Cooperation and Competition Coupled Diffusion of Multi-Feature on Multiplex Networks and Its Control

Dawei Zhao , Shudong Li , Zhen Wang , and Haipeng Peng

Abstract—**Cooperation and competition widely exist in various kinds of network diffusions which however are usually studied separately. Recently, a novel network diffusion model, called multi-feature diffusion (MFD), attracts considerable attentions. The existing works usually assume that each feature diffuses independently and neglects the possible complex interplay between different features. In this paper, we introduce the cooperation and competition into the MFD and propose the *c*ooperation and *c*ompetition coupled diffusion of *m*ulti-*f*eature on multiplex network (CCMF). An unified framework and mathematical analytic theory regarding CCMF are then presented which are applicable and computationally efficient for any number of features and their own different sub-diffusion dynamics. In addition, an interesting finding is obtained in CCMF: compared with the high intensity competition, performing lower intensity competition under weak competition ability is more easier to result in positive effect. Due to the great importance of controlling network diffusion in many diverse contexts, we also propose an optimal allocation strategy of control resource for CCMF which first realizes the promotion and suppression of network diffusion simultaneously under one optimization framework and is also verified to be very efficient.**

Index Terms—**Competition, cooperation, multi-feature diffusion, multiplex network, optimization.**

I. INTRODUCTION

NETWORK diffusion is one of the most widespread natural and social phenomena where the typical examples include

Dawei Zhao is with the Shandong Provincial Key Laboratory of Computer Networks, Shandong Computer Science Center (National Supercomputer Center in Jinan), Qilu University of Technology (Shandong Academy of Sciences), Jinan 250014, China (e-mail: zhaodw@sdas.org).

Shudong Li is with the Cyberspace Institute of Advance Technology, Guangzhou University, Guangzhou 510006, China (e-mail: lishudong@gzhu.edu.cn).

Zhen Wang is with the School of Artificial Intelligence, Optics and Electronics (iOPEN) and School of Cybersecurity, Northwestern Polytechnical University, Xi'an 710072, China (e-mail: w-zhen@nwpu.edu.cn).

Haipeng Peng is with the Information Security Center, State Key Laboratory of Networking and Switching Technology, and National Engineering Laboratory for Disaster Backup and Recovery, Beijing University of Posts and Telecommunications, Beijing 100876, China (e-mail: penghaipeng@bupt.edu.cn).

the spreading of various epidemics in biological networks, the diffusion of information, rumor and virus in communization networks, and the propagation of cascading failures in infrastructure networks [1], [2], [3], [4], [5], [6], [7]. Researches on the evolution mechanism of network diffusion have attracted considerable attentions due to the important potential applications. For example, it can be used to accurately predict the outbreak threshold and the prevalence of a certain diffusion, effectively control the undesired diffusions of disease, virus or rumor, and maximize the influence of positive information and the possibility of acceptance of product and market behaviors, etc [8], [9], [10], [11], [12], [13].

The dynamical model is the core of understanding evolution mechanism of network diffusion. However, the potential risk and difficulty lead to be unfeasible of studying on real network diffusion, mathematical modeling method is applied extensively to be the fundamental framework where to verify the correctness of theories and evaluate the effectiveness of control mechanisms. In the past few decades, several dynamical models, like epidemic model, independent cascading model and linear threshold model, have been proposed to characterize the evolutionary process of various kinds of network diffusions [1]. In addition, the network topology, extracted from real-world, plays a vital role on the outcomes of the diffusion process [14], [15]. The diffusion on the well-known random network, small-world network and scale-free network shows significant differences of dynamic characteristic and evolutionary mechanism [16].

Prior to 2010, the majority of studies on network diffusion are mainly concentrated to the case of single diffusion process on single network. In the past decade, the multiplex network, coupled by several single network layers, has become to be the new research hotspot of complex network [17]. Thus far, numerous research results have been achieved on multiplex network including its structural characteristics and dynamical mechanisms [18], [19], [20], [21], [22], [23], [24]. As a consequence, several network diffusion dynamics on multiplex network were proposed, such as the multiple route diffusion (MRD) and multiple information diffusion (MID) [24], [25], [26], [27]. The MRD characterizes the phenomenon that one event (for example one piece of information, one epidemic, one computer virus, etc) diffuses through multiple routes simultaneously. While the MID is for multiple event and each diffusion is for one event.

Most recently, a novel diffusion model on multiplex network was introduced, i.e. the multi-feature diffusion (MFD), which is for one event with multiple features [28], [29]. In MFD, the outcomes of the event is determined by the diffusion of the

multiple features. In addition, the features could be subdivided into positive and negative categories. The MFD matches to a lot of realistic scenario. For example, one produce usually has its own multiple feature, including the price, appearance, function, quality and so on. Whether the customer will buy the produce or not depends on his/her overall evaluation of all these features [28]. As to one topic in social network, it may contains multiple related messages, both positive and negative. The users accept or deny the topic depending on the comprehensive consideration of all related messages [29].

Research on multi-feature diffusion is still in its infancy, several challenges remain. The number of features in MFD maybe large and different feature may diffuses following its own different dynamical model. In addition, the features can be impossible to diffuse independently since they all around one same event. The interplay between features is also pluralistical rather than monistical, which may include both cooperative relationship between same type of features and competitive relationship between different type of features. The unified framework and mathematical analytic theory regarding MFD with the cooperation and competition interplay between features (CCMF, *c*ooperation and *c*ompetition coupled diffusion of *m*ulti-*f*eature) should be developed.

One of the main purposes of studying network diffusion is to recognize and understand its dynamic characteristic and evolutionary mechanism, and then control it efficiently [8], [30]. Two important problems regarding the control of network diffusion are the optimal selection of node sets either to minimize or maximize the extent of outbreaks under the constraint of limited resources [31], [32], [33], [34]. The minimization problem is equivalent to the network immunization problem which aims to minimize the outbreak size of the diffusion by immunizing a node set of "superblocks" [31], [32]. While the maximization problem is to identify a set of "superspreaders" which are chosen as the initiators of the diffusion, the outbreak size will be maximized [33], [34]. The identification of these "supernodes" is often done by employing the centrality index of the underlying network topology, such as betweenness, degree, pagerank, eigenvector, closeness, k-shell and Collective Influence etc [35]. Another type of solution depends not just on the network topology, but also on the specific dynamics of the diffusion, including the Monte-Carlo (MC) methods, integer/linear programming, greedy algorithms and the message-passing algorithm [33], [36].

Obviously, the minimization problem and the maximization problem of control belong to two opposite problems and thus usually would not be appear in the same application scenarios. Similarly, one of the main control goals of CCMF is also to minimize or maximize the outcomes of the event. However, the realization of control CCMF covers the minimization problem and maximization problem simultaneously due to the coexistence of negative features and positive features around the event. Specifically, the maximization (minimization) of CCMF could be realized through the minimization (maximization) of negative features and the maximization (minimization) of positive features simultaneously. The efficient theory and method regarding such control issue are absent.

In this paper, we focus on the cooperation and competition coupled diffusion of multi-feature on multiplex networks and its control. Our contributions are summarized as follows:

- We introduce the cooperation and competition into the MFD and propose the cooperation and competition coupled diffusion of multi-feature on multiplex network (CCMF). An unified framework and mathematical analytic theory regarding CCMF are then presented which are applicable and computationally efficient for any number of features and their own different sub-diffusion dynamics.
- An interesting finding is obtained in CCMF: compared with the high intensity competition, performing lower intensity competition under weak competition ability is more easier to result in positive effect.
- We propose an optimal allocation strategy of control resource for CCMF which first realizes the promotion and suppression of network diffusion simultaneously under one optimization framework and is also verified to be very efficient.

II. RELATED WORK

The existing diffusion models were developed into two major levels, the macro-level and micro-level. The Bass model is one of the well known macro-model whose most important character is that each network node is assumed to be related to every other nodes, i.e., no knowledge of network topology is needed [37]. The micro-model, in contrast, requires the complete knowledge of the network topology, and the exact probability of mutual influence between any two neighbors. Examples of these models include epidemic model (which can be subdivided into the SI model, SIS model, SIR model, etc.), independent cascading model and linear threshold model. In this sense, the network structures play vital role on the dynamical characteristics of network diffusion, including the macro-level, meso-level and micro-level. From the macro-level, Pastor-Satorras and Vespignani demonstrated the absence of epidemic threshold and critical behavior in a wide range of scale-free network, which indicated that the epidemic can outbreak on its own regardless its spreading ability [16]. At the meso-level, Huang and Li found that networks with strong community structure are helpful for reducing the danger brought by epidemic prevalence [38]. In addition, at the micro-level, it's known that the hub nodes are better able to promote and suppress the network diffusion [35].

From 2010, the research focus of network diffusion was changed from the single diffusion on top of single network to the multiple diffusion on multiplex network, such as the multiple route diffusion (MRD) and multiple information diffusion (MID) [17], [25], [26], [27]. The MRD and MID introduced novel complex dynamical behaviours, the most typical one of which is the interplay between different sub-diffusions. Granell et al. first presented the framework of competing diffusion that the awareness diffusion suppresses the disease spreading on multiplex networks [27]. Wang et al. investigated the coevolution mechanisms between the epidemic spreading and information diffusion on multiplex network. They proved that the epidemic can be significantly suppressed when an optimal information

diffusion rate is adopted [39]. Zhao et al. developed an efficient patch distribution scheme by using the competition mechanism between patch dissemination and virus propagation, an optimal patch dissemination was introduced. [40]. Wang et al. studied the impacts of two kinds of information including the positive information and negative information on the epidemic spreading on two-layers network. They found that the positive information diffusion could suppress the epidemic spreading and the negative information diffusion could promote the epidemic spreading [41]. Wei et al. considered the cooperative diffusion on multiplex network and found that the cooperative spreading processes could promote the epidemic spreading [42]. Wei et al. also proposed an unified analysis framework for different kinds of interplays between two different diffusion processes, including competing diffusion, cooperative diffusion and the combination of the two [43].

In 2019, Li et al. introduced a novel diffusion dynamics of multiple social networking messages associated with the same topic and the cooperative relationship and competitive relationship between messages were also considered [44]. However, the proposed analysis framework lacks extensibility in face of the diffusions of a large number of messages. For the same issue, Wu et al. presented the formal definition, called Multi-Feature diffusion (MFD) [28], [29]. They also discussed the optimal control issues of the MFD in two application scenarios, including the diffusion maximization of product and the minimization of social rumor. But they assumed that the features diffused independently and ignored the possible interplay between the features, such as the cooperative relationship and competitive relationship [45], [46], [47].

The identification of "supernodes" in netowrk can be divided into two categories, structural method and functional method [35], [48]. The former was designed purely based on the topological characteristics of network, the most typical methods include the degree, between, K-shell, CI, and so on. The graph-partitioning, explosive percolation and belief-propagation guided decimation (BPD) algorithm [36], [49], [50], [51] are also representative structural methods. Ribalta et.al. extended several topology indexes of single network to multiplex network and proposed the centrality indexes of betweenness, rigenvector, pagerank and clustering coefficients for multiplex network [52], [53], [54]. Osat et al. reframed the collective influence, explosive immunization and simulated annealing to identify the supernodes of multiplex network [55]. However, there exists great limitations in the structural methods since it fails to take into considerations of the dynamic characteristics of network diffusion. In contrast, the functional method considers both the network topology and dynamics of network diffusion, thus be more universally effective. Altarelli et al. proposed message-passing-based algorithm to identify the optimal immunization nodes for SIR model and SIS model [56]. The discrete particle swarm optimization technique was developed to optimize the local influence criterion for the independent cascade model [57]. Xiao et al. proposed a rumor propagation model based on evolutionary game and data enhancement mechanism to sparse the effective data in the rumor propagation [58]. Lokhov and Saad introduced the dynamic message-passing approach to simulate

the network diffusion processes and proposed an efficient, versatile and principled optimization framework for best distributing the control resource to control the diffusion processes [33]. In addition, Wang et al. designed an extended influence spreading model, based on which a memetic algorithm was developed to find the influential nodes of multiplex networks [59]. However, these existing works on controlling diffusion mainly studied the promotion and suppression of diffusions separately.

III. COOPERATION AND COMPETITION COUPLED DIFFUSION OF MULTI-FEATURE ON MULTIPLEX NETWORK

In this section, we introduce the cooperation and competition into the MFD and propose the novel model of CCMF. An unified framework and mathematical analytic theory regarding CCMF are then presented.

Considering an event $\mathbb{E}$ contains m features $\{A, B, C \ldots\}_m$ $= \Delta_{\mathbb{E}}^+ \bigcup \Delta_{\mathbb{E}}^- = \Delta_{\mathbb{E}}$, where $\Delta_{\mathbb{E}}^+$ and $\Delta_{\mathbb{E}}^-$ represent the sets of positive features and negative features of $\mathbb{E}$ respectively, and $\Delta_{\mathbb{E}}^+ \bigcap \Delta_{\mathbb{E}}^- = \emptyset$. The different features may diffuse through different relationships between network nodes. Therefore, the multiplex network is very suitable to be the underlying network framework of MFD. Assumed $G = \{G^A, G^B, G^C \ldots\}_m$ be a multiplex network coupled by m layers, where $G^A = (V, E^A)$ supports the diffusion of feature A. $V = \{\ldots, i, j, \ldots\}_N$ is the set of N network nodes and $E^A = V \times V$ is the set of edges of layer G^A.

In MFD, different feature may have their own different dynamical behavior, therefore may follow different sub-diffusion dynamical model. For simplicity, we utilize the same Susceptible-Infected-Recovered (SIR) model to characterize the diffusions of all features. But the following presented framework and methods are directly applicable to many different diffusion dynamical model. Specifically, in the SIR model the nodes in a network layer, e.g. G^A, can be in one of the three states: A-susceptible (S_A) state indicates the node is free of the feature A and has not been infected by feature A; A-infected (I_A) state, where the node accepted the feature A and could spread A to its S_A neighbors; and A-recovered (R_A) state means the node still accepted the feature A but has no willing to pass A to other nodes. The discrete time approach is adopted to simulate the evolution process of the above SIR model. Specifically, at each time step t, the I_A state node j can infect all its S_A state neighbor i with transmissibility $\phi_{ji}^A(t)$, and then becomes R_A state node with probability $\varphi_j^A(t)$.

Let $p_i^{S_A}(t)$, $p_i^{I_A}(t)$, $p_i^{R_A}(t)$ be the probabilities of node i be in S_A state, I_A state, R_A state at time step t, respectively. Here we utilize a novel way to represent the transmissibility based on the sigmoid function

$$\phi_{ji}^A(t) = \frac{1}{1 + e^{-(\beta_{ji}^A(t) + \alpha^A r_i^A(t))}} \qquad (1)$$

where

$$r_i^A(t) = \sum_{B \in \Delta_{\mathbb{E}}, B \neq A} (p_i^{I_B}(t) + p_i^{R_B}(t)) h^{BA}. \qquad (2)$$

α^A is a tunable parameter and $\alpha^A > 0$. $p_i^{I_B}(t) + p_i^{R_B}(t)$ indicates the probability of node i has accepted feature B at time step t. h^{BA} is defined as the influence factor of feature B to A, where $h^{BA} > 0$ is called the cooperative factor when A and B belong to the same type of feature, while $h^{BA} < 0$ is called the competitive factor when A and B belong to different type of feature. In this way, $r_i^A(t)$ represents the integrated impact of all the other features to A at time step t. Specifically, $r_i^A(t) = 0$ means feature A diffuses from j to i with transmissibility $\phi_{ji}^A(t) = \frac{1}{1+e^{-\beta_{ji}^A(t)}}$ at time t and do not be affected by other features. When $r_i^A(t) > 0$ ($r_i^A(t) < 0$), the influence of cooperative features regarding A is larger (smaller) than the competitive features, as a consequence i has larger (smaller) probability to accept the feature A. In addition, the sigmoid function-based representation keeps the transmissibility always in the correct range of (0 1).

The evolution process of the CCMF can be characteristiced by the microscopic Markov chain approach (MMCA). Specifically, take the evolution of feature A, for example. It's MMCA equations are given by

$$p_i^{S_A}(t+1) = p_i^{S_A}(t)q_i^A(t), \qquad (3)$$

$$p_i^{I_A}(t+1) = p_i^{S_A}(t)(1 - q_i^A(t)) + p_i^{I_A}(t)(1 - \varphi_i^A(t)), \quad (4)$$

$$p_i^{R_A}(t+1) = p_i^{I_A}(t)\varphi_i^A(t) + p_i^{R_A}(t), \qquad (5)$$

where

$$q_i^A(t) = \prod_{j \in \Gamma_i^A} \left(1 - p_j^{I_A}(t)\phi_{ji}^A(t)\right)$$

$$= \prod_{j \in \Gamma_i^A} \left(1 - p_j^{I_A}(t) \right.$$

$$\left. \frac{1}{1 + e^{-(\beta_{ji}^A(t) + \alpha^A \sum_{B \in \Delta_\mathbb{E}, B \neq A}(p_i^{I_B}(t) + p_i^{R_B}(t))h^{BA})}} \right). (6)$$

$q_i^A(t)$ refers to the probability that node i is not infected by any I_A neighbors at time step t. Note that the above similar definitions and presentations apply to other features.

After the diffusions of all features reaching their stable status (e.g. at time T), the outcomes of the event $\mathbb{E}$ can be calculated through a decision-making function. We say node i is $\mathbb{E}$-active if

$$\sum_{A \in \Delta_\mathbb{E}} w_i^A p_i^{R_A}(T) \geq \theta_i. \qquad (7)$$

Otherwise, i is $\mathbb{E}$-deactivated. θ_i is the decision threshold of node i. w_i^A indicates the weight of feature A for i and $w_i^A > 0$ when A is positive feature, otherwise $w_i^A < 0$. Then the total number of $\mathbb{E}$-active nodes is given by

$$\mathcal{O} = \sum_{i \in V} \Big| \sum_{A \in \Delta_\mathbb{E}} w_i^A p_i^{R_A}(T) \geq \theta_i \Big|. \qquad (8)$$

where $\big| X \big| = 1$ if X holds, otherwise $\big| X \big| = 0$.

The above framework and analytic theory of CCMF is universal and have many advantages. It is suitable for the coevolution of any number of features and computationally efficient without state space explosion issue. The sigmoid function-based expression of transmissibility cleverly and reasonably integrates the impacts of the cooperative and competitive relationships between features, and also keeps its basic attribute at the same time. In addition, such expression allows the diffusions of different features following different sub-diffusion dynamics. Besides, although we use discrete-time model for the evolution of the CCMF in present work, the continuous-time model is also suitable, which can be derived easily. The use of the discrete-time model here mainly due to the requirement for solving the optimal control allocation problem discussed in Section IV. The MMCA equations derived under the assumption of discrete-time model is more easier controlled and handled by the forward-backward propagation algorithm to get the optimal control allocation solutions.

IV. OPTIMAL DEPLOYMENT OF CONTROL RESOURCE

The identification of "supernodes" under the constrain of limited resource is of prime significance in diverse contexts, ranging from the suppression of undesired diffusion and the promotion of beneficial diffusion. To maximize the outcomes of the event in CCMF (due to the similarity of the maximization and minimization of the CCMF under the following proposed control strategy, we here only discuss the maximization problem of the CCMF), three solution can be adopted: purely identify the "superblocks" to minimize the diffusion of negative features, purely identify the "superspreaders" to maximize the diffusion of positive features, and maximize the diffusion of positive features and minimize the diffusion of negative features simultaneously. In this paper, we focus on the third solution due to its innovation and difficulty and the potential best performance.

Therefore, the control objective regarding CCMF is to maximize

$$\mathcal{O} = \sum_{i \in V} \Big| \sum_{A \in \Delta_\mathbb{E}} w_i^A p_i^{R_A}(T) \geq \theta_i \Big|. \qquad (9)$$

However, in order to make the objective function (9) be more appropriately adapted to our optimization algorithm, a approximate representation of (9) is given by

$$\tilde{\mathcal{O}} = \sum_{i \in V} \frac{1}{1 + e^{-\tau(\sum_{A \in \Delta_\mathbb{E}} w_i^A p_i^{R_A}(T) - \theta_i)}}. \qquad (10)$$

τ is a tunable parameter and $\tau > 0$. Obviously, we have $\lim_{\tau \to \infty} \tilde{\mathcal{O}} = \mathcal{O}$.

To maximize the CCMF by maximizing the diffusion of positive features and minimizing the diffusion of negative features simultaneously, two control mechanisms respectively corresponding to the identification of "superspreaders" of the positive feature and "superblocks" of the negative feature can be adopted which are given by

$$i^{S_A}(t) \xrightarrow{v_i^A(t)} i^{I_A}(t+1) \text{ if } A \text{ is positive feature}$$

and

$$i^{S_B}(t) \xrightarrow{m_i^B(t)} i^{M_B}(t+1) \text{ if } B \text{ is negative feature}.$$

TABLE I
THE TOPOLOGY STRUCTURES OF THE MULTIPLEX NETWORKS

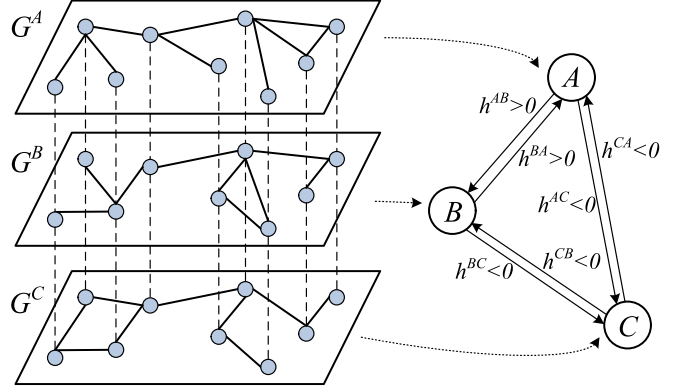| Multiplex network	Layer	Type	N	Average degree
Multiplex SF network	G^A	SF	3000	4
	G^B	SF	3000	8
	G^C	SF	3000	6
Multiplex ER network	G^A	ER	3000	4
	G^B	ER	3000	8
	G^C	ER	3000	6

Fig. 1. The framework of network and features in CCMF. The CCMF model has three features $\{A, B, C\}$ regarding event $\mathbb{E}$, where $A, B \in \Delta_{\mathbb{E}}^+$ and $C \in \Delta_{\mathbb{E}}^-$. The underlying multiplex network has three layers $G = \{G^A, G^B, G^C\}$.

Here, a novel state Immunized (M) is introduced which indicates the node is immunized against the feature and will not be infected by the feature. $v_i^A(t)$ and $m_i^B(t)$ refer to the transition probabilities of node i changed from S_A state to I_A state and S_B to M_B state at time t via external control respectively. Therefore, identifying the "superspreaders" and "superblocks" is converted to construct the optimal deployment of $\{v_i^A(t), m_i^B(t)\}_{i,t,A,B}$.

In this sense, the MMCA equations of the CCMF combined with the control are given by

• If A is a positive feature:

$$p_i^{S_A}(t+1) = p_i^{S_A}(t)(1 - v_i^A(t))q_i^A(t), \tag{11}$$

$$p_i^{I_A}(t+1) = p_i^{S_A}(t)(1 - (1-v_i^A(t))q_i^A(t)) + p_i^{I_A}(t)(1 - \varphi_i^A(t)), \tag{12}$$

$$p_i^{R_A}(t+1) = p_i^{I_A}(t)\varphi_i^A(t) + p_i^{R_A}(t), \tag{13}$$

• If B is a negative feature:

$$\mathcal{L} = \sum_i \frac{1}{1 + e^{-\tau(\sum_{C \in \Delta} w_i^C p_i^{R_C}(T) - \theta_i)}} + \sum_{A \in \Delta_{\mathbb{E}}^+} \left[\sum_{t=0}^{T-1} \lambda_v^A(t) \left(\sum_{i \in V} v_i^A(t) - \mathcal{B}_v^A(t) \right) + \varepsilon \sum_{t=0}^{T-1} \sum_{i \in V} \left(\log \left(v_i^A(t) - \underline{v_i^A(t)} \right) \right. \right.$$

$$+ \log \left(\overline{v_i^A(t)} - v_i^A(t) \right) \right) + \sum_{t=0}^{T-1} \sum_{i \in V} \lambda_i^{S_A}(t+1) \left(p_i^{S_A}(t+1) - p_i^{S_A}(t)(1 - v_i^A(t))q_i^A(t) \right) \sum_{t=0}^{T-1} \sum_{i \in V} \lambda_i^{I_A}(t+1) \left(p_i^{I_A}(t+1) \right.$$

$$- p_i^{S_A}(t)(1 - (1-v_i^A(t))q_i^A(t)) - p_i^{I_A}(t)(1 - \varphi_i^A(t)) \right) + \sum_{t=0}^{T-1} \sum_{i \in V} \lambda_i^{R_A}(t+1) \left(p_i^{R_A}(t+1) - p_i^{I_A}(t)\varphi_i^A(t) - p_i^{R_A}(t) \right)$$

$$+ \sum_{t=0}^{T} \sum_{i \in V} \lambda_i^{q^A}(t) \left(q_i^A(t) - \prod_{j \in \Gamma_i^A} \left(1 - p_j^{I_A}(t) \frac{1}{1 + e^{-(\beta_{ji}^A(t) + r_i^A(t))}} \right) \right) + \sum_{t=0}^{T} \sum_{i \in V} \lambda_i^{r^A}(t) \left(r_i^A(t) - \sum_{C \neq A} (p_i^{I_C}(t) + p_i^{R_C}(t))h^{CA} \right) \right]$$

$$+ \sum_{B \in \Delta_{\mathbb{E}}^-} \left[\sum_{t=0}^{T-1} \lambda_m^B(t) \left(\sum_{i \in V} m_i^B(t) - \mathcal{B}_m^B(t) \right) + \varepsilon \sum_{t=0}^{T-1} \sum_{i \in V} \left(\log \left(m_i^B(t) - \underline{m_i^B(t)} \right) + \log \left(\overline{m_i^B(t)} - m_i^B(t) \right) \right) \right.$$

$$+ \sum_{t=0}^{T-1} \sum_{i \in V} \lambda_i^{S_B}(t+1) \left(p_i^{S_B}(t+1) - p_i^{S_B}(t)(1 - m_i^B(t))q_i^B(t) \right) + \sum_{t=0}^{T-1} \sum_{i \in V} \lambda_i^{I_B}(t+1) \left(p_i^{I_B}(t+1) - p_i^{S_B}(t)(1 - m_i^B(t))(1 \right.$$

$$\left. - q_i^B(t)) - p_i^{I_B}(t)(1 - \varphi_i^B(t)) \right) + \sum_{t=0}^{T-1} \sum_{i \in V} \lambda_i^{R_B}(t+1) \left(p_i^{R_B}(t+1) - p_i^{I_B}(t)\varphi_i^B(t) - p_i^{R_B}(t) \right)$$

$$+ \sum_{t=0}^{T-1} \sum_{i \in V} \lambda_i^{M_B}(t+1) \left(p_i^{M_B}(t+1) - p_i^{M_B}(t) - p_i^{S_B}(t)m_i^B(t) \right) + \sum_{t=0}^{T} \sum_{i \in V} \lambda_i^{q^B}(t) \left(q_i^B(t) \right.$$

$$\left. - \prod_{j \in \Gamma_i^B} \left(1 - p_j^{I_B}(t) \frac{1}{1 + e^{-(\beta_{ji}^B(t) + r_i^B(t))}} \right) \right) + \sum_{t=0}^{T} \sum_{i \in V} \lambda_i^{r^B}(t) \left(r_i^B(t) - \sum_{C \neq B} (p_i^{I_C}(t) + p_i^{R_C}(t))h^{CB} \right) \right] \tag{22}$$

$$p_i^{S_B}(t+1) = p_i^{S_B}(t)(1-m_i^B(t))q_i^B(t), \tag{14}$$

$$p_i^{I_B}(t+1) = p_i^{S_B}(t)(1-m_i^B(t))(1-q_i^B(t))$$
$$+ p_i^{I_B}(t)(1-\varphi_i^B(t)), \tag{15}$$

$$p_i^{R_B}(t+1) = p_i^{R_B}(t) + p_i^{I_B}(t)\varphi_i^B(t), \tag{16}$$

$$p_i^{M_B}(t+1) = p_i^{M_B}(t) + p_i^{S_B}(t)m_i^B(t), \tag{17}$$

In addition, in reality the available control resources are not immutably fixed, but change over time, therefore we define the available total control resource for $v_i^A(t)$ and $m_i^B(t)$ at time t as

$$\sum_i v_i^A(t) \le \mathcal{B}_v^A(t) \tag{18}$$

and

$$\sum_i m_i^B(t) \le \mathcal{B}_m^B(t). \tag{19}$$

The $v_i^A(t)$ and $m_i^B(t)$ may also have their own variation ranges due to the external or internal constraints. That is, $v_i^A(t)$ and $m_i^B(t)$ can increase up to a certain maximum or decrease down to a certain minimum, i.e.,

$$\underline{v_i^A(t)} \le v_i^A(t) \le \overline{v_i^A(t)}. \tag{20}$$

and

$$\underline{m_i^B(t)} \le m_i^B(t) \le \overline{m_i^B(t)}. \tag{21}$$

We now could present the formal definition of optimal deployment of control resource for CCMF

Problem Definition: Given evolutionary process of CCMF (e.g. (11)–(17)) and the constrains of control resource (e.g. (18)–(21)), the optimal control of CCMF is to construct the optimal deployment of $\{v_i^A(t), \ m_i^B(t)\}_{i,t,A,B}$ to maximize $\tilde{\mathcal{O}}$.

From the Problem Definition, it can be found that our proposed optimal control problem of CCMF is universal, targeting arbitrary time window rather than just the initial time, considering both the network topology and diffusion dynamics, containing the optimization of maximization and minimization problems simultaneously.

In order to solve the optimal control problem of CCMF, we propose an optimization method, called OCCMF (Optimization of CCMF), based on the forward-backward propagation theory [33]. OCCMF is initialized by constructing Lagrangian formulation of our optimal control problem of the CCMF which is given by (22), shown at the bottom of the previous page, where "$\lambda_y^X(t)$" are the Lagrange multipliers. The first term of r.h.s. of (22) is the objective function. The second and third terms respectively correspond to the positive features and the negative features, each of which contains the resource constraints and the dynamical model constraints. Setting the derivatives of $\mathcal{L}$ with respect to $\{v_i^A(t), p_i^{S_A}(t), p_i^{I_A}(t), p_i^{R_A}(t), q_i^A(t), r_i^A(t)\}_{i,t,A\in\Delta_{\mathbb{E}}^+}$ and $\{m_i^B(t), p_i^{S_B}(t), p_i^{I_B}(t), p_i^{R_B}(t), p_i^{M_B}(t), q_i^B(t), r_i^B(t)\}_{i,t,B\in\Delta_{\mathbb{E}}^-}$ to zero, we obtain (23), shown at the bottom of the page, and (24) and (25), shown at the bottom of the next page.

Then we perform the forward-backward propagation iteration to derive the optimal deployment of $\{v_i^A(t), \ m_i^B(t)\}_{i,t,A,B}$. The process is described in Algorithm 1 and also formulated with more detail as follows.

$$\partial\mathcal{L}/\partial v_i^A(t) = \lambda_v^A(t) + \frac{\varepsilon}{v_i^A(t)-\underline{v_i^A(t)}} - \frac{\varepsilon}{\overline{v_i^A(t)} - v_i^A(t)} + \left(\lambda_i^{S_A}(t+1)p_i^{S_A}(t)q_i^A(t) - \lambda_i^{I_A}(t+1)p_i^{S_A}(t)q_i^A(t)\right)\Big|t\ne T\Big| = 0$$
$$\tag{23a}$$

$$\partial\mathcal{L}/\partial p_i^{S_A}(t) = \lambda_i^{S_A}(t) + \left(-\lambda_i^{S_A}(t+1)(1-v_i^A(t))q_i^A(t) - \lambda_i^{I_A}(t+1)(1-(1-v_i^A(t))q_i^A(t))\right)\Big|t\ne T\Big| = 0 \tag{23b}$$

$$\partial\mathcal{L}/\partial p_i^{I_A}(t) = \lambda_i^{I_A}(t) + \left(-\lambda_i^{I_A}(t+1)(1-\varphi_i^A(t)) - \lambda_i^{R_A}(t+1)\varphi_i^A(t)\right)\Big|t\ne T\Big|$$
$$+ \sum_{j\in\Gamma_i^A}\lambda_j^{q^A}(t)\frac{1}{1+e^{-(\beta_{ij}^A(t)+r_j^A(t))}}\prod_{k\in\Gamma_j^A,k\ne i}\left(1-p_k^{I_A}(t)\frac{1}{1+e^{-(\beta_{kj}^A(t)+r_j^A(t))}}\right) - \sum_{C\ne A}\lambda_i^{r^C}(t)h^{AC} = 0 \tag{23c}$$

$$\partial\mathcal{L}/\partial p_i^{R_A}(t) = \frac{\tau w_i^A e^{-\tau(\sum_A w_i^A p_i^{R_A}(T)-\theta_i)}}{\left(1+e^{-\tau(\sum_A w_i^A p_i^{R_A}(T)-\theta_i)}\right)^2}[t=T] + \lambda_i^{R_A}(t) - \lambda_i^{R_A}(t+1)\Big|t\ne T\Big| - \sum_{C\ne A}\lambda_i^{r^C}(t)h^{AC} = 0 \tag{23d}$$

$$\partial\mathcal{L}/\partial q_i^A(t) = \left(-\lambda_i^{S_A}(t+1)p_i^{S_A}(t))(1-v_i^A(t)) + \lambda_i^{I_A}(t+1)p_i^{S_A}(t)(1-v_i^A(t))\right)\Big|t\ne T\Big| + \lambda_i^{q^A}(t) = 0 \tag{23e}$$

$$\partial\mathcal{L}/\partial r_i^A(t) = \lambda_i^{r^A}(t) + \sum_{j\in\Gamma_i^A}p_j^{I_A}(t)e^{-(\beta_{ji}^A(t)+r_i^A(t))}\frac{1}{(1+e^{-(\beta_{ji}^A(t)+r_i^A(t))})^2}\lambda_i^{q^A}(t)\prod_{k\in\Gamma_i^A,k\ne j}\left(1-p_k^{I_A}(t)\frac{1}{1+e^{-(\beta_{ki}^A(t)+r_i^A(t))}}\right) = 0$$
$$\tag{23f}$$

1) $\{v_i^A(t), m_i^B(t)\}_{i,t,A,B}$ are initialized to be arbitrary values. Let *Count*=0.

2) Start from the given initial values of $\{p_i^{S_A}(0), p_i^{I_A}(0), p_i^{R_A}(0)\}_{i,A\in\Delta_{\mathbb{E}}^+}$ and $\{p_i^{S_B}(0), p_i^{I_B}(0), p_i^{R_B}(0), p_i^{M_B}(0)\}_{i,B\in\Delta_{\mathbb{E}}^-}$, propagate the MMCA (11)–(17) forward, up to the horizon T (the diffusions of all features have reached their stable status at time T). We obtain the values of $\{p_i^{S_A}(t), p_i^{I_A}(t), p_i^{R_A}(t), q_i^A(t) r_i^A(t)\}_{i,t,A\in\Delta_{\mathbb{E}}^+}$ and $\{p_i^{S_B}(t), p_i^{I_B}(t), p_i^{R_B}(t), p_i^{M_B}(t), q_i^B(t) r_i^B(t)\}_{i,t,B\in\Delta_{\mathbb{E}}^-}$.

3) In turn calculate the boundary values of all Lagrange multipliers at time T

- (23b) and (24b) assign $\lambda_i^{S_A}(T) = 0$ and $\lambda_i^{S_B}(T) = 0$ respectively.
- (24e) gives $\lambda_i^{M_B}(T) = 0$;
- (23e) and (24f) give $\lambda_i^{q^A}(T) = 0$ and $\lambda_i^{q^B}(T) = 0$ respectively;
- (23f) and (24g) assign $\lambda_i^{r^A}(T) = 0$ and $\lambda_i^{r^B}(T) = 0$ based on $\lambda_i^{q^A}(T)$ and $\lambda_i^{q^B}(T)$ respectively;

- (23d) assigns $\lambda_i^{R_A}(T) = -\dfrac{\tau w_i^A e^{-\tau(\sum_{A\in\Delta_{\mathbb{E}}} w_i^A p_i^{R_A}(T)-\theta_i)}}{\left(1+e^{-\tau(\sum_{A\in\Delta_{\mathbb{E}}} w_i^A p_i^{R_A}(T)-\theta_i)}\right)^2}$

  and (24d) assigns $\lambda_i^{R_B}(T) = -\dfrac{\tau w_i^B e^{-\tau(\sum_{A\in\Delta_{\mathbb{E}}} w_i^A p_i^{R_A}(T)-\theta_i)}}{\left(1+e^{-\tau(\sum_{A\in\Delta_{\mathbb{E}}} w_i^A p_i^{R_A}(T)-\theta_i)}\right)^2}$ respectively;

- (23c) and (24c) assign $\lambda_i^{I_A}(T) = 0$ and $\lambda_i^{I_B}(T) = 0$ respectively;
- From (23a), we have where

$$Lv_i^A(T-1) = \lambda_i^{S_A}(T)p_i^{S_A}(T-1)q_i^A(T-1) - \lambda_i^{I_A}(T)p_i^{S_A}(T-1)q_i^A(T-1). \tag{26}$$

Based on the resource constraint $\sum_i v_i^A(T-1) = \mathcal{B}_v^A(T-1)$, we can numerically calculate $\lambda_v^A(T-1)$, then obtain $v_i^A(T-1)$. In same way, we could obtain $\lambda_m^B(T-1)$ and $m_i^B(T-1)$ based on (24a).

$$\partial\mathcal{L}/\partial m_i^B(t) = \lambda_m^B(t) + \frac{\varepsilon}{m_i^B(t) - \underline{m_i^B(t)}} - \frac{\varepsilon}{\overline{m_i^B(t)} - m_i^B(t)} + \left(\lambda_i^{S_B}(t+1)p_i^{S_B}(t)q_i^B(t)\right.$$
$$\left. + \lambda_i^{I_B}(t+1)p_i^{S_B}(t)(1-q_i^B(t)) - \lambda_i^{M_B}(t+1)p_i^{S_B}(t)\right)\Big|t \neq T\Big| = 0 \tag{24a}$$

$$\partial\mathcal{L}/\partial p_i^{S_B}(t) = \lambda_i^{S_B}(t) + \left(-\lambda_i^{S_B}(t+1)(1-m_i^B(t))q_i^B(t) - \lambda_i^{I_B}(t+1)(1-m_i^B(t))(1-q_i^B(t)) - \lambda_i^{M_B}(t+1)m_i^B(t)\right)\Big|t \neq T\Big| = 0 \tag{24b}$$

$$\partial\mathcal{L}/\partial p_i^{I_B}(t) = \lambda_i^{I_B}(t) + \left(-\lambda_i^{I_B}(t+1)(1-\varphi_i^B(t)) - \lambda_i^{R_B}(t+1)\varphi_i^B(t)\right)\Big|t \neq T\Big|$$
$$+ \sum_{j\in\Gamma_i^B} \lambda_j^{q^B}(t)\frac{1}{1+e^{-(\beta_{ij}^B(t)+r_j^B(t))}} \prod_{k\in\Gamma_j^B, k\neq i}\left(1 - p_k^{I_B}(t)\frac{1}{1+e^{-(\beta_{kj}^B(t)+r_j^B(t))}}\right) - \sum_{C\neq B}\lambda_i^{r^C}(t)h^{BC} = 0 \tag{24c}$$

$$\partial\mathcal{L}/\partial p_i^{R_B}(t) = \frac{\tau w_i^B e^{-\tau(\sum_B w_i^B p_i^{R_B}(T)-\theta_i)}}{\left(1+e^{-\tau(\sum_B w_i^B p_i^{R_B}(T)-\theta_i)}\right)^2}[t = T] + \lambda_i^{R_B}(t) - \lambda_i^{R_B}(t+1)\Big|t \neq T\Big| - \sum_{C\neq B}\lambda_i^{r^C}(t)h^{BC} = 0 \tag{24d}$$

$$\partial\mathcal{L}/\partial p_i^{M_B}(t) = \lambda_i^{M_B}(t) - \lambda_i^{M_B}(t+1)\Big|t \neq T\Big| = 0 \tag{24e}$$

$$\partial\mathcal{L}/\partial q_i^B(t) = \left(-\lambda_i^{S_B}(t+1)p_i^{S_B}(t)(1-m_i^B(t)) + \lambda_i^{I_B}(t+1)p_i^{S_B}(t)(1-m_i^B(t))\right)\Big|t \neq T\Big| + \lambda_i^{q^B}(t) = 0 \tag{24f}$$

$$\partial\mathcal{L}/\partial r_i^B(t) = \lambda_i^{r^B}(t) + \sum_{j\in\Gamma_i^B} p_j^{I_B}(t)e^{-(\beta_{ji}^B(t)+r_i^B(t))}\frac{1}{(1+e^{-(\beta_{ji}^B(t)+r_i^B(t))})^2}\lambda_i^{q^B}(t)$$
$$\prod_{k\in\Gamma_i^B, k\neq j}\left(1 - p_k^{I_B}(t)\frac{1}{1+e^{-(\beta_{ki}^B(t)+r_i^B(t))}}\right) = 0 \tag{24g}$$

$$v_i^A(T-1) = \frac{\lambda_v^A(T-1) + Lv_i^A(T-1) - 2\varepsilon \pm \sqrt{(\lambda_v^A(T-1) + Lv_i^A(T-1))^2 + 4\varepsilon^2}}{2(\lambda_v^A(T-1) + Lv_i^A(T-1))} \tag{25}$$
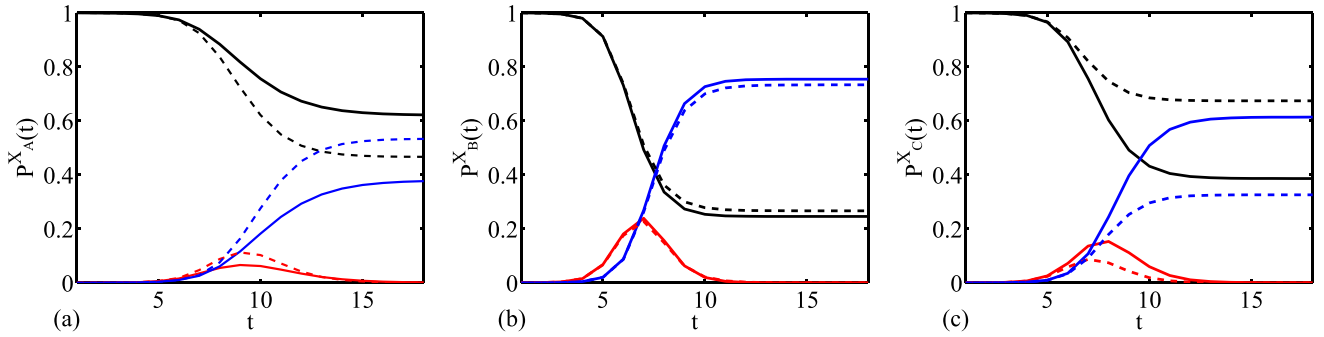
Fig. 2. The densities of susceptible nodes (black line), infected nodes (red line) and recovered nodes (blue line) versus the time. Panels (a), (b), and (c) correspond to features A, B and C respectively. The Figure shows the difference of the MFD with (the dashed lines) and without (the solid lines) the interactions between features. The used network is the multiplex SF network.

Algorithm 1: OCCMF algorithm.

Initialize

$\{v_i^A(t)\}_{i,t,A\in\Delta_{\mathbb{E}}^+} \leftarrow$ random value, $\{m_i^B(t)\}_{i,t,B\in\Delta_{\mathbb{E}}^-} \leftarrow$
random value, Count$\leftarrow$0;

do

  for $t=0$ to T

    Compute $\{p_i^{S_A}(t), p_i^{I_A}(t), p_i^{R_A}(t), q_i^A(t)$
    $r_i^A(t)\}_{i,A\in\Delta_{\mathbb{E}}^+}$ and $\{p_i^{S_B}(t), p_i^{I_B}(t), p_i^{R_B}(t), p_i^{M_B}(t),$
    $q_i^B(t)\ r_i^B(t)\}_{i,B\in\Delta_{\mathbb{E}}^-}$ based on (11)–(17) in turn;

  for $t=T$ to 0

    Compute $\lambda_i^{S_A}(t), \lambda_i^{S_B}(t), \lambda_i^{M_B}(t), \lambda_i^{q^A}(t), \lambda_i^{q^B}(t),$
    $\lambda_i^{r^A}(t), \lambda_i^{r^B}(t), \lambda_i^{R_A}(t), \lambda_i^{R_B}(t), \lambda_i^{I_A}(t)$ and $\lambda_i^{I_B}(t)$
    based on
    (23b), (24b), (24e), (23e), (24f), (23f), (24g), (23d),
    (24d), (23c) and (24c) in turn;
    Compute $\lambda_v^A(t-1)$ and $v_i^A(t-1)$ based on (23a);
    Compute $\lambda_m^B(t-1)$ and $m_i^B(t-1)$ based on (24a);
  Count $\leftarrow$ Count+1;

while Count<*fixed number*;

return $\{v_i^A(t)\}_{i,t,A\in\Delta_{\mathbb{E}}^+}, \{m_i^B(t)\}_{i,t,B\in\Delta_{\mathbb{E}}^-}$.

---

4) Based on the obtained boundary values at step (2) and the values of $v_i^A(T-1)_{i,A\in\Delta_{\mathbb{E}}^+}$ and $m_i^A(T-1)_{i,B\in\Delta_{\mathbb{E}}^-}$, back propagate (23) and (24) to calculate all multipliers and $m_i^A(t)$ and $v_i^A(t)$ regarding all nodes, times and features step by step.

5) *Count=Count*+1. If *Count* equals to the fixed number (the fixed number is 8 in this paper), the algorithm terminates, otherwise go back to Step (2).

The OCCMF is designed based on the MMCA equations of the CCMF and thus is also computationally efficient. It derives the optimal deployment of $\{v_i^A(t), m_i^B(t)\}_{i,t,A,B}$ by considering both the network topology, the diffusion dynamics and the interplays between features, and thus the result is comprehensively.

V. RESULTS

Simulation experiments are performed in this section. The used CCMF model and the multiplex networks are first illustrated. Then we analyze the dynamic characteristics of the CCMF and verify the validity of the proposed control strategy.

A. Experimental Design

Based on the discussions in above two section, it is proved that our proposed framework and analytic theory for the CCMF and the control strategy are all universal. They are suitable for the coevolution of any number of features and computationally efficient for the diffusions of different features following different sub-diffusion dynamics.

Therefore, for simplicity, we consider a CCMF model with three features $\{A, B, C\}$ regarding event $\mathbb{E}$, where $A, B \in \Delta_{\mathbb{E}}^+$ and $C \in \Delta_{\mathbb{E}}^-$. The three features diffuse following the SIR model. We assume the diffusions of each feature have same dynamical parameters, such as $\beta_{ji}^Y(t) = -1$ and $\varphi_i^Y(t) = 1$ for all nodes i and j, feature Y and time t. Corresponding to the CCMF model, the underlying network is a multiplex network with three layers $G = \{G^A, G^B, G^C\}$. Here, we consider two commonly used multiplex networks models, the multiplex scale-free (SF) network and the multiplex Erdős-Rényi (ER) network. Each layers of the multiplex SF networks are all SF networks. Meanwhile, in multiplex ER network each layers are ER networks. Besides, we assume all layers of the multiplex networks have same node sizes, i.e. $N = 3000$. The average degrees of the layers are $\langle k_A \rangle = 4$, $\langle k_B \rangle = 8$ and $\langle k_C \rangle = 6$ respectively. $P^{X_Y}(t) = \sum_i p_i^{X_Y}(t)/N$ indicates the density of X_Y state nodes at time t. Unless otherwise specified, the other corresponding presentation of diffusion model and the network topology are consistent with that in above two sections. The framework of network and CCMF is shown in Fig. 1 and the detailed specifications of the two multiplex networks are listed in Table I.

B. Experimental Analysis

The difference of the MFD with (the dashed lines) and without (the solid lines) the interactions between features is uncovered in Fig. 2. It can be found that the influence of cooperation and competition between features in MFD is pretty obvious. For example, feature A outbreaks easier and diffuses larger scale than it diffuses independently, due to the strong cooperation from B (large average degree) and weak competition from C (small average degree). On the contrary, the outbreak size of B becomes smaller than it diffuses independently, since the influence of cooperation of A is weaker than the competition of C. The feature C is
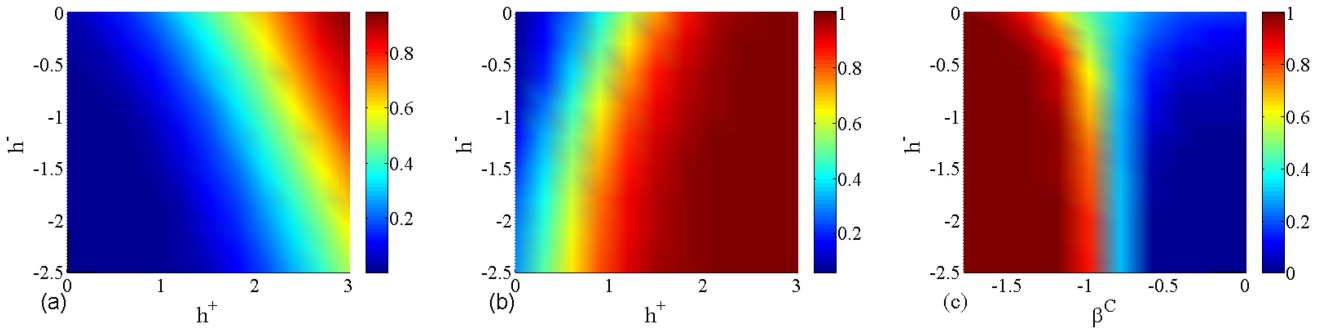
Fig. 3.   The evolutions of $\phi_i^Y(t)$ and $r_i^Y(t)$ of two nodes, $i = 2$ in panel (a) and $i = 2000$ in panel (b). The evolutions of the transmissibility reflects the effect of the cooperation and competition between features. The used network is the multiplex SF network.

significantly suppressed since it just gets competitions from other features.

In essence, the effect of the cooperation and competition between features is mainly featured in changing the transmissibility of feature. We show such effect through two random selected nodes in Fig. 3. One sees that the transmissibility of feature Y between node i and its neighbors ($\phi_{ji}^Y(t) = \phi_i^Y(t)$ for $\forall j$) changes with $r_i^Y(t)$ in same trend. The transmissibility gets bigger when the $r_i^Y > 0$. That is the nodes will be more easier to accepted feature Y when it has accepted more same type of other features, and vice versa.

Fig. 4 shows the impacts of the influence factors and diffusion dynamical parameters on the outcomes of the CCMF. The color of the panel represents the final density of the $\mathbb{E}$-active nodes, i.e. $\rho = \mathcal{O}/N$. We let the cooperative factor $h^{AB} = h^{BA} = h^+$ and the competitive factor $h^{AC} = h^{CA} = h^{BC} = h^{CB} = h^-$. In Fig. 4(a), same parameters and network topology are used with that in Fig. 1 except $\beta_{ji}^C(t)$. Here, we let $\beta_{ji}^C(t) = \beta^C = -0.6$ for any i, j and t. A normal result is obtained that ρ increases with the enhance of cooperation between features and decreases with the enhance of competition. However, if we change β^C to -1 which indicates a smaller transmissibility of feature C is adopted, an opposite result appears. As shown in Fig. 4(b), the density of $\mathbb{E}$-active nodes ρ increases with the enhance of competition, which is contrary to our intuitive cognition. In order to explain such phenomenon, we discuss the relationships of β^C, h^- and ρ. From Fig. 4(c), an very interesting result is obtained: there is an obvious threshold of β^C, denoted as β_c^C, below which, ρ increases with the enhance of competition, otherwise, ρ decreases with the enhance of competition. If we call the β^C the competition ability, an interesting conclusion can be derived: When the feature has weak competition ability, performing competition with high intensity could more easily lead to failure in competition. The high competition intensity require strong competition ability to get positive benefits.

In Fig. 5, we verify the efficiency of our proposed OCCMF optimization algorithm. OCCMF could identify the optimal deployment of $\{v_i^A(t), m_i^B(t)\}_{i,t,A,B}$ under the limited control resource simultaneously. It considers both the network topology, the diffusion dynamics and the interplays between features.
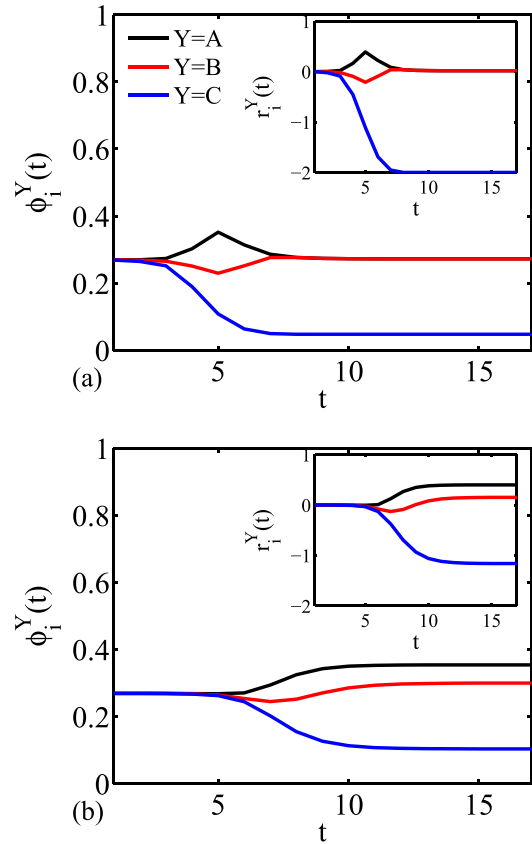

Fig. 4.   Panel (a) and (b): ρ versus h^+ and h^- for different values of β^C. Panel (c): ρ versus β^C and h^-. The used network is the multiplex SF network.

Panel (a) shows the results of different control optimization for the CCMF model on both multiplex scale free networks and multiplex random networks. Here we assume that the control resources are available and unchanged at time 0 to 6 which are given by $\mathcal{B}_v^A(t) = 0.005\,N$ and $\mathcal{B}_m^A(t) = 0.005\,N$ for any A and $t \in [0\ 6]$. Different case is also considered in panel (b) in which the control resources change with time as shown in the inset. The dark blue bar corresponds to no available control resource. The light blue bar refers to the random deployment of all control resource. The green bar only optimizes the features A and B via
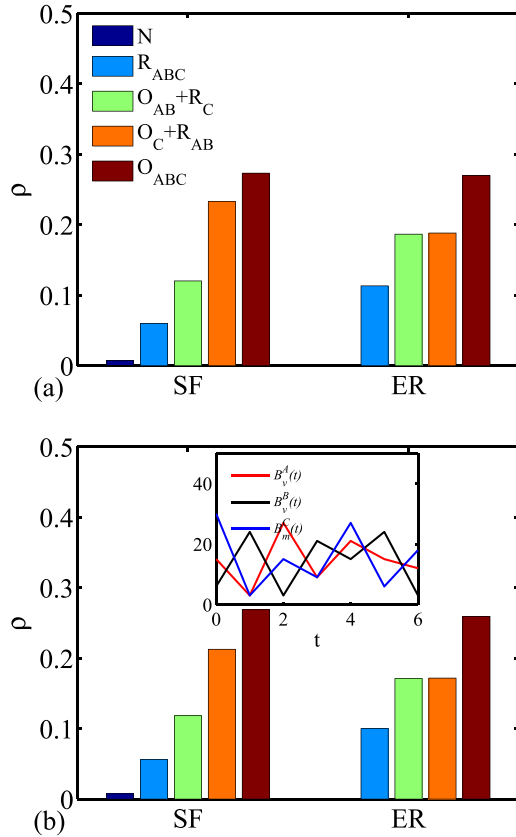
Fig. 5. The comparison of different optimization schemes. N indicates no available control resource. R_{ABC} refers to the random deployment of all control resource. $O_{AB}+R_C$ only optimizes the features A and B via the OCCMF, feature C adopts the random deployment method. O_C+R_{AB} only optimizes the feature C via the OCCMF, features A and B adopts the random deployment method. O_{ABC} optimizes A, B and C via the OCCMF simultaneously. SF and ER indicate the multiplex SF network and the multiplex ER network respectively.

the OCCMF, feature C adopts the random deployment method. Correspondingly, the orange bar only optimizes the feature C via the OCCMF, features A and B adopts the random deployment method. The brown bar optimizes A, B and C via the OCCMF simultaneously. For unchanged control source in panel (a), it can be found that our OCCMF algorithm shows remarkable effect on the optimization of control resources. The $\mathbb{E}$-active nodes obtained from OCCMF is about 5 times the final density of the random deployment method for SF network, and about 2.5 times for ER network. In addition, the simultaneous optimization of A, B and C is also remarkably better than the optimization of parts of features. Panel (b) presents the outcomes of the OCCMF under changed control resources. The OCCMF is still the best which further illustrates its universality and extensive suitability.

VI. CONCLUSION

The MFD well characterized the scenario where the event is determined by the diffusion of multiple features regarding the event. In this paper, we focus on MFD and introduced

the cooperative relationship and the competitive relationship between features into the MFD, and proposed the model of CCMF in multiplex network. Then an MMCA was presented which is applicable and computationally efficient to modeling the evolution process of CCMF with any number of features and different sub-diffusion dynamics. By studying the impacts of influence factor and transmissibility on the CCMF, an interesting founding was obtained. That is performing high intensity competition under weak competition ability is more easier to result in negative effect. To optimal control the CCMF under given resource, an optimization strategy OCCMF was proposed which has universality and extensive suitability. OCCMF could optimally allocate the given control resource and could promote and suppress the network diffusion simultaneously.

In this novel field of multi-feature diffusion dynamics, several challenges remain. In future, the other relationships which are different from the cooperation and competition, between features should be considered [60]. The corresponding analytical framework and control theory should be developed. Besides, we also need to study the continuous-time models of the MDF to meet different practical application demands. The further researches on more effective control strategies of the MDF are also needed.

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Dawei Zhao received the Ph.D. degree in cryptology from the Beijing University of Posts and Telecommunications, Beijing, China, in 2014. He is currently a Professor with the Shandong Computer Science Center (National Supercomputer Center), Jinan, China. His main research interests include network security, complex network and epidemic spreading dynamics.

Shudong Li received the M.S. degree in applied mathematics from Tongji University, Shanghai, China, in June 2005, and the Ph.D. degree in information security from the Beijing University of Posts and Telecommunications, Beijing, China, in July 2012. From 2013-2018, he was a Postdoc with the National University of Defense Technology, Changsha, China. He is currently an Associate Professor with the Cyberspace Institute of Advanced Technology, Guangzhou University, Guangzhou, China. His research interests include Big Data and its security, IoT security, information security, and cryptography, the robustness of complex networks. His was the recipient of the 2015 First prize of scientific and technological progress in Hunan province, China, and 2018 First prize of Chans Chinese information processing science and technology, China. He was also the recipient of the best paper award at CIAT 2020.

Haipeng Peng received the M.S. degree in system engineering from the Shenyang University of Technology, Shenyang, China, in 2006, and the Ph.D. degree in signal and information processing from the Beijing University of Posts and Telecommunications, Beijing, China, in 2010. He is currently a Professor with the School of CyberSpace Security, Beijing University of Posts and Telecommunications. He has coauthored 100 scientific papers. His research interests include information security, network security, complex networks, and control of dynamical systems.

Zhen Wang received the Ph.D. degree from Hong Kong Baptist University, Hong Kong, in 2014. From 2014 to 2016, he was a JSPS Senior Researcher with the Interdisciplinary Graduate School of Engineering Sciences, Kyushu University, Fukuoka, Japan. Since 2017, he has been a Full Professor with Northwestern Polytechnical University, Xi'an, China. He has authored or coauthored more than 100 scientific papers and obtained more than 16000 citations. His research interests include network science, complex system, Big Data, evolutionary game theory, behavior decision, and behavior recognition. He is currently the Editor or Academic Editor for seven journals.