RESEARCH ARTICLE

Semi-Global Leaderless Output Consensus of Heterogeneous Multi-Agent Systems With Input Saturations

KWANG-KYO OH [ID] [1] AND YOUNG-HUN LIM [ID] [2]
[1]Department of Electrical Engineering, Sunchon National University, Sunchon 57922, South Korea
[2]Department of Convergence Electronic Engineering, Gyeongsang National University, Jinju 52828, South Korea

Corresponding author: Young-Hun Lim (yhunlim@gnu.ac.kr)

ABSTRACT This paper studies the semi-global leaderless output consensus problem for heterogeneous multi-agent systems with input saturations. To solve the consensus problem under the input saturations, the consensus trajectory should be bounded such that the control inputs of all agents are not saturated. However, generating a bounded trajectory satisfying this condition is not an easy problem. Accordingly, we construct the modified reference generator to generate the bounded trajectory for any bounded initial states, and two types of distributed controllers depending on the available information are proposed to track the generated trajectory. Specifically, we construct the distributed controllers using the state feedback and the output feedback based on the output regulation approach, respectively. Then, utilizing the low gain method, we show the existence of the solution for the semi-global output consensus problem. Finally, we provide numerical examples to demonstrate the theoretical results.

INDEX TERMS Distributed control, heterogeneous agent, multi-agent system, saturation, semi-global consensus.

I. INTRODUCTION

Consensus problem for multi-agent systems has been widely studied over several decades since it has been extensively applied in many fields, such as formation control [1], [2], cooperative control [3], distributed filtering [4], [5], etc. The goal of consensus is to achieve an agreement among agents using local information exchange [6].

In the consensus problem, most of the pioneer works have focused on homogeneous agents, whose dynamics are identical, for instance, single integrator [6], [7], double integrator [8], and higher order dynamics [9], [10], etc. Moreover, the consensus problem for agents with unknown nonlinearities has been studied in [11] and [18] using the neural network control and the fuzzy adaptive control, respectively. In practice, the agents' dynamics contain different characteristics,

The associate editor coordinating the review of this manuscript and approving it for publication was Haixia Cui [ID].

and, thus, the consensus problem for heterogeneous agents, whose dynamics are nonidentical, has been widely studied in recent years. The necessary and sufficient conditions for the consensus of the heterogeneous agents can be summarized as the internal model principle for synchronization, which implies the existence of a common model in the agents with local controllers [13]. Then, applying the output regulation approach, the dynamic consensus algorithms were proposed in [13], [14], [15], and [16]. Moreover, in [17] and [18], the quasi-synchronization problem has been studied since it is difficult to guarantee the existence of the solution given by the internal model principle.

Meanwhile, the problem of input saturation is an essential issue since input saturations can lead to not only performance degradation but also the instability of systems. In the presence of input saturation, it is well known that global asymptotic stabilization cannot be achieved using a linear feedback control [19], [20]. On the other hand, applying

a low gain approach, a semi-global stabilization can be achieved for a linear system that is asymptotic null controllable with bounded controls(ANCBC) [20]. Therefore, most of the existing works considering the input saturations have studied a semi-global consensus problem for ANCBC heterogeneous agents applying the output regulation theory and the low gain approach [21], [22], [23], [24], [27]. The goal of semi-global consensus is to achieve an agreement for any initial conditions in a priori given arbitrarily large bounded set by low gain feedback such that the control input will not saturate. Note that the distributed controller based on the output regulation approach contains the feedforward control to render the consensus trajectory invariant. Therefore, the consensus trajectory should be bounded such that the control inputs of agents do not saturate. Then, under the assumption that the leader's trajectory is bounded within the prescribed bounded set, the leader-following consensus problem for ANCBC heterogeneous agents with input saturations have been studied in [21], [22], and [23]. In [21] and [22], the semi-global leader-following consensus problem was studied with fixed and switching topologies, respectively. In [23], the control input of the leader agent was considered and the fully distributed controller with the discontinuous functions was proposed. In [24], the suboptimal consensus algorithm was proposed using the model-based and data-driven methods.

Different from the leader-following consensus problem with input saturations, whose reference trajectory given by the leader agent is bounded, we need to generate the bounded trajectory in the leaderless consensus problem since the reference trajectory is not given. However, it is difficult to determine the bounded trajectory such that the control inputs of all agents are not saturated. Therefore, the leaderless consensus problem, which is more challenging, has rarely been addressed. In [25], the agents consisting of single and double integrators were considered. In [26], the second-order heterogeneous agents were considered. In [27], the semi-global leaderless state consensus problem for general ANCBC agents was studied and the modified reference generator based on the anti-windup technique was proposed to generate the bounded trajectory for any bounded initial conditions.

In this paper, we study the semi-global leaderless output consensus problem for heterogeneous multi-agent systems with input saturations. Depending on the available information, two types of distributed controllers are proposed to solve the leaderless output consensus problem by applying the regulation approach and the modified reference generator [27]. Specifically, when the state information is available, the distributed controller using the state feedback is proposed. When the output information is available, the distributed controller with the state observer is designed for each agent. The main contributions of this paper can be summarized as follows. First, the leaderless output consensus problem for the heterogeneous agents with input saturations is investigated. Although [27] studied the leaderless consensus problem with

the relaxed initial conditions of the reference generators, they focused on the state consensus problem, which requires the identical state dimensions of agents. However, in practical applications, the agents may have different dimensions, and, thus, the output consensus problem is more realistic and challenging. Second, by utilizing the low gain approach, we show the existence of the solution for the semi-global output consensus problem. Although the control gain requires global information, by choosing the sufficiently small control gain, we can achieve the output consensus. Moreover, the control gain can be designed in a distributed way by applying the min-consensus algorithm [28].

The rest of this paper is organized as follows. In Section II, preliminaries and problem formulation are presented. In Section III, the semi-global leaderless output consensus problem is solved by constructing two consensus algorithms. In Section IV, the simulation results are presented, and the conclusion and future work are addressed in Section V.

II. PRELIMINARIES AND PROBLEM STATEMENT
A. NOTATIONS AND GRAPH THEORY

For a vector x, we denote the Euclidean norm as $\|x\|$ and the infinity norm as $\|x\|_\infty$. For a matrix A, we denote the largest eigenvalue as $\lambda_M(A)$. For matrices A and B, $(AB)_k$ denote the kth row of AB. $\otimes$ denotes the Kronecker product.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be an weighted undirected graph of order N, with $\mathcal{V} = \{1, 2, \ldots, N\}$ representing the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of undirected edges, and $\mathcal{A} = [a_{ij}] \in \mathbf{R}^{N \times N}$ the underlying weighted adjacency matrix. When $(i, j) \in \mathcal{E}$, the nodes i and j are called adjacent, which means two nodes can exchange information with each other. The weights $a_{ij} = a_{ji} > 0$ if and only if $(i, j) \in \mathcal{E}$; otherwise $a_{ij} = a_{ji} = 0$.

The Laplacian matrix $L = [l_{ij}] \in \mathbf{R}^{N \times N}$ is defined as $l_{ii} = \sum_{j=1, j \neq i}^N a_{ij}$, $l_{ij} = -a_{ij}$, $i \neq j$. The Laplacian matrix of the undirected graph is a positive semi-definite real symmetric matrix, thus all eigenvalues of L are non-negative real. An undirected graph $\mathcal{G}$ is called connected if between any pair of distinct nodes i and j there exists a path between i and j, $i, j \in \mathcal{V}$. An undirected graph is connected if and only if its Laplacian has rank $N - 1$. In this case, the zero eigenvalue has a multiplicity one, and all other eigenvalues are positive. The remaining $N - 1$ eigenvalues are ordered in increasing order as $0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$.

Lemma 1 [8]: For an undirected graph and any $x_i \in \mathbf{R}$, $i = 1, \ldots, N$, we have

$$\sum_{i=1}^N \sum_{i=1}^N a_{ij} x_i (x_j - x_i) = -x^T L x,$$

where $x = [x_1, \ldots, x_N]^T$. Moreover, for a connected undirected graph, $x^T L x = 0$, if and only if $x_i - x_j = 0$, $\forall i, j = 1, \ldots, N$.

B. PROBLEM STATEMENT

In this paper, we consider a group of N agents described by for $i \in \mathcal{V} := \{1, \ldots, N\}$:

$$\dot{x}_i = A_i x_i + B_i \text{sat}(u_i),$$
$$y_i = C_i x_i, \tag{1}$$

where $x_i \in \mathbf{R}^{n_i}$, $u_i \in \mathbf{R}^{m_i}$, and $y_i \in \mathbf{R}^p$ are the state, input, and output of agent i, respectively, and $A_i \in \mathbf{R}^{n_i \times n_i}$, $B_i \in \mathbf{R}^{n_i \times m_i}$, $C_i \in \mathbf{R}^{p \times n_i}$. The function sat : $\mathbf{R}^{m_i} \to \mathbf{R}^{m_i}$ is a normalized vector valued saturation function described by

$$\text{sat}(u_i) = \begin{bmatrix} \text{sat}((u_i)_1), \text{sat}((u_i)_2), \ldots, \text{sat}((u_i)_{m_i}) \end{bmatrix}^T \tag{2}$$

with $\text{sat}((u_i)_k) = \text{sign}((u_i)_k) \min\{|(u_i)_k|, 1\}$.

The communication topology among the agents is described by an weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. Then, in this paper, we construct the distributed controller to solve the output consensus problem for the heterogeneous agents (1), i.e., $\lim_{t \to \infty}(y_i - y_j) = 0$, $\forall i, j \in \mathcal{V}$. The output consensus implies the agents can generate a common trajectory. Therefore, we assume that the existence of common trajectory, which is given by the following internal model principle for synchronization [13]:

Assumption 1: There exist matrices $S \in \mathbf{R}^{q \times q}$, $\Pi_i \in \mathbf{R}^{n_i \times q}$, $\Gamma_i \in \mathbf{R}^{m_i \times q}$, and $R \in \mathbf{R}^{p \times q}$ satisfying the following equations:

$$A_i \Pi_i + B_i \Gamma_i = \Pi_i S,$$
$$C_i \Pi_i = R. \tag{3}$$

Remark 1: Note that *Assumption 1* is a necessary condition for the output consensus of the heterogeneous agents [13]. The condition (3) is known as the regulator equation, which implies that all agents can track the trajectory of the virtual exosystem described by the system matrix S and the output matrix R. In other words, the output consensus can be reached by regulating each agent to track the trajectory of the virtual exosystem, and thus the regulation equation (3) should be satisfied. The solvability of (3) is discussed in [29].

Remark 2: In practice, the equation (3) does not always have a solution. Accordingly, the quasi-synchronization problem without *Assumption 1* has been studied in [17] and [18]. However, the goal of this paper is to achieve the perfect consensus, and thus *Assumption 1* is required.

We next briefly introduce the standard approach to the output consensus problem without the input saturations, i.e., $\text{sat}(u_i) = u_i$. Based on the regulation approach, the output consensus problem can be solved in two steps. First, the distributed reference generators are designed to generate the common trajectory using local information exchanges. Second, the observer-based controllers are designed to track the trajectory of the local reference generator. Then, the following distributed controller can solve the output consensus problem without the input saturations:

$$\dot{w}_i = Sw_i + \sum_{j=1}^{N} a_{ij}(w_j - w_i), \tag{4a}$$

$$\dot{z}_i = A_i z_i + B_i u_i + H_i(C_i z_i - y_i), \tag{4b}$$
$$u_i = K_i(z_i - \Pi_i w_i) + \Gamma_i w_i. \tag{4c}$$

As mentioned above, the distributed reference generator (4a) converges to the common trajectory generated by the following dynamics:

$$\dot{\bar{w}} = S\bar{w}, \quad \bar{w}(t_0) = \frac{1}{N} \sum_{i=1}^{N} w_i(t_0). \tag{5}$$

Then, by regulating the reference signal w_i using the state observer (4b) and the feedback controller (4c), we can achieve the output consensus, i.e., $\lim_{t \to \infty} \|z_i - x_i\| = 0$ and $\lim_{t \to \infty} \|z_i - \Pi_i w_i\| = \|C_i z_i - R w_i\| = 0$, $\forall i \in \mathcal{V}$.

Note that, when the agents achieve the consensus, the control input becomes $u_i = \Gamma_i \bar{w}$ which makes the consensus trajectory invariant from *Assumption 1*. Thus, in the presence of input saturations, the trajectory generated by (5) should be bounded such that the control input of each agent does not saturate, i.e., $\text{sat}(\Gamma_i \bar{w}) = \Gamma_i \bar{w}$. To generate the bounded trajectory, the initial states of the reference generators (4a) should be bounded. However, it is difficult to determine the initial states that satisfy this condition. To relax this condition, [27] proposed the modified reference generator to generate the bounded trajectory for any initial conditions in any priori given bounded set. Inspired by the work [27], this paper studies the semi-global output consensus problem by constructing two types of distributed controllers depending on the available information. Then, the problems studied in this paper can be summarized as follows:

Problem 1 (Semi-Global Output Consensus Using State Feedback): Consider a group of N agents given by (1). Construct a distributed controller using state feedback such that, for any priori given bounded initial sets $\mathbf{X} \subset \mathbf{R}^{nN}$ with $n = \sum_{i=1}^{N} n_i$, $\mathbf{W} \subset \mathbf{R}^{qN}$,

$$\lim_{t \to \infty} \|y_i - y_j\| = 0, \quad \forall i, j \in \mathcal{V}, \tag{6}$$

as long as $(x(t_0), w(t_0)) \in \mathbf{X} \times \mathbf{W}$, where $x = [x_1^T, \ldots, x_N^T]^T$ and $w = [w_1^T, \ldots, w_N^T]^T$.

In *Problem II-B*, it is assumed that each agent can measure the full state information. However, in many real applications, the full state information is not always available, and instead, the output information is most accessible. Therefore, we next consider the consensus problem using output feedback control, which can be summarized as follows:

Problem 2 (Semi-Global Output Consensus Using Output Feedback): Consider a group of N agents given by (1). Construct the distributed controller using output feedback such that, for any priori given bounded initial sets $\mathbf{X}, \mathbf{Z} \subset \mathbf{R}^{nN}$ with $n = \sum_{i=1}^{N} n_i$, $\mathbf{W} \subset \mathbf{R}^{qN}$,

$$\lim_{t \to \infty} \|y_i - y_j\| = 0, \quad \forall i, j \in \mathcal{V}, \tag{7}$$

as long as $(x(t_0), z(t_0), w(t_0)) \in \mathbf{X} \times \mathbf{Z} \times \mathbf{W}$, where $x = [x_1^T, \ldots, x_N^T]^T$, $z = [z_1^T, \ldots, z_N^T]^T$ and $w = [w_1^T, \ldots, w_N^T]^T$.

Note that the global consensus problem under input saturations can not solve in general since the control input can

remain saturated for large initial conditions. However, the consensus can be achieved semi-globally, that is, the initial states of agents and controllers are bounded within a priori given bounded sets such that the control input remains unsaturated based on the low gain approach. It is worth mentioning that the priori given bounded sets can be arbitrarily large.

Throughout this paper, we impose the following assumptions:

Assumption 2: The agents (1) satisfy the following conditions:

1) For all $i \in \mathcal{V}$, (A_i, B_i) is stabilizable and (C_i, A_i) is detectable
2) All the eigenvalues of A_i, $\forall i \in \mathcal{V}$, are in the closed left-half plane

Assumption 3: S is neutrally stable with $S + S^T = 0$.
Assumption 4: The undirected graph $\mathcal{G}$ is connected.

Note that *Assumption 2-4* are the standard assumptions to solve the semi-global consensus in the presence of the input saturations [21], [22], [23].

Before discussing the main results, several preliminary lemmas are provided.

Lemma 2 [20]: Let *Assumption 2* holds. Then, for each $\epsilon \in (0, 1]$, there exists a positive real number $k^* > 0$ such that the following algebraic Riccati equations (AREs) have unique positive definite solutions $P_{(k,\epsilon),i}$ for any $k \le k^*$:

$$A_i^T P_{(k,\epsilon),i} + P_{(k,\epsilon),i} A_i - P_{(k,\epsilon),i} B_i B_i^T P_{(k,\epsilon),i} + k\, P_{(k,\epsilon),i} E E^T P_{(k,\epsilon),i} + \epsilon I = 0. \tag{8}$$

Moreover, E can be arbitrarily chosen, and $A - BB^T P_{(k,\epsilon),i}$ is Hurwitz for any $\epsilon \in (0, 1]$.

Throughout this paper, we will dropout subscript (k, ϵ) and write P_i without ambiguity.

Lemma 3 [27]: For any $x \in \mathbf{R}$ and $a, b \in \mathbf{R}$ with $0 < a \le b$, the dead-zone function defined as $\mathrm{dz}(x) := x - \mathrm{sat}(x)$ satisfies

$$(a\,\mathrm{dz}(x) - bx)\mathrm{dz}(x) \le 0. \tag{9}$$

Lemma 4 [27]: For any $a, b \in \mathbf{R}$ and $\delta \in (0, 1)$, we have

$$\mathrm{sat}(a + \mathrm{sat}_\delta(b)) = \mathrm{sat}_{\mu(t)}(a) + \mathrm{sat}_\delta(b), \tag{10}$$

where $\mathrm{sat}_\delta = \mathrm{sign}(\cdot)\min\{|\cdot|, \delta\}$ and $\mathrm{sat}_{\mu(t)}(\cdot) = \mathrm{sign}(\cdot)\min\{|\cdot|, \mu(t)\}$ with $\mu(t) = 1 - \mathrm{sign}(a)\mathrm{sat}_\delta(b) > 0$.

III. MAIN RESULTS
A. SEMI-GLOBAL OUTPUT CONSENSUS USING STATE FEEDBACK

In this subsection, we solve *Problem II-B*. State feedback distributed controller can be constructed based on the following three steps:

Step 1) Solve the equations in (3) for $\forall i \in \mathcal{V}$.
Step 2) Solve the following AREs for $\forall i \in \mathcal{V}$

$$A_i^T P_i + P_i A_i - P_i B_i B_i^T P_i + k_i^* P_i (\Gamma_i \Pi_i^T)^T \Gamma_i \Pi^T P_i + \epsilon_i I = 0, \tag{11}$$

where $\epsilon_i \in (0, 1]$ and $k_i^* > 0$, $\forall i \in \mathcal{V}$, can be chosen arbitrarily small such that the solution of (11) exists.
Step 3) Construct the following distributed controller for agent i

$$\dot{w}_i = S w_i + k \left(\sum_{j=1}^{N} a_{ij}(w_j - w_i) - \Gamma_i^T \mathrm{dz}_\delta(\Gamma_i w_i) \right), \tag{12a}$$

$$u_i = -B_i^T P_i(x_i - \Pi_i w_i) + \mathrm{sat}_\delta(\Gamma_i w_i), \tag{12b}$$

where $w_i \in \mathbf{R}^q$, $\mathrm{dz}_\delta(\cdot) = (\cdot) - \mathrm{sat}_\delta(\cdot)$, $\delta \in (0, 1)$, $\mathrm{sat}_\delta(\cdot) = \mathrm{sign}(\cdot)\min\{|\cdot|, \delta\}$, and k is a positive constant satisfying

$$k \le \min_{i \in \mathcal{V}} k_i^*. \tag{13}$$

Note that the existence of solution in Step 2) is given by *Lemma 2*. The dead zone function $\mathrm{dz}_\delta(\Gamma_i w_i)$ in (12a) is used to generate the bounded trajectory. When $\|\Gamma_i w_i\|_\infty > \delta$, the dead zone function $\mathrm{dz}_\delta(\Gamma_i w_i)$ becomes active and brings the states of the reference generators w_i back into the bounded region. Therefore, as we will see below, the trajectory generated by the distributed controller (12a) will be bounded as $\lim_{t \to \infty} \|\Gamma_i w_i\|_\infty \le \delta$, and $\delta \in (0, 1)$ determines the maximum value of the consensus trajectory. Moreover, by using $\mathrm{sat}_\delta(\Gamma_i w_i)$ in (12b), the control input can be rewritten from *Lemma 4* as

$$u_i = \mathrm{sat}_{\mu_i(t)}(-B_i^T P_i(x_i - \Pi_i w_i)) + \mathrm{sat}_\delta(\Gamma_i w_i), \tag{14}$$

where $\mu_i(t) = [(\mu_i)_1(t), (\mu_i)_2(t), \ldots, (\mu_i)_{m_i}(t)]^T$ with $(\mu_i)_k = 1 - \mathrm{sign}((-B_i^T P_i(x_i - \Pi_i w_i))_k)\mathrm{sat}_\delta((\Gamma_i w_i)_k) > 0$. Then, we can determine the bounded sets $\mathbf{X}$ and $\mathbf{W}$ to apply the low gain approach such that the control input remains unsaturated, i.e., $\mathrm{sat}_{\mu_i(t)}(-B_i^T P_i(x_i - \Pi_i w_i)) = -B_i^T P_i(x_i - \Pi_i w_i)$.

Then, the following theorem shows the proposed controller solves the semi-global output consensus problem.

Theorem 1: Consider a group of N agents (1) with the distributed controller (12), and let *Assumption 1-3* hold. Then, for any priori given bounded sets $\mathbf{X} \subset \mathbf{R}^{nN}$ and $\mathbf{W} \subset \mathbf{R}^{qN}$, there exist $\epsilon_i^* \in (0, 1]$ such that, for each $\epsilon_i \in (0, \epsilon_i^*]$ and $(x(t_0), w(t_0)) \in \mathbf{X} \times \mathbf{W}$, the distributed controller (12) solves the semi-global output consensus problem.

Proof: We first define the error states as $e_i = x_i - \Pi_i w_i$, $\forall i \in \mathcal{V}$. Then, from (14), we have

$$u_i = \mathrm{sat}_{\mu_i(t)}(-B_i^T P_i e_i) + \mathrm{sat}_\delta(\Gamma_i w_i), \tag{15}$$

where $\mu_i(t) = [(\mu_i)_1(t), (\mu_i)_2(t), \ldots, (\mu_i)_{m_i}(t)]^T$ with $(\mu_i)_k = 1 - \mathrm{sign}((-B_i^T P_i e_i)_k)\mathrm{sat}_\delta((\Gamma_i w_i)_k) > 0$. Then, from *Assumption 1*, we have the error dynamics as follows:

$$\dot{e}_i = A_i x_i + B_i \mathrm{sat}(-B_i^T P_i e_i + \mathrm{sat}_\delta(\Gamma_i w_i))$$
$$- \Pi_i S w_i - k\Pi_i \sum_{j=1}^{N} a_{ij}(w_j - w_i) + k\Pi_i \Gamma_i^T \mathrm{dz}_\delta(\Gamma_i w_i)$$
$$= A_i x_i + B_i \mathrm{sat}_{\mu_i(t)}(-B_i^T P_i e_i) + B_i \mathrm{sat}_\delta(\Gamma_i w_i)$$

$$-A_i\Pi_i w_i - B_i\Gamma_i w_i - k\Pi_i \sum_{j=1}^{N} a_{ij}(w_j - w_i)$$

$$+k\Pi_i\Gamma_i^T \mathrm{dz}_\delta(\Gamma_i w_i)$$

$$= A_i e_i + B_i\mathrm{sat}_{\mu_i(t)}(-B_i^T P_i e_i) + d_i, \qquad (16)$$

where

$$d_i = -B_i\mathrm{dz}_\delta(\Gamma_i w_i) + k\Pi_i\Gamma_i^T \mathrm{dz}_\delta(\Gamma_i w_i)$$

$$-k\Pi_i \sum_{j=1}^{N} a_{ij}(w_j - w_i). \qquad (17)$$

It is clear that the output consensus implies $\lim_{t\to\infty}\|e_i\| = 0$, $\lim_{t\to\infty}\|w_i - w_j\| = 0$ $\forall i, j \in \mathcal{V}$. Then, to prove the output consensus, we next choose the following Lyapunov function candidate:

$$V = \alpha_1 V_e + \alpha_2 V_w,$$

$$V_e = \sum_{i=1}^{N} e_i^T P_i e_i, \quad V_w = \sum_{i=1}^{N} \|w_i\|^2, \qquad (18)$$

where P_i is the solution of AREs (11), and α_1 and α_2 are the positive constants satisfying

$$0 < \alpha_1 < \min_{i \in \mathcal{V}}\left\{\frac{\epsilon_i}{\lambda_M(P_i)^2}, \frac{2\alpha_2 k}{1+k}\right\},$$

$$\alpha_2 > \frac{1}{2}k\lambda_M(L)\lambda_M(\Pi^T\Pi), \qquad (19)$$

with L being the Laplacian matrix and $\Pi = \mathrm{diag}(\Pi_1, \ldots, \Pi_N)$. Let $c > 0$ be such that

$$c \geq \sup_{\substack{\epsilon_i \in (0, 1], \forall i \in \mathcal{V} \\ (x, w) \in \mathbf{X} \times \mathbf{W}}} \alpha_1 V_e + \alpha_2 V_w \qquad (20)$$

Such a c exists since $\mathbf{X}$ and $\mathbf{W}$ are bounded and $\lim_{\epsilon_i \to 0} P_i = 0$, $\forall i \in \mathcal{V}$.

Let $\epsilon_i^* \in (0, 1]$ be such that, for each $\epsilon_i \in (0, \epsilon_i^*]$,

$$(e, w) \in L_V(c) := \{(e, w) : V \leq c\}, \qquad (21)$$

where $e = [e_1^T, \ldots, e_N^T]^T$ and $w = [w_1^T, \ldots, w_N^T]^T$, implies that

$$\left\|B_i^T P_i e_i\right\|_\infty \leq 1 - \delta, \quad \forall i \in \mathcal{V}. \qquad (22)$$

Then, within $L_V(c)$, the input of the error dynamics (16) does not saturate. We next consider the time derivative of V_e for $(e, w) \in L_V(c)$ and $\epsilon_i \in (0, \epsilon_i^*]$ as

$$\dot{V}_e = 2\sum_{i=1}^{N} e_i^T P_i\left(A_i e_i + B_i\mathrm{sat}_{\mu_i(t)}(-B_i^T P_i e_i) + d_i\right)$$

$$= 2\sum_{i=1}^{N} e_i^T P_i\left((A_i - B_iB_i^T P_i)e_i + d_i\right). \qquad (23)$$

Then, from AREs (11), we have

$$\dot{V}_e = \sum_{i=1}^{N}\left(-\epsilon_i\|e_i\|^2 - \|v_i\|^2 - k_i^*\|\Gamma_i\Pi_i^T P_i e_i\|^2 + 2\,e_i^T P_i d_i\right),$$

$$\qquad (24)$$

where $v_i = B_i^T P_i e_i$.

Note that, from (17), it follows that

$$2\sum_{i=1}^{N} e_i^T P_i d_i$$

$$= 2\sum_{i=1}^{N} e_i^T P_i\left(-B_i\mathrm{dz}_\delta(\Gamma_i w_i) + k\Pi_i\Gamma_i^T \mathrm{dz}_\delta(\Gamma_i w_i)\right.$$

$$\left. -k\Pi_i \sum_{j=1}^{N} a_{ij}(w_j - w_i)\right)$$

$$\leq \sum_{i=1}^{N}\left(\|v_i\|^2 + (1+k)\|\mathrm{dz}_\delta(\Gamma_i w_i)\|^2 + k\left\|\Gamma_i\Pi_i^T P_i e_i\right\|^2\right)$$

$$+ 2\,k e^T P\Pi(L \otimes I)w$$

$$\leq \sum_{i=1}^{N}\left(\|v_i\|^2 + (1+k)\|\mathrm{dz}_\delta(\Gamma_i w_i)\|^2 + k\left\|\Gamma_i\Pi_i^T P_i e_i\right\|^2\right)$$

$$+ \alpha_1 e^T PPe + \frac{k^2}{\alpha_1}\|\Pi(L \otimes I)w\|^2$$

$$\leq \sum_{i=1}^{N}\left(\|v_i\|^2 + (1+k)\|\mathrm{dz}_\delta(\Gamma_i w_i)\|^2 + k\left\|\Gamma_i\Pi_i^T P_i e_i\right\|^2\right.$$

$$\left. + \alpha_1\lambda_M(P_i)^2\|e_i\|^2\right) + \frac{k^2}{\alpha_1}\lambda_M(L)\lambda_M(\Pi^T\Pi)w^T(L \otimes I)w,$$

$$\qquad (25)$$

where $P = \mathrm{diag}(P_1, .., P_N)$ and $\Pi = \mathrm{diag}(\Pi_1, \ldots, \Pi_N)$.

Therefore, we have from (24) and (25) that

$$\dot{V}_e \leq \sum_{i=1}^{N}\left\{\left(\alpha_1\lambda_M(P_i)^2 - \epsilon_i\right)\|e_i\|^2 + (1+k)\|\mathrm{dz}_\delta(\Gamma_i w_i)\|^2\right\}$$

$$+ \frac{k^2}{\alpha_1}\lambda_M(L)\lambda_M(\Pi^T\Pi)w^T(L \otimes I)w, \qquad (26)$$

where we have used the fact that $(k - k_i^*)\|\Gamma_i\Pi_i^T P_i e_i\|^2 \leq 0$ since $0 < k \leq \min_{i \in \mathcal{V}} k_i^*$.

We next consider the time derivative of V_w. Since $S + S^T = 0$ from *Assumption 3*, we have

$$\dot{V}_w = 2\sum_{i=1}^{N} w_i^T\left(Sw_i + k\sum_{j=1}^{N} a_{ij}(w_j - w_i) - k\Gamma_i^T \mathrm{dz}_\delta(\Gamma_i w_i)\right)$$

$$= -2\,kw^T(L \otimes I)w - 2\,k\sum_{i=1}^{N} w_i^T\Gamma_i^T \mathrm{dz}_\delta(\Gamma_i w_i). \qquad (27)$$

Therefore, from (26) and (27), the time derivative of V can be written as

$$\dot{V} = \alpha_1\dot{V}_e + \alpha_2\dot{V}_w$$

$$\leq \sum_{i=1}^{N}\alpha_1(\alpha_1\lambda_M(P_i)^2 - \epsilon_i)\|e_i\|^2$$

$$+ k(k\lambda_M(L)\lambda_M(\Pi^T\Pi) - 2\alpha_2)w^T(L \otimes I)w$$

$$+ \sum_{i=1}^{N}(\alpha_1(1+k)\mathrm{dz}_\delta(\Gamma_i w_i) - 2\alpha_2 k\Gamma_i w_i)^T \mathrm{dz}_\delta(\Gamma_i w_i).$$

$$\qquad (28)$$

Note that, from the condition of α_1 in (19), i.e., $\alpha_1(1+k) < 2\alpha_2 k$, and *Lemma 3*, we have

$$(\alpha_1(1+k)\mathrm{dz}_\delta(\Gamma_i w_i) - 2\alpha_2 k\Gamma_i w_i)^T \mathrm{dz}_\delta(\Gamma_i w_i) \le 0. \quad (29)$$

Moreover, the conditions of α_1 and α_2 in (19) give

$$\alpha_1\lambda_M(P_i)^2 - \epsilon_i < 0, \quad \forall i \in \mathcal{V},$$
$$k\lambda_M(L)\lambda_M(\Pi^T\Pi) - 2\alpha_2 < 0. \quad (30)$$

Therefore, the time derivative of V can be rewritten as

$$\dot{V} \le \sum_{i=1}^N \alpha_1(\alpha_1\lambda_M(P_i)^2 - \epsilon_i)\|e_i\|^2$$
$$+ k(k\lambda_M(L)\lambda_M(\Pi^T\Pi) - 2\alpha_2)w^T(L \otimes I)w$$
$$\le 0. \quad (31)$$

Let $\mathcal{M} := \{(e, w) : \dot{V} = 0\}$. It is clear that $\dot{V} = 0$ only when $e_i = 0$, $\forall i \in \mathcal{V}$, and $w_i = w_j$, $\forall i, j \in \mathcal{V}$, from *Lemma 1*. Then, according to LaSalle Invariance Principle, $\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - \Pi_i w_i) = \lim_{t\to\infty}(C_i x_i - C_i\Pi_i w_i) = \lim_{t\to\infty}(y_i - R w_i) = 0$, $\forall i \in \mathcal{V}$. Moreover, $\lim_{t\to\infty}(w_j - w_i) = 0$, $\forall i, j \in \mathcal{V}$. Therefore, the agents achieve the output consensus. Moreover, from the definition of d_i (17), it follows that $\mathrm{dz}_\delta(\Gamma_i w_i) = 0$ in $\mathcal{M}$, which implies $\lim_{t\to\infty} \|\Gamma_i w_i\|_\infty \le \delta$, $\forall i \in \mathcal{V}$. ∎

B. SEMI-GLOBAL OUTPUT CONSENSUS USING OUTPUT FEEDBACK

In this subsection, we solve *Problem II-B*. To construct the distributed controller using the output feedback, the state observer is designed to estimate the states of agents. Then, the proposed controller can be constructed based on the following three steps:

Step 1-2) Step 1-2 are the same as Step 1-2 in Section III-A.

Step 3) Construct the following distributed controller for agent i

$$\dot{w}_i = S w_i + k\left(\sum_{j=1}^N a_{ij}(w_j - w_i) - \Gamma_i^T \mathrm{dz}_\delta(\Gamma_i w_i)\right), \quad (32a)$$

$$\dot{z}_i = A_i z_i + B_i\mathrm{sat}(u_i) + H_i(C_i z_i - y_i), \quad (32b)$$

$$u_i = -B_i^T P_i(z_i - \Pi_i w_i) + \mathrm{sat}_\delta(\Gamma_i w_i), \quad (32c)$$

where $w_i \in \mathbf{R}^q$, $z_i \in \mathbf{R}^{n_i}$, k is a positive constant satisfying (13), and $H_i \in \mathbf{R}^{n_i \times p}$ such that

$$(A_i + H_i C_i)^T Q_i + Q_i(A_i + H_i C_i) < -C_i^T C_i, \quad (33)$$

where $Q_i \in \mathbf{R}^{n_i \times n_i}$ is a positive definite symmetric matrix.

The following theorem shows that *Problem II-B* can be solved based on the output feedback distributed controller (32).

Theorem 2: Consider a group of N agents (1) with the distributed controller (32), and let *Assumption 1 - 3* hold. Then, for any priori given bounded sets $\mathbf{X}, \mathbf{Z} \subset \mathbf{R}^{nN}$ and $\mathbf{W} \subset \mathbf{R}^{qN}$,

there exist $\epsilon_i^* \in (0, 1]$ such that, for each $\epsilon_i \in (0, \epsilon_i^*]$ and $(x(t_0), z(t_0), w(t_0)) \in \mathbf{X} \times \mathbf{Z} \times \mathbf{W}$, the distributed controller (32) solves the semi-global output consensus problem.

Proof: Let us define the error states as $e_i = z_i - \Pi_i w_i$ and $\zeta_i = x_i - z_i$. From (15) and *Assumption 1*, we have the error dynamics as follows:

$$\dot{e}_i = A_i z_i + B_i\mathrm{sat}(-B_i^T P_i e_i + \mathrm{sat}_\delta(\Gamma_i w_i)) - H_i C_i\zeta_i$$
$$- \Pi_i S w_i - k\Pi_i\sum_{j=1}^N a_{ij}(w_j - w_i) + k\Pi_i\Gamma_i^T\mathrm{dz}_\delta(\Gamma_i w_i)$$
$$= A_i z_i + B_i\mathrm{sat}_{\mu_i(t)}(-B_i^T P_i e_i) + B_i\mathrm{sat}_\delta(\Gamma_i w_i) - H_i C_i\zeta_i$$
$$- A_i\Pi_i w_i - B_i\Gamma_i w_i - k\Pi_i\sum_{j=1}^N a_{ij}(w_j - w_i)$$
$$+ k\Pi_i\Gamma_i^T\mathrm{dz}_\delta(\Gamma_i w_i)$$
$$= A_i e_i + B_i\mathrm{sat}_{\mu_i(t)}(-B_i^T P_i e_i) + d_i, \quad (34)$$

where

$$d_i = -B_i\mathrm{dz}_\delta(\Gamma_i w_i) + k\Pi_i\Gamma_i^T\mathrm{dz}_\delta(\Gamma_i w_i) - H_i C_i\zeta_i$$
$$- k\Pi_i\sum_{j=1}^N a_{ij}(w_j - w_i), \quad (35)$$

and

$$\dot{\zeta}_i = A_i x_i + B_i\mathrm{sat}(u_i) - A_i z_i - B_i\mathrm{sat}(u_i) + H_i C_i\zeta_i$$
$$= (A_i + H_i C_i)\zeta_i. \quad (36)$$

We next show the output consensus under the proposed controller, that means $\lim_{t\to\infty}\|e_i\| = 0$, $\lim_{t\to\infty}\|\zeta_i\| = 0$, $\lim_{t\to\infty}\|w_i - w_j\| = 0$, $\forall i, j \in \mathcal{V}$. Then, we choose the following Lyapunov function candidate:

$$V = \alpha_1 V_e + \alpha_2 V_w + \alpha_3 V_\zeta,$$
$$V_e = \sum_{i=1}^N e_i^T P_i e_i, \quad V_w = \sum_{i=1}^N \|w_i\|^2, \quad V_\zeta = \sum_{i=1}^N \zeta_i^T Q_i\zeta_i, \quad (37)$$

where P_i and Q_i are the solutions of AREs (11) and (33), respectively, and $\alpha_1, \alpha_2, \alpha_3$ are the positive constants satisfying

$$0 < \alpha_1 < \min_{i\in\mathcal{V}}\left\{\frac{\epsilon_i}{\lambda_M(P_i)^2}, \frac{2\alpha_2 k}{1+k}\right\},$$
$$\alpha_2 > \frac{1}{2}k\lambda_M(L)\lambda_M(\Pi^T\Pi),$$
$$\alpha_3 > \max_{i\in\mathcal{V}}\lambda_M(H_i^T H_i), \quad (38)$$

with $\Pi = \mathrm{diag}(\Pi_1, \ldots, \Pi_N)$. Let $c > 0$ be such that

$$c \ge \sup_{\substack{\epsilon_i \in (0, 1], \forall i \in \mathcal{V} \\ (x, z, w) \in \mathbf{X} \times \mathbf{Z} \times \mathbf{W}}} \alpha_1 V_e + \alpha_2 V_w + \alpha_3 V_\zeta \quad (39)$$

Such a c exists since $\mathbf{X}$, $\mathbf{Z}$, and $\mathbf{W}$ are bounded and $\lim_{\epsilon_i\to 0} P_i = 0$, $\forall i \in \mathcal{V}$.

Let $\epsilon_i^* \in (0, 1]$ be such that, for each $\epsilon_i \in (0, \epsilon_i^*]$,

$$(e, w, \zeta) \in L_V(c) := \{(e, w, \zeta) : V \le c\}, \quad (40)$$

where $e = [e_1^T, \ldots, e_N^T]^T$ and $\zeta = [\zeta_1^T, \ldots, \zeta_N^T]^T$, implies that implies that

$$\left\| B_i^T P_i e_i \right\|_\infty \leq 1 - \delta, \quad \forall i \in \mathcal{V}. \tag{41}$$

Then, within $L_V(c)$, the input of the error dynamics (34) does not saturate. We next consider the time derivative of V_e for $(e, w, \zeta) \in L_V(c)$ and $\epsilon_i \in (0, \epsilon_i^*]$ as

$$\dot{V}_e = 2 \sum_{i=1}^{N} e_i^T P_i \left(A_i e_i + B_i \mathrm{sat}_{\mu_i(t)}(-B_i^T P_i e_i) + d_i \right)$$
$$= 2 \sum_{i=1}^{N} e_i^T P_i \left((A_i - B_i B_i^T P_i) e_i + d_i \right). \tag{42}$$

Then, from AREs (11), we have

$$\dot{V}_e = \sum_{i=1}^{N} \left(-\epsilon_i \|e_i\|^2 - \|v_i\|^2 - k_i^* \|\Gamma_i \Pi_i^T P_i e_i\|^2 \right.$$
$$\left. + 2 e_i^T P_i d_i \right), \tag{43}$$

where $v_i = B_i^T P_i e_i$.

Note that, from (35), it follows that

$$2 \sum_{i=1}^{N} e_i^T P_i d_i$$
$$= 2 \sum_{i=1}^{N} e_i^T P_i \left(-B_i \mathrm{dz}_\delta(\Gamma_i w_i) + k \Pi_i \Gamma_i^T \mathrm{dz}_\delta(\Gamma_i w_i) \right.$$
$$\left. - H_i C_i \zeta_i - k \Pi_i \sum_{j=1}^{N} a_{ij}(w_j - w_i) \right)$$
$$\leq \sum_{i=1}^{N} \left(\|v_i\|^2 + (1+k) \|\mathrm{dz}_\delta(\Gamma_i w_i)\|^2 + k \left\| \Gamma_i \Pi_i^T P_i e_i \right\|^2 \right.$$
$$\left. + 2\alpha_1 \lambda_M(P_i)^2 \|e_i\|^2 + \frac{1}{\alpha_1} \lambda_M(H_i^T H_i) \zeta_i^T C_i^T C_i \zeta_i \right)$$
$$+ \frac{k^2}{\alpha_1} \lambda_M(L) \lambda_M(\Pi^T \Pi) w^T(L \otimes I)w. \tag{44}$$

Therefore, we have from (43) and (44) that

$$\dot{V}_e \leq \sum_{i=1}^{N} \left(\left(2\alpha_1 \lambda_M(P_i)^2 - \epsilon_i \right) \|e_i\|^2 + (1+k) \|\mathrm{dz}_\delta(\Gamma_i w_i)\|^2 \right.$$
$$\left. + \frac{1}{\alpha_1} \|H_i C_i \zeta\|^2 \right) + \frac{k^2}{\alpha_1} \lambda_M(L) \lambda_M(\Pi^T \Pi) w^T(L \otimes I)w, \tag{45}$$

where we have used the fact that $(k - k_i^*) \|\Gamma_i \Pi_i^T P_i e_i\|^2 \leq 0$ since $0 < k \leq \min_{i \in \mathcal{V}} k_i^*$.

We next consider the time derivative of V_w. Since $S + S^T = 0$ from *Assumption 3*, we have

$$\dot{V}_w = 2 \sum_{i=1}^{N} w_i^T \left(S w_i + k \sum_{j=1}^{N} a_{ij}(w_j - w_i) \right.$$

$$\left. - k \Gamma_i^T \mathrm{dz}_\delta(\Gamma_i w_i) \right)$$
$$= -2 k w^T(L \otimes I)w - 2 k \sum_{i=1}^{N} w_i^T \Gamma_i^T \mathrm{dz}_\delta(\Gamma_i w_i), \tag{46}$$

Moreover, from (33), the time derivative of V_ζ is given by

$$\dot{V}_\zeta = \sum_{i=1}^{N} \zeta_i^T \left((A_i + H_i C_i)^T Q_i + Q_i(A_i + H_i C_i) \right) \zeta_i$$
$$< - \sum_{i=1}^{N} \zeta_i^T C_i^T C_i \zeta_i. \tag{47}$$

Therefore, from (45), (46), and (47), the time derivative of V can be written as

$$\dot{V} = \alpha_1 \dot{V}_e + \alpha_2 \dot{V}_w + \alpha_3 \dot{V}_\zeta$$
$$\leq \sum_{i=1}^{N} \alpha_1(2\alpha_1 \lambda_M(P_i)^2 - \epsilon_i) \|e_i\|^2$$
$$+ k(k\lambda_M(L)\lambda_M(\Pi^T \Pi) - 2\alpha_2)w^T(L \otimes I)w$$
$$+ \sum_{i=1}^{N} (\alpha_1(1+k)\mathrm{dz}_\delta(\Gamma_i w_i) - 2\alpha_2 k \Gamma_i w_i)^T \mathrm{dz}_\delta(\Gamma_i w_i)$$
$$+ \sum_{i=1}^{N} \left(\lambda_M(H_i^T H_i) - \alpha_3 \right) \zeta_i^T C_i^T C_i \zeta_i \tag{48}$$

Note that, from the condition of α_1 in (38), i.e., $\alpha_1(1+k) < 2\alpha_2 k$, and *Lemma 3*, we have

$$(\alpha_1(1+k)\mathrm{dz}_\delta(\Gamma_i w_i) - 2\alpha_2 k \Gamma_i w_i)^T \mathrm{dz}_\delta(\Gamma_i w_i) \leq 0. \tag{49}$$

Moreover, from the conditions of α_1, α_2, and α_3 in (38), it follows for $\forall i \in \mathcal{V}$ that

$$2\alpha_1 \lambda_M(P_i)^2 - \epsilon_i < 0,$$
$$k\lambda_M(L)\lambda_M(\Pi^T \Pi) - 2\alpha_2 < 0,$$
$$\lambda_M(H_i^T H_i) - \alpha_3 < 0. \tag{50}$$

Therefore, the time derivative of V can be rewritten as

$$\dot{V} \leq \sum_{i=1}^{N} \alpha_1(2\alpha_1 \lambda_M(P_i)^2 - \epsilon_i) \|e_i\|^2$$
$$+ k(k\lambda_M(L)\lambda_M(\Pi^T \Pi) - 2\alpha_2)w^T(L \otimes I)w$$
$$+ \sum_{i=1}^{N} \left(\lambda_M(H_i^T H_i) - \alpha_3 \right) \zeta_i^T C_i^T C_i \zeta_i$$
$$\leq 0. \tag{51}$$

Let $\mathcal{M} := \{(e, w, \zeta) : \dot{V} = 0\}$. It is clear that $\dot{V} = 0$ only when $e_i = \zeta_i = 0$, $\forall i \in \mathcal{V}$, and $w_i = w_j$, $\forall i, j \in \mathcal{V}$. Then, according to LaSalle Invariance Principle, $\lim_{t \to \infty} e_i = \lim_{t \to \infty}(z_i - \Pi_i w_i) = 0$, $\lim_{t \to \infty}(w_j - w_i) = 0$, and $\lim_{t \to \infty} \zeta_i = \lim_{t \to \infty}(x_i - z_i) = 0$, $\forall i, j \in \mathcal{V}$, which implies the agents achieve the output consensus. Moreover, from the definition of d_i (35), it follows that $\mathrm{dz}_\delta(\Gamma_i w_i) = 0$ in $\mathcal{M}$, which implies $\lim_{t \to \infty} \|\Gamma_i w_i\|_\infty \leq \delta$. ∎

Remark 3: Note that the condition in (13) requires the global information to choose the control gain k, i.e., k_i^*, $\forall i \in \mathcal{V}$. However, by choosing the sufficiently small control gain k, the condition can always be satisfied. The control gain k also can be chosen in a distributed way by applying the min-consensus algorithm [28].

IV. SIMULATION RESULTS

In this section, we present numerical examples to demonstrate the theoretical results. Consider a group of twelve agents. The communication topology among the agents is given in Fig. 1, and we consider $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The dynamics of agents are described in (1) with

$$i \in \{1, \ldots, 4\} : \begin{cases} A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \end{cases}$$

$$i \in \{5, \ldots, 8\} : \begin{cases} A_i = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, & B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \end{cases}$$

$$i \in \{9, \ldots, 12\} : \begin{cases} A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, & B_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\ C_i = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}. \end{cases} \quad (52)$$

Example 1: We construct the distributed controller using state feedback in Section III-A. Then, solving the equations (3) in Step 1 gives

$$S = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix},$$

$$i \in \{1, \ldots, 4\} : \Pi_i = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \Gamma_i = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}$$

$$i \in \{5, \ldots, 8\} : \Pi_i = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \Gamma_i = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$$

$$i \in \{9, \ldots, 12\} : \Pi_i = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \Gamma_i^T = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}. \quad (53)$$

We next solve the AREs (11) in Step 2 for $\epsilon_i = 0.01$ with $k_i^* = 1$, $\forall i \in \mathcal{V}$, to obtain P_i as follows:

$$i \in \{1, \ldots, 4\} : P_i = \begin{bmatrix} 0.0498 & 0.1117 \\ 0.1117 & 0.4959 \end{bmatrix},$$

$$i \in \{5, \ldots, 8\} : P_i = \begin{bmatrix} 0.1106 & 0.1005 \\ 0.1005 & 0.1005 \end{bmatrix},$$

$$i \in \{9, \ldots, 12\} : P_i = \begin{bmatrix} 0.0579 & 0.1466 & 0.1775 \\ 0.1466 & 0.5320 & 0.7656 \\ 0.1775 & 0.7656 & 1.4586 \end{bmatrix}. \quad (54)$$

Then, we construct the distributed controller (12) in Step 3 with $k = 1 = \min_{i \in \mathcal{V}} k_i^*$ and $\delta = 0.5$.

In this simulation, we randomly choose the initial conditions of each states of the agents and the reference generators
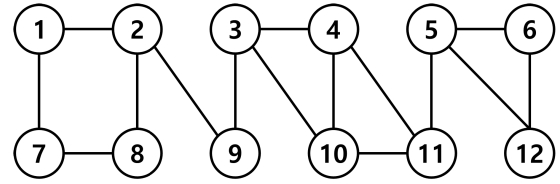

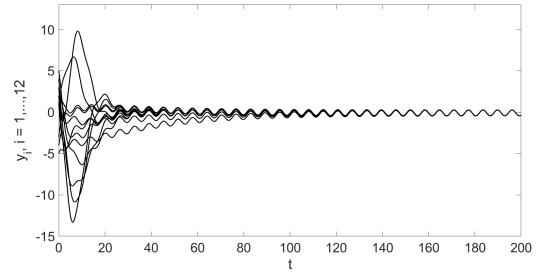**FIGURE 1.** The communication topology among twelve agents.

FIGURE 2. The output trajectories of twelve agents y_i using the distributed controller with state feedback (12).
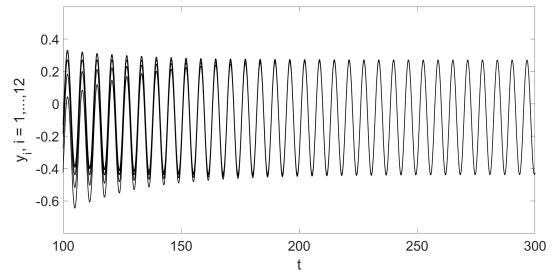

FIGURE 3. The steady-state output trajectories of twelve agents y_i using the distributed controller with state feedback (12).
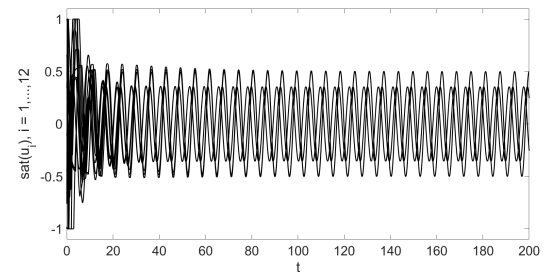

FIGURE 4. The input trajectories of twelve agents sat(u_i) using the distributed controller with state feedback (12).

within the intervals $[-5, 5]$ and $[-10, 10]$, respectively. The simulation results are given in Fig. 2-5. Fig. 2 and 3 show the evolutions of the outputs, and Fig. 4 show the trajectories of the inputs. The simulation results show the outputs converge to the common trajectory generated by the reference generators under the input saturations. Moreover, Fig. 5 shows the proposed reference generator (12a) generates the bounded common trajectory.

We next compare the proposed reference generator (12a) with the standard reference generator (4a) in [13], [14], [15], and [16] under the same initial conditions. As discussed in Section II-B, the feedforward term $\Gamma_i w_i$ generated by the standard reference generator (4a) can exceed the saturation
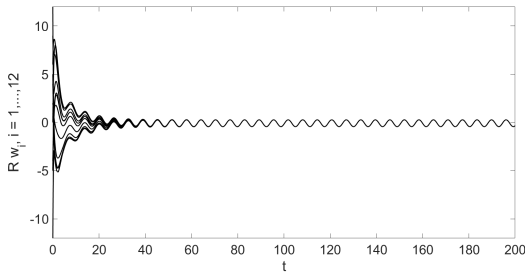
FIGURE 5. The output trajectories of reference generators Rw_i using the distributed controller with state feedback (12).
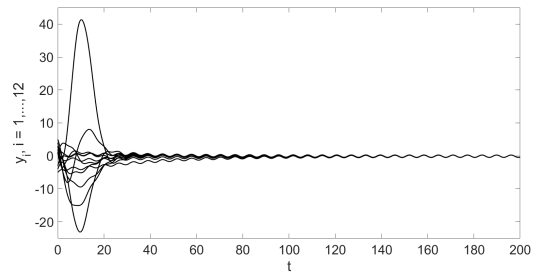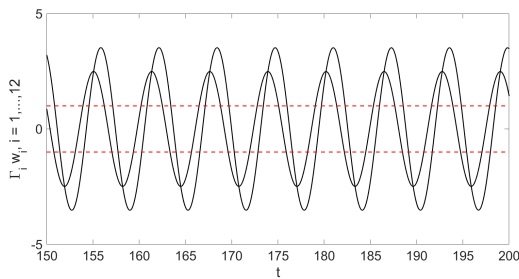

FIGURE 6. The steady-state trajectories of $\Gamma_i w_i$ using the standard reference generator (4a) (solid line) and the saturation level (dashed line).
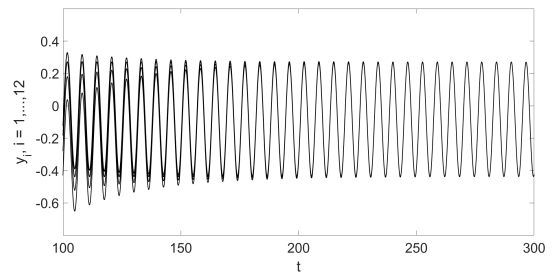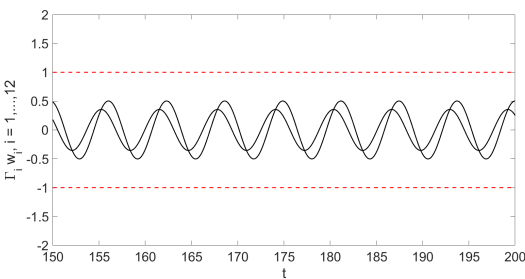

FIGURE 7. The steady-state trajectories of $\Gamma_i w_i$ using the proposed reference generator (4a) (solid line) and the saturation level (dashed line).
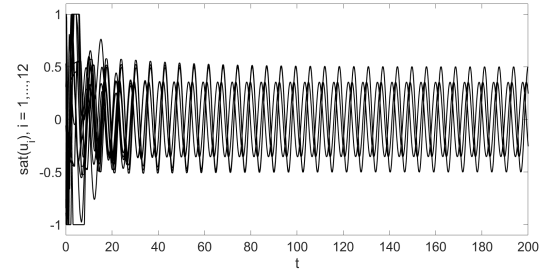

FIGURE 8. The output trajectories of twelve agents y_i using the distributed controller with output feedback (32).

FIGURE 9. The steady-state output trajectories of twelve agents y_i using the distributed controller with output feedback (32).

FIGURE 10. The input trajectories of twelve agents sat(u_i) using the distributed controller with output feedback (32).

FIGURE 11. The trajectories of observer errors $\|\varsigma_i\|$ using the distributed controller with output feedback (32).

level as shown in Fig. 6. Therefore, the control input using the standard distributed controller in the form of (4) remains saturated, and the semi-global consensus problem can not be solved in the presence of input saturations. However, the proposed reference generator (12a) generates the bounded trajectory such that the control inputs do not saturate as shown in Fig. 7.

Example 2: In this example, we construct the distributed controller using output feedback in Section III-B. From *Example 1*, the solutions in Step 1 and 2 are obtained in (53) and (54), respectively. Then, by solving (33) in Step 3, we construct the distributed controller (32) with $k = 1 = \min_{i \in \mathcal{V}} k_i^*$, $\delta = 0.5$, and

$$i \in \{1, \ldots, 4\} : H_i = \begin{bmatrix} -0.9484 \\ -1.3161 \end{bmatrix},$$

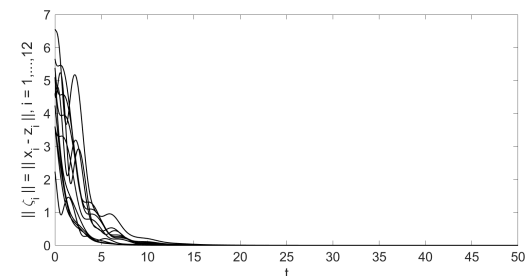$$i \in \{5, \ldots, 8\} : H_i = \begin{bmatrix} -0.4304 \\ -1.0997 \end{bmatrix},$$

$$i \in \{9, \ldots, 12\} : H_i = \begin{bmatrix} -1.4876 \\ -3.1727 \\ -0.9810 \end{bmatrix}. \quad (55)$$

The initial conditions are chosen the same as in *Example 1* with $z_i = 0$, $\forall i \in \mathcal{V}$. Then, the simulation results are shown in Fig. 8-11. We can see from Fig. 8-10 that the proposed

controller can solve the output consensus problem under the input saturations. In Fig. 11, we can see that the observer errors $\zeta_i = x_i - z_i$ converge to zero.

V. CONCLUSION

This paper has studied the leaderless output consensus problem for heterogeneous multi-agent systems with input saturations. We have constructed the modified reference generator which generates the bounded trajectory to avoid input saturation. Moreover, applying the output regulation theory, we have constructed the state feedback and the output feedback controller, respectively, to achieve the output consensus. Then, we have analyzed the existence of the solution that solves the semi-global leaderless output consensus problem under input saturations. Although we have shown that the proposed controllers can solve the leaderless consensus problem under input saturations, the performance of the closed-loop system has not been considered. Therefore, future work will be focused on constructing distributed controllers to improve the convergence speed of overall networked agents.

REFERENCES

[1] Y. Cao, W. Ren, N. Sorensen, L. Ballard, A. Reiter, and J. Kennedy, "Experiments in consensus-based distributed cooperative control of multiple mobile robots," in *Proc. IEEE Int. Conf. Mechatronics Autom.*, Harbin, China, Aug. 2007, pp. 2819–2824.

[2] K. K. Oh, M. C. Park, and H. S. Ahn, "A survey of multi-agent formation control," *Automatica*, vol. 53, pp. 424–440, Mar. 2015.

[3] H.-J. Lee, Y.-H. Lim, and K.-K. Oh, "Consensus based parallel operation of electric motors ensuring speed regulation and load torque sharing," *Int. J. Control, Autom. Syst.*, vol. 19, no. 9, pp. 3111–3121, Sep. 2021.

[4] R. Olfati-Saber and J. S. Shamma, "Consensus filters for sensor networks and distributed sensor fusion," in *Proc. 44th IEEE Conf. Decision Control, Eur. Control (CDC-ECC)*, Dec. 2005, pp. 6698–6703.

[5] M. D. Howard and Z. Qu, "An optimal Kalman-consensus filter for distributed implementation over a dynamic communication network," *IEEE Access*, vol. 9, pp. 66696–66706, 2021.

[6] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[7] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control: Collective group behavior through local interaction," *IEEE Control Syst.*, vol. 27, no. 2, pp. 71–82, Apr. 2007.

[8] W. Ren, "On consensus algorithms for double-integrator dynamics," *IEEE Trans. Autom. Control*, vol. 53, no. 6, pp. 1503–1509, Jul. 2008.

[9] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 57, no. 1, pp. 213–224, Jan. 2010.

[10] W. Liu, H. Dai, S. Yan, Y. Qi, and X. Wu, "Dynamic H_∞ consensus of higher-order nonlinear multi-agent systems with general directed topology," *IEEE Access*, vol. 10, pp. 21316–21326, 2022.

[11] K. Chen, J. Wang, Y. Zhang, and Z. Liu, "Leader-following consensus for a class of nonlinear strick-feedback multiagent systems with state time-delays," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 50, no. 7, pp. 2351–2361, Jul. 2020.

[12] J. Wang, Y. Yan, C. Ma, Z. Liu, K. Ma, and C. L. P. Chen, "Fuzzy adaptive event-triggered finite-time constraint control for output-feedback uncertain nonlinear systems," *Fuzzy Sets Syst.*, vol. 443, pp. 236–257, Aug. 2022.

[13] P. Wieland, R. Sepulchre, and F. Allgöwer, "An internal model principle is necessary and sufficient for linear output synchronization," *Automatica*, vol. 47, no. 5, pp. 1068–1074, May 2011.

[14] Q. Ma and G. Miao, "Output consensus for heterogeneous multi-agent systems with linear dynamics," *Appl. Math. Comput.*, vol. 271, pp. 548–555, Nov. 2015.

[15] T. Han, Z.-H. Guan, B. Xiao, J. Wu, and X. Chen, "Distributed output consensus of heterogeneous multi-agent systems via an output regulation approach," *Neurocomputing*, vol. 360, pp. 131–137, Sep. 2019.

[16] K. Chen, J. Wang, Z. Zhao, G. Lai, and Y. Lyu, "Output consensus of heterogeneous multiagent systems: A distributed observer-based approach," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 52, no. 1, pp. 370–376, Jan. 2022.

[17] Z. Wang, H. He, G.-P. Jiang, and J. Cao, "Distributed tracking in heterogeneous networks with asynchronous sampled-data control," *IEEE Trans. Ind. Informat.*, vol. 16, no. 12, pp. 7381–7391, Dec. 2020.

[18] Z. Wang, X. Jin, L. Pan, Y. Feng, and J. Cao, "Quasi-synchronization of delayed stochastic multiplex networks via impulsive pinning control," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 52, no. 9, pp. 5389–5397, Sep. 2022.

[19] H. J. Sussmann, E. D. Sontag, and Y. Yang, "A general result on the stabilization of linear systems using bounded controls," *IEEE Trans. Autom. Control*, vol. 39, no. 12, pp. 2411–2425, Dec. 1994.

[20] A. Saberi, A. A. Stoorvogel, and P. Sannuti, *Internal and External Stabilization of Linear Systems With Constraints*. Boston, MA, USA: Brikhäuser, 2012.

[21] L. Shi, Y. Li, and Z. Lin, "Semi-global leader-following output consensus of heterogeneous multi-agent systems with input saturation," *Int. J. Robust Nonlinear Control*, vol. 28, no. 16, pp. 4916–4930, 2018.

[22] J. Gao, L. Sun, X. Xiang, H. Song, and Y. Long, "Semi-global cooperative output regulation problem for heterogeneous swarm systems with input saturation under switching network," *IEEE Access*, vol. 7, pp. 36426–36432, 2019.

[23] P. Zhou and B. M. Chen, "Semi-global leader–following output consensus of heterogeneous systems with all agents subject to input saturation," *Int. J. Robust Nonlinear Control*, vol. 32, no. 8, pp. 4648–4664, May 2022.

[24] B. Wang, L. Xu, X. Yi, Y. Jia, and T. Yang, "Semiglobal suboptimal output regulation for heterogeneous multi-agent systems with input saturation via adaptive dynamic programming," *IEEE Trans. Neural Netw. Learn. Syst.*, early access, Jul. 27, 2022, doi: 10.1109/TNNLS.2022.3191673.

[25] H. Geng, Z.-Q. Chen, Z.-X. Liu, and Q. Zhang, "Consensus of a heterogeneous multi-agent system with input saturation," *Neurocomputing*, vol. 166, pp. 382–388, Oct. 2015.

[26] M. Lu, J. Wu, X. Zhan, T. Han, and H. Yan, "Consensus of second-order heterogeneous multi-agent systems with and without input saturation," *ISA Trans.*, vol. 126, pp. 14–20, Jul. 2022.

[27] Y.-H. Lim and H.-S. Ahn, "Semiglobal consensus of heterogeneous multiagent systems with input saturations," *Int. J. Robust Nonlinear Control*, vol. 28, no. 17, pp. 5652–5664, 2018.

[28] J. Cortes, "Analysis and design of distributed algorithms for X-consensus," in *Proc. 45th IEEE Conf. Decis. Control*, Dec. 2006, pp. 3363–3368.

[29] X. Xu, L. Liu, and G. Feng, "Consensus of heterogeneous linear multiagent systems with communication time-delays," *IEEE Trans. Cybern.*, vol. 47, no. 8, pp. 1820–1829, Aug. 2017.

KWANG-KYO OH received the B.S. degree in mineral and petroleum engineering and the M.S. degree in electrical and computer engineering from Seoul National University, in 1998 and 2001, respectively, and the Ph.D. degree in information and mechatronics from the Gwangju Institute of Science and Technology, in 2013. He is currently an Associate Professor with the Department of Electrical Engineering, Sunchon National University. His research interests include cooperative control of multi-agent systems and control applications in power electronics.

YOUNG-HUN LIM received the B.S. degree in electrical engineering from Dankook University, in 2010, and the M.S. and Ph.D. degrees in mechatronics from the Gwangju Institute of Science and Technology, in 2012 and 2016, respectively. He is currently an Associate Professor with the Department of Convergence Electronic Engineering, Gyeongsang National University. His research interests include saturation constraints and multi-agent systems.

● ● ●