Majority Determination in Binary-Valued Communication Networks

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***Abstract*—Majority determination is one of the fundamental problems in multiagent systems. It aims to cooperatively and distributedly determine the majority opinion of agents in a network, where the agents initially vote "in favor of" or "opposed" a proposal. An interesting aspect of this issue is to clarify the lowest resolution of communication required among the agents to determine the majority. In this study, we address this problem with binary-valued communication. To overcome the limitation of the finite capacity of communication channels, we exploit randomized communication, i.e., sending binary values (0 or 1), which are selected according to a probabilistic distribution. Based on this idea, we develop consensus-type algorithms that approximately solve the problem with an arbitrarily prescribed accuracy.**

***Index Terms*—Binary-valued communication, decision making, majority determination, multiagent system.**

I. Introduction

MAJORITY determination is one of the fundamental problems in multiagent systems. The problem is quite simple: When the agents initially vote "in favor of" or "opposed" a proposal, how can they cooperatively and distributedly determine the majority opinion? Such cooperative decision-making is required for several applications, especially in distributed systems, e.g., system-level diagnosis, fault tolerance enhancement, database management, and fault-local mending [1]. Moreover, it typically appears in the decision-making algorithms of duplicated systems.

Majority determination is closely related to the so-called average consensus [2]. If the opinions *in favor* and *opposed* are encoded as 1 and 0, respectively, it is obvious that the majority is determined by comparing the average of the encoded opinions with the value 0.5. Therefore, a promising method for majority determination is to construct an average consensus algorithm for multiagent systems. On the other hand, it should be noted that majority determination is a weaker objective than average consensus.

Here, we are interested in the *lowest* resolution of communication required among agents for majority determination. This is motivated by the fact that the capacity of communication channels is finite in practice and the finite capacity becomes a considerable limitation when the agents have a small memory.

Several solutions to this issue have been obtained as results of *quantized* consensus control. For example, consensus control with quantized states has been examined in [2]–[4]. In these studies, quantized states were exchanged among agents, which resulted in quantized communication. On the other hand, consensus control with real-valued states and quantized communication has been addressed in [5]–[13]. From the viewpoint of the number of quantization levels, i.e., the resolution of communication, the above studies can be classified into [3], [4], [6], [8]–[10], and [13] for an unbounded number, [2], [7], and [11] for a bounded number greater than or equal to three, [5] for a bounded number greater than or equal to two, and [12] for two. Since at least two quantization levels are needed for consensus (and any other purpose), the results in [5] and [12] are notable in terms of the lowest resolution. However, the method employed in [5] is not applicable to the majority determination problems because it achieves the consensus to a quantized value, i.e., 0 or 1 in the case of binary-valued communication, which is not always the majority opinion (though the expected value of the resulting consensus is equal to the average of the initial states). Meanwhile, the method in [12] does not guarantee that the states converge to a constant value, due to which it remains unclear whether the majority opinion has been determined.

On the other hand, majority determination with quantized communication has been studied in [14]–[17]. In [14] and [15], binary-valued communication was considered, but their methods were limited to *complete* graphs or a special class of initial states to guarantee the consensus. Meanwhile, the methods in [16] and [17] were based on four-level communication.

Overall, the existing methods require at least three-level communication (e.g., [2], [7] for three-level communication), which motivates us to clarify the possibility of the majority determination with lower resolution of communication.

Therefore, in this study, we address the majority determination problem for multiagent systems where each agent has a 1-D memory and communicates through a binary-valued signal. To overcome the limitation of communication under the minimal

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memory, we exploit randomized communication, i.e., sending either 0 or 1, selected according to a probabilistic distribution. Based on this idea, we propose consensus-type algorithms, and clarify the relationship among the number of agents, the distribution of opinions, and the accuracy of the method. Furthermore, the previous framework is applied to anomaly detection by a sensor network, which demonstrates its potential for practical use.

Finally, we remark on two things about our contribution. First, if each agent has a sufficiently large memory, then it does not matter that the communication is binary valued. This is because each agent can send any information as a sequence of binary numbers to an agent, and the receiver agent can save the sequence in its large memory. In contrast, in our case with a limited amount of memory, binary-valued communication becomes a considerable limitation. Second, the idea of randomized communication has been employed in some earlier studies [5], [9]–[11]. However, to the best of our knowledge, a method for majority determination with binary-valued communication has never been presented. In this study, we have established a framework with binary-valued communication, which is used to reveal the lowest resolution of communication. In this sense, our result is distinguished from the others.

This article is based on our preliminary version [18], published in a conference proceedings. This journal version contains a full explanation about our result and complete proofs, which are omitted in the conference version.

Notation: Let $\mathbf{R}$, $\mathbf{R}_+$, and $\mathbf{Z}_{0+}$ be a set of real numbers, a set of positive real numbers, and a set of non-negative integers, respectively. We use $0_{n\times m}$ and $\mathbf{1}_n$ to represent an $n \times m$ zero matrix and an n-dimensional vector whose elements are all one, respectively. For a matrix M, $\mathrm{round}(M)$ represents the matrix obtained by rounding off each of its elements to an integer. The matrix M is said to be *irreducible* if there exists no permutation matrix P such that $P^{-1}MP$ is a block upper-triangular matrix. The cardinality of a finite set $\mathbf{S}$ is expressed by $|\mathbf{S}|$. The probability of an event A to occur is represented by $P[A]$. The expectation and variance of a random variable x are denoted by $E[x]$ and $V[x]$, respectively. The conditional expectation of x with respect to a random variable y is expressed as $E[x|y]$. Furthermore, the conditional expectation of x with respect to a filtration $\mathcal{F}$ is denoted by $E[x|\mathcal{F}]$. For a function $f(x)$ of the random variable x, the expectation of $f(x)$ is often denoted by $E_x[f(x)]$ to stress that $E_x[f(x)]$ is the expectation of $f(x)$ with respect to the random variable x. Finally, "w.p.p" stands for "with probability p" in this article.

II. Problem Formulation

A. System Description

Consider a multiagent system with n agents in which the dynamics of agent $i \in \{1, 2, \ldots, n\}$ is governed by

$$\begin{cases} x_i(t+1) = f_i(x_i(t), u_i(t)) \\ \quad y_i(t) = g_i(x_i(t), u_i(t)) \end{cases} \tag{1}$$

where $x_i(t) \in \mathbf{R}$ is the state, $u_i(t) \in \mathbf{Z}_{0+}$ is the input, and $y_i(t) \in \{0, 1\}$ is the *binary*-valued output, and $f_i : \mathbf{R} \times \mathbf{Z}_{0+} \to \mathbf{R}$ and $g_i : \mathbf{R} \times \mathbf{Z}_{0+} \to \{0, 1\}$ are functions.

Agent i receives the information of the sum of the outputs of the neighbors as follows:

$$u_i(t) = \sum_{j\in\mathbf{N}_i} y_j(t) \tag{2}$$

where $\mathbf{N}_i \subseteq \{1, 2, \ldots, n\}$ is the index set of the neighbors of agent i, i.e., the agents connected to agent i. Note that, in general, under (2), agent i cannot know the outputs of individual neighbors, which protects the privacy of the neighbors.

The network structure of the aforementioned multiagent system is represented by a directed graph $G = (\mathbf{V}, \mathbf{E})$ with the node set $\mathbf{V} := \{1, 2, \ldots, n\}$ and the edge set $\mathbf{E} := \{(j, i) \in \mathbf{V} \times \mathbf{V} \mid j \in \mathbf{N}_i\}$. We use $\Delta \in \{0, 1, \ldots, n\}$ to express the maximum in-degree of the network structure G, i.e., $\Delta := \max_{i\in\mathbf{V}} |\mathbf{N}_i|$.

B. Majority Determination Problem

In the multiagent system, we assume that the agents initially vote *opposed* or *in favor* for some proposal, and the opinion of agent i is stored in its initial state $x_i(0)$ as a binary value (0 or 1). The values 0 and 1 represent *opposed* and *in favor*, respectively.

The sets of agents who voted *opposed* and *in favor* are denoted by $\mathbf{I}_0 \subseteq \mathbf{V}$ and $\mathbf{I}_1 \subseteq \mathbf{V}$, respectively, i.e., $\mathbf{I}_0 := \{i \in \mathbf{V} | x_i(0) = 0\}$, $\mathbf{I}_1 := \{i \in \mathbf{V} | x_i(0) = 1\}$, and $|\mathbf{I}_0| + |\mathbf{I}_1| = n$. The sets $\mathbf{I}_0$ and $\mathbf{I}_1$ are called the opposition group and the supportive group, respectively.

We consider the problem of determining the majority opinion for $x_i(0)$ $(i = 1, 2, \ldots, n)$, i.e., determining whether $|\mathbf{I}_0| > |\mathbf{I}_1|$ or $|\mathbf{I}_0| < |\mathbf{I}_1|$, in a distributed manner. The problem is formulated as follows.

Problem 1: For the multiagent system given by (1) and (2), assume that the opinions $x_i(0) \in \{0, 1\}$ $(i = 1, 2, \ldots, n)$ are fixed (but unknown) and n is odd ($|\mathbf{I}_0| > |\mathbf{I}_1|$ or $|\mathbf{I}_0| < |\mathbf{I}_1|$). Find functions f_i and g_i $(i = 1, 2, \ldots, n)$ such that the following statements hold.

(S1) f_i $(i = 1, 2, \ldots, n)$ are of the form $f(x_i(t), u_i(t), |\mathbf{N}_i|)$ with a common function $f : \mathbf{R} \times \mathbf{Z}_{0+} \times \{0, 1, \ldots, n\} \to \mathbf{R}$, and also g_i $(i = 1, 2, \ldots, n)$ are of the form $g(x_i(t), u_i(t), |\mathbf{N}_i|)$ with a common function $g : \mathbf{R} \times \mathbf{Z}_{0+} \times \{0, 1, \ldots, n\} \to \{0, 1\}$.

(S2) There exists a number $\gamma \in \mathbf{R}$ satisfying

$$\begin{cases} \limsup_{t\to\infty} x_i(t) < \gamma, & \text{if } |\mathbf{I}_0| > |\mathbf{I}_1| \\ \liminf_{t\to\infty} x_i(t) > \gamma, & \text{if } |\mathbf{I}_0| < |\mathbf{I}_1| \end{cases} \tag{3}$$

for every $i \in \mathbf{V}$. ■

In this problem, (S1) specifies that the agents basically follow the same type of algorithm for scalability reasons. However, it may be noted that each agent is connected to a different number of neighbors, as mentioned in Section II-A. Consequently, it can design f_i and g_i depending on the number $|\mathbf{N}_i|$ of neighbors. Such a specification is typical in control problems in multiagent systems, such as consensus control and coverage control. Meanwhile, (S2) is concerned with determining the majority opinion. If (S2) holds, the states of all the agents remain either less than or more than γ after a while. Thus, the majority opinion can be determined by checking the relationship between the state of any agent and γ after a certain time, e.g., after a certain period when the state remains either less than or more than γ.

III. Distributed Algorithm for Majority Determination

A. Algorithm Based on Randomized Communication

If a real-valued output is available for (1), Problem 1 can be easily solved by constructing an average consensus algorithm. In fact, the average consensus for $x_i(0) \in \{0, 1\}$ $(i = 1, 2, \dots, n)$ indicates (3) for $\gamma = 0.5$. However, the output has a binary value in our problem. Thus, we propose an algorithm with *randomized* communication, which results in an approximation of the typical consensus algorithm.

The proposed algorithm is given as follows:

$$\begin{cases} x_i(t+1) = & x_i(t) + \varepsilon(t)\left(u_i(t) - |\mathbf{N}_i| x_i(t)\right) \\ \quad y_i(t) \quad = & \begin{cases} 0 \text{ w.p. } 1 - x_i(t) \\ 1 \text{ w.p. } x_i(t) \end{cases} \end{cases} \tag{4}$$

where $\varepsilon(t) \in [0, \Delta]$ is the time-varying gain of this algorithm, $u_i(t)$ is the neighbors' information given in (2), and $y_i(t)$ is a random variable obtained from the Bernoulli distribution. Note that (S1) in Problem 1 holds for this algorithm. This algorithm is called the majority determination algorithm.

This algorithm is in a similar form as that of the typical consensus algorithm. In fact, (2) is equivalent to

$$u_i(t) - |\mathbf{N}_i| x_i(t) = \sum_{j \in \mathbf{N}_i} (y_j(t) - x_i(t))$$

which enables us to rewrite the state equation in (4) as

$$x_i(t+1) = x_i(t) + \varepsilon(t) \sum_{j \in \mathbf{N}_i} (y_j(t) - x_i(t)).$$

However, it should be noted that $y_j(t)$ $(j \in \mathbf{N}_i)$ are binary-valued and random.

In this algorithm, $x_i(t)$ must take a value from [0,1] because the probability distribution of the random variable $y_j(t)$ is specified by $x_i(t)$ and $1 - x_i(t)$ in (4). Under the aforementioned condition $\varepsilon(t) \in [0, \Delta]$, it is guaranteed that $x_i(t) \in [0, 1]$ for every $i \in \mathbf{V}$ and $t \in \{1, 2, \dots\}$.

Lemma 1: Consider the multiagent system with the majority determination algorithm in (4). If $x_i(0) \in \{0, 1\}$ for every $i \in \mathbf{V}$, then $x_i(t) \in [0, 1]$ for every $i \in \mathbf{V}$ and $t \in \{1, 2, \dots\}$.

Proof: This lemma can be proved by mathematical induction. In particular, by noting that $x_i(0) \in \{0, 1\}$ $(i = 1, 2, \dots, n)$, we show that $x_i(t+1) \in [0, 1]$ $(i = 1, 2 \dots, n)$ under $x_i(t) \in [0, 1]$ $(i = 1, 2 \dots, n)$.

For an arbitrary $i \in \mathbf{V}$, consider the state equation in (4). If $x_j(t) \in [0, 1]$ for every $j \in \{1, 2, \dots, n\}$, then $u_i(t) \le |\mathbf{N}_i|$ from (2). Moreover, $\varepsilon(t) \in [0, \Delta]$ implies that $\varepsilon(t)|\mathbf{N}_i| \le 1$. Applying these facts to the state equation in (4), we obtain

$$\begin{aligned} x_i(t+1) &= x_i(t) + \varepsilon(t)\left(u_i(t) - |\mathbf{N}_i| x_i(t)\right) \\ &\le x_i(t) + \varepsilon(t)\left(|\mathbf{N}_i| - |\mathbf{N}_i| x_i(t)\right) \\ &\le x_i(t) + (1 - x_i(t)) \\ &\le 1. \end{aligned}$$

On the other hand

$$\begin{aligned} x_i(t+1) &= x_i(t) + \varepsilon(t)\left(u_i(t) - |\mathbf{N}_i| x_i(t)\right) \\ &= (1 - \varepsilon(t)|\mathbf{N}_i|) x_i(t) + \varepsilon(t) u_i(t) \\ &\ge 0 \end{aligned}$$

because $\varepsilon(t)|\mathbf{N}_i| \le 1$ as shown earlier, $x_i(t) \in [0, 1]$, $\varepsilon(t) > 0$, $|\mathbf{N}_i| > 0$, and $u_i(t) \ge 0$ by definition.

Thus, $x_i(t+1) \in [0, 1]$. ■

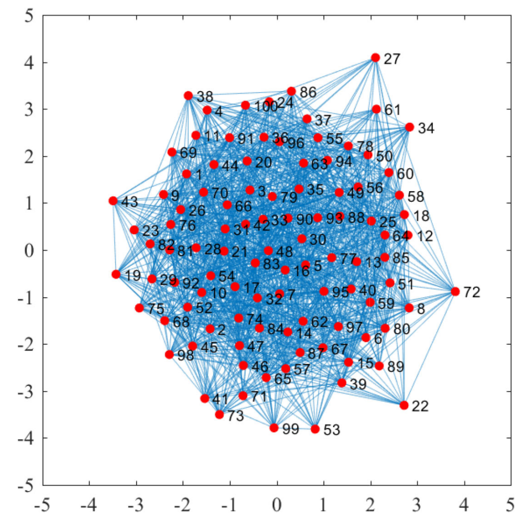

Fig. 1. Network structure in example ($n = 100$ and $\Delta = 42$).

B. Example

Before showing the theoretical results for the majority determination algorithm in (4), we demonstrate the performance of the algorithm through simulations.

Consider a multiagent system with $n = 100$ and network structure G, as shown in Fig. 1. For G, the maximum in-degree Δ is 42 and the average in-degree $(1/100)\sum_{i=1}^{100} |\mathbf{N}_i|$ is 27.74. This graph is randomly generated in such a way that its adjacent matrix is given by $\mathrm{round}(M + M^\top)$ for a random matrix $M \in \mathbf{R}^{100\times 100}$ whose diagonal elements are zero and off-diagonal elements are independently drawn from the uniform distribution on [0, 0.4]. Note that the matrix $M + M^\top$ is symmetric and the graph is undirected.

For this system, we apply the majority determination algorithm in (4) with the gain

$$\varepsilon(t) = \frac{1.5}{\Delta(2 + t)}.$$

Fig. 2 shows the result for the case where 59% of the agents are in favor and 41% oppose. It is clear that the states $x_i(t)$ $(i = 1, 2, \dots, 100)$ converge to a value between 0.5 and 0.6. On the other hand, Fig. 3 shows the result for the case where 37% agents in favor and 63% oppose. Here, the states $x_i(t)$ $(i = 1, 2, \dots, 100)$ converge to a value between 0.35 and 0.45.

These results suggest that (3) is satisfied for $\gamma = 0.5$, and the algorithm solves Problem 1 in the earlier cases.

IV. Performance Analysis of Majority Determination Algorithms

Here, we present theoretical results for validating the performance of the majority determination algorithm in (4).

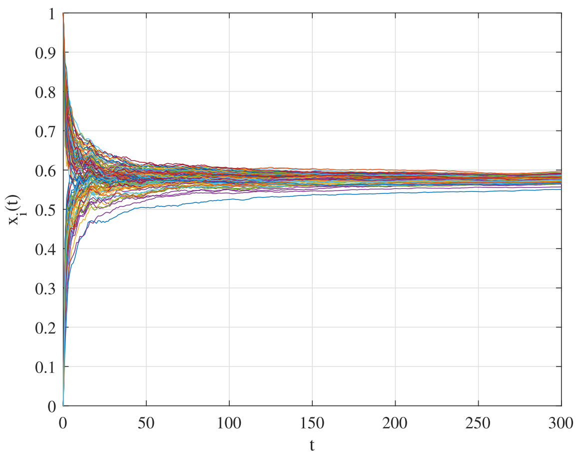

Fig. 2. Majority determination for the case where 59% of the agents are in favor.

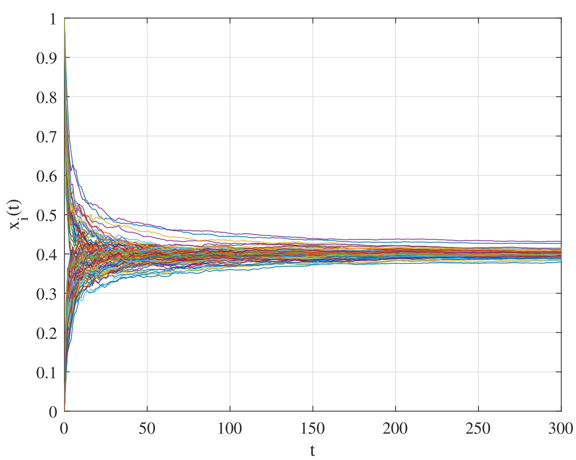

Fig. 3. Majority determination for the case where 37% of the agents are in favor.

A. Collective Dynamics and Properties

Let us first derive the collective dynamics of the multiagent system. For the network structure G, the adjacency matrix, the degree matrix, and the graph Laplacian are denoted by $A \in \mathbf{R}^{n\times n}$, $D \in \mathbf{R}^{n\times n}$, and $L \in \mathbf{R}^{n\times n}$, respectively, i.e., $L = D - A$ [19]. Let also $x(t) := [x_1(t)\, x_2(t)\, \cdots\, x_n(t)]^\top$.

We introduce the following variables to represent the difference between $x_i(t)$ and $y_i(t)$:

$$w_i(t) := y_i(t) - x_i(t). \tag{5}$$

Using these variables and (2), the state equation of (4) can be rewritten as

$$x_i(t+1) = x_i(t) + \varepsilon(t) \sum_{j\in\mathbf{N}_i} (x_j(t) - x_i(t) + w_j(t)).$$

Combining the aforementioned equations for $i = 1, 2, \ldots, n$, we eventually obtain

$$x(t+1) = (I - \varepsilon(t)L)x(t) + \varepsilon(t)Aw(t) \tag{6}$$

where $w(t) := [w_1(t)\, w_2(t)\, \cdots\, w_n(t)]^\top$. It is clear that (6) corresponds to the collective dynamics of the typical discrete-time consensus algorithm, but with a time-varying gain $\varepsilon(t)$ and an additional term $\varepsilon(t)Aw(t)$.

By considering the probabilistic properties of $w(t)$ (given in Appendix II), we can derive several results on the steady-state behavior of the multiagent system with the majority determination algorithm in (4).

First, the following result is related to the consensus of $x_i(t)$ $(i = 1, 2, \ldots, n)$.

Lemma 2: For the multiagent system with the majority determination algorithm in (4), suppose that the opinions $x_i(0) \in \{0, 1\}$ $(i = 1, 2, \ldots, n)$ are fixed. If the following statement holds.

(A1) The network structure G is strongly connected.

(A2) The gain $\varepsilon(t)$ is given by

$$\varepsilon(t) = \frac{1}{\Delta} c(t)$$

for a sequence $c(t) \in \mathbf{R}_+$ such that

$$\sup_{t\in\mathbf{Z}_{0+}} c(t) < 1,\ \sum_{t=0}^{\infty} c(t) = \infty,\ \sum_{t=0}^{\infty} (c(t))^2 < \infty$$

then $x(t)$ converges to the set[1] $\mathrm{span}(\mathbf{1}_n)$ w.p.1. ■

Proof: See Appendix III. ■

Note that (A2) implies that $\varepsilon(t) \in [0, \Delta]$. This lemma guarantees that the states $x_i(t)$ $(i = 1, 2, \ldots, n)$ reach the consensus almost surely, i.e.,

$$\lim_{t\to\infty} (x_i(t) - x_j(t)) = 0 \ \text{ w.p.1} \tag{7}$$

for every $(i, j) \in \mathbf{V} \times \mathbf{V}$. However, the result does not indicate that the states converge to a constant value.

Now, we present the results for the average of the states $x_i(t)$ $(i = 1, 2, \ldots, n)$, i.e.,

$$\mu(t) := \frac{1}{n} \sum_{i=1}^{n} x_i(t). \tag{8}$$

Lemma 3: For the multiagent system with the majority determination algorithm in (4), suppose that the opinions $x_i(0) \in \{0, 1\}$ $(i = 1, 2, \ldots, n)$ are given. If

(A1') the network structure G is balanced,

and (A2) holds, then

$$E[\mu(t)] = \mu(0) \tag{9}$$

$$V[\mu(t)] \leq \frac{\delta}{4n} \sum_{\tau=0}^{t} (c(\tau))^2 \tag{10}$$

hold for every $t \in \{1, 2, \ldots\}$, where

$$\delta := \frac{1}{n} \sum_{i=1}^{n} \left(\frac{|\mathbf{N}_i|}{\Delta} \right)^2 \tag{11}$$

is the average of the squares of the ratio of in-degree to the maximum in-degree.

Proof: See Appendix IV. ■

[1]Consider a vector sequence $x(t) \in \mathbf{R}^n$ $(t = 0, 1, \ldots)$ and a set $\mathbf{S} \subseteq \mathbf{R}^n$. For each $\epsilon \in \mathbf{R}_+$, if there exists a $\tau \in \mathbf{Z}_{0+}$ satisfying $\inf_{y\in\mathbf{S}} \|x(t) - y\| < \epsilon$ for every $t \in \{\tau, \tau+1, \ldots\}$, then we say that the vector sequence $x(t)$ $(t = 0, 1, \ldots)$ converges to the set $\mathbf{S}$.

This lemma provides the expectation and variance of the average of the states. If the states $x_i(t)$ $(i = 1, 2, \ldots, n)$ reach the consensus [i.e., (7) holds for every $(i, j) \in \mathbf{V} \times \mathbf{V}$], this result facilitates the characterization of the limit of the states $x_i(t)$ $(i = 1, 2, \ldots, n)$.

We comment on the tightness of the upper bound in (10). As easily seen in the proof in Appendix IV-B, the variance $V[\mu(t)]$ is expressed as

$$V[\mu(t)] = \frac{1}{n}\sum_{\tau=0}^{t-1}\left((c(\tau))^2\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{|\mathbf{N}_i|}{\Delta}\right)^2 E[(w_i(\tau))^2]\right)\right).$$

Then, $0 \le E[(w_i(\tau))^2] \le 1/4$ for every $i \in \mathbf{V}$ and $\tau \in \mathbf{Z}_{0+}$. The upper bound $1/4$ will be given in Lemma 5(iii) and it is a tight bound in the sense that $E[(w_i(\tau))^2] = 1/4$ if $x_i(\tau) = 0.5$ [see (4) and (5)]. Note that $x_i(\tau) = 0.5$ often occurs, e.g., as illustrated in Figs. 2 and 3. Thus, the upper bound in (10) is loose for the replacement of $E[(w_i(\tau))^2]$ by $1/4$.

B. Performance of Majority Determination Algorithm

Based on Lemmas 2 and 3, the following result is obtained.

Theorem 1: For the multiagent system with the majority determination algorithm in (4), suppose that the opinions $x_i(0) \in \{0, 1\}$ $(i = 1, 2, \ldots, n)$ are fixed. Moreover, suppose that $\sigma \in (0, 1]$ and $\theta \in (0, 1)$ are arbitrarily given. If (A1), (A1'), and (A2) are true, and

(A3) either the opposition group $\mathbf{I}_0$ or the supportive group $\mathbf{I}_1$ outnumbers the other in the sense that

$$\frac{\big||\mathbf{I}_1| - |\mathbf{I}_0|\big|}{n} \ge \sigma \quad (12)$$

(A4) the number n of agents is sufficiently large such that

$$n > \frac{\delta}{(1-\theta)\sigma^2}\sum_{t=1}^{\infty}(c(t))^2 \quad (13)$$

then there exists a time $T \in \mathbf{Z}_{0+}$ such that

$$\begin{cases} P\left[\sup_{t\ge T} x_i(t) < 0.5\right] \ge \theta, \ \text{if } |\mathbf{I}_0| > |\mathbf{I}_1| \\ P\left[\inf_{t\ge T} x_i(t) > 0.5\right] \ge \theta, \ \text{if } |\mathbf{I}_0| < |\mathbf{I}_1| \end{cases} \quad (14)$$

for every $i \in \mathbf{V}$.

Proof: See Section 1. ■

Theorem 1 provides an approximate solution to Problem 1 in the sense that the states of all the agents remain less than or more than 0.5 after some time T in a probabilistic sense. In fact, $\sup_{t\ge T} x_i(t) < 0.5$ implies that $\limsup_{t\to\infty} x_i(t) < 0.5$ and its converse is also true. Therefore, (14) means that (3) holds with probability greater than or equal to θ. Thus, if θ is sufficiently large, there is a sufficiently high probability that the majority opinion of $x_i(0)$ $(i = 1, 2, \ldots, n)$ is determined. Hence, the parameter θ corresponds to the level of confidence of the solution. Meanwhile, the parameter σ corresponds to the resolution of distinguishing two groups: If σ is smaller, the algorithm can be applied to the case in which the two groups have more identical sizes.

For this result, three remarks are given.

TABLE I
LOWER BOUNDS OF σ SATISFYING (13) FOR $\delta = 0.75^2$ AND $\sum_{t=1}^{\infty}(c(t))^2 = 0.9$

	$n = 10^2$	$n = 10^3$	$n = 10^4$	$n = 10^5$
$\theta = 0.99$	0.7116	0.2250	0.0712	0.0225
$\theta = 0.90$	0.2250	0.0712	0.0225	0.0072
$\theta = 0.70$	0.1300	0.0411	0.0130	0.0042
$\theta = 0.50$	0.1007	0.0319	0.0101	0.0032

First, the parameters σ and θ, which specify the accuracy of this algorithm, can be arbitrarily selected. However, as shown in (A4), if the gain sequence $\varepsilon(t)$ $(t = 0, 1, \ldots,)$ is fixed [i.e., $\sum_{t=1}^{\infty}(c(t))^2$ is fixed in (13)], the applicable size (i.e., n) of the multiagent system becomes more limited as $\sigma \to 0$ and $\theta \to 1$. Table I shows the lower bounds of σ satisfying (13) for $\delta = 0.75^2$ and $\sum_{t=1}^{\infty}(c(t))^2 = 0.9$. If (A4) does not hold for given parameters n, δ, σ, θ, and $c(t)$, an alternative is to reselect the design parameter $c(t)$ so as to satisfy (13). In fact, $c(t)$ in the form of

$$c(t) = \frac{c_1}{c_0 + t} \quad (15)$$

satisfies the three conditions in (A2) and $\sum_{t=1}^{\infty}(c(t))^2 < 2c_1$ subject to $0 < c_1 < c_0$. Thus, by appropriately selecting c_1, we can satisfy (13) for the given parameters. In this sense, the proposed method approximately solves the problem with an arbitrarily prescribed accuracy.

Second, the convergence rate of $x_i(t)$, which is useful for the (rough) estimation of the length of T, is given as follows. If $c(t)$ is given by (15) and $x(t)$ converges to the set $\mathrm{span}(\mathbf{1}_n)$ w.p.1. as shown in Lemma 2, the convergence rate is $O(1/\sqrt{t})$, more precisely, $E[\|x_i(t) - x_j(t)\|_2] = O(1/\sqrt{t})$ for every $(i, j) \in \mathbf{V} \times \mathbf{V}$. This is the straightforward consequence from the convergence rate analysis of classical stochastic approximation (Robbins–Monro algorithms) for finding roots (see, e.g., [20] and [21]) and the fact that (6) is a Robbins–Monro algorithm, as shown in Appendix III. It should be remarked that the convergence rate does not depend on the network structure unlike the typical consensus algorithm whose convergence rate depends on the network structure (i.e., the eigenvalues of graph Laplacian). This property may be useful in the sense that the convergence rate is known in advance even when we do not have the exact information on the network structure.

Finally, in Figs. 2 and 3, it seems that the states do not reach the consensus, although the consensus is guaranteed by Lemma 2. However, it is not the case. This is because the convergence rate is $O(1/\sqrt{t})$ as mentioned earlier and it is not so fast. In fact, we can observe the consensus after a long time in both the cases; for example, in the case of Fig. 2, we obtain $\max_{(i,j)\in\{1,2,\ldots,n\}} |x_i(300) - x_j(300)| \simeq 0.047$ and $\max_{(i,j)\in\{1,2,\ldots,n\}} |x_i(3000) - x_j(3000)| \simeq 0.011$.

C. Proof of Theorem 1

Theorem 1 is a straightforward consequence of the following three facts.

(i) For every $t \in \{1, 2, \ldots\}$, $P[|\mu(t) - \mu(0)| < \sigma/2] \ge \theta$.

(ii) For every $t \in \{1, 2, \ldots\}$, the following relation holds: If $|\mathbf{I}_0| > |\mathbf{I}_1|$, $|\mu(t) - \mu(0)| < \sigma/2$ implies that $\mu(t) < 0.5$; if $|\mathbf{I}_0| < |\mathbf{I}_1|$, it implies that $\mu(t) > 0.5$.

(iii) For every $i \in \mathbf{V}$, $\lim_{t\to\infty}(\mu(t) - x_i(t)) = 0$ w.p.1.

Note that (iii) implies that if there exists a time $T_1 \in \mathbf{Z}_{0+}$ such that $\mu(t) < 0.5$ for every $t \in \{T_1, T_1 + 1, \ldots\}$, there exists a time $T_2 \in \mathbf{Z}_{0+}$ such that $x_i(t) < 0.5$ for every $t \in \{T_2, T_2 + 1, \ldots\}$. A similar proposition holds for $\mu(t) > 0.5$ and $x_i(t) > 0.5$.

Now, we prove (i)–(iii).

(i) For the random variable $\mu(t)$, Chebyshev's inequality is given by

$$P\left[\left|\mu(t) - E[\mu(t)]\right| < \omega\sqrt{V[\mu(t)]}\right] \geq 1 - \frac{1}{\omega^2} \tag{16}$$

where $\omega \in [1, \infty)$ is an arbitrary number. Applying Lemma 3 to (16) with $\omega := \sqrt{1/(1-\theta)}$ (≥ 1), we have

$$P\left[|\mu(t) - \mu(0)| < \sqrt{\frac{1}{1-\theta}}\sqrt{\frac{\delta}{4n}\sum_{\tau=0}^{t}(c(\tau))^2}\right] \geq 1 - (1-\theta). \tag{17}$$

The right-hand side is equal to θ. In addition, (A2) and (A4) (in particular, $c(t) \in \mathbf{R}_+$) imply that

$$\sqrt{\frac{1}{1-\theta}}\sqrt{\frac{\delta}{4n}\sum_{\tau=0}^{t}(c(\tau))^2} < \frac{\sigma}{2}$$

on the left-hand side of (17). Hence, we obtain (i).

(ii) First, consider the case $|\mathbf{I}_0| < |\mathbf{I}_1|$. The inequality $|\mu(t) - \mu(0)| < \sigma/2$ implies that

$$\mu(t) > \mu(0) - \frac{\sigma}{2}. \tag{18}$$

On the other hand, if $|\mathbf{I}_0| < |\mathbf{I}_1|$, the left-hand side of (12) is equal to

$$\frac{|\mathbf{I}_1| - |\mathbf{I}_0|}{n} = \frac{|\mathbf{I}_1| - (n - |\mathbf{I}_1|)}{n} = \frac{2|\mathbf{I}_1|}{n} - 1 = 2\mu(0) - 1$$

because $|\mathbf{I}_0| = n - |\mathbf{I}_1|$ and $|\mathbf{I}_1|/n = \mu(0)$. Applying this result and (12)–(18), we have

$$\mu(t) > \mu(0) - \frac{\sigma}{2} > \mu(0) - \frac{2\mu(0) - 1}{2} = 0.5.$$

The other case, i.e., $|\mathbf{I}_0| > |\mathbf{I}_1|$, can be proved in a similar way.

(iii) According to Lemma 2, (7) holds for every $(i, j) \in \mathbf{V} \times \mathbf{V}$ under (A1) and (A2). From (7) and (8), we have

$$\begin{aligned}\lim_{t\to\infty}(\mu(t) - x_i(t)) &= \lim_{t\to\infty}\left(\left(\frac{1}{n}\sum_{j=1}^{n}x_j(t)\right) - x_i(t)\right)\\ &= \lim_{t\to\infty}\frac{1}{n}\sum_{j=1}^{n}(x_j(t) - x_i(t))\\ &= \frac{1}{n}\sum_{j=1}^{n}\lim_{t\to\infty}(x_j(t) - x_i(t))\\ &= 0 \ \text{ w.p.1.}\end{aligned}$$

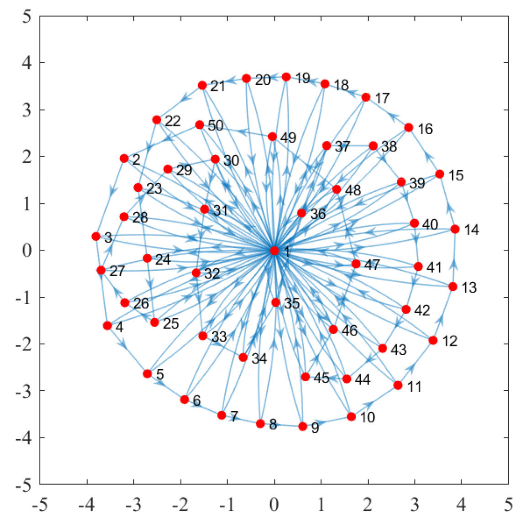

Fig. 4. Network structure of the sensor network ($n = 50$ and $\Delta = 49$).

V. Application to Anomaly Detection by Sensor Network

An application of majority determination is anomaly detection by a sensor network. In this section, we demonstrate our framework through this application.

Consider a sensor network composed of n sensor nodes, which aims at detecting anomaly of a target system. In this network, binary-valued communication is available among nodes. At a certain moment, each node measures the state of the system, which is either normal or anomaly; however, the measurements are not accurate, e.g., in the sense that the measurement is false with a probability. In this case, it is reasonable to exploit the majority of the measurements as the output of the sensor network. This is exactly our case.

Now, let us illustrate how the above anomaly detection is performed by the majority determination algorithm in (4). Consider a sensor network with $n = 50$ and network structure G whose adjacency matrix is given by

$$A := \begin{bmatrix} 0 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & 0 & \vdots & & \ddots & 1 \\ 1 & 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}. \tag{19}$$

The network structure G is illustrated in Fig. 4, where node 1 bidirectionally communicate with all other nodes, node $i + 1$ is connected to node i ($i = 2, 3, \ldots, 49$), and node 50 is connected to node 1. Note that G is directed but balanced, $\Delta = 49$, and $\delta \simeq 0.0216$. The measurement of sensor node i is stored in its initial state $x_i(0)$ as a binary value. The values 0 and 1 represent *normal* and *anomaly*, respectively. It is assumed that we have the prior information $\sigma = 0.65$ for the collective measurements.

For determining the majority in the sense of (14) with the confidence level $\theta = 0.95$, we apply the majority determination

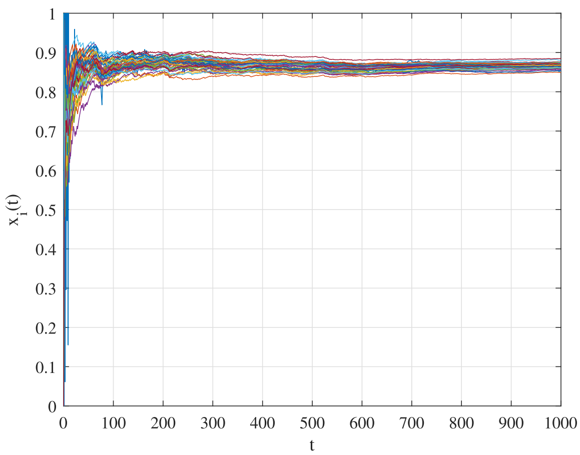

Fig. 5. Majority determination for the case where 90% of the measurements are anomaly.

algorithm in (4) with the gain

$$\varepsilon(t) = \frac{23}{\Delta(1+t)}.$$

In this case, the right-hand side of (13) is less than 47.11, which implies that (13) holds.

Fig. 5 shows the result for the case where 10% of the measurements are normal and 90% anomaly. It is observed that the states $x_i(t)$ $(i = 1, 2, \ldots, 50)$ are more than 0.5, from which the sensor network outputs *anomaly*. Note that the consensus among nodes is achieved as guaranteed by Lemma 2 but the consensus value is not always equal to the average of $x_i(0)$ $(i = 1, 2, \ldots, n)$ [see (9) and (10)].

In this way, our framework is useful for cooperative decision making in sensor networks.

VI. Conclusion

We developed distributed algorithms for a majority determination problem with binary-valued communication. Based on randomized communication, our algorithms approximately solved the problem with an arbitrarily prescribed accuracy. We also clarified the relationship among the number of agents, the distribution of opinions, and the accuracy of the method. Our results revealed that the lowest resolution required for majority determination is two-level.

In future, we hope to extend our framework to the case with malicious agents. Moreover, it may be interesting to handle time-varying opinions.

Appendix I
Robbins–Monro Algorithm

A. Cooperativity and Irreducibility of Vector-Valued Functions

We introduce the notions of *cooperativity* and *irreducibility* for vector-valued functions, according to [22, Sec. 4].

Definition 1: Consider a function $h : \mathbf{R}^n \to \mathbf{R}^n$.

(i) The function h is said to be cooperative if it is continuously differentiable and the off-diagonal elements of the Jacobian matrix $\partial h(x)/\partial x$ are non-negative for every $x \in \mathbf{R}^n$.

(ii) The function h is said to be irreducible if the Jacobian matrix $\partial h(x)/\partial x$ is irreducible for every $x \in \mathbf{R}^n$. ■

B. Robbins–Monro Algorithm and Convergence

The Robbins–Monro algorithm is given as

$$x(t+1) = x(t) + a(t)\left(h(x(t)) + e(t)\right) \tag{20}$$

where $x(t) \in \mathbf{R}^n$ is the state, $a(t) \in \mathbf{R}$ is the time-varying gain, $e(t) \in \mathbf{R}^n$ is a random vector, and $h : \mathbf{R}^n \to \mathbf{R}^n$ is a function.

The following result is a straightforward consequence of [23, Th. 2], and [22, Th. A, Th. 4.4, and Corollary 4.6].

Lemma 4: Consider the Robbins–Monro algorithm in (20). If the following conditions hold, then the state $x(t)$ converges to the set of zeros of the function h.

(B1) h is Lipschitz, cooperative, and irreducible.

(B2) $a(t) > 0$ for every $t \in \mathbf{Z}_{0+}$, $\sum_{t=0}^{\infty} a(t) = \infty$, and $\sum_{t=0}^{\infty}(a(t))^2 < \infty$.

(B3) The stochastic process $e(t)$ $(t = 0, 1, \ldots)$ is a martingale difference sequence with respect to the filtration $\mathcal{F}_{t-1}$ generated by $\{x(0), e(0), e(1), \ldots, e(t-1)\}$, i.e.,

$$E[e(t)|\mathcal{F}_{t-1}] = 0 \text{ w.p.1.}$$

Moreover, the stochastic process $e(t)$ is square-integrable martingale, i.e.,

$$E[\|e(t)\|^2|\mathcal{F}_{t-1}] \le d(1 + \|x(t)\|^2) \text{ w.p.1}$$

for some $d \in \mathbf{R}_+$.

(B4) $\sup_{t \in \mathbf{Z}_{0+}} \|x(t)\| < \infty$ w.p.1. ■

Appendix II
Probabilistic Properties of Variables $w_i(t)$

It is important to clarify the probabilistic properties of the variables $w_i(t)$ $(i = 1, 2, \ldots, n)$ in (5). The following lemma provides the expectations of $w_i(t)$ and their products.

Lemma 5: For the multiagent system with the majority determination algorithm in (4), suppose that the opinions $x_i(0) \in \{0, 1\}$ $(i = 1, 2, \ldots, n)$ are fixed. Then, the following statements hold.

(i) $E[w_i(t)] = 0$ for every $i \in \mathbf{V}$ and $t \in \mathbf{Z}_{0+}$.

(ii) $E[w_i(t_1)w_j(t_2)]$

$$= \begin{cases} E[(1 - x_i(t_1))x_i(t_1)] & \text{if } i = j \text{ and } t_1 = t_2, \\ 0 & \text{otherwise} \end{cases}$$

for every $(i, j) \in \mathbf{V} \times \mathbf{V}$ and $(t_1, t_2) \in \mathbf{Z}_{0+} \times \mathbf{Z}_{0+}$.

(iii) $E[(w_i(t))^2] \le 1/4$ for every $i \in \mathbf{V}$ and $t \in \mathbf{Z}_{0+}$.

Proof: (i) Considering that $w_i(t)$ only depends on $x_i(t)$ according to (4), it is straightforward to calculate $E[w_i(t)]$ using (4) and (5). In fact

$$E[w_i(t)] = E_{x_i(t)}[E[w_i(t)|x_i(t)]]$$

$$E[w_i(t)|x_i(t)] = E[y_i(t) - x_i(t)|x_i(t)]$$

$$= (0 - x_i(t))(1 - x_i(t)) + (1 - x_i(t))x_i(t)$$
$$= 0$$

which implies that $E[w_i(t)] = E_{x_i(t)}[0] = 0$.

(ii) We first consider the former case, i.e., $i = j$ and $t_1 = t_2$. In this case, we have $E[w_i(t_1)w_j(t_2)] = E[(w_i(t_1))^2]$ and $w_i(t_1)$ only depends on $x_i(t_1)$ as in (4). Thus

$$E[(w_i(t_1))^2] = E_{x_i(t_1)}[E[(w_i(t_1))^2|x_i(t_1)]],$$
$$\begin{aligned} &E[(w_i(t_1))^2|x_i(t_1)] \\ &= E[(y_i(t_1) - x_i(t_1))^2|x_i(t_1)] \\ &= (0 - x_i(t_1))^2(1 - x_i(t_1)) + (1 - x_i(t_1))^2 x_i(t_1) \\ &= (1 - x_i(t_1))x_i(t_1) \end{aligned}$$

according to (4) and (5), which prove that

$$E[w_i(t_1)w_j(t_2)] = E[(1-x_i(t_1))x_i(t_1)].$$

We next consider the case where $i \neq j$ or $t_1 \neq t_2$. Using straightforward calculation based on (4) and (5), we have

$$\begin{aligned} &E[w_i(t_1)w_j(t_2)|x_i(t_1), x_j(t_1)] \\ &= (0 - x_i(t_1))(0 - x_j(t_2))(1 - x_i(t_1))(1 - x_j(t_2)) \\ &\quad + (0 - x_i(t_1))(1 - x_j(t_2))(1 - x_i(t_1))x_j(t_2) \\ &\quad + (1 - x_i(t_1))(0 - x_j(t_2))x_i(t_1)(1 - x_j(t_2)) \\ &\quad + (1 - x_i(t_1))(1 - x_j(t_2))x_i(t_1)x_j(t_2) \\ &= 0. \end{aligned}$$

Thus

$$\begin{aligned} &E[w_i(t_1)w_j(t_2)] \\ &= E_{x_i(t_1),x_j(t_2)}[E[w_i(t_1)w_j(t_2)|x_i(t_1), x_j(t_2)]] \\ &= E_{x_i(t_1),x_j(t_2)}[0] \\ &= 0. \end{aligned}$$

(iii) By assumption, $x_i(0) \in \{0, 1\}$ for every $i \in \mathbf{V}$. Furthermore, Lemma 1 implies that $x_i(t) \in [0, 1]$ for every $i \in \{1, 2, \ldots, n\}$ and $t \in \{1, 2, \ldots\}$. On the other hand, the maximum value of $(1 - x_i(t))x_i(t)$ with respect to $x_i(t) \in [0, 1]$ is $1/4$. Therefore, it follows from (ii) that $E[(w_i(t))^2] \leq 1/4$. ■

Appendix III
Proof of Lemma 2

Here, we prove Lemma 2.

The collective dynamics in (6) corresponds to the Robbins–Monro algorithm in (20) (see Appendix I) by considering $-Lx(t)$, $Aw(t)$, and $\varepsilon(t)$ as $h(x(t))$, $e(t)$, and $a(t)$, respectively. Thus, we show that the conditions (B1)–(B4) hold for $h(x(t)) := -Lx(t)$, $e(t) := Aw(t)$, and $a(t) := \varepsilon(t)$.

(B1) Since the function $-Lx$ is linear with respect to x, $-Lx$ is Lipschitz. Next, the off-diagonal elements of any graph Laplacian are nonpositive because $L = D - A$, D is diagonal, and the elements of A are non-negative [19]. Thus, the off-diagonal elements of the Jacobian of $-Lx$, i.e., $-L$, are nonnegative for every $x \in \mathbf{R}^n$, which proves that the Jacobian of $-Lx$ is cooperative. Finally, it is well established that the matrix $-L$ is irreducible under (A1). Consequently, (B1) holds for $h(x(t)) := -Lx(t)$.

(B2) Condition (A2) implies (B2).

(B3) For $e(t) := Aw(t)$, the first half of (B3) is given by Lemma 5 (i). Next, it follows from (5), $y_i(t) \in \{0, 1\}$, and Lemma 1 that $w(t) \in [-1, 1]^n$ for every $x_i(0) \in \{0, 1\}$ ($i = 1, 2, \ldots, n$) and $t \in \mathbf{Z}_{0+}$. Hence, $\|Aw(t)\| \leq \|A\|\|w(t)\| \leq \|A\|\sqrt{n}$. This indicates that $E[\|Aw(t)\|^2|\mathcal{F}_{t-1}] \leq \|A\|^2 n$, which establishes the second half of (B3).

(B4) This follows from Lemma 1 and the assumption that $x_i(0) \in \{0, 1\}$ ($i = 1, 2, \ldots, n$).

Appendix IV
Proof of Lemma 3

Lemma 3 can be proved as follows.

A. Proof of (9)

Multiplying both sides of (6) by $(1/n)\mathbf{1}_n^\top$ from left, we obtain

$$\frac{1}{n}\mathbf{1}_n^\top x(t+1) = \frac{1}{n}\mathbf{1}_n^\top x(t) - \varepsilon(t)\frac{1}{n}\mathbf{1}_n^\top Lx(t) + \varepsilon(t)\frac{1}{n}\mathbf{1}_n^\top Aw(t). \tag{21}$$

Then, $\mu(t) = (1/n)\mathbf{1}_n^\top x(t)$ from (8), $\mathbf{1}_n^\top L = 0_{1\times n}$ under (A1') [19], and $A = D - L$. Thus, (21) is equivalent to

$$\begin{aligned} \mu(t+1) &= \mu(t) + \varepsilon(t)\frac{1}{n}\mathbf{1}_n^\top Dw(t) - \varepsilon(t)\frac{1}{n}\mathbf{1}_n^\top Lw(t) \\ &= \mu(t) + \varepsilon(t)\frac{1}{n}\mathbf{1}_n^\top Dw(t). \end{aligned}$$

Therefore

$$\mu(t) = \mu(0) + \sum_{\tau=0}^{t-1} \varepsilon(\tau)\frac{1}{n}\mathbf{1}_n^\top Dw(\tau). \tag{22}$$

As only $w(\tau)$ ($\tau = 0, 1, \ldots, t-1$) are random variables, it follows from Lemma 5 (i) that

$$\begin{aligned} E[\mu(t)] &= \mu(0) + E\left[\sum_{\tau=0}^{t-1} \varepsilon(\tau)\frac{1}{n}\mathbf{1}_n^\top Dw(\tau)\right] \\ &= \mu(0) + \sum_{\tau=0}^{t-1} \varepsilon(\tau)\frac{1}{n}\mathbf{1}_n^\top DE[w(\tau)] \\ &= \mu(0). \end{aligned}$$

B. Proof of (10)

The matrix D is diagonal with the diagonal elements $|\mathbf{N}_i|$ ($i = 1, 2, \ldots, n$). Considering this fact along with (22), $(\mu(t) - \mu(0))^2$ can be calculated as

$$\begin{aligned} (\mu(t) - \mu(0))^2 &= \left(\sum_{\tau=0}^{t-1} \varepsilon(\tau)\frac{1}{n}\mathbf{1}_n^\top Dw(\tau)\right)^2 \\ &= \left(\frac{1}{n}\sum_{\tau=0}^{t-1}\sum_{i=1}^{n} \varepsilon(\tau)|\mathbf{N}_i|w_i(\tau)\right)^2 \end{aligned} \tag{23}$$

subject to (A1'). It follows from (9), (11), (23), Lemma 5 (ii) and (iii), and (A2) that

$$\begin{aligned} V[\mu(t)] &= E[(\mu(t)-E[\mu(t)])^2] \\ &= E[(\mu(t)-\mu(0))^2] \\ &= E\left[\left(\frac{1}{n}\sum_{\tau=0}^{t-1}\sum_{i=1}^{n}\varepsilon(\tau)|\mathbf{N}_i|w_i(\tau)\right)^2\right] \\ &= \frac{1}{n^2}\sum_{\tau=0}^{t-1}\sum_{i=1}^{n}(\varepsilon(\tau))^2|\mathbf{N}_i|^2E[(w_i(\tau))^2] \\ &\le \frac{1}{n^2}\sum_{\tau=0}^{t-1}\sum_{i=1}^{n}\left(\frac{1}{\Delta}c(\tau)\right)^2|\mathbf{N}_i|^2\frac{1}{4} \\ &= \frac{1}{4n^2}\sum_{\tau=0}^{t-1}\left((c(\tau))^2\frac{1}{\Delta^2}\sum_{i=1}^{n}|\mathbf{N}_i|^2\right) \\ &= \frac{\delta}{4n}\sum_{\tau=0}^{t-1}(c(\tau))^2. \end{aligned} \tag{24}$$

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