theorem 2. If there exist positive constants τ_a, τ'_a such that $\frac{d}{\tau_a} - \frac{d'}{\tau'_a} + \frac{\Theta}{T_c^2} \le 0$, then system (1) is predefined-time stable over the class $\mathcal{F}[\tau_a, N_0, \tau'_a, N'_0]$, where $\Theta = \frac{e^{(1-\alpha\delta)(dN_0+d'N'_0)}}{p^\delta(1-\alpha\delta)^2} + \frac{e^{(\beta\delta-1)(dN_0+d'N'_0)}}{q^\delta(\beta\delta-1)^2}$.

Proof: Without loss of generality, we also assume that $x_0 \neq 0$. Denote $\lambda = -\frac{d}{\tau_a} + \frac{d'}{\tau'_a} > 0$. If $V(x_0) \le 1$, skip to Step 2. If $V(x_0) > 1$, the proof is partitioned into two steps: $V(x(t))$ approaches 1 in fixed time; $V(x(t))$ approaches 0 from 1 in fixed time.

Step 1: Let $t_0 = 0$, $\bar{q} = q^\delta$, and $\bar{\beta} = \beta\delta$. From the definition of Lyapunov function, consider the following comparison system:

$$\begin{cases} D^+y(t) = -\bar{q}y^{\bar{\beta}}(t), & t \in \mathbb{R}_{\ge t_0} \setminus (\mathcal{T}_S \cup \mathcal{T}_D) \quad (5)\\ y(t) = e^d y(t^-), & t \in \mathcal{T}_D \quad (6)\\ y(t) = e^{-d'} y(t^-), & t \in \mathcal{T}_S \quad (7)\end{cases}$$

with initial condition $y(t_0) = y_0 \triangleq V(x_0)$. According to the comparison principle, it obtains that $0 \le V(x(t)) \le y(t)$ for $t \ge t_0$. Then, it will show that the solution of system (5)−(7) approaches 1 in fixed time. For $t \in [t_0, t_1)$, it follows from (5) that:

$$y^{1-\bar{\beta}}(t) = y^{1-\bar{\beta}}(t_0) + \bar{q}(\bar{\beta}-1)(t-t_0). \quad (8)$$

Then, it will show that for $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}_+$,

$$y^{1-\bar{\beta}}(t) = y^{1-\bar{\beta}}(t_0)e^{(1-\bar{\beta})(dN_D(t,t_0)-d'N_S(t,t_0))} + \bar{q}(\bar{\beta}-1) \times \Big\{(t-t_k) + \sum_{i=1}^{k}(t_i - t_{i-1})e^{(1-\bar{\beta})(dN_D(t,t_{i-1})-d'N_S(t,t_{i-1}))}\Big\} \quad (9)$$

where $\sum_{i=1}^{0}(\cdot) = 0$. From (8), the assertion (9) is true for $k = 0$. Suppose that the assertion (9) is true for $k = 0, 1, \ldots, m-1$, then it follows from (5)−(7) that for $t \in [t_m, t_{m+1})$:

$$\begin{aligned} y^{1-\bar{\beta}}(t) &= y^{1-\bar{\beta}}(t_m^-)e^{d^*(1-\bar{\beta})} + \bar{q}(\bar{\beta}-1)(t-t_m)\\ &= y^{1-\bar{\beta}}(t_0)e^{(1-\bar{\beta})(dN_D(t_m^-,t_0)-d'N_S(t_m^-,t_0)+d^*)}\\ &\quad + \bar{q}(\bar{\beta}-1)\Big\{(t-t_m) + \sum_{i=1}^{m}(t_i - t_{i-1})\\ &\quad \times e^{(1-\bar{\beta})(dN_D(t_m^-,t_{i-1})-d'N_S(t_m^-,t_{i-1})+d^*)}\Big\}\\ &= y^{1-\bar{\beta}}(t_0)e^{(1-\bar{\beta})(dN_D(t,t_0)-d'N_S(t,t_0))}\\ &\quad + \bar{q}(\bar{\beta}-1)\Big\{(t-t_m) + \sum_{i=1}^{m}(t_i - t_{i-1})\\ &\quad \times e^{(1-\bar{\beta})(dN_D(t,t_{i-1})-d'N_S(t,t_{i-1}))}\Big\}\end{aligned}$$

where $d^* = d$ if $t_m \in \mathcal{T}_D$ and $d^* = -d'$ if $t_m \in \mathcal{T}_S$. Thus, the assertion (9) is true for $k = m$ and for $k \in \mathbb{N}$. Then, it can be derived from (9) that

$$\begin{aligned} y^{1-\bar{\beta}}(t) &= y^{1-\bar{\beta}}(t_0)e^{(1-\bar{\beta})(dN_D(t,t_0)-d'N_S(t,t_0))}\\ &\quad + \bar{q}(\bar{\beta}-1)\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} e^{(1-\bar{\beta})(dN_D(t,s)-d'N_S(t,s))}ds\\ &\quad + \bar{q}(\bar{\beta}-1)\int_{t_k}^{t} e^{(1-\bar{\beta})(dN_D(t,s)-d'N_S(t,s))}ds\\ &\ge \frac{1}{\sigma_1}y^{1-\bar{\beta}}(t_0)e^{-\lambda(1-\bar{\beta})(t-t_0)}\\ &\quad + \frac{1}{\sigma_1}\bar{q}(\bar{\beta}-1)\int_0^{t-t_0} e^{-\lambda(1-\bar{\beta})s}ds\\ &= \frac{1}{\sigma_1}\Big(y^{1-\bar{\beta}}(t_0) + \frac{\bar{q}}{\lambda}\Big)e^{-\lambda(1-\bar{\beta})(t-t_0)} - \frac{\sigma_1\bar{q}}{\lambda} \end{aligned} \quad (10)$$

for $t \in [t_k, t_{k+1})$ where $\sigma_1 = e^{(dN_0+d'N'_0)(\bar{\beta}-1)}$. Therefore, there exists $T_1 = \frac{1}{\lambda(\bar{\beta}-1)}\ln(1+\frac{\lambda\sigma_1}{\bar{q}})$ such that $y^{1-\bar{\beta}}(T_1) \ge 1$, which indicates that $V(x(t))$ approaches 1 in fixed time.

Step 2: Let $t_0 = T_1$, $\bar{p} = p^\delta$, and $\bar{\alpha} = \alpha\delta$. Consider the following comparison system:

$$\begin{cases} D^+y(t) = -\bar{p}y^{\bar{\alpha}}(t), & t \in \mathbb{R}_{\ge t_0} \setminus (\mathcal{T}_S \cup \mathcal{T}_D)\\ y(t) = e^d y(t^-), & t \in \mathcal{T}_D\\ y(t) = e^{-d'} y(t^-), & t \in \mathcal{T}_S\end{cases} \quad (11)$$

with $y(t_0) = y_0 \le 1$. Similar to (10), it can be derived that

$$\begin{aligned} y^{1-\bar{\alpha}}(t) &= y^{1-\bar{\alpha}}(t_0)e^{(1-\bar{\alpha})(dN_D(t,t_0)-d'N_S(t,t_0))}\\ &\quad - \bar{p}(1-\bar{\alpha})\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} e^{(1-\bar{\alpha})(dN_D(t,s)-d'N_S(t,s))}ds\\ &\quad - \bar{p}(1-\bar{\alpha})\int_{t_k}^{t} e^{(1-\bar{\alpha})(dN_D(t,s)-d'N_S(t,s))}ds\\ &= e^{(1-\bar{\alpha})(dN_D(t,t_0)-d'N_S(t,t_0))}\max\Big\{0, y^{1-\bar{\alpha}}(t_0)\\ &\quad - \bar{p}(1-\bar{\alpha})\int_{t_0}^{t} e^{(1-\bar{\alpha})(-dN_D(s,t_0)+d'N_S(s,t_0))}ds\Big\}\\ &\le e^{(1-\bar{\alpha})(dN_D(t,t_0)-d'N_S(t,t_0))}\max\Big\{0, 1\\ &\quad - \sigma_2\bar{p}(1-\bar{\alpha})\int_0^{t-t_0} e^{\lambda(1-\bar{\alpha})(s-t_0)}ds\Big\}\\ &= e^{(1-\bar{\alpha})(dN_D(t,t_0)-d'N_S(t,t_0))}\\ &\quad \times \max\Big\{0, 1 - \frac{\bar{p}}{\lambda\sigma_2}[e^{\lambda(1-\bar{\alpha})(t-t_0)} - 1]\Big\} \end{aligned} \quad (12)$$

where $\sigma_2 = e^{(dN_0+d'N'_0)(1-\bar{\alpha})}$. Therefore, there exists $T_2 = \frac{1}{\lambda(1-\bar{\alpha})}\ln(1+\frac{\lambda\sigma_2}{\bar{p}})$ such that $y(T_1+T_2) = 0$, which indicates that $V(x(t))$ approaches 0 in fixed time.

Finally, combining the radial unboundedness of V, it concludes that there exists a constant $T^* = T_1 + T_2$, which is independent of initial condition x_0, such that $x(t) = 0$ for $t \ge T^*$. Due to the fact that $\ln(1+a) \le \sqrt{a}$ for $a \ge 0$, it follows that $T^* \le \sqrt{\Theta/\lambda} \le T_c$. Thus, system (1) is predefined-time stable over the class $\mathcal{F}[\tau_a, N_0, \tau'_a, N'_0]$. ■

Remark 3: In [3], the predefined-time controllers were designed for first-order and second-order systems. Then, the Lyapunov-like characterization was proposed for PTS of autonomous system in [5]. But most of existing results for PTS are invalid for impulsive systems. In contrast, the inequality of Theorem 3 indicates that the PTS of impulsive systems with hybrid impulses involving multiple jump maps can be ascertained if the hybrid impulses have stabilizing accumulative effect, and subsequently the systems with impulsive disturbance can be stabilized in predefined-time sense by fixed-time controller with impulsive regulation. See the next section for details.

Predefined-time allocation of mobile agents: In order to show the applicability of established results, an allocation algorithm based on fixed-time control with impulsive regulation is designed for the predefined-time allocation of mobile agents. Consider the equidistant allocation of n mobile agents on a segment. The 3D model of each agent is expressed by the following integrator system:

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{z} = w \quad (13)$$

where $x = [x_1, \ldots, x_n]^T$, $y = [y_1, \ldots, y_n]^T$, $z = [z_1, \ldots, z_n]^T$, $u = [u_1, \ldots, u_n]^T$, $v = [v_1, \ldots, v_n]^T$, $w = [w_1, \ldots, w_n]^T$, $\pi_i = (x_i, y_i, z_i)$ is the position of ith agent and $\zeta_i = (u_i, v_i, w_i)$ is the allocation algorithm to drive agents to the segment with endpoints (x_0, y_0, z_0) and $(x_{n+1}, y_{n+1}, z_{n+1})$. The equilibrium point for ith agent is $\pi_i^* = \pi_0 + \frac{i}{n+1}(\pi_{n+1} - \pi_0)$. To achieve the allocation on the segment in fixed time, the allocation algorithm is designed by $\zeta_i = \Phi(\frac{1}{2}(\pi_{i-1} - \pi_i) + \frac{1}{2}(\pi_{i+1} - \pi_i))$ where $\Phi(s) = \text{sign}(s)(\kappa_1|s|^{\mu_1} + \kappa_2|s|^{\mu_2})$, $\kappa_1, \kappa_2 > 0$, $0 \le \mu_1 < 1 < \mu_2 < \infty$. According to Section 4 of [2] or Theorem 1 of [15], each mobile agent is allocated equidistantly in fixed time as shown in Fig. 1(a). However, the mobile agents are usually subject to external disturbance in real word. Here, we consider the impulsive disturbance $\pi_i(t) = e^d\pi_i(t^-) + (1-e^d)\pi^*$ for $t \in \mathcal{T}_{\mathcal{D}} \subset \mathcal{F}_-[\tau_a, N_0]$ where $d, \tau_a > 0$, $N_0 \ge 0$. Then, the conditions for fixed-time allocation established in [2] and [15] fails. Fig. 1(b) illustrates the trajectories of agents which do not move towards the segment. To achieve the fixed-time allocation and further predefined-time allocation, the impulsive regulation is added into the system by $\pi_i(t) = e^{-d'}\pi_i(t^-) + (1-e^{-d'})\pi^*$ for $t \in \mathcal{T}_{\mathcal{S}} \subset$

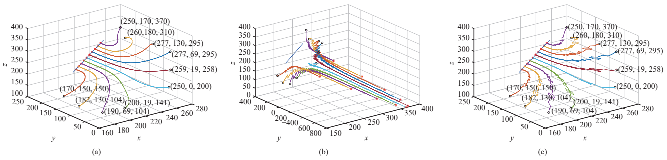

Fig. 1. Trajectories of mobile agents. (a) Agents without impulsive disturbance under fixed-time control ($\kappa_1 = 1.8$, $\kappa_2 = 1$, $\mu_1 = \frac{2}{3}$, and $\mu_2 = \frac{4}{3}$); (b) Agents with impulsive disturbance under fixed-time control ($d = 0.2$, $\tau_a = 1$, and $N_0 = 0$); (c) Agents with impulsive disturbance under fixed-time control with impulsive regulation ($d' = 0.1$, $\tau'_a = 0.4$, and $N'_0 = 1$).

$\mathcal{F}_+[\tau'_a, N'_0]$ where $d', \tau'_a > 0$, $N'_0 \geq 0$. Then, the dynamics of mobile agents can be expressed by

$$\begin{cases} \dot{\eta}(t) = \Phi(\eta(t)), & t \in \mathbb{R}_{\geq 0} \setminus (\mathcal{T}_{\mathcal{S}} \cup \mathcal{T}_{\mathcal{D}}) \\ \eta(t) = e^{d} \eta(t^-), & t \in \mathcal{T}_{\mathcal{D}} \\ \eta(t) = e^{-d'} \eta(t^-), & t \in \mathcal{T}_{\mathcal{S}} \end{cases} \tag{14}$$

with appropriate initial condition, where $\eta = [(Ax + b_x)^T, (Ay + b_y)^T, (Az + b_z)^T]^T$, $b_x = [0.5x_0, 0, \ldots, 0, 0.5x_{n+1}]^T$, $0.5y_{n+1}]^T$, $b_y = [0.5y_0, 0, \ldots, 0, 0.5y_{n+1}]^T$, $b_z = [0.5x_0, 0, \ldots, 0, 0.5z_{n+1}]^T$, and

$$A = \begin{bmatrix} -1 & 0.5 & 0 & \cdots & 0 \\ 0.5 & -1 & 0.5 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0.5 & -1 \end{bmatrix}.$$

According to Theorem 3, the predefined-time allocation of mobile agents can be achieved, if $\frac{d}{\tau_a} - \frac{d'}{\tau'_a} + \frac{\Theta}{T_c^2} \leq 0$. Fig. 1(c) shows the trajectories of agents which are allocated on the segment, and Fig. 2 presents the trajectories versus time where the settling time of allocation is less than the predefined time $T_c = 40$.

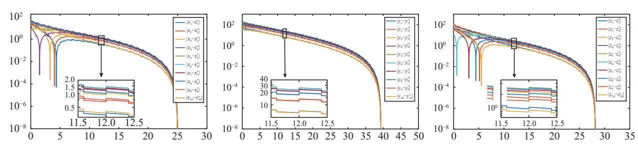

Fig. 2. Trajectories versus time of mobile agents.

Conclusion: In this letter, the FXS and PTS of impulsive systems have been investigated. It shows that the FXS of impulsive systems can be achieved by fixed-time control without linear feedback regulation. And the fixed-time stable nonlinear systems are robust against destabilizing impulses and hybrid impulses. The PTS of impulsive systems can also be ensured by fixed-time control with impulsive regulation, which is verified by the predefined-time allocation of mobile agents. In the future, the FXS and PTS of impulsive nonlinear systems with delayed impulses will be studied.

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