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Oscillator-Inspired Dynamical Systems to Solve Boolean Satisfiability

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ABSTRACT Dynamical systems can offer a novel non-Boolean approach to computing. Specifically, the natural minimization of energy in the system is a valuable property for minimizing the objective functions of combinatorial optimization problems, many of which are still challenging to solve using conventional digital solvers. In this work, we design two oscillator-inspired dynamical systems to solve quintessential computationally intractable problems in Boolean satisfiability (SAT). The system dynamics are engineered such that they facilitate solutions to two different flavors of the SAT problem. We formulate the first dynamical system to compute the solution to the 3-SAT problem, while for the second system, we show that its dynamics map to the solution of the Max-not-all-equal (NAE)-3-SAT problem. Our work advances our understanding of how this physics-inspired approach can be used to address challenging problems in computing.

INDEX TERMS Boolean satisfiability (SAT), combinatorial optimization, dynamical system, Max-not-all-equal (NAE)-SAT, oscillator.

I. INTRODUCTION

DYNAMICAL systems offer a unique "toolbox" for solving combinatorial optimization problems [1], [2], [3], [4]. The intrinsic energy minimization in such systems provides a natural analog to the minimization of an objective function associated with combinatorial optimization problems [5]. The exploration of new computing paradigms for solving such problems, as in the dynamical system-based approach considered here, is motivated by the fact that computing the solutions to such problems using traditional digital algorithms continues to present a significant challenge [6], [7]. As a case in point, solving Boolean satisfiability (SAT) is an archetypal combinatorial optimization problem that is still considered fundamentally intractable for conventional digital hardware. The SAT problem is defined as the challenge of evaluating if there exists a Boolean assignment for the variables in a given Boolean expression (in the conjunctive normal form) that would make the expression TRUE. Besides being the first known NP-complete problem [8], the SAT problem is considered particularly relevant since many practical combinatorial optimization problems can be easily reduced to the solution of the SAT problem. Here, we specifically consider the case of the 3-SAT problem, a constrained but NP-complete version of SAT, where each clause contains no more than three literals.

In this work, we design and analyze two oscillator-inspired dynamical systems and show that their dynamics can be directly used to compute solutions to the 3-SAT (System I) and the Max-NAE-3-SAT (System II) problems. The Not-all-Equal (NAE)-SAT problem is an NP-complete variant of the SAT problem, which imposes the additional constraint that every clause must contain a literal that is true and another literal that is false. The Max-NAE-SAT problem is the optimization version of the problem where the objective is to maximize the number of clauses that meet this constraint. We note that Ercsey-Ravasz et al. [9] proposed an analog computational model for solving the SAT problem which was formulated using nonoscillating (analog) variables; furthermore, in our prior work, we have also proposed computational models for many combinatorial problems (e.g., NAE-SAT, integer factorization among others) with nonoscillating analog variable [10]. While we draw many important insights from these works, our effort here is fundamentally different in that our dynamical systems use oscillating (analog) variables and consequently exhibits a different set of dynamics. We would also like to point out that there have been multiple prior works that have explored the formulation of oscillator-based computational models for solving combinatorial optimization problems such as Maximum Cut [11], [12], [13], [14], [15], Maximum Independent Set [16], [17], Graph coloring [18], [19], [20], [21], and Max-K-Cut [22] among others. However, all these problems, unlike Boolean SAT, have objective functions with a quadratic degree [23]. Subsequently, the oscillator-based computational models can be developed using the Kuramoto framework that cannot be directly applied here.

II. RESULTS

A. SYSTEM I

To formulate System I, we represent every variable x_i in the Boolean expression using an analog variable α_i, where $x_i = ((1+\cos(t+\alpha_i))/2)$, which can be considered as a level-shifted oscillator; the oscillator's angular frequency (ω) is assumed to be $\omega = 1$ in this theoretical analysis. The relationship between x_i and α_i ($x_i = ((1+\cos(t+\alpha_i))/2)$) is defined such that the maximum (or minimum) value of the analog variable equals the Boolean assignment for $x_i \in \{0, 1\}$, respectively. For each clause C_m, we define $K_{m,\text{osc}}(t,\alpha) = \prod_{i=1}^{N}(1 - ((1+c_{mi}\cos(t+\alpha_i))/2))$, where $c_{mi} = 1(-1)$, if the ith variable appears in the mth clause in the normal (negated) form; $c_{mi} = 0$, if the variable is absent from the mth clause; $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N]$; N is the number of variables in the SAT problem. It can be observed that $K_{m,\text{osc}}(t,\alpha) = 0$, if and only if the clause is satisfied. We define the dynamical system: $(-\nabla_\alpha V)_i = 1 + (d\alpha_i/dt)$. The energy function for the system is defined as

$$V = \sum_{m=1}^{M} A\left(K_{m,\text{osc}}(t,\alpha)\right)^2. \tag{1}$$

Here, M is the total number of clauses in the problem. $V = 0$ when all the clauses are satisfied and consequently corresponds to the solution of the SAT problem (if the problem is satisfiable). To evaluate the temporal evolution of the system energy, we calculate (dV/dt), which is given by

$$\frac{dV}{dt} = \sum_{i=1}^{N}\left(\frac{\partial V}{\partial \alpha_i}\right)\left(\frac{d\alpha_i}{dt}\right) + \frac{\partial V}{\partial t}. \tag{2}$$

Using (1) and the definition of $K_{m,\text{osc}}(t,\alpha)$, we can calculate $(\partial V/\partial t)$ as

$$\begin{aligned}\frac{\partial V}{\partial t} &= \sum_{m=1}^{M}\left(2AK_{m,\text{osc}}(t,\alpha)\frac{\partial\left(K_{m,\text{osc}}(t,\alpha)\right)}{\partial t}\right)\\ &= \sum_{m=1}^{M}\left(2AK_{m,\text{osc}}(t,\alpha)\left(\sum_{i=1}^{N}\frac{c_{mi}K_{m,\text{osc}}(t,\alpha)}{1-c_{mi}\cos(t+\alpha_i)}\right.\right.\\ &\qquad \left.\left.\times\sin(t+\alpha_i)\right)\right)\\ &= \sum_{i=1}^{N}\sum_{m=1}^{M}\left(2AK_{m,\text{osc}}(t,\alpha)\frac{c_{mi}K_{m,\text{osc}}(t,\alpha)}{1-c_{mi}\cos(t+\alpha_i)}\right.\\ &\qquad \left.\times\sin(t+\alpha_i)\right).\end{aligned} \tag{3}$$

Furthermore, $(\partial V/\partial \alpha_i)$ can be calculated as

$$\begin{aligned}\frac{\partial V}{\partial \alpha_i} &= \sum_{m=1}^{M}\left(2AK_{m,\text{osc}}(t,\alpha)\frac{\partial\left(K_{m,\text{osc}}(t,\alpha)\right)}{\partial \alpha_i}\right)\\ &= \sum_{m=1}^{M}\left(2AK_{m,\text{osc}}(t,\alpha)\frac{c_{mi}K_{m,\text{osc}}(t,\alpha)}{1-c_{mi}\cos(t+\alpha_i)}\right.\\ &\qquad \left.\sin(t+\alpha_i)\right).\end{aligned} \tag{4}$$

Substituting (4) into (3), $(\partial V/\partial t)$ can be expressed as

$$\frac{\partial V}{\partial t} = \sum_{i=1}^{N}\frac{\partial V}{\partial \alpha_i}. \tag{5}$$

By substituting the expression for $(\partial V/\partial t)$ from (5) into (2), (dV/dt) can be calculated as

$$\begin{aligned}\frac{dV}{dt} &= \sum_{i=1}^{N}\left(\frac{\partial V}{\partial \alpha_i}\right)\left(\frac{d\alpha_i}{dt}\right) + \frac{\partial V}{\partial t}\\ &= \sum_{i=1}^{N}\left(\frac{\partial V}{\partial \alpha_i}\right)\left(\frac{d\alpha_i}{dt}\right) + \sum_{i=1}^{N}\frac{\partial V}{\partial \alpha_i}\\ &= \sum_{i=1}^{N}\left(\frac{\partial V}{\partial \alpha_i}\right)\left(1+\frac{d\alpha_i}{dt}\right).\end{aligned} \tag{6a}$$

Furthermore, utilizing the system dynamics $(-\nabla_\alpha V)_i = 1 + (d\alpha_i/dt)$ (defined above), (6a) can be expressed as

$$\begin{aligned}\frac{dV}{dt} &= \sum_{i=1}^{N}\left(\frac{\partial V}{\partial \alpha_i}\right)\left(1+\frac{d\alpha_i}{dt}\right)\\ &= -\sum_{i=1}^{N}\left(1+\frac{d\alpha_i}{dt}\right)\left(1+\frac{d\alpha_i}{dt}\right)\\ &= -\sum_{i=1}^{N}\left(1+\frac{d\alpha_i}{dt}\right)^2.\end{aligned} \tag{6b}$$

It can be observed from the above equation that V is a decreasing function with time since $(dV/dt) \leq 0$. Consequently, this implies that the corresponding system dynamics will evolve to reduce the system energy (V).

To formulate the system dynamics $(d\alpha_i/dt)$, we express (dV/dt) as

$$\frac{dV}{dt} = \sum_{m=1}^{M}\left(2AK_{m,\text{osc}}(t,\alpha)\frac{d\left(K_{m,\text{osc}}(t,\alpha)\right)}{dt}\right) \tag{7a}$$

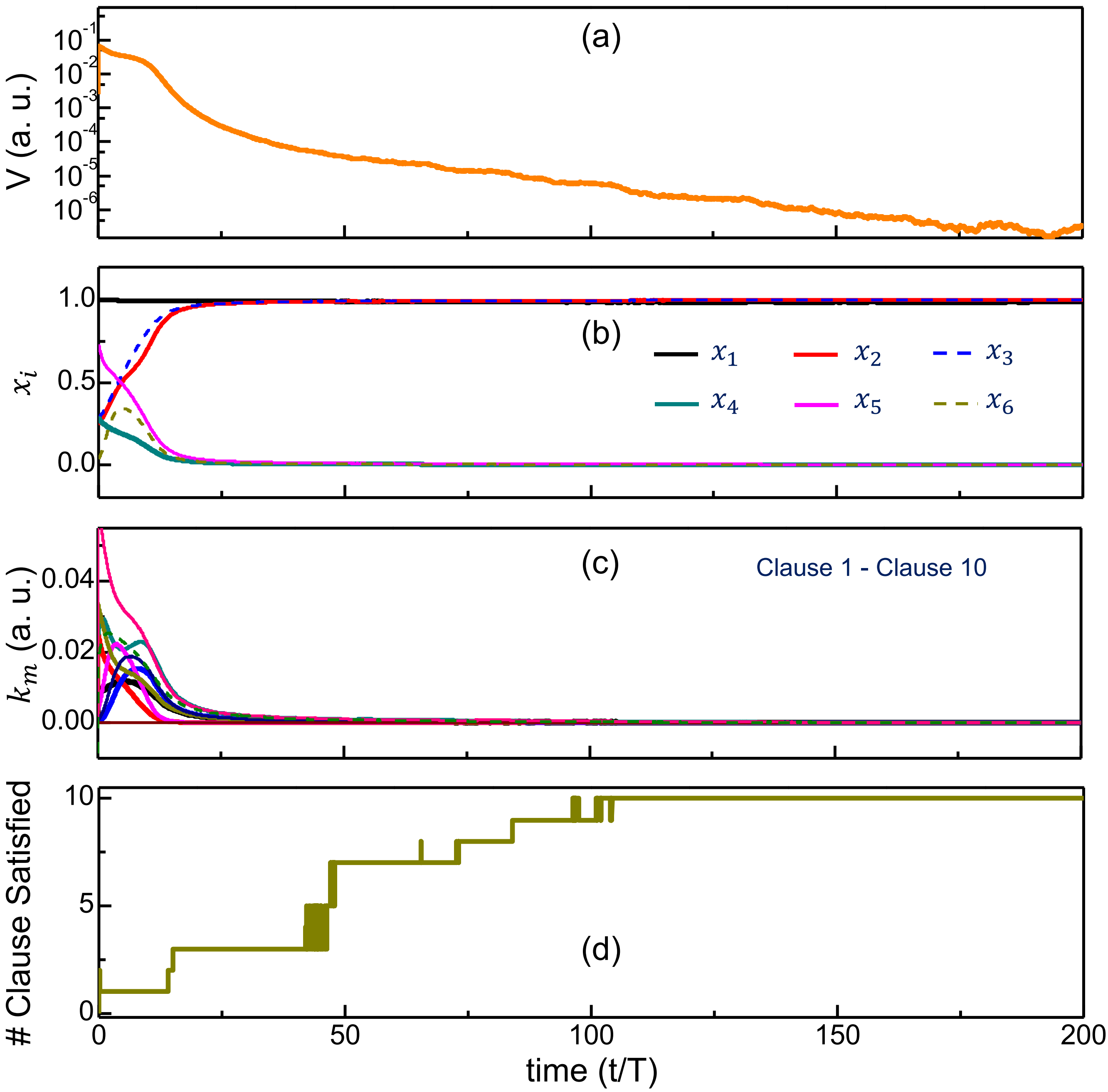

FIGURE 1. Evolution of (a) V, (b) x_i, (c) k_m, and (d) number of clauses satisfied with time for an illustrative 3-SAT problem with six variables and ten clauses that is computed using the System I dynamics. Here, $\omega = 2\pi$ is used such that $T = 1$. The simulation is performed using a stochastic differential equation framework (details in Supplement 1).

$$\frac{dV}{dt} = \sum_{m=1}^{M}\left(2AK_{m,\mathrm{osc}}\sum_{i=1}^{N}\left(c_{mi}\frac{K_{m,\mathrm{osc}}}{1-c_{mi}\cos\left(t+\alpha_i\right)} \times \sin\left(t+\alpha_i\right)\left(1+\frac{d\alpha_i}{dt}\right)\right)\right) \tag{7b}$$

$$\frac{dV}{dt} = \sum_{i=1}^{N}\left(\sum_{m=1}^{M}\left(2AK_{m,\mathrm{osc}}c_{mi}\frac{K_{m,\mathrm{osc}}}{1-c_{mi}\cos\left(t+\alpha_i\right)} \times \sin\left(t+\alpha_i\right)\left(1+\frac{d\alpha_i}{dt}\right)\right)\right). \tag{7c}$$

Equating (6b) and (7c), we get

$$-\left(1+\frac{d\alpha_i}{dt}\right) = \sum_{m=1}^{M}\left(2AK_{m,\mathrm{osc}}\left(t,\alpha\right)\frac{c_{mi}K_{m,\mathrm{osc}}\left(t,\alpha\right)}{1-c_{mi}\cos\left(t+\alpha_i\right)} \times \sin\left(t+\alpha_i\right)\right). \tag{8a}$$

The above equation can be rewritten as

$$\frac{d\alpha_i}{dt} = -\left(\sum_{m=1}^{M}\left(2AK_{m,\text{osc}}(t,\alpha)\,\frac{c_{mi}K_{m,\text{osc}}(t,\alpha)}{1-c_{mi}\cos(t+\alpha_i)}\right.\right.$$
$$\left.\left. sin\,(t+\alpha_i)\right)+1\right). \quad (8b)$$

The above equation describes the phase dynamics of the system, which compute the SAT solutions. The first term on the right-hand side (RHS) in (8b) represents the dissipative component of the system dynamics. The RHS in (8b) is 2π periodic in time. At a steady state, $V = 0$; $(d\alpha_i/dt) = -1$, which implies that $\alpha_i = -t + c_i$, with c_i being a constant offset in the time-varying phase that assumes a value in $\{0, \pi\}$ in a way that minimizes the total system energy and solves the SAT problem. A node i (defined by $((1+\cos(t+\alpha_i))/2)$) will eventually settle to 1 (when $c_i = 0$) or 0 (when $c_i = \pi$). Thus, the system is designed such that the out-of-phase feedback essentially "cancels" out the oscillations when the system achieves the ground-state energy. This corresponds to all the clauses being satisfied (if the problem is satisfiable). Fig. 1 illustrates the system dynamics for a representative SAT problem. Details of the simulation framework have been discussed in Supplement 1.

B. SYSTEM II

For this implementation, we formulate the system dynamics as $(-\nabla_\alpha E)_i = (d\alpha_i/dt)$, where E is the potential energy function of the system. In contrast to the prior approach, here, we will first define the system dynamics and subsequently aim to show that there exists a Lyapunov (energy) function which can directly be mapped to the solution to the Max-NAE-3-SAT problem. We consider a system whose dynamics are defined by

$$\begin{aligned}\frac{d\alpha_i}{dt} &= \sin(t+\alpha_i)\\ &\times\left(-\sum_{m=1}^{M}\left(2AK_{m,\text{osc}}(t,\alpha)\,\frac{c_{mi}K_{m,\text{osc}}(t,\alpha)}{1-c_{mi}\cos(t+\alpha_i)}\right)\right)\\ &- \sin(2t+2\alpha_i)\,A_s\cos(2t)\\ &\equiv \chi\,(t+\alpha_i(t))\,B_i(t) + \chi\,(2t+2\alpha_i(t))\,B^{(2)}(t). \end{aligned} \quad (9)$$

The above equation can be interpreted as a (sinusoidal) oscillator under perturbation $[B_i(t)]$, and second harmonic signal injection $B^{(2)}(t) \equiv A_s\cos(2t)$ which helps binarize the phases to $(0, \pi)$ [24], [25], as illustrated further on. $\chi(t+\alpha_i)$ and $\chi(2t+2\alpha_i)$ are the first and the second harmonics of the perturbation projection vectors (PPVs) of the oscillator, respectively. A and A_s are positive constants. It can be observed that the dynamics described in (9) are a modified version of the dynamics derived in (8b) for System I and essentially help us formulate the dynamics of System II. However, it must be emphasized here that we do not use the potential energy function V defined for System I since it does not decrease monotonically for the System II dynamics. Instead, using the dynamics described above, we will formulate a new energy function E whose ground state maps to the solution to the Max-NAE-3-SAT problem.

To define E, we first reformulate (9) in terms of the relative phase difference. Substituting the definition of $K_{m,\text{osc}}(t,\alpha)$, (9) can be rewritten as

$$\begin{aligned}\frac{d\alpha_i}{dt} &= -A\sin(t+\alpha_i)\\ &\times\sum_{m=1}^{M}\left(c_{mi}\left(\prod_{j=1;j\neq i}^{N}\left(\frac{1-c_{mj}\cos(t+\alpha_j)}{2}\right)\right)^2\right.\\ &\left.\times\left(\frac{1-c_{mi}\cos(t+\alpha_i)}{2}\right)\right)\\ &- \sin(2t+2\alpha_i)A_s\cos(2t). \end{aligned} \quad (10)$$

Expanding the above equation, we obtain

$$\begin{aligned}\frac{d\alpha_i}{dt} &= -\frac{A}{2}\left(\sum_{m=1}^{M}\left(c_{mi}\sin(t+\alpha_i)\right.\right.\\ &\left.\times\left(\prod_{j=1;j\neq i}^{N}\left(\frac{1-c_{mj}\cos(t+\alpha_j)}{2}\right)\right)^2\right)\\ &-\sum_{m=1}^{M}\left(\frac{1}{2}c_{mi}^2\sin(2(t+\alpha_i))\right.\\ &\left.\left.\times\left(\prod_{j=1;j\neq i}^{N}\left(\frac{1-c_{mj}\cos(t+\alpha_j)}{2}\right)\right)^2\right)\right)\\ &- \sin(2t+2\alpha_i)A_s\cos(2t). \end{aligned} \quad (11)$$

Furthermore, using trigonometric identities to express all the product terms in $(\prod_{j=1;j\neq i}^{N}(((1-c_{mj}\cos(t+\alpha_j))/2)))^2$ as the sum of $\cos(\cdot)$ terms, we rewrite the expression as

$$\sum_{\mu_N=-2}^{2}\cdots\sum_{\mu_2=-2}^{2}\sum_{\mu_1=-2}^{2} C_{\mu_1,\mu_2,\ldots,\mu_N;\neq\mu_i}$$
$$\times\cos\left(\left(\sum_{j=1;j\neq i}^{N}\left|c_{mj}\right|\mu_j\right)t + \sum_{j=1;j\neq i}^{N}\left|c_{mj}\right|\mu_j\alpha_j\right).$$

Using the approach described by Wang and Roychowdhury [26], a differential equation such as equation (11) can be formulated as a Multitime Partial Differential Equation (MPDE), wherein the fundamental oscillation is assumed to happen in fast time t_1 while the phases evolve in slow time t_2. Subsequently, (11) can then be approximated as

$$\frac{d\alpha_i}{dt}$$
$$= -A\sum_{m=1}^{M}\left(\sum_{\mu_N=-2}^{2}\cdots\sum_{\mu_2=-2}^{2}\sum_{\mu_1=-2}^{2} c_{mi}Q_1C^1_{\mu_1,\mu_2,\ldots,\mu_N;\neq\mu_i}\right.$$

$$\times \sin\left(\alpha_i - \sum_{j=1;j\neq i}^{N} \left|c_{mj}\right| \mu_j\alpha_j(t)\Big|_{Q_1}\right)\Bigg)\Bigg)$$
$$+A\sum_{m=1}^{M}\Bigg(\sum_{\mu_N=-2}^{2}\cdots\sum_{\mu_2=-2}^{2}\sum_{\mu_1=-2}^{2} c_{mi}^2 Q_2 C^2_{\mu_1,\mu_2,\ldots,\mu_N;\neq\mu_i}$$
$$\times \sin\left(2\alpha_i - \sum_{j=1;j\neq i}^{N} \left|c_{mj}\right| \mu_j\alpha_j(t)\Big|_{Q_2}\right)\Bigg)\Bigg)$$
$$-A_{s1}\sin(2\alpha_i). \quad (12)$$

Here, $Q_1 = 1$ when $\sum_{j=1;j\neq i}^{N}|c_{mj}|\mu_j = 1$, else $Q_1 = 0$; $Q_2 = 1$ when $\sum_{j=1;j\neq i}^{N}|c_{mj}|\mu_j = 2$, else $Q_2 = 0$. Additional details regarding the derivation of (12) can be found in Supplement 2. Remarkably, there is a Lyapunov function $E(\alpha(t))$ which can be defined for these dynamics as

$$E(\alpha(t)) = \sum_{i=1}^{N}\Bigg[-A\sum_{m=1}^{M}\sum_{\mu_N=-2}^{2}\cdots\sum_{\mu_2=-2}^{2}\sum_{\mu_1=-2}^{2} c_{mi}Q_1 \times C^1_{\mu_1,\mu_2,\ldots,\mu_N;\neq\mu_i} \cos\left(\alpha_i(t) - \sum_{j=1;j\neq i}^{N}\left|c_{mj}\right|\mu_j\alpha_j(t)\Big|_{\sum_{j=1;j\neq i}^{N}|c_{mj}|\mu_j=1}\right)\Bigg]$$
$$+\sum_{i=1}^{N}\Bigg[\frac{A}{2}\sum_{m=1}^{M}\sum_{\mu_N=-2}^{2}\cdots\sum_{\mu_2=-2}^{2}\sum_{\mu_1=-2}^{2} c_{mi}^2 Q_2 \times C^2_{\mu_1,\mu_2,\ldots,\mu_N;\neq\mu_i} \cos\left(2\alpha_i(t) - \sum_{j=1;j\neq i}^{N}\left|c_{mj}\right|\mu_j\alpha_j(t)\Big|_{\sum_{j=1;j\neq i}^{N}|c_{mj}|\mu_j=2}\right)\Bigg]$$
$$-\frac{A_{s1}}{2}\cos(2\alpha_i(t)). \quad (13)$$

Unlike V (defined for System I), $E(\alpha(t))$ is defined in terms of relative phase difference (and not in terms of the absolute phase). To show that $E(\alpha(t))$ is a decreasing function in time, that is, $((dE(\alpha(t)))/dt) \leq 0$, we express $((dE(\alpha(t)))/dt) = \sum_{i=1}^{N}[((\partial E(\alpha(t)))/(\partial\alpha_i(t)))((d\alpha_i(t))/dt)]$, where $((\partial E(\alpha(t)))/(\partial\alpha_i(t)))$ can be calculated as

$$\frac{\partial E(\alpha(t))}{\partial\alpha_i(t)} = A\sum_{m=1}^{M}\sum_{\mu_N=-2}^{2}\cdots\sum_{\mu_2=-2}^{2}\sum_{\mu_1=-2}^{2} c_{mi}Q_1 C^1_{\mu_1,\mu_2,\ldots,\mu_N;\neq\mu_i} \times \sin\left(\alpha_i - \sum_{j=1;j\neq i}^{N}\left|c_{mj}\right|\mu_j\alpha_j(t)\Big|_{Q_1}\right)$$
$$-\frac{2A}{2}\sum_{m=1}^{M}\sum_{\mu_N=-2}^{2}\cdots\sum_{\mu_2=-2}^{2}\sum_{\mu_1=-2}^{2} c_{mi}^2 Q_2 C^2_{\mu_1,\mu_2,\ldots,\mu_N;\neq\mu_i} \times \sin\left(2\alpha_i - \sum_{j=1;j\neq i}^{N}\left|c_{mj}\right|\mu_j\alpha_j(t)\Big|_{Q_2}\right) + A_{s1}\sin(2\alpha_i)$$
$$\equiv -\frac{d\alpha_i(t)}{dt}. \quad (14)$$

Thus,

$$\frac{\partial E(\alpha(t))}{\partial\alpha_i(t)} = -\frac{d\alpha_i(t)}{dt}. \quad (15)$$

It can be observed that (15) represents the system dynamics described earlier. Subsequently,

$$\frac{dE(\alpha(t))}{dt} = \sum_{i=1}^{N}\left[\left(\frac{\partial E(\alpha(t))}{\partial\alpha_i(t)}\right)\left(\frac{d\alpha_i(t)}{dt}\right)\right] \quad (16)$$
$$= -\sum_{i=1}^{N}\left[\left(\frac{d\alpha_i(t)}{dt}\right)^2\right] \leq 0. \quad (17)$$

Equation (17) reveals that $E(\alpha(t))$ is decreasing in time.

While (13) represents a general form, we will specifically define the energy E for the case when each clause contains exactly three literals and subsequently show that its ground state can be used to find the solution of the NAE-3-SAT problem. When a clause contains three literals (corresponding to variables i, j, k), E can be expressed as

$$E(\alpha) = \sum_{i=1}^{N}\Bigg(\pi A 2^{-2N+1}\sum_{\substack{m=1;i\neq j\neq k;c_{mi}\neq 0\\ c_{mj}\neq 0,c_{mk}\neq 0}}^{M}\Bigg(2c_{mi}c_{mj}\left(1+\frac{1}{2}c_{mk}^2\right)\times\cos(\alpha_i-\alpha_j)$$
$$+2c_{mi}c_{mk}\left(1+\frac{1}{2}c_{mj}^2\right)\cos(\alpha_i-\alpha_k)$$
$$+\frac{1}{2}c_{mi}c_{mj}c_{mk}^2\cos(\alpha_i+\alpha_j-2\alpha_k)$$
$$+\frac{1}{2}c_{mi}c_{mk}c_{mj}^2\cos(\alpha_i+\alpha_k-2\alpha_j)$$
$$+\frac{1}{8}c_{mi}^2c_{mk}^2\left(1+\frac{1}{2}c_{mj}^2\right)\cos(2\alpha_i-2\alpha_k)$$
$$+\frac{1}{2}c_{mi}^2c_{mj}c_{mk}\cos(2\alpha_i-\alpha_j-\alpha_k)$$
$$+\frac{1}{8}c_{mi}^2c_{mj}^2\left(1+\frac{1}{2}c_{mk}^2\right)\cos(2\alpha_i-2\alpha_j)\Bigg)\Bigg)$$
$$-\sum_{i=1}^{N}\frac{\pi A_s}{2}\cos(2\alpha_i). \quad (18)$$

The details of this derivation are shown in Supplement 3. The output variables are defined by the oscillator phases α which settle to $\{0, \pi\}$ owing to the second harmonic injection. We note that if a clause contains literals corresponding to only one or two distinct variables, $i \neq j \neq k$ constraint will not be imposed for that specific clause in (18). The specific nature

NAE-SAT Clause $x = (x_i, x_j, x_k)$	$E(\alpha_i, \alpha_j, \alpha_k)$ for a single clause $\left(\propto (T_{ij} + T_{jk} + T_{ki})\right)$			
	$\alpha = (0,0,0)$ $\equiv x = (1,1,1)$	$\alpha = (0,0,\pi)$ $\equiv x = (1,1,0)$	$\alpha = (0,\pi,\pi)$ $\equiv x = (1,0,0)$	$\alpha = (\pi,\pi,\pi)$ $\equiv x = (0,0,0)$
$C_{NAE} = (x_i \vee x_j \vee x_k) . (\overline{x_i} \vee \overline{x_j} \vee \overline{x_k})$	$\left(\frac{189}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 0$	$\left(\frac{-51}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 1$	$\left(\frac{-51}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 1$	$\left(\frac{189}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 0$
$C_{NAE} = (x_i \vee x_j \vee \overline{x_k}) . (\overline{x_i} \vee \overline{x_j} \vee x_k)$	$\left(\frac{-51}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 1$	$\left(\frac{189}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 0$	$\left(\frac{-51}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 1$	$\left(\frac{-51}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 1$
$C_{NAE} = (x_i \vee \overline{x_j} \vee \overline{x_k}) . (\overline{x_i} \vee x_j \vee x_k)$	$\left(\frac{-51}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 1$	$\left(\frac{-51}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 1$	$\left(\frac{189}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 0$	$\left(\frac{-51}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 1$
$C_{NAE} = (\overline{x_i} \vee \overline{x_j} \vee \overline{x_k}) . (x_i \vee x_j \vee x_k)$	$\left(\frac{189}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 0$	$\left(\frac{-51}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 1$	$\left(\frac{-51}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 1$	$\left(\frac{189}{256}\right)\pi A - \frac{3}{2}\pi A_s$ $C_{NAE} = 0$

FIGURE 2. $E(\alpha_i, \alpha_j, \alpha_k)$ for a single NAE-3-SAT clause computed for different combinations of the literals. It can be observed that the energy is minimum only when the NAE-SAT clause is satisfied. Only selected combinations have been shown here; a detailed table considering all combinations has been shown in Supplement 4.

of the arguments of the cos(·) functions shown in (18) arise from the characteristics of the cross-correlation operation performed in (12). The corresponding dynamics associated with (18) can be defined as

$$\begin{aligned}\frac{d\alpha_i}{dt} &= \pi A 2^{-2N+1} \sum_{\substack{m=1; i\neq j\neq k; c_{mi}\neq 0\\ c_{mj}\neq 0, c_{mk}\neq 0}}^{M} \\ &\Bigg(2c_{mi}c_{mj}\left(1+\frac{1}{2}c_{mk}^2\right)\sin(\alpha_i-\alpha_j) \\ &+2c_{mi}c_{mk}\left(1+\frac{1}{2}c_{mj}^2\right)\sin(\alpha_i-\alpha_k) \\ &+\frac{1}{2}c_{mi}c_{mj}c_{mk}^2\sin(\alpha_i+\alpha_j-2\alpha_k) \\ &+\frac{1}{2}c_{mi}c_{mk}c_{mj}^2\sin(\alpha_i+\alpha_k-2\alpha_j) \\ &+\frac{1}{4}c_{mi}^2c_{mk}^2\left(1+\frac{1}{2}c_{mj}^2\right)\sin(2\alpha_i-2\alpha_k) \\ &+\frac{1}{4}c_{mi}^2c_{mj}^2\left(1+\frac{1}{2}c_{mk}^2\right)\sin(2\alpha_i-2\alpha_j) \\ &+c_{mi}^2c_{mj}c_{mk}\sin(2\alpha_i-\alpha_j-\alpha_k)\Bigg) \\ &-\pi A_s\sin(2\alpha_i). \qquad (19)\end{aligned}$$

The above equation describes the phase dynamics of the system which compute the solution to the NAE-3-SAT Problem. The second harmonic injection signal $-\sum_{i=1}^{N}(\pi A_s/2)\cos(2\alpha_i)$ (for an appropriate injection strength A_s) essentially lowers the energy of the system corresponding to $\alpha \in \{0, \pi\}$, since the minimization of $-\sum_{i=1}^{N}(\pi A_s/2)\cos(2\alpha_i)$ to $-\sum_{i=1}^{N}(\pi A_s/2)$ forces the oscillators to take these binary phase values; this concept was also exploited in designing oscillator-based Ising machines [26]. Thus, when the system achieves ground state, each 2α term in (18) induces a phase difference of 0 or 2π, and hence the corresponding $\cos(\alpha_i + \alpha_j - 2\alpha_k)$ term can be simplified to $\cos(\alpha_i + \alpha_j)$. Furthermore, $\cos(2\alpha_i - 2\alpha_j)$ will take constant values at these specific phase points (represented as C). Additionally, $c_{mi}^2 = c_{mj}^2 = c_{mk}^2 = 1$. Thus, at these discrete phase points, $E(\alpha)$ for a problem in which each clause consists of three literals can be reduced to

$$\begin{aligned}E(\alpha) = \pi A 2^{-2N+1} \sum_{i=1}^{N} &\sum_{\substack{m=1; i\neq j\neq k; c_{mi}\neq 0\\ c_{mj}\neq 0, c_{mk}\neq 0}}^{M} \\ &\Big(3c_{mi}c_{mj}\cos(\alpha_i-\alpha_j) \\ &+3c_{mi}c_{mk}\cos(\alpha_i-\alpha_k)\end{aligned}$$

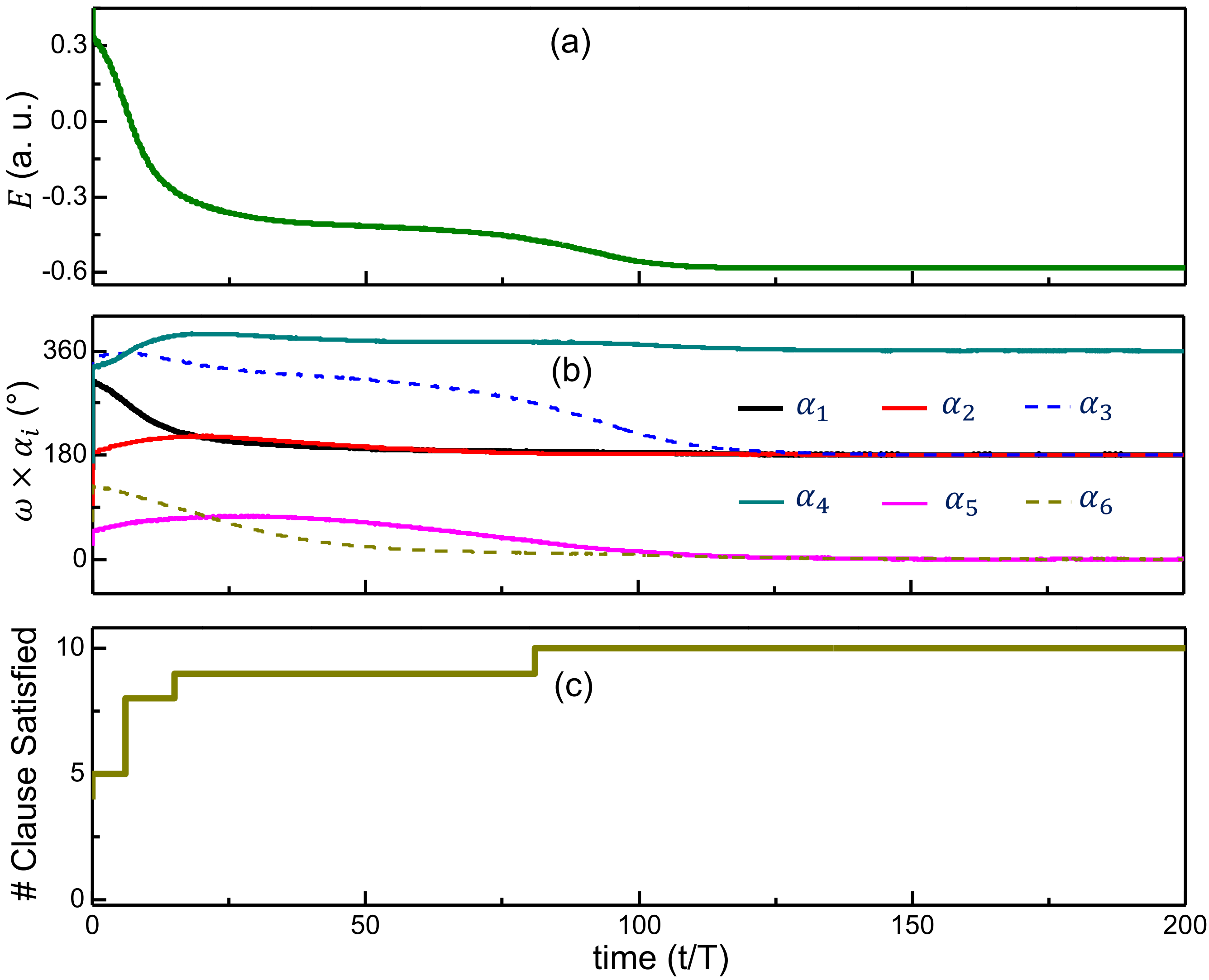

FIGURE 3. Evolution of (a) E, (b) $\omega\alpha_i$, and (c) number of satisfied clauses with time, for an illustrative NAE-3-SAT problem with six variables and ten clauses that is solved using the System II dynamics. The dynamics are obtained by simulating (10). In this simulation, $\omega = 2\pi$ is used such that $T = 1$. Details of the SDE simulation are described in Supplement 1.

$$+\frac{1}{2}c_{mi}c_{mj}\cos\left(\alpha_i+\alpha_j\right)$$
$$+\frac{1}{2}c_{mi}c_{mk}\cos\left(\alpha_i+\alpha_k\right)$$
$$+\frac{1}{2}c_{mj}c_{mk}\cos\left(\alpha_j+\alpha_k\right)\Bigg)$$
$$+C-\sum_{i=1}^{N}\frac{\pi A_s}{2}\cos\left(2\alpha_i\right). \tag{20}$$

Rearranging the above equation, we obtain

$$E\left(\alpha\right)=\pi A2^{-2N+1}\sum_{\substack{m=1;i\neq j\neq k;c_{mi}\neq 0\\ c_{mj}\neq 0,c_{mk}\neq 0}}^{M}\sum_{i=1}^{N}\Bigg(3c_{mi}c_{mj}\cos\left(\alpha_i-\alpha_j\right)$$
$$+3c_{mi}c_{mk}\cos\left(\alpha_i-\alpha_k\right)$$
$$+\frac{1}{2}c_{mi}c_{mj}\cos\left(\alpha_i+\alpha_j\right)$$
$$+\frac{1}{2}c_{mi}c_{mk}\cos\left(\alpha_i+\alpha_k\right)$$
$$+\frac{1}{2}c_{mj}c_{mk}\cos\left(\alpha_j+\alpha_k\right)\Bigg)+C$$
$$-\sum_{i=1}^{N}\frac{\pi A_s}{2}\cos\left(2\alpha_i\right)$$
$$=\sum_{m=1}^{M}\beta_m\left(\alpha_i,\alpha_j,\alpha_k\right)+C-\sum_{i=1}^{N}\frac{\pi A_s}{2}\cos\left(2\alpha_i\right)$$
$$=\sum_{m=1}^{M}\beta_m\left(\alpha_i,\alpha_j,\alpha_k\right)+C-C_s. \tag{21}$$

Both C and $C_s(=\sum_{i=1}^{N}(\pi A_s/2)\cos(2\alpha_i))$ are constants at the phase points, $\alpha\in\{0,\pi\}$. Consequently, $E(\alpha)$ is minimized when $\sum_{m=1}^{M}\beta_m(\alpha_i,\alpha_j,\alpha_k)$ is minimum. Now, for a single clause consisting of three literals corresponding to three variables x_i, x_j, x_k (here, x_i, x_j, x_k can appear in normal or negated form in the clause), $\beta_m(\alpha_i,\alpha_j,\alpha_k)$ can be written as

$$\begin{aligned}\beta_m\left(\alpha_i,\alpha_j,\alpha_k\right)&\\ =\pi A2^{-2N+1}&\\ \times\Bigg(c_{mi}c_{mj}&\left(6\cos\left(\alpha_i-\alpha_j\right)+\frac{3}{2}\cos\left(\alpha_i+\alpha_j\right)\right)\\ +c_{mj}c_{mk}&\left(6\cos\left(\alpha_j-\alpha_k\right)+\frac{3}{2}\cos\left(\alpha_j+\alpha_k\right)\right)\\ +c_{mk}c_{mi}&\left(6\cos\left(\alpha_k-\alpha_i\right)+\frac{3}{2}\cos\left(\alpha_k+\alpha_i\right)\right)\Bigg)\\ +C-C_s&\end{aligned}\tag{22a}$$

$$\beta_m\left(\alpha_i,\alpha_j,\alpha_k\right)=\pi A2^{-2N+1}\left(T_{ij}+T_{jk}+T_{ki}\right)+C-C_s\tag{22b}$$

where

$$T_{ij}=c_{mi}c_{mj}\left(6\cos\left(\alpha_i-\alpha_j\right)+\frac{3}{2}\cos\left(\alpha_i+\alpha_j\right)\right).\tag{23}$$

Equation (22b) reveals that $\beta_m(\alpha_i,\alpha_j,\alpha_k)$ is minimum when $T_{ij}+T_{jk}+T_{ki}$ is minimum.

At the phase points $\alpha_i,\alpha_j,\alpha_k\in\{0,\pi\}$, T_{ij}, T_{jk}, T_{ki}, and $T_{ij}+T_{jk}+T_{ki}$ are binary in nature and exhibit the property that $T_{ij}+T_{jk}+T_{ki}$, and thus $\beta_m(\alpha_i,\alpha_j,\alpha_k)$, is minimized when $(x_i\oplus x_j)\vee(x_j\oplus x_k)\vee(x_k\oplus x_i)=1$. This is illustrated in the following paragraph. However, first, we simplify $(x_i\oplus x_j)\vee(x_j\oplus x_k)\vee(x_k\oplus x_i)$ as

$$\begin{aligned}&\left(x_i\oplus x_j\right)\vee\left(x_j\oplus x_k\right)\vee\left(x_k\oplus x_i\right)\\ &\quad=\left(x_i\overline{x}_j\vee\overline{x}_ix_j\right)\vee\left(x_j\overline{x}_k\vee\overline{x}_jx_k\right)\\ &\qquad\vee\left(x_k\overline{x}_i\vee\overline{x}_kx_i\right)\\ &\quad=\left(x_i\vee x_j\vee x_k\right).\left(\overline{x}_i\vee\overline{x}_j\vee\overline{x}_k\right).\end{aligned}\tag{24}$$

Remarkably, the above equation corresponds to a clause of the NAE-3-SAT problem. The terms within the first parentheses in (24) implement the standard SAT constraint, while the terms in the second parentheses implement the constraint that at least one literal must be false. Here, we again emphasize that x_i can appear in both normal or negated form; for example, if the clause is $(x_i\vee\overline{x}_j\vee x_k)$, the corresponding NAE-SAT clause will be $(x_i\vee\overline{x}_j\vee x_k).(\overline{x}_i\vee x_j\vee\overline{x}_k)$.

To show that the energy corresponding to a clause, $T_{ij}+T_{jk}+T_{ki}$, is minimized when an NAE-3-SAT clause is satisfied, we consider the table in Fig. 2. It can be observed from the table that an NAE-SAT clause is satisfied only when $T_{ij}+T_{jk}+T_{ki}$ assumes the minimum value. Considering the inherent symmetry in the expression, only selected cases have been presented here. However, the complete table has been shown in Supplement 4.

Consequently, as the system evolves toward the global minimum of $E=\sum_{m=1}^{M}\beta_m(\alpha_i,\alpha_j,\alpha_k)+C-C_s$, it aims to maximize the number of satisfied NAE-3-SAT clauses (defined by $(x_i\vee x_j\vee x_k).(\overline{x}_i\vee\overline{x}_j\vee\overline{x}_k)$). In other words, it computes the solution to the Max-NAE-3-SAT problem. Fig. 3 shows the solution for an illustrative NAE-3-SAT problem having six variables and ten clauses. The oscillator dynamics are simulated using (10). However, (19) can also be used to compute the solution, as shown in Supplement 5.

III. CONCLUSION

In summary, we proposed two oscillator-inspired dynamical systems and demonstrated their ability to compute two different categories of the Boolean SAT problem. Our work helps advance the understanding of how oscillator-inspired computational approaches can be designed and evaluated, specifically, for solving combinatorial optimization problems that have objective functions with degrees greater than two.

Our oscillator-based formulation complements the level-based analog dynamical systems that have been proposed for solving SAT [9]. We note that in the Ising machine/Hopfield networks, there are well-known formulations that lead to an oscillatory version of a level-based analog dynamical system. Exploring such relationships between the level-based and oscillator formulations for the above SAT solvers is an interesting challenge.

We also emphasize that the present work focuses on developing computational models, inspired by the principles of oscillator-based dynamical systems, and results in algorithms to solve multiple forms of Boolean SAT. The physical implementation of such models in hardware, where the system "naturally computes" the dynamics, can provide additional performance benefits and presents a potential future direction for this work.

The results presented here help expand the application of oscillator-based dynamical systems and the physics-inspired computing approach to a larger class of combinatorial optimization problems.

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