IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received 14 October 2022, accepted 11 December 2022, date of publication 16 December 2022, date of current version 22 December 2022.

Digital Object Identifier 10.1109/ACCESS.2022.3230281

RESEARCH ARTICLE

A Study on the Class of Non-Symmetric 3-Point Relaxed Quaternary Subdivision Schemes

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ABSTRACT In this paper, a 3-point relaxed non-symmetric approximating stationary quaternary subdivision scheme with two parameters is presented. The limiting curves generated by the scheme are C^3-continuous. The roles of the parameters in controlling the shapes are discussed. A little attention to the construction of non-symmetric subdivision schemes has been paid. This paper discusses in depth the construction of the scheme and the shape-preserving properties of the constructed scheme. The scheme is suitable for fitting the monotone, convex and nonlinear data that appear frequently in digital signal processing as well as in engineering and industrial shapes. The fitting of important shapes and functions used in engineering and industry is also presented.

INDEX TERMS Quaternary, approximating, interpolating, monotonicity, convexity.

I. INTRODUCTION

A popular branch of computational geometry is computer-aided geometric design, which is used for animation, graphical modelling, and the construction of curves and surfaces.

Subdivision is an emerging efficient technique for generating smooth curves and surfaces. Subdivision schemes are recursive processes for the fast generation of refined sequences that finally represent curves or surfaces. The traditional research for subdivision schemes has focused on the construction of new schemes and the properties of the generated curves (smoothness, monotonicity, convexity, etc.). Shape-preserving approximations are valuable methods that predict or control the shape of curves or surfaces by the shape of control points. For instance, convexity or monotonicity is a useful method to construct shape-preserving approximations starting from the initial data sequence.

Reconstruction operators also have a great role in the subdivision scheme. If the reconstruction operator is non-linear, it will produce a non-linear subdivision scheme. Non-linear subdivision schemes are data-dependent, whereas the linear scheme is data-independent. Both schemes are related to each other. The general form of a non-linear scheme can be written as

$$(S_N)_n = (S_L f)_n + \mathfrak{F}(\Lambda f)_n, \quad \forall n \in \mathbb{Z},\ \forall f \in l^\infty(\mathbb{Z})$$

where $\mathfrak{F} : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ is a nonlinear operator, $\Lambda : (l^\infty \mathbb{Z}) \to l^\infty(\mathbb{Z})$ is linear and continuous. S_L is convergent linear subdivision scheme.

The use of shape parameters in subdivision schemes is also very popular to improve flexibility in the shape of curves and surfaces. In the literature, most of the work has been done on binary, uniform, stationary i.e (mask is independent of level k) schemes with symmetric mask. While moving towards higher arities, the smoothness order of the basic limit function increases with smaller support width.

Mustafa and Faheem [1] were the first to investigate the 4-point quaternary approximating subdivision scheme with one shape parameter having C^3 limit curves. Siddique and Younis [2] introduced a general algorithm for quaternary m-point approximating subdivision schemes. Ko [3] analyzed the convergence and regularity of the quaternary approximating subdivision scheme. Later on, Pervez [4] investigated shape-preserving properties of schemes presented by Ko in [3]. Bari et al. [5] constructed an algorithm for $3n$-point quaternary approximating subdivision schemes. Siddiqi and Ahmad [6] presented a 5-point approximating subdivision scheme that allows the construction of a

The associate editor coordinating the review of this manuscript and approving it for publication was Hongwei Du.

C^4-continuous curve. Hussain et al. [7] proposed the generalized condition for 5-point approximating subdivision scheme having arbitrary arity. In addition, Zheng and Song [8] proposed a class of n-point p-ary interpolatory subdivision schemes. Shahzad et al. [9] worked on the error bounds and subdivision depth of quaternary subdivision schemes. Mustafa et al. [10] investigated 3-point binary relaxed subdivision scheme with two parameters.

Conti et al. [11] presented an effective technique for creating de Rham-type dual m-ary approximation subdivision schemes. Romani et al. [12] gave a thorough characterization of the mask of dual interpolatory univariate schemes of varying arity in terms of trigonometric polynomial identities linked with the schemes. Cai [13] described the convexity-preserving characteristics of the interpolating four-point subdivision scheme of arity 3. They described conditions on the scheme's parameters to maintain convexity. Tan et al. [14] analyzed a five-point subdivision scheme with two parameters and analyzed the uniform convergence and the shape-preserving properties of the four-point scheme. In order to ensure that the convexity of the limit curve is preserved, Novara and Romani [15] specify the requirements that the free parameter of the interpolating 5-point ternary subdivision scheme and the corners of a strictly convex original polygon must meet. Pitolli [16] examined how well ternary subdivision schemes produced by bell-shaped masks preserve shape. Their scheme maintains monotonicity. They demonstrated that the first order divided difference mask must also be bell-shaped in order to maintain convexity.

Amat and Liandrat [17] investigated the quaternary 4-point nonlinear approximating subdivision scheme's convergence and regularity. They demonstrated that their scheme eliminates the Gibbs phenomenon in traditional linear schemes.

Zhao et al. [18] presented the estimation problem of varying-coefficient models having discontinuous coefficient functions. They demonstrated that the resulting estimator provides smooth estimates of the continuity of the coefficients in such models.

In the literature, most attention has been given to symmetric subdivision schemes. However, there exist only a few numbers of non-symmetric subdivision schemes. This area of research is less explored. Non-symmetric schemes can also be charming for designers to adjust the shape according to their requirements by keeping the appearance of the curve smooth. Conti and Hormann [19] investigated the reproduction property of symmetric and non-symmetric schemes. They extended their results to derive a unified condition for polynomial reproduction that covers symmetric and non-symmetric schemes. Mustafa et al. [20] used generalized divided difference techniques to analyze some non-symmetric ternary subdivision schemes. The smoothness of the limit function can easily be improved by giving up symmetry and increasing the arity. This has motivated us to study a scheme with an arity of more than 3, having a non-symmetric mask.

This paper is arranged as follows: Section 2 contains the preliminaries needed for this paper. We build the masks of quaternary non-symmetric subdivision schemes in Section 3. Sections 4 and 5 are devoted to the analysis of the proposed scheme. Numerical examples reflecting the performance of our schemes are also presented in Section 5. The conclusion is presented in Section 6.

II. PRELIMINARIES

In this section, we recall some well-known basic results.

A quaternary subdivision scheme for curve is defined as the limit of refined control polygon or meshes, according to some refinement rules recursively, in which set of control points $f^k = \{f_i^k \in \mathbb{R} : i \in \mathbb{Z}\}$ of polygon at k-th level is mapped to refined polygon $\{f^{k+1} = f_i^{k+1} \in \mathbb{R} : i \in \mathbb{Z}\}$ at the $(k+1)$-th level by applying the following subdivision rule

$$f_i^{k+1} = \sum_{j\in\mathbb{Z}} \vartheta_{i-4j}^k f_j^k \quad j = 0, 1, 2, 3.$$

with the sequence of finite set of real coefficient $\{\vartheta_i^k : k \geq 0, i \in \mathbb{Z}^c\}$, ($c = 1$ for curve and $c = 2$ for surfaces) is called the mask at kth level of refinement scheme. If mask is independent of k i.e ($a^{(k)} = a$ for all $k \geq 0$), then it is called stationary, otherwise it is called non stationary.

A subdivision scheme is called non-symmetric if $\vartheta_i \neq \vartheta_{-i}$ and $\vartheta_{i-1} \neq \vartheta_{-i}$, $i \in \mathbb{Z}$.

Arity of the scheme is defined as the number of points that are inserted at level $k+1$ in between two consecutive points of level k. If the number of inserted points is 4, then the subdivision scheme is called a quaternary scheme.

The complexity of the subdivision scheme is how many control points at the k-th subdivision level are used for inserting new points at the $(k+1)$-th level.

Definition 1 (Hussain et al. [21]): A univariate data (x_s, f_s), $s = 0, 1, 2, \ldots, n$ is monotonically increasing if $f_s < f_{s+1}$ $\forall\, s = 0, 1, 2, \ldots, n$ and the derivative at the data points obey the condition $D_s > 0$ $\forall\, s = 0, 1, 2, \ldots, n$.

Definition 2 (Mehaute and Uteras [22]): Convexity is defined as "Given a set of control points $P_i^k \in \mathbb{Z}$ with $P_i^k = \{x_i^k, f_i^k\}$, f_i^k is strictly convex at a point x_i^k, if second order divided difference $\mathbb{D}_i^k = f[x_{i-1}^k, x_i^k, x_{i+1}^k] > 0$".

Definition 3: A stencil which gives a point on the limit curve in the form of the original control points is called a limit stencil. The limit stencil evaluates points on the limit curve itself with a relatively small number of calculations. We obtain the limit stencil by using a formula,

$$R^\infty = V\left(\lim_{j\to\infty} \sigma^j\right) V^{-1} f^0,$$

where

$$\lim_{j\to\infty} \sigma^j = \sigma^\infty,$$

so

$$R^\infty = V\sigma^\infty V^{-1} f^0,$$

where V is the matrix of eigenvectors corresponding to eigenvalues and σ is the diagonal matrix of eigenvalues from subdivision matrix of the scheme.

Theorem 1: The necessary conditions for a quaternary subdivision scheme to be convergent is that its corresponding mask $\vartheta_i : i \in \mathbb{Z}$ necessarily satisfies basic sum rules.

$$\sum_{j\in\mathbb{Z}} \vartheta_{4_j} = 1, \quad \sum_{j\in\mathbb{Z}} \vartheta_{4_j+1} = 1,$$
$$\sum_{j\in\mathbb{Z}} \vartheta_{4_j+2} = 1, \quad \sum_{j\in\mathbb{Z}} \vartheta_{4_j+3} = 1. \tag{1}$$

or satisfies the conditions

$$\vartheta(1) = 4, \quad \vartheta(e^{\frac{2\pi i}{4}j}) = 0 \quad for \quad j = 1, 2, 3. \tag{2}$$

Algorithm: The following algorithm [23] is used to prove the smoothness of quaternary 4-point subdivision scheme to be C^3-continuous.

Input: Insert the mask $\vartheta_{j,q}$, $j = 0, 1 \ldots m-1$, $q = 0, 1, \ldots n-1$ of the scheme where m and n indicate the complexity and arity of the scheme respectively.

Step 1: If $\sum_{j=0}^{m-1} \vartheta_{j,q} = 1, \qquad q = 0, 1, ..n-1.$

i.e. the necessary conditions are satisfied, move to next step otherwise stop.

Step 2: Use the following recursive formula by doing the calculations on the right-hand sides of the following equations and allocating the results to the left-hand sides of the equations:

$$\begin{aligned}\vartheta_{j,n} &= \vartheta_{j,0}, j = 0, 1 \ldots m-1,\\ \vartheta_{j,r} &= \vartheta_{j-1,r} + \vartheta_{j,r} - \vartheta_{j,r+1}, r = 0, \ldots n-2,\\ \vartheta_{j,n-1} &= \vartheta_{j-1,n-1} + \vartheta_{j,n-1} - \vartheta_{j-1,n}, j = 0, 1 \ldots m-1.\end{aligned} \tag{3}$$

Step 3: In this step, sufficient conditions are satisfied by using the following relation and using the latest values $\vartheta_{j,r}$, $\vartheta_{j,n-1}$ obtained in step 2,

$$\begin{cases} \gamma_r = \sum_{j=0}^{m-1} \mid \vartheta_{j,r} \mid; r = 0, .., n-2. \\ \gamma_{n-1} = \sum_{j=0}^{m-1} \mid \vartheta_{j,n-1} \mid; r = 0, .., n-2. \end{cases} \tag{4}$$

If $\gamma = \max\{\gamma_r, \gamma_{n-1}\} < 1$, then move to next step, otherwise stop.

Step 4: Compute the latest values of $\vartheta_{j,r}$ and $\vartheta_{j,n-1}$ obtained in step 2, of 1^{st} iteration in the right hand side of the following equation and assign the obtained values to the left hand side of the equation,

$$\begin{aligned}\vartheta_{j,r} &= n\vartheta_{j,r}, \quad r = 0, \ldots, n-2, j = 0, .., m-1,\\ \vartheta_{j,n-1} &= n\vartheta_{j,n-1}, \quad j = 0, 1, 2, 3.., m-1.\end{aligned} \tag{5}$$

Move to step 1.

Result: The $\bar{s}$-times completion of steps 1-4 mean original scheme is $C^{\bar{s}}$-continuous.

III. CONSTRUCTION OF THE 3-POINT RELAXED NON-SYMMETRIC SUBDIVISION SCHEME

In this section, we present the construction process of a 3-point relaxed non-symmetric parametric quaternary subdivision scheme with two parameters. The refinement rules of the proposed scheme are obtained by using displacement vectors. The process of constructing the scheme is given as follows.

The four subdivision rules of quaternary B-spline subdivision scheme can be written as

$$\begin{aligned}g_{4i+1}^{k+1} &= f_i^k + \frac{5}{16}\left[\frac{\triangle f_{i+1} - \triangle f_i}{2}\right],\\ g_{4i+2}^{k+1} &= \zeta_1 f_{i-1}^k + \zeta_2 f_i^k + \zeta_3 f_{i+1}^k + \zeta_4 f_{i+2}^k\\ &\quad + \frac{5}{64}\left[\frac{\triangle f_i + 2\triangle f_{i+1} + \triangle f_{i+2}}{2}\right],\\ g_{4i+3}^{k+1} &= \zeta_5 f_{i-1}^k + \zeta_6 f_i^k + \zeta_6 f_{i+1}^k + \zeta_5 f_{i+2}^k\\ &\quad + \frac{5}{32}\left[\frac{\triangle f_{i+2} - \triangle f_i}{2}\right],\\ g_{4i+4}^{k+1} &= \zeta_4 f_{i-1}^k + \zeta_3 f_i^k + \zeta_2 f_{i+1}^k + \zeta_1 f_{i+2}^k\\ &\quad + \frac{5}{64}\left[\frac{3\triangle f_{i+2} - 2\triangle f_{i+1} - \triangle f_i}{2}\right],\end{aligned} \tag{6}$$

where $\triangle f_{i+1} = f_{i+1} - f_i$ and the weights ζ_j are given by

$$\zeta_1 = -\frac{7}{128}, \quad \zeta_2 = \frac{105}{128}, \quad \zeta_3 = \frac{35}{128},$$
$$\zeta_4 = -\frac{5}{128}, \quad \zeta_5 = -\frac{1}{16}, \quad \zeta_6 = \frac{9}{16}.$$

The 3-point relaxed quaternary approximating scheme (6) can also be written as:

$$\begin{pmatrix} g_{4i+1}^{k+1} \\ g_{4i+2}^{k+1} \\ g_{4i+3}^{k+1} \\ g_{4i+4}^{k+1} \end{pmatrix} = \frac{1}{64}\begin{pmatrix} 10 & 44 & 10 & 0 \\ 4 & 40 & 20 & 0 \\ 1 & 31 & 31 & 1 \\ 0 & 20 & 40 & 4 \end{pmatrix}\begin{pmatrix} f_{i-1}^k \\ f_i^k \\ f_{i+1}^k \\ f_{i+2}^k \end{pmatrix}. \tag{7}$$

The symbol of the 3-point relaxed quaternary approximating scheme (6) is

$$R(z) = \frac{z\left(1 + z + z^2 + z^3\right)^{3+1}}{4^3}. \tag{8}$$

Now we introduce four displacement vectors as

$$d_{4i+l-1}^{k+1} = X_1(g_{4i+l}^{k+1} - g_{4i+l+1}^{k+1}), \quad l = 1, 2, 3, 4, \; X_1 = \frac{5}{2}. \tag{9}$$

The displacement vectors defined in (9) can also be written as

$$\begin{pmatrix} d_{4i}^{k+1} \\ d_{4i+1}^{k+1} \\ d_{4i+2}^{k+1} \\ d_{4i+3}^{k+1} \end{pmatrix} = \frac{1}{128}\begin{pmatrix} 30 & 20 & -50 & 0 \\ 15 & 45 & -55 & -5 \\ 5 & 55 & -45 & -15 \\ 0 & 50 & -20 & -30 \end{pmatrix}\begin{pmatrix} f_{i-1}^k \\ f_i^k \\ f_{i+1}^k \\ f_{4i+2}^k \end{pmatrix}. \tag{10}$$

The symbol of the displacement vectors in (10) is

$$B(z)=\frac{\left(1+z+z^2+z^3\right)^{3+1}}{4^3}\left[\frac{5(z-z^2)}{2}\right]. \tag{11}$$

Now we define four other displacement vectors as:

$$dd_{4i+l}^{k+1}=X_2(g_{4i+l}^{k+1}-g_{4i+l+1}^{k+1})\quad l=0,1,2,3,\quad X_2=\frac{1}{4}. \tag{12}$$

Which can also be written as

$$\begin{pmatrix} dd_{4i}^{k+1}\\ dd_{4i+1}^{k+1}\\ dd_{4i+2}^{k+1}\\ dd_{4i+3}^{k+1}\end{pmatrix}=\frac{1}{256}\begin{pmatrix}10 & -4 & -6 & 0\\ 6 & 4 & -10 & 0\\ 3 & 9 & -11 & -1\\ 1 & 11 & -9 & -3\end{pmatrix}\begin{pmatrix} f_{i-1}^{k}\\ f_{i}^{k}\\ f_{i+1}^{k}\\ f_{4i+2}^{k}\end{pmatrix}. \tag{13}$$

The symbol of the displacement vectors (13) can be written as

$$D(z)=\frac{\left(1+z+z^2+z^3\right)^{3+1}}{4^3}\left[\frac{(1-z)}{4}\right]. \tag{14}$$

By using (9) and (12) in the following relation we get the following subdivision rules

$$f_{4i+j}^{k+1}=g_{4i+j+1}^{k+1}+\alpha d_{4i+j}^{k+1}+\beta dd_{4i+j}^{k+1},\quad j=0,1,2,3.$$

Now we obtain a new combined quaternary subdivision scheme with two shape parameters by moving the points $g_{4i}^{k+1}, g_{4i+1}^{k+1}, g_{4i+2}^{k+1}, g_{4i+3}^{k+1}$ of (9) to the new position according to the displacement vectors.

$$\begin{pmatrix} f_{4i}^{k+1}\\ f_{4i+1}^{k+1}\\ f_{4i+2}^{k+1}\\ f_{4i+3}^{k+1}\end{pmatrix}=\begin{pmatrix} g_{4i+1}^{k+1}\\ g_{4i+2}^{k+1}\\ g_{4i+3}^{k+1}\\ g_{4i+4}^{k+1}\end{pmatrix}+\begin{pmatrix}\alpha & 0 & 0 & 0\\ 0 & \alpha & 0 & 0\\ 0 & 0 & \alpha & 0\\ 0 & 0 & 0 & \alpha\end{pmatrix}\times\begin{pmatrix} d_{4i}^{k+1}\\ d_{4i+1}^{k+1}\\ d_{4i+2}^{k+1}\\ d_{4i+3}^{k+1}\end{pmatrix}+\begin{pmatrix}\beta & 0 & 0 & 0\\ 0 & \beta & 0 & 0\\ 0 & 0 & \beta & 0\\ 0 & 0 & 0 & \beta\end{pmatrix}\begin{pmatrix} dd_{4i}^{k+1}\\ dd_{4i+1}^{k+1}\\ dd_{4i+2}^{k+1}\\ dd_{4i+3}^{k+1}\end{pmatrix}.$$

Hence, we get the following combined subdivision scheme with two shape parameters, as shown in the equation at the bottom of the page.

In matrix notation, this can be written as:

$$\begin{pmatrix} f_{4i}^{k+1}\\ f_{4i+1}^{k+1}\\ f_{4i+2}^{k+1}\\ f_{4i+3}^{k+1}\end{pmatrix}=\begin{pmatrix}\vartheta_{-4} & \vartheta_{0} & \vartheta_{4} & 0\\ \vartheta_{-5} & \vartheta_{-1} & \vartheta_{3} & \vartheta_{7}\\ \vartheta_{-6} & \vartheta_{-2} & \vartheta_{2} & \vartheta_{6}\\ \vartheta_{-7} & \vartheta_{-3} & \vartheta_{1} & \vartheta_{5}\end{pmatrix}\begin{pmatrix} f_{i-1}^{k}\\ f_{i}^{k}\\ f_{i+1}^{k}\\ f_{i+2}^{k}\end{pmatrix}, \tag{15}$$

where

$$\vartheta_{-4}=\frac{5\beta}{128}+\frac{15\alpha}{64}+\frac{5}{32},\quad \vartheta_{0}=\frac{5\alpha}{32}+\frac{11}{16}-\frac{\beta}{64},$$
$$\vartheta_{4}=-\frac{25\alpha}{64}+\frac{5}{32}-\frac{3\beta}{128},\quad \vartheta_{-5}=\frac{1}{16}+\frac{3\beta}{128}+\frac{15\alpha}{128}, \tag{16}$$

$$\vartheta_{-1}=\frac{45\alpha}{128}+\frac{5}{8}+\frac{\beta}{64},\quad \vartheta_{3}=-\frac{55\alpha}{128}+\frac{5}{16}-\frac{5\beta}{128},$$
$$\vartheta_{7}=-\frac{5\alpha}{128},\quad \vartheta_{-6}=\frac{3\beta}{256}+\frac{5\alpha}{128}+\frac{1}{64},$$
$$\vartheta_{-2}=\frac{55\alpha}{128}+\frac{31}{64}+\frac{9\beta}{256},\quad \vartheta_{2}=\frac{31}{64}-\frac{45\alpha}{128}-\frac{11\beta}{256},$$
$$\vartheta_{6}=-\frac{15\alpha}{128}+\frac{1}{64}-\frac{\beta}{256},\quad \vartheta_{-7}=\frac{\beta}{256},$$
$$\vartheta_{-3}=\frac{5}{16}+\frac{11\beta}{256}+\frac{25\alpha}{64},\quad \vartheta_{1}=-\frac{5\alpha}{32}-\frac{9\beta}{256}+\frac{5}{8},$$
$$\vartheta_{5}=-\frac{3\beta}{256}-\frac{15}{64}\alpha+\frac{1}{16}. \tag{17}$$

The Laurent polynomial corresponding to (15) is:

$$\vartheta_i(z)=\sum_{i=-7}^{7}\vartheta_i z^i, \tag{18}$$

where ϑ_i: $i=-7,\ldots,7$ are defined in (17).

IV. PROPERTIES OF THE PROPOSED NON-SYMMETRIC SCHEME

In this section, we focus on some desirable properties of the proposed subdivision scheme (15).

A. CONTINUITY ANALYSIS

The smoothness of the scheme depends upon the continuity. In this section, we analyze the continuity of the proposed 3-point relaxed quaternary approximating subdivision scheme by using numerical algorithms for divided differences method. The following result shows that the scheme is C^3-continuous.

$$\begin{cases} f_{4i}^{k+1}=\left(\frac{5}{128}\beta+\frac{15}{64}\alpha+\frac{5}{32}\right)f_{i-1}^{k}+\left(\frac{5}{32}\alpha+\frac{11}{16}-\frac{1}{64}\beta\right)f_{i}^{k}+\left(-\frac{25}{64}\alpha+\frac{5}{32}-\frac{3}{128}\beta\right)f_{i+1}^{k},\\ f_{4i+1}^{k+1}=\left(\frac{1}{16}+\frac{3}{128}\beta+\frac{15}{128}\alpha\right)f_{i-1}^{k}+\left(\frac{45\alpha}{128}+\frac{5}{8}+\frac{\beta}{64}\right)f_{i}^{k}+\left(-\frac{55\alpha}{128}+\frac{5}{16}-\frac{5\beta}{128}\right)f_{i+1}^{k}+\left(-\frac{5\alpha}{128}\right)f_{i+2}^{k},\\ f_{4i+2}^{k+1}=\left(\frac{3\beta}{256}+\frac{5\alpha}{128}+\frac{1}{64}\right)f_{i-1}^{k}+\left(\frac{55\alpha}{128}+\frac{31}{64}+\frac{9\beta}{256}\right)f_{i}^{k}+\left(\frac{31}{64}-\frac{45\alpha}{128}-\frac{11\beta}{256}\right)f_{i+1}^{k}+\left(-\frac{15\alpha}{128}+\frac{1}{64}-\frac{\beta}{256}\right)f_{i+2}^{k},\\ f_{4i+3}^{k+1}=\left(\frac{\beta}{256}\right)f_{i-1}^{k}+\left(\frac{5}{16}+\frac{11\beta}{256}+\frac{25\alpha}{64}\right)f_{i}^{k}+\left(-\frac{5\alpha}{32}-\frac{9\beta}{256}+\frac{5}{8}\right)f_{i+1}^{k}+\left(-\frac{3\beta}{256}-\frac{15\alpha}{64}+\frac{1}{16}\right)f_{i+2}^{k}.\end{cases}$$

Theorem 2: The 3-point relaxed quaternary approximating subdivision scheme (15) is C^3-continuous for the parametric interval $\beta \in (-4, 4)$ and $\alpha \in (-\frac{4}{5}+\frac{\beta}{10}, \frac{\beta}{10}) \bigcap (-\frac{2}{5}, \frac{2}{5})$.

Proof: Scheme (15) can be rewritten as

$$\begin{cases} f_{4i}^{k+1} = \vartheta_{0,0} f_{i-1}^k + \vartheta_{1,0} f_i^k + \vartheta_{2,0} f_{i+1}^k + \vartheta_{3,0} f_{i+2}^k, \\ f_{4i+1}^{k+1} = \vartheta_{0,1} f_{i-1}^k + \vartheta_{1,1} f_i^k + \vartheta_{2,1} f_{i+1}^k + \vartheta_{3,1} f_{i+2}^k, \\ f_{4i+2}^{k+1} = \vartheta_{0,2} f_{i-1}^k + \vartheta_{1,2} f_i^k + \vartheta_{2,2} f_{i+1}^k + \vartheta_{3,2} f_{i+2}^k, \\ f_{4i+3}^{k+1} = \vartheta_{0,3} f_{i-1}^k + \vartheta_{1,3} f_i^k + \vartheta_{2,3} f_{i+1}^k + \vartheta_{3,3} f_{i+2}^k. \end{cases}$$

Since the complexity and arity of scheme is 4, i.e $m = n = 4$. The coefficients of points f_i^k of scheme (15) are defined below

$$\begin{aligned}
\vartheta_{0,0} &= \frac{5\beta}{128} + \frac{15}{64}\alpha + \frac{5}{32}, \quad \vartheta_{1,0} = \frac{5\alpha}{32} + \frac{11}{16} - \frac{\beta}{64}, \\
\vartheta_{2,0} &= -\frac{25\alpha}{64} + \frac{5}{32} - \frac{3\beta}{128}, \quad \vartheta_{3,0} = 0, \\
\vartheta_{0,1} &= \frac{1}{16} + \frac{3}{128}\beta + \frac{15\alpha}{128}, \quad \vartheta_{1,1} = \frac{45\alpha}{128} + \frac{5}{8} + \frac{\beta}{64}, \\
\vartheta_{2,1} &= -\frac{55\alpha}{128} + \frac{5}{16} - \frac{5\beta}{128}, \quad \vartheta_{3,1} = -\frac{5\alpha}{128}, \\
\vartheta_{0,2} &= \frac{3}{256}\beta + \frac{5\alpha}{128} + \frac{1}{64}, \quad \vartheta_{1,2} = \frac{55\alpha}{128} + \frac{31}{64} + \frac{9\beta}{256}, \\
\vartheta_{2,2} &= \frac{31}{64} - \frac{45\alpha}{128} - \frac{11\beta}{256}, \quad \vartheta_{3,2} = -\frac{15\alpha}{128} + \frac{1}{64} - \frac{\beta}{256}, \\
\vartheta_{0,3} &= \frac{\beta}{256}, \quad \vartheta_{1,3} = \frac{5}{16} + \frac{11\beta}{256} + \frac{25\alpha}{64}, \\
\vartheta_{2,3} &= -\frac{5\alpha}{32} - \frac{9\beta}{256} + \frac{5}{8}, \quad \vartheta_{3,3} = -\frac{3\beta}{256} - \frac{15\alpha}{64} + \frac{1}{16}.
\end{aligned}$$

1st Iteration: Here, in the first step, we will check whether the necessary condition for convergence is satisfied.

Step 1: Since $\sum_{j=0}^{m-1} \vartheta_{j,q} = 1, q = 0, 1; , ..n-1.$

Hence the necessary conditions are satisfied, so we will move to the next step.

Step 2 Using (3), we obtain the following results

$$\begin{aligned}
\vartheta_{0,4} &= \vartheta_{0,0} = \frac{5\beta}{128} + \frac{15\alpha}{64} + \frac{5}{32}, \quad \vartheta_{3,4} = \vartheta_{3,0} = 0, \\
\vartheta_{2,4} &= \vartheta_{2,0} = -\frac{25}{64}\alpha + \frac{5}{32} - \frac{3\beta}{128}, \quad \vartheta_{3,3} = 0. \\
\vartheta_{0,0} &= \frac{3}{32} + \frac{\beta}{64} + \frac{15\alpha}{128}, \quad \vartheta_{3,1} = 0, \\
\vartheta_{0,1} &= \frac{3\beta}{256} + \frac{5\alpha}{64} + \frac{3}{64}, \quad \vartheta_{2,0} = -\frac{5\alpha}{128}, \\
\vartheta_{2,1} &= \frac{1}{64} - \frac{5\alpha}{64} - \frac{\beta}{256}, \quad \vartheta_{1,1} = \frac{3}{16} - \frac{\beta}{128}, \\
\vartheta_{0,2} &= \frac{1}{128}\beta + \frac{5\alpha}{128} + \frac{1}{64}, \quad \vartheta_{1,2} = \frac{3}{16} + \frac{5\alpha}{64}, \\
\vartheta_{2,2} &= -\frac{15\alpha}{128} - \frac{\beta}{128} + \frac{3}{64}, \quad \vartheta_{3,2} = 0, \\
\vartheta_{1,3} &= \frac{\beta}{128} + \frac{5\alpha}{32} + \frac{5}{32}, \quad \vartheta_{0,3} = \frac{\beta}{256} \\
\vartheta_{2,3} &= \frac{-3\beta}{256} - \frac{5}{32}\alpha + \frac{3}{32}, \\
\vartheta_{1,0} &= \frac{-5\alpha}{64} + \frac{5}{32} - \frac{\beta}{64}, \\
\vartheta_{1,4} &= \vartheta_{1,0} = \frac{5\alpha}{32} + \frac{11}{16} - \frac{\beta}{64}.
\end{aligned}$$

Step 3: In this step, sufficient conditions are satisfied by using (4) and using the latest values $\vartheta_{j,r}$, $\vartheta_{j,n-1}$ obtained in step 2,

$$\gamma_0 = \sum_{j=0}^{3} \mid \vartheta_{j,0} \mid .$$

This implies that

$$\begin{aligned}
\gamma_0 &= \left| \frac{3}{32} + \frac{1}{64}\beta + \frac{15}{128}\alpha \right| + \left| \frac{-5}{64}\alpha + \frac{5}{32} - \frac{1}{64}\beta \right| \\
&\quad + \left| -\frac{5}{128}\alpha \right|, \\
&= \left| \frac{\beta}{32} + \frac{25\alpha}{128} - \frac{3}{128} \right| < 1.
\end{aligned}$$

Similarly,

$$\gamma_1 = \sum_{j=0}^{3} \mid \vartheta_{j,1} \mid .$$

This implies

$$\begin{aligned}
\gamma_1 &= \left| \frac{3}{256}\beta + \frac{5}{64}\alpha + \frac{3}{64} \right| + \left| \frac{3}{16} - \frac{1}{128}\beta \right| \\
&\quad + \left| \frac{1}{64} - \frac{5}{64}\alpha - \frac{1}{256}\beta \right|, \\
\gamma_1 &= \left| \frac{3\beta}{128} + \frac{5\alpha}{32} - \frac{5}{32} \right| < 1.
\end{aligned}$$

And

$$\gamma_2 = \sum_{j=0}^{3} \left| \vartheta_{j,2} \right| .$$

This implies

$$\begin{aligned}
\gamma_2 &= \left| \frac{1}{128}\beta + \frac{5}{128}\alpha + \frac{1}{64} \right| + \left| \frac{3}{16} + \frac{5}{64}\alpha \right| \\
&\quad + \left| -\frac{15}{128}\alpha - \frac{1}{128}\beta + \frac{3}{64} \right|, \\
\gamma_2 &= \left| \frac{\beta}{64} + \frac{15\alpha}{64} + \frac{5}{32} \right| < 1.
\end{aligned}$$

Furthermore,

$$\gamma_3 = \sum_{j=0}^{3} \left| \vartheta_{j,3} \right| .$$

This implies that

$$\begin{aligned}
\gamma_3 &= \left| \frac{1}{256}\beta \right| + \left| \frac{1}{128}\beta + \frac{5}{32}\alpha + \frac{5}{32} \right| \\
&\quad + \left| \frac{-3}{256}\beta - \frac{5}{32}\alpha + \frac{3}{32} \right|,
\end{aligned}$$

$$\gamma_3 = \left|\frac{3\beta}{128} + \frac{5\alpha}{16} + \frac{1}{16}\right| < 1.$$

Since $\gamma = \max\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\} < 1$, therefore scheme (15) is C^0-continuous.

Now for C^1-continuity, we move to the next step.

Step 4: Compute the latest values of $\vartheta_{j,r}$ and $\vartheta_{j,n-1}$ obtained in step 2, of 1^{st} iteration in the right hand side of (5) and assign the obtained values to the left hand side of the equation. Hence, we obtain the following results.

$$\begin{aligned}
&\vartheta_{0,0} = \frac{3}{8} + \frac{\beta}{16} + \frac{15}{32}\alpha, \quad \vartheta_{1,0} = \frac{-5\alpha}{16} + \frac{5}{8} - \frac{\beta}{16},\\
&\vartheta_{2,0} = -\frac{5\alpha}{32}, \quad \vartheta_{0,1} = \frac{3\beta}{64} + \frac{5\alpha}{16} + \frac{3}{16},\\
&\vartheta_{1,1} = \frac{3}{4} - \frac{\beta}{32}, \quad \vartheta_{2,1} = \frac{1}{16} - \frac{5\alpha}{16} - \frac{\beta}{64},\\
&\vartheta_{1,3} = \frac{\beta}{32} + \frac{5}{8}\alpha + \frac{5}{8}, \quad \vartheta_{2,3} = \frac{-3\beta}{64} - \frac{5\alpha}{8} + \frac{3}{8},\\
&\vartheta_{0,3} = \frac{\beta}{64}, \quad \vartheta_{3,0} = 0,\\
&\vartheta_{0,2} = \frac{\beta}{32} + \frac{5\alpha}{32} + \frac{1}{16}, \quad \vartheta_{1,2} = \frac{5\alpha}{16} + \frac{3}{4},\\
&\vartheta_{2,2} = \frac{-\beta}{32} - \frac{15\alpha}{32} + \frac{3}{16}, \quad \vartheta_{3,1} = 0,\\
&\vartheta_{3,3} = 0, \quad \vartheta_{3,2} = 0.
\end{aligned}$$

Move to next iteration.

2nd iteration:

Now for C^1-continuity, use the values of step 4 as inputs, and follow the similar procedure of the 1st iteration again.

Step 1: $\sum_{j=0}^{m-1} \vartheta_{j,q} = 1, q = 0, 1, ..n-1.$

The necessary conditions are satisfied, we will move to the next step.

Step 2: Again using the recursive formula (3) we obtain,

$$\begin{aligned}
&\vartheta_{0,4} = \vartheta_{0,0} = \frac{\beta}{16} + \frac{15\alpha}{32} + \frac{3}{8}, \quad \vartheta_{2,4} = \vartheta_{2,0} = -\frac{5\alpha}{32},\\
&\vartheta_{1,4} = \vartheta_{1,0} = \frac{-5\alpha}{16} + \frac{5}{8} - \frac{\beta}{16}, \quad \vartheta_{3,4} = \vartheta_{3,0} = 0,\\
&\vartheta_{0,0} = \frac{3}{16} + \frac{\beta}{64} + \frac{5\alpha}{32}, \quad \vartheta_{2,0} = 0,\\
&\vartheta_{1,0} = \frac{-5\alpha}{32} + \frac{1}{16} - \frac{\beta}{64}, \quad \vartheta_{3,0} = 0,\\
&\vartheta_{0,1} = \frac{\beta}{64} + \frac{5\alpha}{32} + \frac{1}{8}, \quad \vartheta_{3,1} = 0,\\
&\vartheta_{1,1} = \frac{1}{8} - \frac{\beta}{64} - \frac{5\alpha}{32}, \quad \vartheta_{2,1} = 0,\\
&\vartheta_{0,2} = \frac{\beta}{64} + \frac{5\alpha}{32} + \frac{1}{16}, \quad \vartheta_{1,2} = 0,\\
&\vartheta_{1,2} = -\frac{5\alpha}{32} - \frac{1}{64}\beta + \frac{3}{16}, \quad \vartheta_{3,2} = \vartheta_{2,2} = 0,\\
&\vartheta_{0,3} = \frac{\beta}{64}, \quad \vartheta_{3,3} = 0,\\
&\vartheta_{1,3} = \frac{-1\beta}{64} + \frac{5}{32}\alpha + \frac{1}{4}, \quad \vartheta_{2,3} = -\frac{5\alpha}{32}.
\end{aligned}$$

Step 3: Again satisfying the sufficient condition

$$\gamma_0 = \sum_{j=0}^{3} \mid \vartheta_{j,0} \mid .$$

This implies that

$$\begin{aligned}
\gamma_0 &= \left|\frac{3}{16} + \frac{1}{64}\beta + \frac{5}{32}\alpha\right| + \left|\frac{-5}{32}\alpha + \frac{1}{16} - \frac{1}{64}\beta\right|\\
&= \left|\frac{\beta}{32} + \frac{5\alpha}{16} + \frac{1}{8}\right| < 1.
\end{aligned}$$

Similarly

$$\gamma_1 = \sum_{j=0}^{3} \left|\vartheta_{j,1}\right|.$$

This implies

$$\begin{aligned}
\gamma_1 &= \left|\frac{1}{64}\beta + \frac{5}{32}\alpha + \frac{1}{8}\right| + \left|\frac{1}{8} - \frac{1}{64}\beta - \frac{5}{32}\alpha\right|,\\
&= \left|\frac{\beta}{32} + \frac{5\alpha}{16}\right| < 1,
\end{aligned}$$

Furthermore,

$$\gamma_2 = \sum_{j=0}^{3} \mid \vartheta_{j,2} \mid .$$

This implies that

$$\begin{aligned}
\gamma_2 &= \left|\frac{1}{64}\beta + \frac{5}{32}\alpha + \frac{1}{16}\right| + \left|\frac{5}{32}\alpha + \frac{1}{64}\beta - \frac{3}{16}\right|,\\
&= \left|\frac{\beta}{32} + \frac{5\alpha}{16} - \frac{1}{8}\right| < 1.
\end{aligned}$$

And

$$\gamma_3 = \sum_{j=0}^{3} \mid \vartheta_{j,2} \mid .$$

This implies that

$$\begin{aligned}
\gamma_3 &= \left|\frac{1}{64}\beta\right| + \left|\frac{-1}{64}\beta + \frac{5}{32}\alpha + \frac{1}{4}\right| + \left|-\frac{5}{32}\alpha\right|,\\
&= \left|\frac{5\alpha}{16} + \frac{1}{4}\right| < 1.
\end{aligned}$$

Hence for $\alpha \in (-4, \frac{12}{5})$, $\beta \in \max\{\gamma_0, \gamma_1, \gamma_2\} < 1$. Therefore first order divided difference scheme is C^0-continuous while the scheme (15) is C^1-continuous.

Step 4 Substitute the values of $\vartheta_{j,r}$ and $\vartheta_{j,n-1}$ obtained from step 2, of 2nd iteration in (5).

$$\begin{aligned}
&\vartheta_{0,0} = \frac{3}{4} + \frac{1}{16}\beta + \frac{5}{8}\alpha, \quad \vartheta_{1,0} = \frac{-5}{8}\alpha + \frac{1}{4} - \frac{1}{16}\beta,\\
&\vartheta_{2,0} = 0, \quad \vartheta_{3,0} = 0,\\
&\vartheta_{0,1} = \frac{1}{16}\beta + \frac{5}{8}\alpha + \frac{1}{2}, \quad \vartheta_{3,1} = 0,\\
&\vartheta_{1,1} = \frac{1}{2} - \frac{1}{16}\beta - \frac{5}{8}\alpha, \quad \vartheta_{2,1} = 0,
\end{aligned}$$

$$\vartheta_{0,2} = \frac{1}{16}\beta + \frac{5}{8}\alpha + \frac{1}{4}, \quad \vartheta_{1,2} = -\frac{5}{8}\alpha - \frac{1}{16}\beta + \frac{3}{4},$$
$$\vartheta_{2,2} = 0, \quad \vartheta_{3,2} = 0,$$
$$\vartheta_{1,3} = \frac{-1}{16}\beta + \frac{5}{8}\alpha + 1, \quad \vartheta_{2,3} = -\frac{5}{8}\alpha,$$
$$\vartheta_{0,3} = \frac{1}{16}\beta, \quad \vartheta_{3,3} = 0.$$

Using the above updated values as input for 3rd iteration.

3rd iteration:

Step 1: As the necessary conditions for uniform convergence are satisfied, i.e.

$$\sum_{j=0}^{m-1}\vartheta_{j,q} = 1, \quad q = 0, 1, ..n-1.$$

Move to next step.

Step 2: Now again using the relation (3), we have

$$\vartheta_{0,4} = \vartheta_{0,0} = \frac{3}{4} + \frac{1}{16}\beta + \frac{5}{8}\alpha,$$
$$\vartheta_{1,4} = \vartheta_{0,0} = \frac{1}{4} - \frac{1}{16}\beta - \frac{5}{8}\alpha,$$
$$\vartheta_{2,4} = 0, \quad \vartheta_{3,4} = 0,$$
$$\vartheta_{0,0} = \frac{1}{4}, \quad \vartheta_{1,0} = 0,$$
$$\vartheta_{2,0} = 0, \quad \vartheta_{3,0} = 0,$$
$$\vartheta_{1,1} = 0, \quad \vartheta_{2,1} = 0,$$
$$\vartheta_{3,1} = 0, \quad \vartheta_{1,2} = -\frac{5}{8}\alpha,$$
$$\vartheta_{2,2} = 0, \quad \vartheta_{3,2} = 0,$$
$$\vartheta_{1,3} = \frac{-1}{16}\beta + \frac{1}{4}, \quad \vartheta_{2,3} = 0,$$
$$\vartheta_{3,3} = 0, \quad \vartheta_{0,1} = \frac{1}{4},$$
$$\vartheta_{0,2} = \frac{5}{8}\alpha + \frac{1}{4}, \quad \vartheta_{0,3} = \frac{1}{16}\beta.$$

Step 3 Using the relation (4), we have

$$\gamma_0 = \sum_{j=0}^{3} |\ \vartheta_{j,0}\ |\ .$$

This implies that

$$\gamma_0 = \frac{1}{4} < 1.$$

Similarly

$$\gamma_1 = \sum_{j=0}^{3} |\ \vartheta_{j,1}\ |\ .$$

This implies

$$\gamma_1 = \frac{1}{4} < 1.$$

Furthermore

$$\gamma_2 = \sum_{j=0}^{3} |\ \vartheta_{j,2}\ |\ .$$

This implies

$$\gamma_2 = \left|\frac{5}{8}\alpha + \frac{1}{4}\right| + \left|-\frac{5}{8}\alpha\right| = \left|\frac{5}{4}\alpha + \frac{1}{4}\right| < 1,$$

And

$$\gamma_3 = \sum_{j=0}^{3} |\ \vartheta_{j,3}\ |\ .$$

This implies

$$\gamma_3 = \left|\frac{1}{16}\beta\right| + \left|\frac{-1}{16}\beta + \frac{1}{4}\right| = \left|\frac{1}{8}\beta + \frac{1}{4}\right| < 1.$$

Therefore, first order divided difference scheme is C^1-continuous while scheme (15) is C^2-continuous for this interval.

$$\alpha \in \Big(-1, \frac{3}{5}\Big) \quad \text{and} \quad \beta \in \Big(-6, 10\Big).$$

Step 4: Using (5), we get

$$\vartheta_{0,0} = 1, \quad \vartheta_{1,0} = 0, \quad \vartheta_{2,0} = 0, \quad \vartheta_{3,0} = 0,$$
$$\vartheta_{0,1} = 1, \quad \vartheta_{1,1} = 0, \quad \vartheta_{2,1} = 0, \quad \vartheta_{3,1} = 0,$$
$$\vartheta_{0,2} = \frac{5}{2}\alpha + 1, \quad \vartheta_{1,2} = -\frac{5}{2}\alpha,$$
$$\vartheta_{2,2} = 0, \quad \vartheta_{3,2} = 0, \quad \vartheta_{0,3} = \frac{1}{4}\beta,$$
$$\vartheta_{1,3} = \frac{-\beta}{4} + 1, \quad \vartheta_{2,3} = 0, \quad \vartheta_{3,3} = 0.$$

Now for C^2-continuity use the values obtained in step 4 as input for 4th iteration.

4th iteration

Step 1:

$$\sum_{j=0}^{m-1}\vartheta_{j,q} = 1, \quad q = 0, 1, ..n-1.$$

Now again

Step 2: Using (3), we have

$$\vartheta_{0,4} = \vartheta_{0,0} = 1, \quad \vartheta_{1,4} = \vartheta_{1,0} = 0, \quad \vartheta_{2,4} = \vartheta_{2,0} = 0,$$
$$\vartheta_{0,0} = 0, \quad \vartheta_{1,0} = 0, \quad \vartheta_{2,0} = 0,$$
$$\vartheta_{3,0} = 0, \quad \vartheta_{1,1} = 0, \quad \vartheta_{2,1} = 0,$$
$$\vartheta_{3,1} = 0, \quad \vartheta_{0,2} = \frac{5}{2}\alpha - \frac{\beta}{4} + 1, \quad \vartheta_{1,2} = 0,$$
$$\vartheta_{2,2} = 0, \quad \vartheta_{3,2} = 0, \quad \vartheta_{0,3} = \frac{\beta}{4}$$
$$\vartheta_{1,3} = 0, \quad \vartheta_{2,3} = 0, \quad \vartheta_{3,3} = 0,$$
$$\vartheta_{3,4} = \vartheta_{3,0} = 0, \quad \vartheta_{0,1} = -\frac{5}{2}\alpha.$$

Step 3 Using (4), we have

$$\gamma_0 = \sum_{j=0}^{3} \mid \vartheta_{j,0} \mid .$$
$$\gamma_0 = 0 < 1.$$

Similarly,

$$\gamma_1 = \sum_{j=0}^{3} \mid \vartheta_{j,1} \mid .$$
$$\gamma_1 = \left|-\frac{5}{2}\alpha\right| < 1.$$
$$\alpha \in \left(-\frac{2}{5}, \frac{2}{5}\right).$$

And

$$\gamma_2 = \sum_{j=0}^{3} \mid \vartheta_{j,2} \mid .$$
$$\gamma_2 = \left|\frac{5}{2}\alpha - \frac{1}{4}\beta + 1\right| < 1,$$

By simplifying

$$\alpha \in \left(-\frac{4}{5} + \frac{\beta}{10}, \frac{\beta}{10}\right).$$

Also

$$\gamma_3 = \sum_{j=0}^{3} \mid \vartheta_{j,3} \mid; \qquad n = 4,$$
$$\gamma_3 = \left|\frac{1}{4}\beta\right| < 1.$$

Therefore

$$\beta \in (-4, 4).$$

Therefore, first order divided difference scheme is C^2-continuous while scheme (15) is C^3-continuous for the interval $\beta \in (-4, 4)$, $\alpha \in (-\frac{4}{5} + \frac{\beta}{10}, \frac{\beta}{10}) \bigcap (-\frac{2}{5}, \frac{2}{5})$.

Step 4 Using (5), we get

$$\vartheta_{0,0} = 0, \quad \vartheta_{1,0} = 0, \vartheta_{2,0} = 0,$$
$$\vartheta_{3,0} = 0, \quad \vartheta_{0,1} = -10\alpha, \quad \vartheta_{1,1} = 0,$$
$$\vartheta_{2,1} = 0, \quad \vartheta_{3,1} = 0, \quad \vartheta_{0,2} = 10\alpha - \beta + 4,$$
$$\vartheta_{1,2} = 0, \quad \vartheta_{2,2} = 0, \quad \vartheta_{3,2} = 0,$$
$$\vartheta_{0,3} = \beta, \quad \vartheta_{1,3} = 0, \quad \vartheta_{2,3} = 0,$$
$$\vartheta_{3,3} = 0.$$

5th iteration

Step 1:

$$\sum_{j=0}^{m-1} \vartheta_{j,q} \neq 1, \quad q = 0, 1, ..n - 1.$$

The necessary conditions are not satisfied. Hence, the scheme (15) is not C^4-continuous. □

Remark 1: If $\beta = -10$, $\alpha = 1$, the scheme (15) reduces to the 4-point interpolatory quaternary subdivision scheme known as quaternary Dubuc- Deslauriers 4-point scheme [24].

B. MONOTONICITY PRESERVATION

In this section, we will examine the monotonicity property of the 4-point quaternary approximating subdivision scheme (15). If the first order divided differences $D_i^k = 4^k(f_{i+1}^k - f_i^k)$ of scheme are positive for all iterations of the scheme, then the property of monotonicity preservation can be achieved for the subdivision scheme. To derive the condition for monotonicity preservation of the subdivision scheme (15) from the initial sequence to the limit curves, we can write the first order divided difference for the scheme (15) as,

$$\begin{cases} D_{4i}^{k+1} = \left(\frac{3}{8} + \frac{\beta}{16} + \frac{15\alpha}{32}\right) D_{i-1}^k + \left(\frac{5}{8} - \frac{\beta}{16} - \frac{5\alpha}{16}\right) \\ \qquad D_i^k + \left(\frac{5\alpha}{32}\right) D_{i+1}^k, \\ D_{4i+1}^{k+1} = \left(\frac{3}{16} + \frac{3\beta}{64} + \frac{5\alpha}{16}\right) D_{i-1}^k + \left(\frac{3}{4} - \frac{\beta}{32}\right) D_i^k \\ \qquad + \left(\frac{1}{16} - \frac{\beta}{64} - \frac{5\alpha}{16}\right) D_{i+1}^k, \\ D_{4i+2}^{k+1} = \left(\frac{1}{16} + \frac{\beta}{32} + \frac{5\alpha}{32}\right) D_{i-1}^k + \left(\frac{5\alpha}{16} + \frac{3}{4}\right) D_i^k \\ \qquad + \left(\frac{3}{16} - \frac{\beta}{32} - \frac{15\alpha}{32}\right) D_{i+1}^k, \\ D_{4i+3}^{k+1} = \left(\frac{\beta}{64}\right) D_{i-1}^k + \left(\frac{5}{8} + \frac{\beta}{32} + \frac{5\alpha}{8}\right) D_i^k \\ \qquad + \left(\frac{3}{8} - \frac{3\beta}{64} - \frac{5\alpha}{8}\right) D_{i+1}^k. \end{cases} \tag{19}$$

To derive a condition that can guarantee the monotonicity of 4-point quaternary scheme (15), we rewrite the refinement rules for the first order divided difference scheme (19) as.

$$\begin{cases} D_{4i}^{k+1} = b_1 D_{i-1}^k + b_2 D_i^k + b_3 D_{i+1}^k, \\ D_{4i+1}^{k+1} = b_4 D_{i-1}^k + b_5 D_i^k + b_6 D_{i+1}^k, \\ D_{4i+2}^{k+1} = b_7 D_{i-1}^k + b_8 D_i^k + b_9 D_{i+1}^k, \\ D_{4i+3}^{k+1} = b_{10} D_{i-1}^k + b_{11} D_i^k + b_{12} D_{i+1}^k. \end{cases} \tag{20}$$

Theorem 3: Let $p_i^k = \frac{D_{i+1}^k}{D_i^k}$, $q_i^k = \frac{D_i^k}{D_{i+1}^k} = \frac{1}{p_i^k}$ and let $Q^k = \max_i\{p_i^k, q_i^k\}$, $k \geq 0$, $i \in \mathbb{Z}$. If the initial polygon is monotonically increasing, i.e. $D_i^0 \geq 0$, $i \in \mathbb{Z}$, then for all $\alpha = 0, \beta \in (0, 4)$ and $Q^0 < \lambda = \frac{-\beta+24}{12+2\beta}$ and for all $\beta = 0$, $\alpha \in (-\frac{2}{5}, 0)$ and $Q^0 < \lambda = \frac{12+5\alpha}{4}$ then the quaternary 3- point relaxed subdivision scheme (15) preserves the monotonicity of the initial data.

Lemma 4: Let $b_i : i = 1, \ldots 12$ be the coefficient defined in (20) and λ be the value defined in Theorem (3). It follows that the following inequalities hold

(i) $\left(\frac{b_1-b_3}{\lambda}\right)+b_2>0$;

(ii) $(b_4+b_6)\lambda+b_5>0$;

(iii) $b_7\lambda+b_8+\frac{b_9}{\lambda}>0$;

(iv) $\frac{b_{10}}{\lambda}+b_{11}+\frac{b_{12}}{\lambda}>0$;

(v) $(b_4+b_6-b_2)\lambda+b_5-\lambda^2b_1+\lambda^2b_3<0$;

(vi) $b_1\lambda+b_2-\frac{b_3}{\lambda}-b_4-\lambda b_5-b_6<0$;

(vii) $\frac{b_7}{\lambda}+b_8+\frac{b_9}{\lambda}-b_4-b_5\lambda-b_6<0$;

(viii) $\left(\frac{b_4+b_6}{\lambda}+(b_5-b_7-b_9)-b_8\lambda\right)<0$;

(ix) $\frac{b_{10}}{\lambda}+b_{11}+\frac{b_{12}}{\lambda}-b_7-b_8\lambda-b_9<0$;

(x) $b_7\lambda+b_8+b_9\lambda-b_{11}\lambda-b_{12}\lambda^2-b_{10}\lambda^2<0$;

(xi) $(b_{10}-b_1)\lambda+(b_{11}-b_2)+\frac{(b_{12}-b_3)}{\lambda}<0$.

Proof: Let $p_i^k=\frac{D_{i+1}^k}{D_i^k}$, $q_i^k=\frac{D_i^k}{D_{i+1}^k}$, $Q^k=\max_i\{p_i^k,q_i^k\}$ start by observing that, from the definition of p_i^k, q_i^k and Q_i^k, we immediately have $Q^0<\lambda$ and $Q^0>\frac{1}{\lambda}$ as well as p_i^0, $q_i^0<\lambda$ and $-p_i^0,-q_i^0<\frac{-1}{\lambda}$.

Then we continue via mathematical induction on k that, for all $k\in\mathbb{N}$, $D_i^k>0$ for all $i\in\mathbb{Z}$ and $Q^k<\lambda$. Therefore, assuming $D_i^k>0$ for all $i\in\mathbb{Z}$ and $Q^k<\lambda$, the proof consist in showing that $D_{4i+j}^{k+1}>0$ for $j=0,1,2,3$ and $Q^{k+1}<\lambda$, namely p_{4i+j}^{k+1}, $q_{4i+j}^{k+1}<\lambda$ for $j=0,1,2,3$.

Now, we show that $D_{4i}^{k+1}>0$, $D_{4i+1}^{k+1}>0$, $D_{4i+2}^{k+1}>0$ and $D_{4i+3}^{k+1}>0$.

Considering Lemma 4 (i), from first rule of (20) we get

$$\begin{aligned}D_{4i}^{k+1}&=D_i^k\left[b_1(q_{i-1}^k)+b_2-b_3(p_i^k)\right]\\&>D_i^k\left[\frac{b_1}{\lambda}+b_2-\frac{b_3}{\lambda}\right]\\&>D_i^k\left[\frac{b_1-b_3}{\lambda}+b_2\right]\\&>0.\end{aligned}$$

Analogously, using second rule from (20) and using Lemma 4(*ii*), we have

$$\begin{aligned}D_{4i+1}^{k+1}&=D_i^k\left[-b_4(-q_{i-1}^k)+b_5-b_6(-p_i^k)\right]\\&>D_i^k\left[-b_4(-\lambda)+b_5-b_6(-\lambda)\right]\\&>D_i^k\left[b_4\lambda+b_5+b_6(\lambda)\right]\\&=D_i^k\left[(b_4+b_6)\lambda+b_5\right]\\&>0.\end{aligned}$$

Similarly, using third rule from (20), using Lemma 4(*iii*)

$$D_{4i+2}^{k+1}=D_i^k\left[-b_7(-q_{i-1}^k)+b_8+b_9(p_i^k)\right]$$

This implies that

$$\begin{aligned}D_{4i+2}^{k+1}&>D_i^k\left[-b_7(-\lambda)+b_8+\frac{b_9}{\lambda}\right]\\&=D_i^k\left[b_7\lambda+b_8+\frac{b_9}{\lambda}\right]\\&>0.\end{aligned}$$

Additionally, using Lemma 4(*iv*), the fourth rule in (20) yields,

$$\begin{aligned}D_{4i+3}^{k+1}&=D_i^k\left[b_{10}(q_{i-1}^k)+b_{11}+b_{12}(p_i^k)\right]\\D_{4i+3}^{k+1}&>D_i^k\left[\frac{b_{10}}{\lambda}+b_{11}+\frac{b_{12}}{\lambda}\right]\\&>0.\end{aligned}$$

Which implies that $D_i^{k+1}>0$ for all $i\in\mathbb{Z}$. Therefore, by induction $D_i^k>0$, for all $k\geq 0$, $i\in\mathbb{Z}$, $k\in\mathbb{Z}$. Now we prove that $Q^{k+1}<\lambda$.

Using Lemma 4(*v*)

$$\begin{aligned}D_{4i+1}^{k+1}-\lambda D_{4i}^{k+1}&=D_i^k\left[b_4q_{i-1}^k+b_5+b_6p_i^k-b_1\lambda q_{i-1}^k\right.\\&\qquad\left.-\lambda b_2+\lambda b_3p_i^k\right]\\&<D_i^k\left[(b_4+b_6-b_2)\lambda+b_5-b_1\lambda^2+b_3\lambda^2\right]\\&<D_i^k\frac{-1}{32}(\lambda-1)\left[(12+2\beta+10\alpha)\lambda\right.\\&\quad\left.-\beta+24\right]\\&=0.\end{aligned}$$

due to definition of λ. Thus $D_{4i+1}^{k+1}-\lambda D_{4i}^{k+1}<0$. Moreover, using Lemma 4(*vi*), we have

$$\begin{aligned}D_{4i}^{k+1}-\lambda D_{4i+1}^{k+1}&=D_i^k\left[b_1q_{i-1}^k+b_2-b_3p_i^k-\lambda b_4q_{i-1}^k-\right.\\&\qquad\left.\lambda b_5-\lambda b_6p_i^k\right]\\&<D_i^k\left[b_1\lambda+b_2-\frac{b_3}{\lambda}-b_4-b_5\lambda-b_6\right]\\&=-D_i^k\frac{1}{32\lambda}(\lambda-1)\left[(-12+3\beta+15\alpha)\lambda\right.\\&\quad\left.+5\alpha\right].\\&=0.\end{aligned}$$

Thus $D_{4i}^{k+1}-\lambda D_{4i+1}^{k+1}<0$.

Analogously, considering Lemma 4(*vii*), we obtain

$$\begin{aligned}D_{4i+2}^{k+1}-\lambda D_{4i+1}^{k+1}&=D_i^k\left[b_7q_{i-1}^k+b_8+b_9p_i^k-\lambda b_4q_{i-1}^k\right.\\&\qquad\left.-\lambda b_5-\lambda b_6p_i^k\right]\\&<D_i^k\left[\frac{b_7}{\lambda}+b_8+\frac{b_9}{\lambda}-b_4-b_5\lambda-b_6\right]\\&=-D_i^k\frac{1}{32\lambda}(\lambda-1)\left[(-24+\beta)\lambda-8+10\alpha\right].\\&=0.\end{aligned}$$

Hence, the latter yields $D_{4i+2}^{k+1}-\lambda D_{4i+1}^{k+1}<0$.

Using Lemma 4(*viii*)

$$\begin{aligned} D^{k+1}_{4i+1} - \lambda D^{k+1}_{4i+2} &= D^k_i \Big[b_4 q^k_{i-1} + b_5 + b_6 p^k_i - b_7 \lambda q^k_{i-1} \\ &\qquad - \lambda b_8 - \lambda b_9 p^k_i\Big] \\ &< D^k_i \Big[(-b_4)(-q^k_{i-1}) + b_5 - (b_6)(-p^k_i) \\ &\qquad - b_7 \lambda q^k_{i-1} - \lambda b_8 - \lambda b_9 p^k_i\Big] \\ &< D^k_i \left[\frac{b_4}{\lambda} + b_5 + \frac{b_6}{\lambda} - b_7 - \lambda\, b_8 - b_9\right] \\ &< D^k_i \left[\frac{b_4 + b_6}{\lambda} + (b_5 - b_7 - b_9) - b_8 \lambda\right] \\ &= D^k_i \frac{-1}{32\lambda}(\lambda - 1)\left[(24 + 10\alpha)\lambda + 8 + \beta\right]. \\ &= 0. \end{aligned}$$

In view of Lemma 4(*ix*), we get

$$\begin{aligned} D^{k+1}_{4i+3} - \lambda D^{k+1}_{4i+2} &= D^k_i \Big[b_{10} q^k_{i-1} + b_{11} + b_{12} p^k_i - \lambda b_7 q^k_{i-1} \\ &\qquad - \lambda b_8 - \lambda b_9 p^k_i\Big] \\ &< D^k_i \left[\frac{b_{10}}{\lambda} + b_{11} + \frac{b_{12}}{\lambda} - b_7 - b_8 \lambda - b_9\right] \\ &< D^k_i \left[\frac{b_{10}}{\lambda} + b_{11} + \frac{b_{12}}{\lambda} - b_7 - b_8 \lambda - b_9\right] \\ &= D^k_i \frac{-1}{32\lambda}(\lambda - 1)\left[(24 + 10\alpha)\lambda - \beta - 20\alpha \right. \\ &\qquad \left. + 12\right] = 0. \end{aligned}$$

Using Lemma 4(*x*), we get

$$\begin{aligned} D^{k+1}_{4i+2} - \lambda D^{k+1}_{4i+3} &= D^k_i \Big[b_7 q^k_{i-1} + b_8 + b_9 p^k_i - \lambda b_{10} q^k_{i-1} \\ &\qquad - \lambda b_{11} - \lambda b_{12} p^k_i\Big] \\ &< D^k_i \Big[b_7 \lambda + b_8 + b_9 \lambda - b_{10} \lambda^2 - \lambda\, b_{11} \\ &\qquad - b_{12} \lambda^2\Big] \\ &< D^k_i \Big[(b_7 + b_9 - b_{11})\lambda + b_8 \\ &\qquad - (b_{10} - b_{12})\lambda^2\Big] \\ &= D^k_i \frac{1}{32\lambda}(\lambda - 1)\left[(20\alpha - 12 + \beta)\lambda \right. \\ &\qquad \left. - 24 - 10\alpha\right] = 0. \end{aligned}$$

Hence, $D^{k+1}_{4i+2} - \lambda D^{k+1}_{4i+3} < 0$.

Finally, using Lemma 4(*xi*), we can write

$$\begin{aligned} D^{k+1}_{4i+3} - \lambda D^{k+1}_{4i+4} &= D^k_i \Big[b_{10} q^k_{i-1} + b_{11} + b_{12} p^k_i - \lambda b_1 \\ &\qquad - \lambda b_2 p^k_i - \lambda b_3 p^k_i p^k_{i+1}\Big] \\ &< D^k_i \,\Big[b_{10}\lambda + b_{11} - b_1 \lambda \\ &\qquad + (b_{12} - \lambda b_2) p^k_i + \lambda\, b_3 p^k_i (-p^k_{i+1})\Big] \\ &< D^k_i \left[b_{10}\lambda + b_{11} - b_1 \lambda + \frac{(b_{12} - b_3)}{\lambda} - b_2\right] \\ &= -D^k_i \frac{3}{64\lambda}(\lambda - 1)\left[(8 + \beta + 10\alpha)\lambda \right. \\ &\qquad \left. - 10\alpha + 8 - \beta\right] = 0. \end{aligned}$$

So, using the definition of λ $D^{k+1}_{4i+3} - \lambda D^{k+1}_{4i+4} < 0$.

Here, we see that $Q^{k+1} < \lambda$, also obviously $Q^{k+1} > \frac{1}{\lambda}$, which completes the proof. □

C. CONVEXITY PRESERVATION OF 3-POINT RELAXED QUATERNARY SUBDIVISION SCHEME

If the scheme produces a limit curve that preserves the strict convexity of initial data from a strictly convex polygon, then a subdivision scheme is said to satisfy the property of strict convexity preservation. We can attain the property of strict convexity preservation by considering the following conditions.

- Denote the second order divided difference by $\mathbb{D}^k_j = 4^{2k}[h^k_{j-1} - 2h^k_j + h^k_{j+1}]$. i.e. Suppose the initial control points are strictly convex i.e. $\mathbb{D}^0_i > 0$ for all $i \in \mathbb{Z}$.
- Finding a constant $\eta > 1$ (η depends upon selected value of the free parameter α and β) such that, if we assume $\mathbb{D}^0_i > 0$ for all $i \in \mathbb{Z}$, it is verified that $\max_i \left\{\frac{\mathbb{D}^0_{i+1}}{\mathbb{D}^0_i}, \frac{\mathbb{D}^0_i}{\mathbb{D}^0_{i+1}}\right\} < \eta$.
- The value of η also satisfies the necessary condition for inequality $\frac{1}{\eta} < R^0 < \eta$ i.e. $\forall k \in \mathbb{N}, \mathbb{D}^k_i > 0, \forall i \in \mathbb{Z}$, $\max_i \left\{\frac{\mathbb{D}^k_{i+1}}{\mathbb{D}^k_i}, \frac{\mathbb{D}^k_i}{\mathbb{D}^k_{i+1}}\right\} < \eta$.

D. CONVEXITY PRESERVATION

We can write second order divided difference for the scheme (15) as,

$$\left\{\begin{aligned} \mathbb{D}^{k+1}_{4i} &= \left(\frac{\beta}{16}\right)\mathbb{D}^k_{i-1} + \left(1 - \frac{\beta}{16} - \frac{5\alpha}{8}\right)\mathbb{D}^k_i \\ &\quad - \left(\frac{5\alpha}{8}\right)\mathbb{D}^k_{i+1}, \\ \mathbb{D}^{k+1}_{4i+1} &= \left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{3}{4}\right)\mathbb{D}^k_i \\ &\quad + \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{1}{4}\right)\mathbb{D}^k_{i+1}, \\ \mathbb{D}^{k+1}_{4i+2} &= \left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{1}{2}\right)\mathbb{D}^k_i \\ &\quad + \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{1}{2}\right)\mathbb{D}^k_{i+1}, \\ \mathbb{D}^{k+1}_{4i+3} &= \left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{1}{4}\right)\mathbb{D}^k_i \\ &\quad + \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{3}{4}\right)\mathbb{D}^k_{i+1}. \end{aligned}\right. \tag{21}$$

To prove Theorem 6 by direct calculations, some inequalities are established in the following Lemma.

Lemma 5: . If $\beta = 0$, $\alpha \in (-\frac{2}{5}, 0)$ and $\eta := \frac{6-5\alpha}{5\alpha+4}$ then the following inequalities hold, also assuming $\alpha = 0$, $\beta \in (0, 4)$ and $\eta := \frac{12+\beta}{\beta}$, then the following inequalities are also satisfied.

(i) $\left[\frac{2\beta-5\alpha}{16\eta}+\left(1-\frac{\beta}{16}+\frac{5\alpha}{8}\right)\right]>0,$

(ii) $\left[\left(\frac{\beta}{16}+\frac{5\alpha}{8}+\frac{3}{4}\right)+\left(\frac{-\beta}{16}-\frac{5\alpha}{8}+\frac{1}{4}\right)\frac{1}{\eta}\right]>0,$

(iii) $\left[\left(\frac{\beta}{16}+\frac{5\alpha}{8}+\frac{1}{2}\right)+\left(\frac{-\beta}{16}-\frac{5\alpha}{8}+\frac{1}{2}\right)\eta\right]>0,$

(iv) $\left[\left(\frac{\beta}{16}+\frac{5\alpha}{8}+\frac{1}{4}\right)+\left(\frac{-\beta}{16}-\frac{5\alpha}{8}+\frac{3}{4}\right)\frac{1}{\eta}\right]>0,$

(v) $\left[\left(\frac{3}{4}+\frac{5\alpha}{4}\right)+\left(\frac{-5\alpha}{4}-\frac{3}{4}\right)\eta\right]<0,$

(vi) $\frac{-1}{16\eta}(\eta-1)[(12+10\alpha+\beta)\eta+(\beta-10\alpha)]<0,$

(vii) $\frac{1}{16\eta}(\eta-1)[(-8+10\alpha+\beta)\eta+(-\beta-10\alpha-12)]<0,$

(viii) $\frac{1}{16\eta}(\eta-1)[(-4+10\alpha+\beta)\eta-(\beta+10\alpha+8)]<0,$

(ix) $\frac{1}{16\eta}(\eta-1)[(-12+10\alpha+\beta)\eta-(\beta+10\alpha+8)]<0.$

Theorem 6: Assuming $\hbar_i^k=\frac{\mathbb{D}_{i+1}^k}{\mathbb{D}_i^k}$, $y_i^k=\frac{\mathbb{D}_i^k}{\mathbb{D}_{i+1}^k}=\frac{1}{\hbar_i^k}$ *and* $R^k=\max_i\{\hbar_i^k,y_i^k\}$. *If the initial polygon is strictly convex, i.e.* $\mathbb{D}_i^0>0,\forall i\in\mathbb{Z}$, *then for all* $\beta=0$, $\alpha\in(-\frac{2}{5},0)$ *and* $R^0<\eta:=\frac{6-5\alpha}{5\alpha+4}$ *and then assuming for* $\alpha=0$, $\beta\in(0,4)$ *and* $R^0<\eta:=\frac{12+\beta}{\beta}$, *then quaternary* 3*-point relaxed scheme with refinement rules define in (15) preserves the convexity of the given data.*

Proof: Use mathematical induction, to prove $\mathbb{D}_i^k>0$ and $R^k<\eta$. By the definition of $\hbar_i^k$, y_i^k and R_i^k, we have $R^0<\eta$ and $R^0>\frac{1}{\eta}$ also $\hbar_i^0<\eta$, $y_i^0<\eta$ and $-p_i^0<\frac{-1}{\eta}$, $-y_i^0<\frac{-1}{\eta}$. Then applying mathematical induction on k, $\forall k\in\mathbb{N}$, $\mathbb{D}_i^k>0,\forall i\in\mathbb{Z}$ and $R^k<\eta$. Assuming $\mathbb{D}_i^k>0,\forall i\in\mathbb{Z}$ and $R^k<\eta$.

We have to show that $\mathbb{D}_{4i+j}^{k+1}>0,\forall j=0,1,2,3.$ and $R^{k+1}<\eta$, namely $\hbar_{4i+j}^{k+1}<\eta$, $y_{4i+j}^{k+1}<\eta\ \forall j=0,1,2,3.$ i.e. $\hbar_{4i}^{k+1}=\frac{\mathbb{D}_{4i+1}^{k+1}}{\mathbb{D}_{4i}^{k+1}}<\eta$ or $\mathbb{D}_{4i+1}^{k+1}-\eta\mathbb{D}_{4i}^{k+1}<0$. Also $y_{4i}^{k+1}=\frac{\mathbb{D}_{4i}^{k+1}}{\mathbb{D}_{4i+1}^{k+1}}<\eta$ or $\mathbb{D}_{4i}^{k+1}-\eta\mathbb{D}_{4i+1}^{k+1}<0$.

Considering Lemma 5(i) and using first rule of (21) we obtain

$$\begin{aligned}\mathbb{D}_{4i}^{k+1}&=\mathbb{D}_i^k\left[\left(\frac{\beta}{16}\right)q_{i-1}^k+\left(1-\frac{\beta}{16}+\frac{5\alpha}{8}\right)-\left(\frac{5\alpha}{8}\right)\hbar_i^k\right]\\&>\mathbb{D}_i^k\left[\frac{2\beta-5\alpha}{16\eta}+\left(1-\frac{\beta}{16}+\frac{5\alpha}{8}\right)\right]\\&>0.\end{aligned}$$

Analogously using second rule from (21) and Lemma 5(ii), we get

$$\begin{aligned}\mathbb{D}_{4i+1}^{k+1}&=\mathbb{D}_i^k\left[\left(\frac{\beta}{16}+\frac{5\alpha}{8}+\frac{3}{4}\right)+\left(\frac{-\beta}{16}-\frac{5\alpha}{8}+\frac{1}{4}\right)\hbar_i^k\right]\\&>\mathbb{D}_i^k\left[\left(\frac{\beta}{16}+\frac{5\alpha}{8}+\frac{3}{4}\right)+\left(\frac{-\beta}{16}-\frac{5\alpha}{8}+\frac{1}{4}\right)\frac{1}{\eta}\right]\\&>0.\end{aligned}$$

Additionally, considering Lemma 5(iii) and the third rule in (21) yields

$$\begin{aligned}\mathbb{D}_{4i+2}^{k+1}&=\mathbb{D}_i^k\left[\left(\frac{\beta}{16}+\frac{5\alpha}{8}+\frac{1}{2}\right)+\left(\frac{-\beta}{16}-\frac{5\alpha}{8}+\frac{1}{2}\right)\hbar_i^k\right]\\&>\mathbb{D}_i^k\left[\left(\frac{\beta}{16}+\frac{5\alpha}{8}+\frac{1}{2}\right)+\left(\frac{-\beta}{16}-\frac{5\alpha}{8}+\frac{1}{2}\right)\eta\right]\\&>0.\end{aligned}$$

Also from fourth rule of (21), and Lemma 5(iv), we get

$$\begin{aligned}\mathbb{D}_{4i+3}^{k+1}&=\mathbb{D}_i^k\left[\left(\frac{\beta}{16}+\frac{5\alpha}{8}+\frac{1}{4}\right)+\left(\frac{-\beta}{16}-\frac{5\alpha}{8}+\frac{3}{4}\right)\hbar_i^k\right]\\&>\mathbb{D}_i^k\left[\left(\frac{\beta}{16}+\frac{5\alpha}{8}+\frac{1}{4}\right)+\left(\frac{-\beta}{16}-\frac{5\alpha}{8}+\frac{3}{4}\right)\frac{1}{\eta}\right]\\&>0.\end{aligned}$$

Moreover, using Lemma 5(v) we have

$$\begin{aligned}&\mathbb{D}_{4i+1}^{k+1}-\eta\mathbb{D}_{4i}^{k+1}\\&\quad=\mathbb{D}_i^k\Bigg[\left(\frac{\beta}{16}+\frac{5\alpha}{8}+\frac{3}{4}\right)+\left(\frac{-\beta}{16}-\frac{5\alpha}{8}+\frac{1}{4}\right)\hbar_i^k\\&\qquad-\eta\left(\left(\frac{\beta}{16}\right)q_{i-1}^k+\left(1-\frac{\beta}{16}+\frac{5\alpha}{8}\right)\right.\\&\qquad\left.-\left(\frac{5\alpha}{8}\right)\hbar_i^k\right)\Bigg].\end{aligned}$$

This implies that

$$\begin{aligned}&\mathbb{D}_{4i+1}^{k+1}-\eta\mathbb{D}_{4i}^{k+1}\\&\quad<\mathbb{D}_i^k\Bigg[\left(\frac{\beta}{16}+\frac{5\alpha}{8}+\frac{3}{4}\right)+\left(\frac{-\beta}{16}-\frac{5\alpha}{8}+\frac{1}{4}\right)(\eta)\\&\qquad-\eta\left(\frac{\beta}{16}\left(\frac{1}{\eta}\right)\right)-\eta\left(1-\frac{\beta}{16}+\frac{5\alpha}{8}\right)\\&\qquad+\eta\left(\frac{5\alpha}{8}\right)\frac{1}{\eta}\Bigg].\end{aligned}$$

Further implies that

$$\mathbb{D}_{4i+1}^{k+1}-\eta\mathbb{D}_{4i}^{k+1}<\mathbb{D}_i^k\left[\left(\frac{3}{4}+\frac{5\alpha}{4}\right)+\left(\frac{-5\alpha}{4}-\frac{3}{4}\right)\eta\right]$$

for $\beta=0$, $\alpha\in(-\frac{2}{5},0)$ and $\eta:=\frac{6-5\alpha}{5\alpha+4}$ we have

$$\mathbb{D}_{4i+1}^{k+1}-\eta\mathbb{D}_{4i}^{k+1}=-\frac{1}{4}\mathbb{D}_i^k(\eta-1)(3+5\alpha)=0.$$

In the same way for$\alpha=0$, $\beta\in(0,4)$ and $\eta:=\frac{12+\beta}{\beta}$ we have

$$\mathbb{D}_{4i+1}^{k+1}-\eta\mathbb{D}_{4i}^{k+1}=\mathbb{D}_i^k\left(\frac{3}{4}-\frac{3\eta}{4}\right)=0.$$

Due to definition of η, $\mathbb{D}_{4i+1}^{k+1}-\eta\mathbb{D}_{4i}^{k+1}<0$, i.e. $\hbar_{4i}^{k+1}<\eta$. Now consider

$$\begin{aligned}\mathbb{D}_{4i}^{k+1}-\eta\mathbb{D}_{4i+1}^{k+1}&=\mathbb{D}_i^k\Bigg[\left(\left(\frac{\beta}{16}\right)q_{i-1}^k+\left(1-\frac{\beta}{16}+\frac{5\alpha}{8}\right)\right.\\&\qquad\left.-\left(\frac{5\alpha}{8}\right)\hbar_i^k\right)-\eta\left(\left(\frac{\beta}{16}+\frac{5\alpha}{8}+\frac{3}{4}\right)\right.\\&\qquad\left.+\left(\frac{-\beta}{16}-\frac{5\alpha}{8}+\frac{1}{4}\right)\hbar_i^k\right)\Bigg].\end{aligned}$$

This implies that

$$\mathbb{D}_{4i}^{k+1} - \eta \mathbb{D}_{4i+1}^{k+1} < \mathbb{D}_i^k \left[\left(\left(\frac{\beta}{16} \right) \frac{1}{\eta} + \left(1 - \frac{\beta}{16} + \frac{5\alpha}{8} \right) - \left(\frac{5\alpha}{8} \right) \frac{1}{\eta} \right) - \eta \left(\left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{3}{4} \right) + \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{1}{4} \right) \frac{1}{\eta} \right) \right]$$
$$< \mathbb{D}_i^k \frac{-1}{16\eta} (\eta - 1)[(12 + 10\alpha + \beta)\eta + (\beta - 10\alpha)].$$

for $\alpha = 0$, $\beta \in (0, 4)$ and $\eta := \frac{12+\beta}{\beta}$.

$$\mathbb{D}_{4i}^{k+1} - \eta \mathbb{D}_{4i+1}^{k+1} < \mathbb{D}_i^k \frac{-1}{16\eta} (\eta - 1)[(12 + \beta)\eta + (\beta)]$$

for $\beta = 0$, $\alpha \in (-\frac{2}{5}, 0)$ and $\eta := \frac{6-5\alpha}{5\alpha+4}$.

$$\mathbb{D}_{4i}^{k+1} - \eta \mathbb{D}_{4i+1}^{k+1} < \mathbb{D}_i^k \frac{-1}{16\eta} (\eta - 1)[(12 + 10\alpha)\eta + (-10\alpha)].$$

Thus, using Lemma 5(vi) we have $\mathbb{D}_{4i}^{k+1} - \eta \mathbb{D}_{4i+1}^{k+1} < 0$, i.e $y_{4i}^{k+1} < \eta$.

Similarly considering Lemma 5(vii), we get

$$\mathbb{D}_{4i+1}^{k+1} - \eta \mathbb{D}_{4i+2}^{k+1} = \mathbb{D}_i^k \left[\left(\left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{3}{4} \right) + \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{1}{4} \right) \hbar_i^k \right) - \eta \left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{1}{2} \right) + \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{1}{2} \right) \hbar_i^k \right].$$

This implies that

$$\mathbb{D}_{4i+1}^{k+1} - \eta \mathbb{D}_{4i+2}^{k+1} < \mathbb{D}_i^k \left[\left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{3}{4} \right) + \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{1}{4} \right) \eta - \eta \left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{1}{2} \right) - \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{1}{2} \right) \eta^2 \right].$$

Further implies that

$$\mathbb{D}_{4i+1}^{k+1} - \eta \mathbb{D}_{4i+2}^{k+1} < \mathbb{D}_i^k \frac{1}{16\eta} (\eta - 1) [(-8 + 10\alpha + \beta)\eta + (-\beta - 10\alpha - 12)] .$$

for $\beta = 0$, $\alpha \in (-\frac{2}{5}, 0)$ and $\eta := \frac{6-5\alpha}{5\alpha+4}$.

$$\mathbb{D}_{4i+1}^{k+1} - \eta \mathbb{D}_{4i+2}^{k+1} < \mathbb{D}_i^k \frac{1}{8\eta} (\eta - 1)[(-4 + 5\alpha)\eta - (5\alpha + 6)]$$

for $\alpha = 0$, $\beta \in (0, 4)$ and $\eta := \frac{12+\beta}{\beta}$.

$$\mathbb{D}_{4i+1}^{k+1} - \eta \mathbb{D}_{4i+2}^{k+1} < \mathbb{D}_i^k \frac{1}{16\eta} (\eta - 1)[(-8+\beta)\eta + (-\beta - 12)].$$

From the definition of η, $\mathbb{D}_{4i+1}^{k+1} - \eta \mathbb{D}_{4i+2}^{k+1} < 0$, i.e $y_{4i+1}^{k+1} < \eta$.

Using Lemma 5($viii$), we have

$$\mathbb{D}_{4i+2}^{k+1} - \eta \mathbb{D}_{4i+1}^{k+1} = \mathbb{D}_i^k \left[\left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{1}{2} \right) + \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{1}{2} \right) \hbar_i^k - \eta \left(\left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{3}{4} \right) + \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{1}{4} \right) \hbar_i^k \right) \right].$$

This implies that

$$\mathbb{D}_{4i+2}^{k+1} - \eta \mathbb{D}_{4i+1}^{k+1} < \mathbb{D}_i^k \left[\left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{1}{2} \right) + \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{1}{2} \right) \eta - \eta \left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{3}{4} \right) - \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{1}{4} \right) \eta^2 \right].$$

Further implies that

$$\mathbb{D}_{4i+2}^{k+1} - \eta \mathbb{D}_{4i+1}^{k+1} < \mathbb{D}_i^k \frac{1}{16\eta} (\eta - 1) \times [(-4 + 10\alpha + \beta)\eta - (\beta + 10\alpha + 8)]$$

for $\alpha = 0$, $\beta \in (0, 4)$ and $\eta := \frac{12+\beta}{\beta}$.

$$\mathbb{D}_{4i+2}^{k+1} - \eta \mathbb{D}_{4i+1}^{k+1} < \mathbb{D}_i^k \frac{1}{16\eta} (\eta - 1)[(-4 + \beta)\eta - (\beta + 8)]$$

for $\beta = 0$, $\alpha \in (-\frac{2}{5}, 0)$ and $\eta := \frac{6-5\alpha}{5\alpha+4}$.

$$\mathbb{D}_{4i+2}^{k+1} - \eta \mathbb{D}_{4i+1}^{k+1} < \mathbb{D}_i^k \frac{1}{16\eta} (\eta - 1) \times [(-4 + 10\alpha)\eta - (10\alpha + 8)] .$$

Due to the definition of η, $\mathbb{D}_{4i+2}^{k+1} - \eta \mathbb{D}_{4i+1}^{k+1} < 0$, i.e $\hbar_{4i+1}^{k+1} < \eta$.

Using Lemma 5(ix), we get

$$\mathbb{D}_{4i+2}^{k+1} - \eta \mathbb{D}_{4i+3}^{k+1} = \mathbb{D}_i^k \left[\left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{1}{2} \right) + \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{1}{2} \right) \hbar_i^k - \eta \left(\left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{1}{4} \right) + \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{3}{4} \right) \hbar_i^k \right) \right],$$

$$\mathbb{D}_{4i+2}^{k+1} - \eta \mathbb{D}_{4i+3}^{k+1} < \mathbb{D}_i^k \left[\left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{1}{2} \right) + \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{1}{2} \right) \eta - \eta \left(\frac{\beta}{16} + \frac{5\alpha}{8} + \frac{1}{4} \right) - \left(\frac{-\beta}{16} - \frac{5\alpha}{8} + \frac{3}{4} \right) \eta^2 \right]$$

for $\alpha = 0$, $\beta \in (0, 4)$, $\eta := \frac{12+\beta}{\beta}$.

$$\mathbb{D}_{4i+2}^{k+1} - \eta \mathbb{D}_{4i+3}^{k+1} < \mathbb{D}_i^k \frac{1}{16\eta} (\eta - 1) [(-12 + 10\alpha + \beta)\eta - (\beta + 10\alpha + 8)]$$
$$< \mathbb{D}_i^k \frac{1}{16\eta} (\eta - 1)[(-12 + \beta)\eta - (\beta + 8)]$$

for $\beta = 0$, $\alpha \in (-\frac{2}{5}, 0)$ and $\eta := \frac{6-5\alpha}{5\alpha+4}$.

$$\mathbb{D}_{4i+2}^{k+1} - \eta\mathbb{D}_{4i+3}^{k+1} < \mathbb{D}_i^k \frac{1}{16\eta}(\eta-1)\left[(-12+10\alpha)\eta -(10\alpha+8)\right].$$

So, $\mathbb{D}_{4i+2}^{k+1} - \eta\mathbb{D}_{4i+3}^{k+1} < 0$, i.e. $y_{4i+2}^{k+1} < \eta$.

Here, we see that $R^{k+1} < \eta$. Since $R^{k+1} = \max_i\{\hbar_i^k, y_i^k\}$, then obviously $R^{k+1} > \frac{1}{\eta}$. □

Theorem 7: The support of 3-point relaxed quaternary subdivision scheme (15) is $[-\frac{7}{3}, \frac{7}{3}]$.

Proof: Support size shows the property of a subdivision curve in which the area of the limit curve will be affected by moving a single control point from its initial place. i.e., the effect of one vertex on its neighbouring points. Support size can be found by calculating the distance between two corresponding parameters which belong to the most right and most left vertex.

First we calculate all non-zero point for $k = 0$ and $i = -1, 0, 1, -2$ in 4-point parametric quaternary subdivision scheme. Hence, all non-zero points at the first iteration level make a sequence as

$$f_{-7}^1, f_{-6}^1, f_{-5}^1, f_{-4}^1, f_{-3}^1, \ldots, f_0^1, f_1^1, f_2^1, f_3^1, f_4^1, f_5^1, f_6^1, f_7^1.$$

From the above sequence, a set can be defined as

$$D_k = \left\{\frac{i}{4^k} \mid i \in \mathbb{Z}\right\}.$$

Such that

$$\phi\left(\frac{i}{4^k}\right) = f_i^k \quad \forall \quad i \in \mathbb{Z}.$$

The leftmost non-zero vertex is f_{-7}^1 with $\phi\left(\frac{-7}{4^1}\right) = f_{-7}^1$ and the rightmost non-zero vertex is f_7^1 with $\phi\left(\frac{7}{4^1}\right) = f_7^1$.

Now for second iteration level, we calculate all non-zero points for $k = 1$ and $i = -9, -8, -7, \ldots, 5, 6, 7, 8$. Hence, all the non-zero points at second iteration level make the sequence as $f_{-35}^2, f_{-34}^2, \ldots, f_{34}^2, f_{35}^2$. The leftmost non-zero vertex is f_{-35}^2 with $\phi\left(\frac{-7(1+4)}{4^2}\right) = f_{-35}^2$ and the rightmost non-zero vertex is f_{35}^2 with $\phi\left(\frac{7(1+4)}{4^2}\right) = f_{35}^2$.

Now we calculate all the non-zero points at third iteration level for $k = 2$ and $i = -36, -35, -34, \ldots, 0, 1, 2, 3, \ldots, 35, 36$.

Hence, all the non-zero points at the third iteration level make the sequence as $P_{-147}^3, P_{-146}^3, \ldots, P_{-2}^3, P_{-1}^3, P_0^3, \ldots, P_{146}^3, P_{147}^3$. The leftmost non-zero vertex is P_{-147}^3 with $\phi\left(\frac{-7(1+4+4^2)}{4^3}\right) = P_{-7(1+4+4^2)}^3$ and the rightmost non-zero vertex is P_{147}^3 with $\phi\left(\frac{7(1+4+4^2)}{4^2}\right) = P_{7(1+4+4^2)}^3$. Similarly, after k subdivision step, the left most non-zero vertex is $P_{-7(1+4+4^2+\ldots+4^{k-1})}^k$ with $\phi\left(\frac{-7(1+4+4^2+\ldots+4^{k-1})}{4^k}\right) = P_{-7(1+4+4^2+\ldots+4^{k-1})}^k$ and the rightmost non-zero vertex is $P_{7(1+4+4^2+\ldots+4^{k-1})}^k$ with $\phi\left(\frac{7(1+4+4^2+\ldots+4^{k-1})}{4^k}\right) = P_{7(1+4+4^2+\ldots+4^{k-1})}^k$.

Hence, the support of size is

$$\text{Support size} = \lim_{k\to\infty}\left[\frac{7\zeta}{4^k} - \frac{-7\zeta}{4^k}\right],$$

where

$$\zeta = 1 + 4 + 4^2 + \ldots + 4^{k-1}.$$

This implies that

$$\text{Support size} = \lim_{k\to\infty}\left[14\left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^k}\right)\right].$$

$$\text{Support size} = 14\left(\frac{\frac{1}{4}}{1-\frac{1}{4}}\right) = \frac{14}{3}.$$

Hence, the support region is $\left[-\frac{7}{3}, \frac{7}{3}\right]$.

Figure 1, shows the basic limit function of proposed scheme for different values of parameters. □

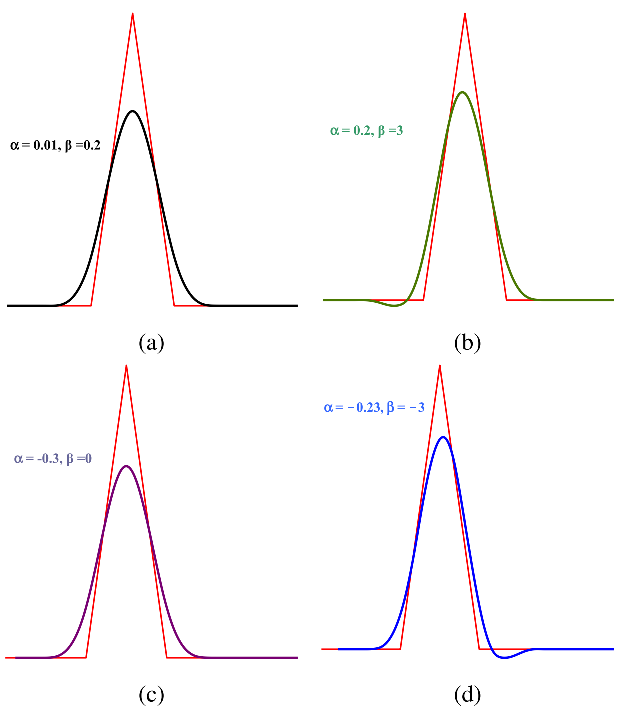


FIGURE 1. **The basic limit functions of the 3-point relaxed quaternary scheme and their support for different values of parameters. The solid red lines are initial control polygons made by the data of the Kronecker delta.**

Theorem 8: The limit stencils of 3-point relaxed quaternary subdivision scheme (15) is $[\zeta_1, \zeta_2, \zeta_3, \zeta_4]$, *where*

$$\zeta_1 = \frac{1}{181440}\Big[1000\alpha^3 + 300\alpha^2\beta + 13200\alpha^2 + 30\alpha\beta^2 +3360\alpha\beta + 24720\alpha + \beta^3 + 204\beta^2 +4488\beta + 8960\Big],$$

$$\zeta_2 = \frac{1}{60480}\Big[29040\alpha + 189\beta - 1680\alpha\beta^2 - 30\alpha\beta^2 - 300\alpha^2\beta - 4800\alpha^2 - 1000\alpha^3 - 120\beta^2 - \beta^3 + 34720\Big],$$

$$\zeta_3 = \frac{1}{60480}\Big[1000\alpha^3 + 300\alpha^2\beta - 3600\alpha^2 + 30\alpha\beta^2 - 32400\alpha + \beta^3 + 36\beta^2 - 3240\beta + 22400\Big],$$

and

$$\zeta_4 = \frac{1}{181440}\Big[-1000\alpha^3 - 300\alpha^2\beta + 12000\alpha^2 - 30\alpha\beta^2 + 1680\alpha\beta - 14640\alpha - \beta^3 + 48\beta^2 - 456\beta + 1120\Big].$$

Proof: To find out the limit stencils of (15), we substitute $i = -1$ in refinement rules of (15). Hence we obtain the following refined points.

$$\begin{cases} f_{-4}^{k+1} = \left(\frac{5}{128}\beta + \frac{15}{64}\alpha + \frac{5}{32}\right)f_{-2}^k + \left(\frac{5}{32}\alpha + \frac{11}{16} - \frac{1}{64}\beta\right)f_{-1}^k + \left(-\frac{25}{64}\alpha + \frac{5}{32} - \frac{3}{128}\beta\right)f_0^k, \\ f_{-3}^{k+1} = \left(\frac{1}{16} + \frac{3}{128}\beta + \frac{15}{128}\alpha\right)f_{-2}^k + \left(\frac{45}{128}\alpha + \frac{5}{8} + \frac{1}{64}\beta\right)f_{-1}^k + \left(-\frac{55}{128}\alpha + \frac{5}{16} - \frac{5}{128}\beta\right)f_0^k + \left(-\frac{5}{128}\alpha\right)f_1^k, \\ f_{-2}^{k+1} = \left(\frac{3}{256}\beta + \frac{5}{128}\alpha + \frac{1}{64}\right)f_{-2}^k + \left(\frac{55}{128}\alpha + \frac{31}{64} + \frac{9}{256}\beta\right)f_{-1}^k + \left(\frac{31}{64} - \frac{45}{128}\alpha - \frac{11}{256}\beta\right)f_0^k + \left(-\frac{15}{128}\alpha + \frac{1}{64} - \frac{1}{256}\beta\right)f_1^k, \\ f_{-1}^{k+1} = \left(\frac{1}{256}\beta\right)f_{-2}^k + \left(\frac{5}{16} + \frac{11}{256}\beta + \frac{25}{64}\alpha\right)f_{-1}^k + \left(-\frac{5}{32}\alpha - \frac{9}{256}\beta + \frac{5}{8}\right)f_0^k + \left(-\frac{3}{256}\beta - \frac{15}{64}\alpha + \frac{1}{16}\right)f_1^k. \end{cases}$$

It can be written as

$$\begin{pmatrix} f_{-4}^{k+1} \\ f_{-3}^{k+1} \\ f_{-2}^{k+1} \\ f_{-1}^{k+1} \end{pmatrix} = \begin{pmatrix} \vartheta_{-4} & \vartheta_0 & \vartheta_4 & 0 \\ \vartheta_{-5} & \vartheta_{-1} & \vartheta_3 & \vartheta_7 \\ \vartheta_{-6} & \vartheta_{-2} & \vartheta_2 & \vartheta_6 \\ \vartheta_{-7} & \vartheta_{-3} & \vartheta_1 & \vartheta_5 \end{pmatrix} \begin{pmatrix} f_{-2}^k \\ f_{-1}^k \\ f_0^k \\ f_1^k \end{pmatrix}.$$

□

Symbolically, it can be written as,

$$f^{k+1} = R_1 f^k.$$

Hence local subdivision matrix of (15) is

$$R_1 = \begin{pmatrix} \vartheta_{-4} & \vartheta_0 & \vartheta_4 & 0 \\ \vartheta_{-5} & \vartheta_{-1} & \vartheta_3 & \vartheta_7 \\ \vartheta_{-6} & \vartheta_{-2} & \vartheta_2 & \vartheta_6 \\ \vartheta_{-7} & \vartheta_{-3} & \vartheta_1 & \vartheta_5 \end{pmatrix}.$$

Now we calculate eigenvalues of subdivision matrix R_1 and denote them by λ_i; $i = 0, 1, 2, 3$.

$$\lambda_i = 1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}.$$

The matrix of eigenvector corresponding to the eigenvalues is

$$V = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 1 \\ x_{21} & x_{22} & x_{23} & 1 \\ x_{31} & x_{32} & x_{33} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

where

$$x_{11} = \frac{Z_1}{Z_2}, \quad x_{21} = \frac{Z_3}{Z_4}, \quad x_{31} = \frac{Z_5}{Z_6},$$
$$x_{12} = \frac{Z_7}{Z_8}, \quad x_{22} = \frac{Z_9}{Z_{10}}, \quad x_{32} = \frac{Z_{11}}{Z_{12}},$$
$$x_{13} = \frac{Z_{13}}{Z_{14}}, \quad x_{23} = \frac{Z_{15}}{Z_{16}}, \quad x_{33} = \frac{Z_{17}}{Z_{18}},$$

where

$$Z_1 = \big[8000\alpha^3 + 2400\alpha^2\beta - 14100\alpha^2 + 240\alpha\beta^2 - 3360\alpha\beta + 11280\alpha + 8\beta^3 - 195\beta^2 + 1632\beta - 3920\big],$$
$$Z_2 = \big[8000\alpha^3 + 2400\alpha^2\beta + 23700\alpha^2 + 240\alpha\beta^2 + 4200\alpha\beta + 26400\alpha + 8\beta^3 + 183\beta^2 + 2010\beta + 11200\big],$$
$$Z_3 = \big[-(-8000\alpha^3 - 2400\alpha^2\beta + 1500\alpha^2 - 240\alpha\beta^2 + 840\alpha\beta + 2580\alpha - 8\beta^3 + 69\beta^2 + 132\beta - 1120)\big],$$
$$Z_4 = \big[8000\alpha^3 + 2400\alpha^2\beta + 23700\alpha^2 + 240\alpha\beta^2 + 4200\alpha\beta + 26400\alpha + 8\beta^3 + 183\beta^2 + 2010\beta + 11200\big],$$
$$Z_5 = \big[(8000\alpha^3 + 2400\alpha^2\beta + 11100\alpha^2 + 240\alpha\beta^2 + 1680\alpha\beta + 2460\alpha + 8\beta^3 + 57\beta^2 - 6\beta - 1400)\big],$$
$$Z_6 = \big[(8000\alpha^3 + 2400\alpha^2\beta + 23700\alpha^2 + 240\alpha\beta^2 + 4200\alpha\beta + 26400\alpha + 8\beta^3 + 183\beta^2 + 2010\beta + 11200)\big],$$
$$Z_7 = \big[(200\alpha^2 + 40\alpha\beta - 370\alpha + 2\beta^2 - 43\beta + 260)\big],$$
$$Z_8 = \big[(200\alpha^2 + 40\alpha\beta + 530\alpha + 2\beta^2 + 47\beta + 440)\big],$$
$$Z_9 = \big[-(-200\alpha^2 - 40\alpha\beta + 70\alpha - 2\beta^2 + 13\beta + 40)\big],$$
$$Z_{10} = \big[(200\alpha^2 + 40\alpha\beta + 530\alpha + 2\beta^2 + 47\beta + 440)\big],$$
$$Z_{11} = \big[(200\alpha^2 + 40\alpha\beta + 230\alpha + 2\beta^2 + 17\beta + 20)\big],$$
$$Z_{12} = \big[(200\alpha^2 + 40\alpha\beta + 530\alpha + 2\beta^2 + 47\beta + 440)\big],$$
$$Z_{13} = \big[(10\alpha + \beta - 16)\big], \quad Z_{14} = \big[(10\alpha + \beta + 20)\big],$$
$$Z_{15} = \big[(10\alpha + \beta - 4)\big], \quad Z_{16} = \big[(10\alpha + \beta + 20)\big],$$
$$Z_{17} = \big[(10\alpha + \beta + 8)\big], \quad Z_{18} = \big[(10\alpha + \beta + 20)\big].$$

By taking inverse of above matrix, we have

$$V^{-1}=\begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14}\\ y_{21} & y_{22} & y_{23} & y_{24}\\ y_{31} & y_{32} & y_{33} & y_{34}\\ y_{41} & y_{42} & y_{43} & y_{44}\end{pmatrix}.$$

where

$$y_{11}=\frac{1}{22680}\big[-26400\alpha-2010\beta-4200\alpha\beta-240\alpha\beta^2 -2400\alpha^2\beta-23700\alpha^2-8000\alpha^3-183\beta^2 -8\beta^3-11200\big],$$

$$y_{12}=\frac{1}{98280}\big[104000\alpha^3+31200\alpha^2\beta+308100\alpha^2 +3120\alpha\beta^2+54600\alpha\beta+343200\alpha+104\beta^3 +2379\beta^2+26130\beta+145600\big],$$

$$y_{13}=\frac{1}{7560}\big[-26400\alpha-2010\beta-4200\alpha\beta-240\alpha\beta^2 -2400\alpha^2\beta-23700\alpha^2-8000\alpha^3-183\beta^2 -8\beta^3-11200\big],$$

$$y_{14}=\frac{1}{7560}\big[8000\alpha^3+2400\alpha^2\beta+23700\alpha^2+240\alpha\beta^2 +4200\alpha\beta+26400\alpha+8\beta^3+183\beta^2+2010\beta +11200\big],$$

$$y_{21}=\frac{1}{4320}\big[2000\alpha^3+600\alpha^2\beta+6900\alpha^2+60\alpha\beta^2 +1320\alpha\beta+8640\alpha+2\beta^3+63\beta^2 +816\beta+3520\big],$$

$$y_{22}=\frac{1}{1440}\big[-6520\alpha-628\beta-1160\alpha\beta-60\alpha\beta^2 -600\alpha^2\beta-6100\alpha^2-2000\alpha^3-55\beta^2 -2\beta^3-1760\big],$$

$$y_{23}=\frac{1}{1440}\big[2000\alpha^3+600\alpha^2\beta+5300\alpha^2+60\alpha\beta^2 +1000\alpha\beta+4400\alpha+2\beta^3+47\beta^2+440\beta\big],$$

$$y_{24}=\frac{1}{4320}\big[1760-252\beta-840\alpha\beta-60\alpha\beta^2 -600\alpha^2\beta-4500\alpha^2-2000\alpha^3-39\beta^2-2\beta^3 -2280\alpha\big],$$

$$y_{31}=\frac{1}{8640}\big[-(10\alpha+\beta+20)(100\alpha^2+20\alpha\beta +340\alpha+\beta^2+46\beta+160)\big],$$

$$y_{32}=\frac{1}{2880}\big[1000\alpha^3+300\alpha^2\beta+3400\alpha^2+30\alpha\beta^2 +800\alpha\beta+1600\alpha+\beta^3+46\beta^2+400\beta-2400\big],$$

$$y_{33}=\frac{1}{28840}\big[2800\alpha+40\beta-400\alpha\beta-30\alpha\beta^2 -300\alpha^2\beta-1400\alpha^2-1000\alpha^3-26\beta^2 -\beta^3+3200\big],$$

$$y_{34}=\frac{1}{8640}\big[1000\alpha^3+300\alpha^2\beta-600\alpha^2+30\alpha\beta^2 -4800\alpha+\beta^3+6\beta^2-240\beta+800\big],$$

$$y_{41}=\frac{1}{181440}\big[1000\alpha^3+300\alpha^2\beta+13200\alpha^2 +30\alpha\beta^2+3360\alpha\beta+24720\alpha+\beta^3+204\beta^2 +4488\beta+8960\big],$$

$$y_{42}=\frac{1}{60480}\big[29040\alpha+1896\beta-1680\alpha\beta-30\alpha\beta^2 -300\alpha^2\beta-4800\alpha^2-1000\alpha^3-120\beta^2 -\beta^3+34720\big],$$

$$y_{43}=\frac{1}{60480}\big[1000\alpha^3+300\alpha^2\beta-3600\alpha^2+30\alpha\beta^2 -32400\alpha+\beta^3+36\beta^2-3240\beta+22400\big],$$

$$y_{44}=\frac{1}{181440}\big[-1000\alpha^3-300\alpha^2\beta+12000\alpha^2 -30\alpha\beta^2+1680\alpha\beta-14640\alpha-\beta^3+48\beta^2 -456\beta+1120\big].$$

Let σ be the diagonal matrix of eigenvalues, then

$$\sigma=\begin{pmatrix} (1)^j & 0 & 0 & 0\\ 0 & (\frac{1}{4})^j & 0 & 0\\ 0 & 0 & (\frac{1}{16})^j & 0\\ 0 & 0 & 0 & (\frac{1}{64})^j\end{pmatrix}.$$

Since $R_1=V\sigma V^{-1}$, therefore $R_1^k=V\sigma^k V^{-1}$. Also we have

$$f^{k+1}=R_1 f^k=R_1^2 f^{k-1}=R_1^3 f^{k-2}=\ldots=R_1^k f^0.$$

Which implies that

$$f^{k+1}=V\delta^k V^{-1}q^0.$$

Hence

$$\lim_{k\to\infty} f^{k+1}=V(\lim_{k\to\infty}\delta^k)V^{-1}f^0.$$

After simplification, we get

$$\begin{pmatrix} f_{-3}^{\infty}\\ f_{-2}^{\infty}\\ f_{-1}^{\infty}\\ f_0^{\infty}\end{pmatrix}=\begin{pmatrix} \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4\\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4\\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4\\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4\end{pmatrix}\begin{pmatrix} f_{-3}^{0}\\ f_{-2}^{0}\\ f_{-1}^{0}\\ f_0^{0}\end{pmatrix}.$$

Here we check whether new and old vertices lie on the limit curve. We use the technique described by Hormann and Sabin [25] considering the scheme (15) as a function, the ordinate is obtained from the scheme, and control points are used as abscissa that is equally spaced.

Theorem 9: The 3-point relaxed quaternary approximating subdivision scheme (15) *has cubic precision.*

Proof: Consider the origin in the middle of original span with ordinate$\ldots,(-5)^n,(-3)^n,(-1)^n,(1)^n,(3)^n,(5)^n,\ldots$. If $y=x^n$, then we have,

$$[y]=\ldots,\vartheta_{-4}(-5)^n+\vartheta_0(-3)^n+\vartheta_4(-1)^n,$$

$$\vartheta_{-5}(-5)^n + \vartheta_{-1}(-3)^n + \vartheta_3(-1)^n + \vartheta_7(1)^n,$$
$$\vartheta_{-6}(-5)^n + \vartheta_{-2}(-3)^n + \vartheta_2(-1)^n + \vartheta_6(1)^n,$$
$$\vartheta_{-7}(-5)^n + \vartheta_{-3}(-3)^n + \vartheta_1(-1)^n + \vartheta_2(1)^n,$$
$$\vartheta_{-4}(-3)^n + \vartheta_{-1}(-1)^n + \vartheta_3(1)^n + \vartheta_7(3)^n,$$
$$\vartheta_{-5}(-3)^n + \vartheta_{-1}(-1)^n + \vartheta_3(1)^n + \vartheta_7(3)^n,$$
$$\vartheta_{-6}(-3)^n + \vartheta_{-2}(-1)^n + \vartheta_2(1)^n + \vartheta_6(3)^n,$$
$$\vartheta_{-7}(-3)^n + \vartheta_{-3}(-1)^n + \vartheta_1(1)^n + \vartheta_2(3)^n,$$
$$\vdots$$
$$\vartheta_{-6}(-1)^n + \vartheta_{-2}(1)^n + \vartheta_2(3)^n + \vartheta_6(5)^n,$$
$$\vartheta_{-7}(-1)^n + \vartheta_{-3}(1)^n + \vartheta_1(3)^n + \vartheta_2(5)^n, \ldots,$$

where ϑ_c: $c = -7, -6, \ldots .6, 7.$ defined in (17).

If $y = x^1$, then

$$[y] = \ldots, \left(\frac{-\beta}{8} - \frac{5\alpha}{4} - 3\right), \left(\frac{-5}{2} - \frac{-\beta}{8} - \frac{5\alpha}{4}\right), \left(\frac{-\beta}{8} - \frac{5\alpha}{4} - 2\right), \left(\frac{-\beta}{8} - \frac{5\alpha}{4} - \frac{3}{2}\right), \left(\frac{-\beta}{8} - \frac{5\alpha}{4} - 1\right), \left(\frac{-\beta}{8} - \frac{5\alpha}{4} - \frac{1}{2}\right) \left(\frac{-\beta}{8} - \frac{5\alpha}{4}\right), \left(\frac{-\beta}{8} - \frac{5\alpha}{4} + \frac{1}{2}\right), \left(\frac{-\beta}{8} - \frac{5\alpha}{4} + 1\right), \left(\frac{-\beta}{8} - \frac{5\alpha}{4} + \frac{3}{2}\right), \left(\frac{-\beta}{8} - \frac{5\alpha}{4} + 2\right), \left(\frac{-\beta}{8} - \frac{5\alpha}{4} + \frac{5}{2}\right), \ldots$$

$$[\Delta y] = \ldots, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots$$

$$[\Delta^2 y] = 0,$$

where Δ represents the differences of the vertices.

If $y = x^2$, then

$$[y] = \ldots, \left(\frac{13\beta}{16} - \frac{55\alpha}{8} + \frac{41}{4}\right), \left(\frac{11\beta}{16} - \frac{45\alpha}{8} + \frac{15}{2}\right), \left(\frac{9\beta}{16} - \frac{35\alpha}{8} + \frac{21}{4}\right) \left(\frac{7\beta}{16} - \frac{25\alpha}{8} + \frac{7}{2}\right), \left(\frac{5\beta}{16} - \frac{15\alpha}{8} + \frac{9}{4}\right), \left(\frac{3\beta}{16} - \frac{5\alpha}{8} + \frac{3}{2}\right), \left(\frac{1\beta}{16} - \frac{5\alpha}{8} + \frac{5}{4}\right) \left(\frac{-\beta}{16} - \frac{15\alpha}{8} + \frac{3}{2}\right), \left(\frac{-3\beta}{16} - \frac{25\alpha}{8} + \frac{9}{4}\right), \left(\frac{-5\beta}{16} - \frac{35\alpha}{8} + \frac{7}{2}\right), \left(\frac{-7\beta}{16} - \frac{45\alpha}{8} + \frac{21}{4}\right), \left(\frac{-9\beta}{16} - \frac{55\alpha}{8} + \frac{15}{2}\right), \ldots$$

By taking differences, we get $[\Delta^3 y] = 0$.

If $y = x^3$, then

$$[y] = \ldots, \left(\frac{-71\beta}{16} - \frac{265\alpha}{8} - \frac{153}{4}\right) \left(\frac{-53\beta}{16} - \frac{95\alpha}{4} - 25\right), \left(\frac{-19\beta}{8} - \frac{65\alpha}{4} - \frac{31}{2}\right), \left(\frac{-13\beta}{8} - \frac{85\alpha}{8} - 9\right), \left(\frac{-17\beta}{16} - \frac{55\alpha}{8} - \frac{19}{4}\right), \left(\frac{-11\beta}{16} - 5\alpha - 2\right), \left(\frac{-1\beta}{2} - 5\alpha\right), \left(\frac{-1\beta}{2} - \frac{55\alpha}{8} + 2\right), \left(\frac{-11\beta}{16} - \frac{85\alpha}{8} + \frac{19}{4}\right), \left(\frac{-17\beta}{16} - \frac{65\alpha}{4} + 9\right), \left(\frac{-13\beta}{8} - \frac{95\alpha}{4} + \frac{31}{2}\right), \left(\frac{-19\beta}{8} - \frac{265\alpha}{8} + 25\right), \ldots$$

By taking differences, we get $[\Delta^4 y] = 0$.

If $y = x^4$, then

$$[y] = \ldots, \left(\frac{185\beta}{8} + \frac{635\alpha}{4} + \frac{307}{2}\right), \left(\frac{127\beta}{8} + \frac{405\alpha}{4} + 90\right), \left(\frac{81\beta}{8} + \frac{235\alpha}{4} + \frac{99}{2}\right), \left(\frac{47\beta}{8} + \frac{125\alpha}{4} + 26\right), \left(\frac{25\beta}{8} + \frac{75\alpha}{4} + \frac{27}{2}\right), \left(\frac{15\beta}{8} + \frac{25\alpha}{4} + 6\right), \left(\frac{5\beta}{8} - \frac{25\alpha}{4} + \frac{7}{2}\right). \left(\frac{-5\beta}{8} - \frac{75\alpha}{4} + 6\right), \left(\frac{-15\beta}{8} - \frac{125\alpha}{4} + \frac{27}{2}\right), \left(\frac{-25\beta}{8} - \frac{235\alpha}{4} + 26\right), \left(\frac{-47\beta}{8} - \frac{405\alpha}{4} + \frac{99}{2}\right), \left(\frac{-81\beta}{8} - \frac{635\alpha}{4} + 90\right).$$

By taking differences we have,

$$[\Delta^5 y] = (-15\alpha), \left(\frac{-3\beta}{2} + 15\alpha + 6\right), \frac{3\beta}{2}, 0, (-15\alpha)$$

Hence $[\Delta^5 y] \neq 0$.

□

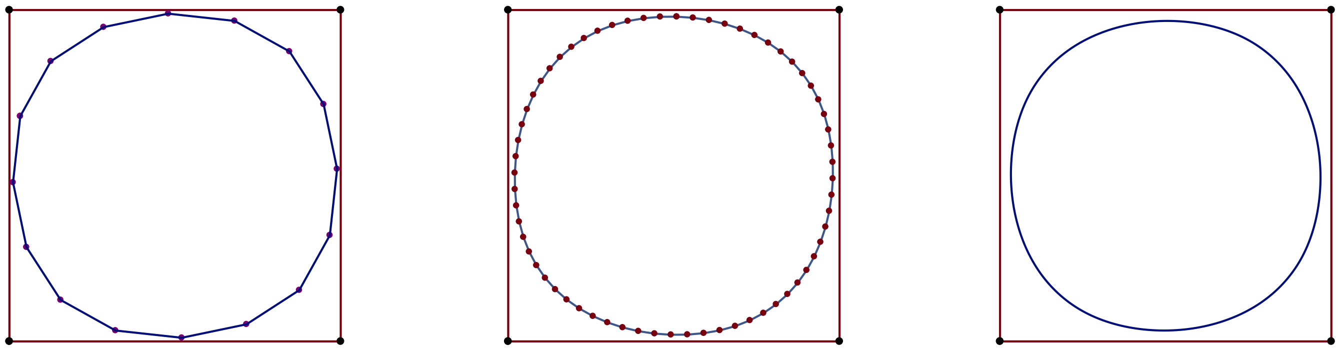
Thus the scheme (15) has cubic precision for any value of α and β.

V. COMPARISON AND APPLICATION OF THE SCHEME

In this section, few examples are considered to illustrate the behaviour of the proposed scheme by setting appropriate parameters. In Table 1, we compare the continuity and support of the proposed scheme with existing quaternary schemes.

A. EXAMPLE 1

We get an initial unrefined polygon by connecting the *2D* points f_i^0. This polygon is the input of the scheme. The four refinement rules that comprise the refinement scheme is defined in (15). The new points are inserted by using these rules in the previous unrefined polygon. Then all the new points are connected by straight lines to get a refined polygon. This refined polygon is the output of the scheme.



(a) **First refinement level** (b) **Second refinement level** (c) **Third refinement level**

FIGURE 2. Graphical representation of the scheme: Initial polygon (outer square) is made by connecting four points and inner polygons are created by using the scheme (15).

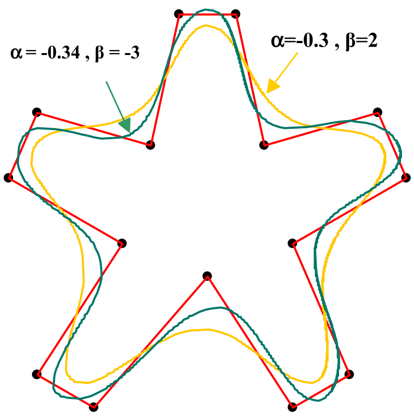


FIGURE 3. Parametric controlled shapes: The Black bullets show the initial control points. The solid Red lines are initial control polygon. Other coloured lines show the curves fitted by our subdivision scheme after four refinement steps.

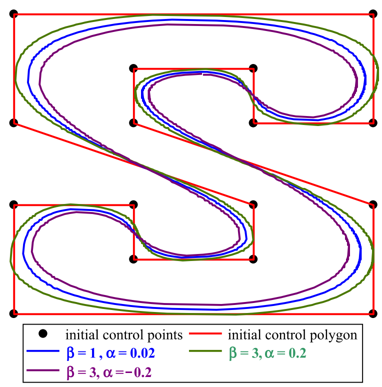


FIGURE 4. Parametric controlled shapes: The Black bullets show the initial control points. The solid red lines are initial control polygon. Other coloured lines show the curves fitted by our subdivision scheme after four refinement steps.

In the next step, the scheme takes the refined polygon as an input and generates the most refined polygon as an output. At each refinement level, more points are inserted, and the points become denser and denser. This process is repeatedly applied to get the required smooth curve. This procedure is graphically presented in Figure 2.

B. EXAMPLE 2

The shape of the curve can be adjusted with shape parameters. Here we focus on how we can adjust the shape of the curve by adjusting the values of shape parameters without disturbing the position of control points. Figure 3 and Figure 4 illustrates the effect of parameters α and β on the shape of curve for $\alpha = -0.34, -0.3$ and $\beta = -3, 2$ and $\alpha = 0.02, 0.2, -0.2$ and $\beta = 1, 3, 3$ respectively.

C. EXAMPLE 3

Here, we take monotonic data from the monotonic function $y = \frac{x^4}{2} + 2x^3$. The initial control points are $\{(-1, \frac{-7}{4}), (0, 0), (1, \frac{9}{4}), (2, 20), (3, \frac{297}{4}), (4, 192)\}$. Which satisfy the conditions of monotonicity. The limit curve is shown in Figure 5, it shows that the proposed scheme (15) preserves monotonicity.

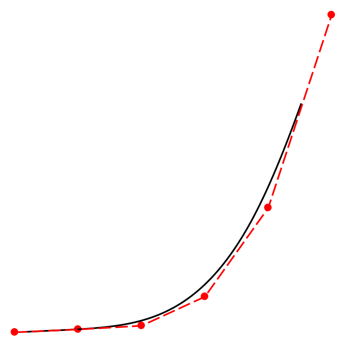


FIGURE 5. Monotonicity preservation: The red bullets show the initial control points, while the dashed lines show the initial polygon. The black solid lines are the limit curve obtained by using monotonic increasing data for $\beta = 0, \alpha = -0.1$.

D. EXAMPLE 4

Here, we take convex data from the convex function $y = x^4 + x^2 + 2$. The initial set of control points are $\{(-4, 274), (-3, 92), (-2, 22), (-1, 4), (0, 2), (1, 4), (2, 22),$

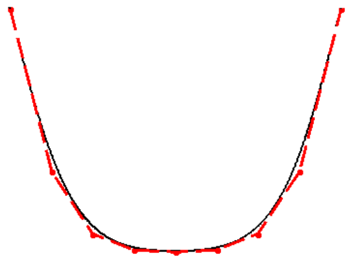


FIGURE 6. **Convexity preservation: The red bullets show the initial control points, while the dashed lines show the initial polygon. The Black solid lines is the limit curve obtained by using convex data for $\beta = 2$, $\alpha = 0$.**

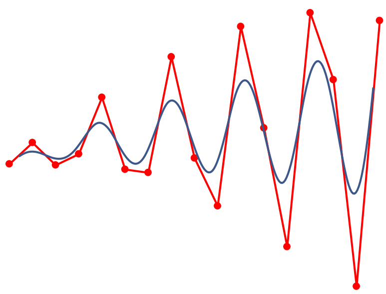


FIGURE 7. **Fitting of sine/nonlinear function: The red bullets show the initial control points, while the solid Red lines show the initial polygon. The Blue solid lines is the limit curve obtained by using nonlinear data with sine function for $\beta = 2$, $\alpha = -0.31$.**

$(3, 92), (4, 274)\}$. These points satisfy the condition of convexity. The limit curve shown in Figure 6 shows that the proposed scheme (15) preserves convexity. Convex shapes are important in engineering and industry. It is observed in Figures 6 that our scheme preserves the convexity of the shapes, so our scheme is suitable for fitting of convex shapes.

E. EXAMPLE 5

Here, we take nonlinear data from the nonlinear function $y = xsin(2x) + x^{1/2}$. The initial set of control points are $\{(0, 0), (1, 1.04), (2, 1.55), (3, 2.05), (4, 2.56), (5, 3.10), (6, 4.71), (7, 4.34), (8, 5.03), (9, 5.78), (10, 6.58), (11,7.43), (12, 8.34), (13, 9.30), (14, 10.31, 11.31), (15, 11.37), (16, 12.48)\}$. The limit curve shown in Figure 7 shows that the proposed scheme (15) is suitable for the fitting of nonlinear data including sine function.

F. EXAMPLE 6

In this example, we use a discontinuous function given by Zhao et al. [18]. Our proposed scheme provides a smooth estimate of a discontinuous function. There is no oscillation and no unwanted features in the curve fitted by our scheme.

$$F(\varrho) = \begin{cases} -3\varrho + 2, & \text{if} \quad x \in [0, 0.3] \\ -3\varrho + 3 - \sin\left(\dfrac{(\varrho - 0.3)\pi}{0.2}\right), & \text{if } x \in [0.3, 0.7] \\ \dfrac{\varrho}{2} + 1.55. & \text{if } x \in [0.7, 1] \end{cases}$$

The sine function appears frequently in digital signal processing as well as in engineering and industrial shapes. The breakage of a signal can be represented by a discontinuous function. The fitting of this type of function is presented in Figure 8. From this figure, it is observed that the scheme gives a better fit for this type of function.

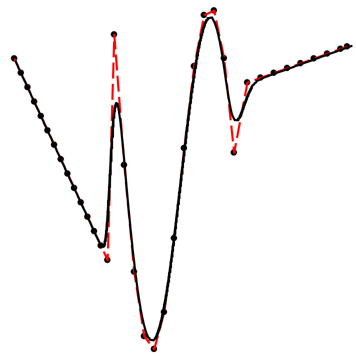


FIGURE 8. **Fitting of discontinuous function: The Navy bullets show the initial control points, while the dashed lines show the initial polygon. The solid Red lines is limit curve obtained by using a discontinuous function for $\beta = 2$, $\alpha = -\frac{8}{25}$.**

TABLE 1. **The comparison of continuity and support of proposed scheme with existing schemes.**

Complexity of Schemes	Nature	Continuity	Support
4-point	approximating [3]	C^2	$\frac{14}{3}$
4-point	approximating [1]	C^3	5
3-point	interpolating [5]	C^2	$\frac{14}{3}$
3-point	approximating [2]	C^2	$\frac{11}{3}$
4-point	approximating [2]	C^3	$\frac{11}{3}$
5-point	interpolating [8]	C^3	$\frac{19}{3}$
4-point	interpolating [26]	C^2	5
3-point relaxed	approximating proposed	C^3	$\frac{14}{3}$

G. CONCLUSION

In this paper, a new 3-point relaxed non-symmetric subdivision scheme has been introduced with two parameters to control the shape and smoothness of limit curves. The limiting function is supported on $[\frac{-7}{3}, \frac{7}{3}]$. The scheme can produce the C^3-continuous curve/shape with less computational complexity compared to the existing schemes in the literature, as shown in Table 1. Limit stencils and shape-preserving properties of the scheme are also discussed. By fitting the convex and monotone shapes, as well as the sine functions and nonlinear function employed in engineering and industry, we have demonstrated the applications of the scheme. Furthermore, it is observed that the fitting of shapes is controlled by parameters. In the future, we are interested in analyzing the general compact form of a non-symmetric scheme to improve smoothness with a smaller support width. Of course, extension of this work to surface fitting is a possible future research direction.

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