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RESEARCH ARTICLE

Unpaired Many-to-Many Disjoint Path Covers in Nonbipartite Torus-Like Graphs With Faulty Elements

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ABSTRACT One of the key problems in parallel processing is finding disjoint paths in the underlying graph of an interconnection network. The *disjoint path cover* of a graph is a set of pairwise vertex-disjoint paths that altogether cover every vertex of the graph. Given disjoint source and sink sets, $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$, in graph G, an *unpaired many-to-many k-disjoint path cover* joining S and T is a disjoint path cover $\{P_1, \ldots, P_k\}$, in which each path P_i runs from source s_i to some sink t_j. In this paper, we reveal that a nonbipartite torus-like graph, if built from lower dimensional torus-like graphs that have good disjoint-path-cover properties of the unpaired type, retains such a good property. As a result, an m-dimensional nonbipartite torus, $m \geq 2$, with at most f vertex and/or edge faults has an unpaired many-to-many k-disjoint path cover joining arbitrary disjoint sets S and T of size k each, subject to $k \geq 1$ and $f + k \leq 2m - 2$. The bound of $2m - 2$ on $f + k$ is nearly optimal.

INDEX TERMS Disjoint path, path cover, path partition, torus, toroidal grid, interconnection network.

I. INTRODUCTION

Interconnection networks play a crucial role in the performance of a supercomputing system. Given the internal processor and memory structures in each node, a distributed-memory architecture is primarily characterized by the network used to interconnect the nodes [1]. One of the central issues in the study of interconnection networks is finding parallel paths, which are naturally related to routing among nodes and the fault tolerance of the network [2], [3]. An interconnection network is frequently modeled as a graph, in which the vertices and edges represent nodes and links, respectively. Parallel paths correspond to the disjoint paths of the underlying graph.

The problems of building disjoint paths in a graph have received significant attention in the literature. Refer to, for example, [4], [5], [6], [7], and [8] for details. It is often important to find disjoint paths that collectively pass through all vertices. The disjoint path cover of a graph is a set of vertex-disjoint paths that altogether cover every vertex of the graph. Disjoint path cover problems are applicable in many areas such as software testing, database design, and code optimization [6], [9]. In addition, the problem is related to applications in which full utilization of network nodes is important [10].

Let G be a finite, simple undirected graph, where its vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively. A *path* from s to t is a sequence $\langle u_1, \ldots, u_l \rangle$ of distinct vertices of G such that $u_1 = s$, $u_l = t$, and $(u_i, u_{i+1}) \in E(G)$ for all $i \in \{1, \ldots, l-1\}$. If $l \geq 3$ and $(u_l, u_1) \in E(G)$, then the sequence is called a *cycle*. An *s–t path* refers to a path that runs from s to t; an *s-path* refers to a path starting at vertex s. The *path cover* of graph G is a set of paths in G such that every vertex of G is contained in at least one path. The *disjoint path cover* (DPC) of G is a path cover in which every vertex of G is covered by exactly one path. This study is concerned with a disjoint path cover in which each path runs from a prescribed source to a prescribed sink.

Given disjoint subsets $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ of $V(G)$ for a positive integer k, a *many-to-many*

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k-disjoint path cover is a DPC composed of k paths that collectively join S and T. If each source $s_i \in S$ must be joined to a specific sink $t_i \in T$, the many-to-many k-DPC is called *paired*, and it is *unpaired* if no such constraint is imposed. As is intuitively clear, we call the vertices in S and T *sources* and *sinks*, respectively, which together form a set of *terminals*.

Definition 1 (see [11]): A graph G is called *f-fault paired* (resp. *unpaired*) *k-disjoint path coverable* if $f + 2k \leq |V(G)|$ and G has a paired (resp. unpaired) k-DPC joining arbitrary disjoint set S of k sources and set T of k sinks in $G - F$ for any fault set $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$.

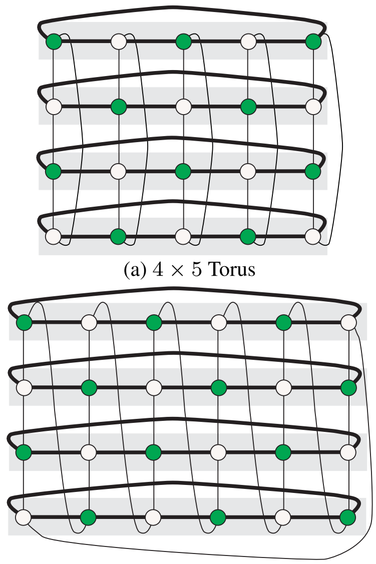
Among the interconnection networks proposed in the literature, torus is one of the widely recognized networks. An *m-dimensional torus* is defined as a Cartesian product of m cycles, $C_{d_1} \times \cdots \times C_{d_m}$, where C_{d_j} represents a cycle of length $d_j \geq 3$ for $j \in \{1, \ldots, m\}$. Given two graphs, G_0 and G_1, of the same order and a bijection ϕ from $V(G_0)$ to $V(G_1)$, we denote by $G_0 \oplus_\phi G_1$ the graph whose vertex set is $V(G_0) \cup V(G_1)$ and edge set is $E(G_0) \cup E(G_1) \cup \{(v, \phi(v)) : v \in V(G_0)\}$. To simplify the notation, we often omit the bijection ϕ from $\oplus_\phi$. Given d graphs $G_0, \ldots, G_{d-1}$ of the same order n, if we apply the graph constructor $\oplus$ to each pair G_i and $G_{(i+1) \bmod d}$ for $i \in \{0, \ldots, d-1\}$, then we obtain a graph with nd vertices. This graph is said to be obtained through the *cycle-based recursive construction*.

Definition 2 (see [12]): An *m-dimensional torus-like graph*, $m \geq 1$, is a graph obtained through the cycle-based recursive construction from $(m-1)$-dimensional torus-like graphs $G_0, \ldots, G_{d-1}$, $d \geq 3$, of the same order, where the 0-dimensional torus-like graph is a one-vertex graph K_1.

Here, the graphs $G_0, \ldots, G_{d-1}$ are called the *components* of the torus-like graph. Figure 1 shows examples of torus-like graphs. Each vertex v in component G_i has two neighbors outside G_i: one in $G_{(i+1) \bmod d}$, denoted by v^+, and the other in $G_{(i-1) \bmod d}$, denoted by v^-. Contracting the components of the torus-like graph into single vertices results in a cycle C_d of length d.

Disjoint path cover problems have been studied for various classes of graphs, including recent studies on dense graphs [13], cube of connected graphs [14], balanced hypercubes [15], [16], hypercube-like networks [17], [18], recursive circulants [19], directed graphs [20], k-ary n-cubes [21], and torus networks [22]. In particular, the paired disjoint path cover problem for torus-like graphs was investigated in [12] for a nonbipartite case and in [23] for a bipartite case. In addition, a study on unpaired disjoint path covers of a bipartite k-ary n-cube, which is a special form of torus, can be found in [24].

In this study, we investigate the unpaired disjoint path cover problem for nonbipartite torus-like graphs, following the approach taken in [12] for the paired DPC problem. We reveal that a torus-like graph has a good disjoint-path-cover property of the unpaired type if every component of the graph has good disjoint-path-cover and Hamiltonian properties. Specifically, we prove that an m-dimensional nonbipartite torus-like graph,



(a) 4×5 Torus

(b) Double loop network

FIGURE 1. Examples of 2-dimensional nonbipartite torus-like graphs, where an intra-component edge is indicated by a thick edge.

$m \geq 3$, composed of d components $G_0, \ldots, G_{d-1}$ is f-fault unpaired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq 2m - 2$ if each component G_i is f-fault unpaired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq 2m - 4$ and moreover, G_i is $(2m-5)$-fault Hamiltonian-connected and $(2m-4)$-fault Hamiltonian.

As a result, we obtain that an m-dimensional nonbipartite torus, $m \geq 2$, is f-fault unpaired k-disjoint path coverable for any f and $k \geq 1$ subject to $f + k \leq 2m - 2$. To the best of our knowledge, no studies on unpaired disjoint path covers in a nonbipartite torus or in a nonbipartite torus-like graph can be found in the literature. Moreover, the bound of $2m - 2$ on $f + k$ is nearly optimal, specifically, one less than the bound, $\delta(G) - 1$, of the necessary condition shown in Lemma 1 below, where $\kappa(G)$ and $\delta(G)$ denote the connectivity and degree of graph G, respectively. Note that the degree of the m-dimensional torus is $2m$.

Lemma 1: (see [11]) *Let G be an f-fault unpaired k-disjoint path coverable graph, where $k \geq 2$. Then, $f + k \leq \kappa(G)$. Furthermore, if G has $f + 2k + 1$ or more vertices, then $f + k \leq \delta(G) - 1$.*

II. PRELIMINARIES

The disjoint path cover problems of a graph are closely related to the Hamiltonian properties, as well as the vertex connectivity, of the graph. For example, an unpaired 1-DPC joining two vertices is the Hamiltonian path that connects them. A path that visits each vertex exactly once is a *Hamiltonian path*, and a cycle that visits each vertex exactly once is a *Hamiltonian cycle*. A graph is *traceable* if a Hamiltonian path exists, a graph is *Hamiltonian* if a Hamiltonian cycle exists, and a graph is *Hamiltonian-connected* if every two distinct vertices are joined by a Hamiltonian path. The Hamiltonian properties of the torus networks are as follows:

Lemma 2: (see [12], [25]) *Every m-dimensional nonbipartite torus, $m \geq 2$, is $(2m-3)$-fault Hamiltonian-connected and $(2m-2)$-fault Hamiltonian.*

As mentioned above, studies on unpaired DPCs of a nonbipartite torus cannot be found in the literature. However, we can see from Lemma 2 that an m-dimensional nonbipartite torus, $m \geq 2$, is $(2m-3)$-fault unpaired 1-disjoint path coverable. In addition, we can refer to the studies on paired DPCs because a paired k-DPC joining S and T is, by definition, an unpaired k-DPC joining the two. Some studies on paired disjoint path covers in a nonbipartite torus can be summarized as follows:

Lemma 3: (Kronenthal et al. [26] and Park [27]) *A 2-dimensional nonbipartite torus is paired 2-disjoint path coverable.*

Lemma 4: Let G be an m-dimensional nonbipartite torus $C_{d_1} \times \cdots \times C_{d_m}$, where $m \geq 2$.

(a) G is f-fault paired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + 2k \leq 2m$ (Park [12]).

(b) If $m \geq 3$ and at most one d_j is even, G is $(2m-3)$-edge-fault paired 2-disjoint path coverable (Li et al. [22]).

Lemma 3 leads to that a 2-dimensional nonbipartite torus is unpaired 2-disjoint path coverable. However, not every 2-dimensional nonbipartite torus is unpaired 3-disjoint path coverable nor 1-fault unpaired 2-disjoint path coverable. For example, consider the 4×5 torus shown in Figure 1(a), which has 20 vertices, half of them are colored green and the other half are colored white. Except for the four edges, two joining pairs of green vertices and two joining pairs of white vertices, all other edges join two vertices with different colors. It can be seen that an unpaired 3-DPC joining S and T cannot exist if S and T both contain white vertices only. Also, an unpaired 2-DPC joining S and T of two white vertices cannot exist if a white vertex is faulty, or if an edge joining a pair of green vertices is faulty.

Let us now consider some topological properties of a torus-like graph, which were discovered in [12].

Lemma 5:(see [12]) *Let G be an m-dimensional torus-like graph composed of d components $G_0, \ldots, G_{d-1}$. (a) G is a regular graph of degree 2m, which has at least 3^m vertices. (b) The connectivity of G is 2m. (c) The diameter of G is no more than $\lfloor \frac{d}{2} \rfloor$ plus the maximum diameter over all components. (d) G has no triangle (cycle of length three) if $d \geq 4$ and every G_i has no triangle. (e) There are at most three common neighbors for any pair of vertices in G. Moreover, if $d \geq 4$ and any pair of vertices in each component have at most two common neighbors, then any pair of vertices in G have at most two common neighbors.*

Lemma 6: There is at most one common neighbor for two adjacent vertices in a torus-like graph.

Proof: The proof is by induction on the dimension m of a torus-like graph. Suppose two vertices u and v are adjacent. It is obvious that the two have at most one common neighbor if $m = 1$. Let $m \geq 2$ for the inductive step. If (u, v) is an edge of some component G_i, then there is at most one common neighbor belonging to G_i by the induction hypothesis. If (u, v) is an inter-component edge, then there is at most one common neighbor in a component other than the components to which u or v belongs, proving the lemma. ■

It is useful to extend the notion of an *unpaired k-disjoint* path cover on not necessarily disjoint sets, S and T, of sources and sinks in a way that a vertex that belongs to both sets is considered as a valid, one-vertex path. Note that a disjoint path cover joining disjoint terminal sets contains no one-vertex path. A *generalized k-disjoint path cover* [28] joining S and T in graph G is defined as a set of k disjoint paths of G composed of

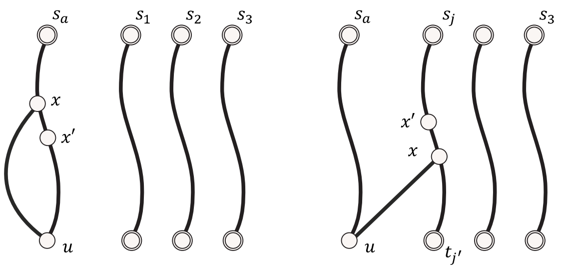
- $|S \cap T|$ one-vertex paths for terminals in $S \cap T$, and
- $k - |S \cap T|$ paths that form an unpaired $(k - |S \cap T|)$-DPC joining $S \setminus (S \cap T)$ and $T \setminus (S \cap T)$ in $G - (S \cap T)$.

Lemma 7: Let G_i be an $(m-1)$-dimensional torus-like graph that is f-fault unpaired k-disjoint path coverable for any f and $k \geq 1$ subject to $f + k \leq 2m-4$, where $m \geq 3$. Then, there exists a generalized k-DPC joining arbitrary distinct set S_i of k sources and set T_i of k sinks for any fault set F_i with $|F_i| \leq f$ subject to $f + k \leq 2m-4$.

Proof: Given distinct terminal sets S_i and T_i of size k each in G_i, along with a fault set F_i such that $|F_i| \leq f$ and $f+k \leq 2m-4$, we are to build a generalized k-DPC joining S_i and T_i in $G_i - F_i$. A vertex in $S_i \cap T_i$ can be seen as a one-vertex path, which runs from a vertex in S_i to itself also in T_i. So, it suffices to build an unpaired $(k-f')$-DPC joining $S_i \setminus F'$ and $T_i \setminus F'$ in $G_i - (F_i \cup F')$ where $F' = S_i \cap T_i$ and $f' = |F'|$. The unpaired $(k-f')$-DPC exists by the hypothesis of the lemma, because $k - f' \geq 1$ and $(f + f') + (k - f') = f + k \leq 2m-4$. Thus, the lemma is proven. ■

Lemma 8: Let G_i be an $(m-1)$-dimensional torus-like graph, $m \geq 3$, such that G_i is f-fault unpaired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq 2m-4$ and moreover, G_i is $(2m-5)$-fault Hamiltonian-connected and $(2m-4)$-fault Hamiltonian. Suppose we are given a fault set F_i, arbitrary sets S_i of k' sources and T_i of k sinks in G_i, where $|F_i| + k \leq 2m-4$ and $k' = k+1$ or $k+2$. (a) If $k' = k+1$, then there is a vertex u such that S_i and $T_i \cup \{u\}$ are joined by a generalized $(k+1)$-DPC of $G_i - F_i$. Also, there is another vertex u' than u such that S_i and $T_i \cup \{u'\}$ are joined by a generalized $(k+1)$-DPC. (b) If $k' = k+2$, there is a vertex subset $\{u, v\}$ such that S_i and $T_i \cup \{u, v\}$ are joined by a generalized $(k+2)$-DPC of $G_i - F_i$. Also, there is a vertex subset $\{u', v'\}$ other than $\{u, v\}$ such that S_i and $T_i \cup \{u', v'\}$ are joined by a generalized $(k+2)$-DPC.

Proof: Let $S_i = \{s_1, \ldots, s_{k'}\}$ and $T_i = \{t_1, \ldots, t_k\}$. We assume w.l.o.g. $s_1 \notin T_i$ if $k' = k+1$ and $s_1, s_2 \notin T_i$ if $k' = k+2$. For the proof of (a), let $k' = k+1$. Firstly, suppose $T_i \not\subseteq S_i$. Then, there exists a generalized k-DPC joining $S_i \setminus \{s_1\}$ and T_i in $G_i - F_i$ by Lemma 7 and the hypothesis of this lemma. A path in the DPC, say the s_a–$t_{a'}$ path, passes through s_1 as an intermediate vertex. Dividing the s_a–$t_{a'}$ path, represented as $(s_a, \ldots, u, s_1, \ldots, t_{a'})$, into two path segments, $(s_a, \ldots, u)$ and $(s_1, \ldots, t_{a'})$, results in a generalized $(k+1)$-DPC joining S_i and $T_i \cup \{u\}$, as required. We now show that another vertex $u' \neq u$ exists.



(a) x lies on the s_a–u path. (b) x lies on the s_j–$t_{j'}$ path.

FIGURE 2. Illustrations of the proof of lemma 8.

There exists a neighbor, x, of u such that x and (u, x) are both fault-free, x is different from any source in $S_i \setminus \{s_a\}$ and also different from the predecessor of u in the s_a–u path. This is because there are $\delta(G_i)$ candidates for x in G_i whereas at most $|F_i| + |S_i \setminus \{s_a\}| + 1$ of them could be blocked (by $|F_i|$ faults, $|S_i \setminus \{s_a\}|$ sources other than s_a, and the predecessor of u), for which $|F_i|+|S_i\setminus\{s_a\}|+1 = |F_i|+k+1 \leq 2m-3 < 2m-2 = \delta(G_i)$. As shown in Figure 2(a), if x lies on the s_a–u path and is different from the immediate predecessor of u, representing the s_a–u path as $(s_a, \ldots, x, x', \ldots, u)$ where x' is the immediate successor of x, it suffices to replace the s_a–u path with a s_a–x' path $(s_a, \ldots, x, u \ldots, x')$. If x lies on the s_j–$t_{j'}$ path and $x \neq s_j$, representing the s_j–$t_{j'}$ path as $(s_j, \ldots, x', x, \ldots, t_{j'})$ where x' is the immediate predecessor of x, it suffices to divide the s_j–$t_{j'}$ path into two path segments $(s_j, \ldots, x')$ and $(x, \ldots, t_{j'})$ and then redefine $(s_j, \ldots, x')$ as a new s_j-path and $(s_a, \ldots, u, x, \ldots, t_{j'})$ as a new s_a-path. Secondly, suppose $T_i \subseteq S_i$. There exists a Hamiltonian cycle C in $G_i-(F_i\cup T_i)$, from which we can extract a Hamiltonian path of the graph that runs from s_1 to some vertex u. The Hamiltonian path and $|T_i|$ one-vertex paths together form a generalized $(k+1)$-DPC joining S_i and $T_i \cup \{u\}$. We can also extract a Hamiltonian s_1–u' path from C for some $u' \neq u$ by traversing C in reverse order, meaning there is a generalized $(k+1)$-DPC joining S_i and $T_i \cup \{u'\}$.

For the proof of (b), let $k' = k+2$. Firstly, suppose $T_i \not\subseteq S_i$. Then, there exists a generalized k-DPC joining $S_i \setminus \{s_1, s_2\}$ and T_i in G_i-F_i. So, we can build a generalized $(k+2)$-DPC joining S_i and $T_i\cup\{u, v\}$ for some vertices u and v by dividing each of the paths in the DPC that passes through s_1 and/or s_2 into path segments, similar to the proof of (a). Let s_a–u and s_b–v paths denote the paths in the generalized $(k+2)$-DPC that run to u and v, respectively. If there is a neighbor of u on the s_a–u path different from the immediate predecessor of u, or if there is a neighbor of u on the s_j-path, $j \neq a$, different from s_j, we can build a generalized $(k+2)$-DPC joining S_i and $T_i \cup \{u', v\}$ for some vertex u' in the same manner as the proof of (a). Also, we can build a required $(k+2)$-DPC joining S_i and $T_i \cup \{u, v'\}$ for some v' in the same way if there is a neighbor of v on the s_b–v path different from the immediate predecessor of v, or if there is a neighbor of v on the s_j-path, $j \neq b$, different from s_j. It remains to consider the case when the sets of neighbors of u and v in $G_i - F_i$ are $(S_i \setminus \{s_a\}) \cup \{x\}$ and $(S_i \setminus \{s_b\}) \cup \{y\}$, where x and y respectively are the immediate predecessors of u and v, and F_i consists of common neighbors of u and v and possibly an extra edge (u, v). It suffices to merge the s_a–u and s_b–v paths into a cycle through the edges (u, s_b) and (v, s_a), and then extract two paths, s_a–u' and s_b–v' paths, from the cycle for some $\{u', v'\}$ different from $\{u, v\}$. This is possible because the cycle contains five or more vertices, i.e., $x \neq s_a$ or $y \neq s_b$. (Suppose $x = s_a$ and $y = s_b$ for a contradiction. If $(u, v) \notin F_i$, then the neighbor sets of u and v in G_i are both equal to $S_i\cup F_i$, meaning u and v have $2m-2 \geq 4$ common neighbors, which contradicts Lemma Lemma 5(e). If $(u, v) \in F_i$, then u and v have $2m-3 \geq 3$ common neighbors, which also contradicts Lemma 6) Secondly, suppose $T_i \subseteq S_i$. There is a Hamiltonian cycle C in $G_i-(F_i\cup T_i)$, which passes through both s_1 and s_2. Similar to the proof of (a), we can extract s_1–u and s_2–v paths that cover all vertices of C for some vertices u, v. The two paths and $|T_i|$ one-vertex paths together form a generalized $(k+2)$-DPC joining S_i and $T_i \cup \{u, v\}$. Also, the s_1–u' and s_2–v' paths extracted from C for some $\{u', v'\}$ different from $\{u, v\}$ can be used to build a generalized $(k+2)$-DPC joining S_i and $T_i \cup \{u', v'\}$. Note that cycle C has a length of at least $|V(G_i)|-(|F_i|+k) \geq 3^{m-1}-(2m-4) \geq 7$ for $m \geq 3$; thus, $\{u', v'\}$ exists. This completes the entire proof. ■

III. CHAIN OF TORUS-LIKE GRAPHS

Let G be an m-dimensional torus-like graph built from d components $G_0, \ldots, G_{d-1}$, where each G_i is an $(m-1)$-dimensional torus-like graph. The subgraph of G in which consecutive components, say $G_0, \ldots, G_r$, are connected by the edges between G_i and G_{i+1} for $i \in \{0, \ldots, r-1\}$ forms a *chain of torus-like graphs*, and will be denoted by $G_0\oplus\cdots\oplus G_r$ or simply by $G_{0,r}$. The chain $G_0\oplus\cdots\oplus G_r$ is obtained from G by removing components $G_{r+1}, \ldots, G_{d-1}$ if $r \in \{0, \ldots, d-2\}$, or by removing all edges connecting G_{d-1} and G_0 if $r = d-1$. The Hamiltonian properties of a chain of torus-like graphs were studied in [12], as shown below.

Lemma 9: (see [12]) *Let G_i, $i \in \{0, \ldots, r\}$, be an $(m-1)$-dimensional torus-like graph of the same order, $m \geq 3$, such that G_i is $(2m-5)$-fault Hamiltonian-connected, $(2m-4)$-fault Hamiltonian, and unpaired 2-disjoint path coverable. Then, the graph H defined as $G_0\oplus\cdots\oplus G_r$, $r \geq 1$, is $(2m-4)$-fault Hamiltonian-connected and $(2m-3)$-fault Hamiltonian.*

In this section, we show that chain $H := G_0 \oplus \cdots \oplus G_r$, $r \geq 1$, has a good disjoint-path-cover property if every component G_i has. Let $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ be the source and sink sets given in chain H, respectively. We denote by S_i and T_i the sets of sources and sinks contained in G_i, respectively, that is, $S_i = S\cap V(G_i)$ and $T_i = T\cap V(G_i)$; $S_{i,j}$ and $T_{i,j}$ denote the source and sink sets contained in $G_{i,j}$. Let $k_i = \min\{|S_i|, |T_i|\}$ and $k_{i,j} = \min\{|S_{i,j}|, |T_{i,j}|\}$. In addition, F denotes a set of faults, faulty vertices and/or edges, so that $F \subseteq V(G) \cup E(G)$. Let F_i and $F_{i,j}$ denote the fault sets of G_i and $G_{i,j}$, respectively. Also, let $f = |F|$,

$f_i = |F_i|$ and $f_{i,j} = |F_{i,j}|$. We assume w.l.o.g. that

$$\begin{aligned} &k_0 > k_r, \\ \text{or } &k_0 = k_r \text{ and } f_0 > f_r, \\ \text{or } &k_0 = k_r \text{ and } f_0 = f_r \text{ and } |S_0 \cup T_0| \geq |S_r \cup T_r|. \end{aligned} \quad (1)$$

The source and sink sets are interchangeable, so we can further assume

$$|S_0| \geq |T_0|. \quad (2)$$

Note that chain H is composed of a subchain H' defined as $G_0 \oplus \cdots \oplus G_{r-1}$, possibly G_0 if $r = 1$, and a single component G_r. Thus, we have $S = S_{0,r-1} \cup S_r$, $T = T_{0,r-1} \cup T_r$, and $k_{0,r-1} + k_r + k'_{r-1,r} = k$, where $k'_{r-1,r} = \max\{|S_{0,r-1}|, |T_{0,r-1}|\} - k_{0,r-1} = \max\{|S_r|, |T_r|\} - k_r$. In addition, the fault set of H is $F = F_{0,r-1} \cup F_r \cup F'_{r-1,r}$, where $F'_{i,i+1}$ denotes the set of edge faults bridging G_i and G_{i+1}.

Theorem 1: Let G_i, $i \in \{0, \dots, r\}$, be an $(m-1)$-dimensional torus-like graph of the same order, $m \geq 3$, such that G_i is f-fault unpaired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq 2m - 4$ and moreover, G_i is $(2m-5)$-fault Hamiltonian-connected and $(2m-4)$-fault Hamiltonian. Suppose we are given disjoint sets S and T of sources and sinks, and a fault set F in chain H defined as $G_0 \oplus \cdots \oplus G_r$, $r \geq 1$, such that $k \geq 2$ and $f + k \leq 2m - 3$. Then, there exists an unpaired k-DPC joining S and T in $H - F$ except for the case when $k = 2m - 3$, $S \subseteq V(G_0)$ and $T \subseteq V(G_r)$. For the exceptional configuration, there exist vertex subsets $\{u_1, u_2\}$ of G_0 and $\{v_1, v_2\}$ of G_r such that a generalized $(2m-2)$-DPC joining $S \cup \{v_j\}$ and $T \cup \{u_i\}$ exists for every pair $i, j \in \{1, 2\}$.

Proof: An unpaired k-DPC with respect to fault set F can be obtained from an unpaired k-DPC with respect to a virtual fault set $F \cup F'$, where F' is a set of arbitrary $(2m-3) - (f + k)$ fault-free edges. Consequently, we can assume that

$$f + k = 2m - 3. \quad (3)$$

The proof proceeds by induction on r. Suppose $r = 1$ for the base step, where $H = G_0 \oplus G_1$, $H' = G_0$, $k = k_0 + k_1 + k'_{0,1}$, $|S_0| \geq |T_0|$, and $|S_1| \leq |T_1|$. Cases 1a, 1b, and 1c below deal with the base step of $r = 1$; the inductive step of $r \geq 2$ is addressed later in Cases 2a and 2b.

***Case 1a:** $k_1 \geq 1$ or $f_0 \leq f - 1$ ($r = 1$).* First, we introduce a basic procedure for building an unpaired k-DPC in this case.

Procedure Find-UDPC-A(S, T, F, $G_0 \oplus G_1$)

/* $k_1 \geq 1$ or $f_0 \leq f - 1$. See Figure 3. */

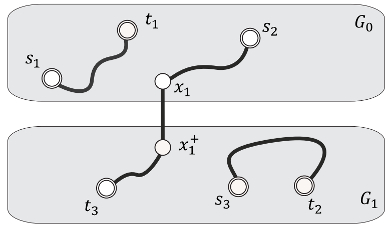
1: Pick up $k'_{0,1}$ free edges between G_0 and G_1. Let X_i denote the set of endvertices of the free edges in G_i, $i \in \{0, 1\}$.

2: Build an unpaired $(k_0 + k'_{0,1})$-DPC joining S_0 and $T_0 \cup X_0$ in $G_0 - F_0$.

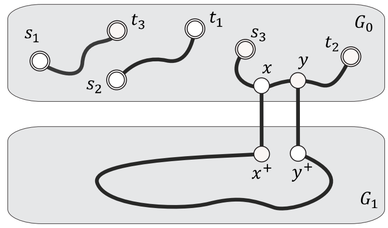
3: Case when $k_1 + k'_{0,1} \geq 1$:

  a: Build an unpaired $(k_1+k'_{0,1})$-DPC joining $S_1 \cup X_1$ and T_1 in $G_1 - F_1$.

  b: Merge the two DPCs through the $k'_{0,1}$ free edges.



(a) Case when $k_1 + k'_{0\,1} \geq 1$.



(b) Case when $k_1 + k'_{0\,1} = 0$.

FIGURE 3. Illustrations of procedure Find-UDPC-A.

4: Case when $k_1 + k'_{0,1} = 0$:

  a: Pick up an edge (x, y) on a path in the DPC of G_0 such that x^+, (x, x^+), y^+, and (y, y^+) are all fault-free.

  b: Replace the edge (x, y) with a Hamiltonian x^+–y^+ path of $G_1 - F_1$.

Claim 1: When $k_1 \geq 1$ or $f_0 \leq f - 1$, Procedure Find-UDPC-A builds an unpaired k-DPC in $G_0 \oplus G_1 - F$.

Proof. The $k'_{0,1}$ free edges of Step 1 exist, because there are $|V(G_0)|$ candidate edges whereas at most $f + 2k$ of them could be blocked (by f faults and $2k$ terminals), for which $|V(G_0)| - (f+2k) \geq 3^{m-1} - (f+k) - k \geq 3^{m-1} - (2m-3) - (2m-3) \geq 2m - 3 \geq k \geq k'_{0,1}$ for $m \geq 3$. The unpaired $(k_0 + k'_{0,1})$-DPC of Step 2 exists by the hypothesis of the theorem, because $f_0 + (k_0 + k'_{0,1}) = f_0 + (k - k_1) \leq f + k - 1 = 2m - 4$. Also, the unpaired $(k_1 + k'_{0,1})$-DPC of Step 3 exists because $f_1 + (k_1 + k'_{0,1}) = f_1 + (k - k_0) \leq f + k - 1 = 2m - 4$. Finally, the Hamiltonian path of Step 4 exists because $f_1 \leq f = 2m - 3 - k \leq 2m - 5$. Thus, this claim is proven. □

***Case 1b:** $k_1 = 0$, $f_0 = f$, and $k_0 \geq 1$ or $f_0 \geq 1$ ($r = 1$).* There are two basic procedures in this case, Find-UDPC-B and Find-UDPC-C, depending on whether $k_0 = k$ or not.

Procedure Find-UDPC-B(S, T, F, $G_0 \oplus G_1$)

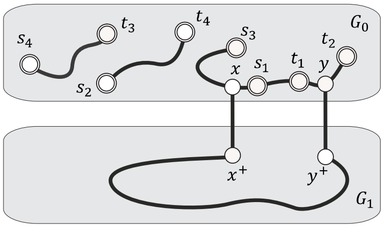
/* $k_1 = 0, f_0 = f$, and $k_0 = k$. See Figure 4. */

1: Build an unpaired $(k_0 - 1)$-DPC joining $S_0 \setminus \{s_1\}$ and $T_0 \setminus \{t_1\}$ in $G_0 - F_0$, where s_1 and t_1 are regarded as nonterminals temporarily.

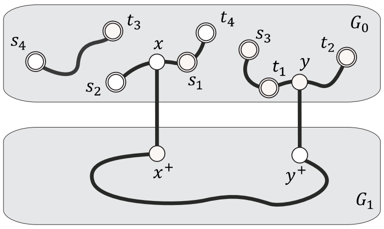
2: Case when there exists a path P_i in the DPC that passes through both s_1 and t_1, say $P_i = \langle s_i, P_x, x, P_1, y, P_y, t_{\sigma_i} \rangle$ for some s_1–t_1 path P_1:

  a: Divide P_i into three path segments $\langle s_i, P_x, x \rangle$, P_1, $\langle y, P_y, t_{\sigma_i} \rangle$.

  b: Combine $\langle s_i, P_x, x \rangle$ with $\langle y, P_y, t_{\sigma_i} \rangle$ through a Hamiltonian x^+–y^+ path of G_1.

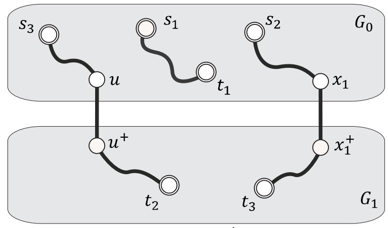


(a) Case when there is a single DPC path that passes through s_1 and t_1.

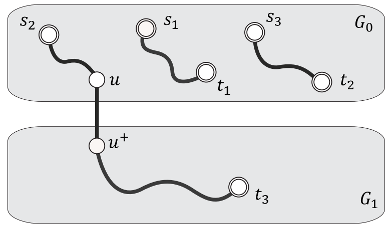


(b) Case when two DPC paths collectively pass through s_1 and t_1.

FIGURE 4. Illustrations of procedure Find-UDPC-B.



(a) Case when $k'_{0,1} \geq 2$.



(b) Case when $k'_{0,1} = 1$.

FIGURE 5. Illustrations of procedure Find-UDPC-C.

3: Case when P_i and P_j, $j \neq i$, in the DPC pass through s_1 and t_1, respectively, say $P_i = \langle s_i, P_x, x, s_1, P_a, t_{\sigma_i} \rangle$ and $P_j = \langle s_j, P_b, t_1, y, P_y, t_{\sigma_j} \rangle$:
   a: Divide P_i into two path segments $\langle s_i, P_x, x \rangle$ and $\langle s_1, P_a, t_{\sigma_i} \rangle$. Also, divide P_j into $\langle s_j, P_b, t_1 \rangle$ and $\langle y, P_y, t_{\sigma_j} \rangle$.
   b: Combine $\langle s_i, P_x, x \rangle$ with $\langle y, P_y, t_{\sigma_j} \rangle$ through a Hamiltonian x^+–y^+ path of G_1.

Claim 2: When $k_1 = 0, f_0 = f$, and $k_0 = k$, Procedure Find-UDPC-B builds an unpaired k-DPC in $G_0 \oplus G_1 - F$.

Proof. The unpaired $(k_0 - 1)$-DPC of Step 1 exists by the hypothesis of the theorem, because $f_0 + (k_0 - 1) = f + k - 1 = 2m - 4$. The Hamiltonian paths of Steps 2(b) and 3(b) also exist because $f_1 = 0$, proving this claim. □

In the remainder of Case 1b, we assume $k_0 < k$, implying $k'_{0,1} = k - k_0 \geq 1$.

Procedure Find-UDPC-C($S, T, F, G_0 \oplus G_1$)
/* $k_1 = 0, f_0 = f$, and $k_0 < k$. See Figure 5. */
1: Pick up $k'_{0,1} - 1$ free edges between G_0 and G_1. Let X_i be the set of endvertices of the free edges in G_i, $i \in \{0, 1\}$.
2: Build a generalized k-DPC in $G_0 - F_0$ joining S_0 and $T_0 \cup X_0 \cup \{u\}$ for some vertex $u \in V(G_0)$ such that u^+ is not a sink if $k'_{0,1} = 1$.
3: Build a generalized $k'_{0,1}$-DPC joining $X_1 \cup \{u^+\}$ and T_1 in G_1.
4: Merge the two DPCs through the free edges and edge (u, u^+).

Claim 3: When $k_1 = 0, f_0 = f$, and $k_0 < k$, Procedure Find-UDPC-C builds an unpaired k-DPC in $G_0 \oplus G_1 - F$.

Proof. The existence of $k'_{0,1} - 1$ free edges in Step 1 can be shown in the same way as the proof of Claim 1. The generalized k-DPC of Step 2 exists by Lemma 8, because $f_0 + (k - 1) = f + k - 1 = 2m - 4$. Also, the generalized DPC of Step 3 exists by Lemma 7, because $0 + k'_{0,1} = (f + k) - (f_0 + k_0) \leq f + k - 1 = 2m - 4$. Note that if $k'_{0,1} = 1$, the generalized $k'_{0,1}$-DPC of Step 3 is a Hamiltonian path of G_1 that joins u^+ and the unique sink in G_1. Thus, this claim is proven. □

Case 1c: *$k'_{0,1} = k$ and $f = 0$ ($r = 1$).* Notice that this is the exceptional configuration of the theorem, where $k = 2m - 3$, $S \subseteq V(G_0)$ and $T \subseteq V(G_r)$. A generalized $(2m - 2)$-DPC joining $S \cup \{v_j\}$ and $T \cup \{u_i\}$ can be built as follows: (i) Pick up $2m - 4$ free edges bridging G_0 and G_1. Let X_i denote the set of endvertices of the free edges in G_i, $i \in \{0, 1\}$. (ii) There exists a subset $\{u_1, u_2\}$ of vertices in G_0 such that G_0 has a generalized $(2m - 3)$-DPC joining S and $X_0 \cup \{u_i\}$ for each u_i by Lemma 8(a). (iii) Also, there exists a subset $\{v_1, v_2\}$ of vertices in G_1 such that G_1 has a generalized $(2m - 3)$-DPC joining $X_1 \cup \{v_j\}$ and T for each v_j again by Lemma 8(a). (iv) It suffices to combine the DPC of G_0 joining S and $X_0 \cup \{u_i\}$ with the DPC of G_1 joining $X_1 \cup \{v_j\}$ and T through the free edges.

Hereafter, suppose $r \geq 2$ for the inductive step, where H', defined as $G_0 \oplus \cdots \oplus G_{r-1}$, is made of two or more components. Chain H' always contains a terminal by the assumption (1), whereas G_r may not contain a terminal.

Case 2a: *G_r contains no terminals ($r \geq 2$).* The induction hypothesis can be applied to chain H' to build an unpaired k-DPC joining S and T in $H' - F_{0,r-1}$ unless S and T form the exceptional configuration of the theorem.

Procedure Find-UDPC-D($S, T, F, G_0 \oplus \cdots \oplus G_r$)
/* $G_r, r \geq 2$, contains no terminals. See Figure 6. */
1: Build an unpaired k-DPC joining S and T in $H' - F_{0,r-1}$.
2: Pick up an edge $(x, y) \in E(G_{r-1})$ on a path in the k-DPC such that $x^+, (x, x^+), y^+, (y, y^+)$ are all fault-free.
3: Replace the edge (x, y) with a Hamiltonian x^+–y^+ path of $G_r - F_r$.

Claim 4: When $G_r, r \geq 2$, contains no terminals, Procedure Find-UDPC-D builds an unpaired k-DPC in $H - F$ unless $k = 2m - 3, S \subseteq V(G_0)$ and $T \subseteq V(G_{r-1})$.

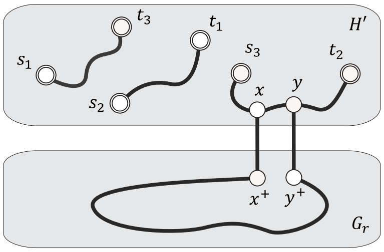
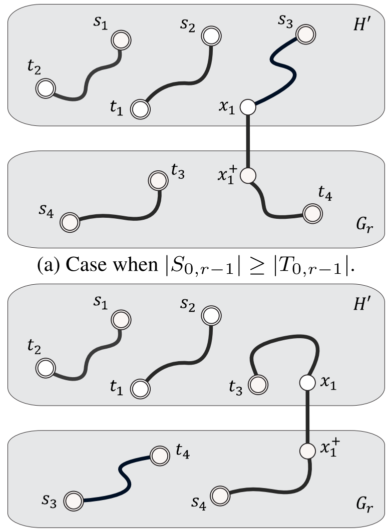


FIGURE 6. **Illustrations of procedure Find-UDPC-D.**

Proof. The unpaired k-DPC of Step 1 exists, by the induction hypothesis, unless $k = 2m-3$, $S \subseteq V(G_0)$ and $T \subseteq V(G_{r-1})$. The Hamiltonian path of Step 3 exists because $f_r \le f = (2m-3)-k \le 2m-5$. Therefore, it suffices to show that the edge (x, y) of Step 2 exists. The paths in the k-DPC collectively pass through at least $|V(G_{r-1}) \setminus F_{r-1}| - 2k \ge 3^{m-1} - f_{r-1} - 2k$ vertices of G_{r-1} as intermediate vertices (excluding the terminals). So, there are at least $\left\lceil \frac{1}{2}\left(3^{m-1} - f_{r-1} - 2k\right)\right\rceil$ candidate edges for (x, y), whereas at most $2\left(f_r + f'_{r-1,r}\right)$ of them could be blocked (two for each fault in $F_r \cup F'_{r-1,r}$), for which $\left\lceil \frac{1}{2}\left(3^{m-1} - f_{r-1} - 2k\right)\right\rceil - 2\left(f_r + f'_{r-1,r}\right) \ge \left\lceil \frac{1}{2}\left(3^{m-1} - 2(f+k) - 2f\right)\right\rceil \ge \left\lceil \frac{1}{2}\left(3^{m-1} - 2(2m-3) - 2(2m-5)\right)\right\rceil \ge 1$ for $m \ge 3$. Thus, the edge (x, y) exists, thereby proving the claim. $\square$

We are to build an unpaired k-DPC for the remaining case where $k = 2m-3$, $S \subseteq V(G_0)$ and $T \subseteq V(G_{r-1})$, utilizing a generalized $(2m-3)$-DPC of a component shown in Lemma 8(a). Note that $f = 0$ by the assumption (3). If we pick up $2m-4$ edges $\{(y_i, y_i^+) : i = 1, \ldots, 2m-4\}$ bridging G_0 and G_1 and build a generalized $(2m-3)$-DPC joining S and $\{y_1, \ldots, y_{2m-3}\}$ for some vertex $y_{2m-3} \in V(G_0)$, then we can extend the $2m-3$ paths, each of which runs from a source, to vertices in G_1 through the edges (y_i, y_i^+), $i \in \{1, \ldots, 2m-3\}$. Repeating this process $r-1$ times, we can extend the $2m-3$ paths to the vertices in G_{r-1}. Let the paths be s_1–s'_1, …, s_{2m-3}–s'_{2m-3} paths for some $S' := \{s'_1, \ldots, s'_{2m-3}\} \subseteq V(G_{r-1})$. We can assume $|S' \cap T| \le 1$, because when picking up $2m-4$ edges between G_{r-2} and G_{r-1}, we only need to select those whose endvertices are different from the sinks. It remains to build a generalized $(2m-3)$-DPC joining S' and T in $G_{r-1} \oplus G_r$. Procedure Find-UDPC-B can be recycled for our purpose, except that G_{r-1} and G_r respectively are used instead of G_0 and G_1, and that if $|S' \cap T| = 1$, then $S' \cap T$ is regarded as a fault set temporarily and $S' \setminus (S' \cap T)$ and $T \setminus (S' \cap T)$ are used instead of S' and T, respectively.

Case 2b: *G_r contains a terminal ($r \ge 2$).* In this case, G_0 contains a fault, or contains a terminal, even a source by the assumptions (1) and (2). That is, $f_0 \ge 1$ or $|S_0| \ge 1$. Recall that $S = S_{0,r-1} \cup S_r$ and $T = T_{0,r-1} \cup T_r$. Furthermore, $k = k_{0,r-1} + k_r + k'_{r-1,r}$, where $k'_{r-1,r}$ is equal to the difference between $|S_{0,r-1}|$ and $|T_{0,r-1}|$ and also equal to the difference between $|S_r|$ and $|T_r|$. Note that $|T_r| \ge |S_r|$ does not always hold.



(a) Case when $|S_{0,r-1}| \ge |T_{0,r-1}|$.

(b) Case when $|S_{0,r-1}| < |T_{0,r-1}|$.

FIGURE 7. **Illustrations of procedure Find-UDPC-E.**

Procedure Find-UDPC-E($S, T, F, G_0 \oplus \cdots \oplus G_r$)
/* G_r, $r \ge 2$, contains a terminal. See Figure 7. */

1: Pick up $k'_{r-1,r}$ free edges between G_{r-1} and G_r. The sets of endvertices of the free edges in G_{r-1} and G_r, respectively, are denoted by X_{r-1} and X_r.
2: Build an unpaired $(k_{0,r-1} + k'_{r-1,r})$-DPC in $H' - F_{0,r-1}$ joining $S_{0,r-1}$ and $T_{0,r-1} \cup X_{r-1}$ if $|S_{0,r-1}| \ge |T_{0,r-1}|$; otherwise, build an unpaired $(k_{0,r-1} + k'_{r-1,r})$-DPC joining $S_{0,r-1} \cup X_{r-1}$ and $T_{0,r-1}$.
3: Build an unpaired $(k_r + k'_{r-1,r})$-DPC in $G_r - F_r$ joining S_r and $T_r \cup X_r$ if $|S_r| \ge |T_r|$; otherwise, build an unpaired $(k_r + k'_{r-1,r})$-DPC joining $S_r \cup X_r$ and T_r.
4: Combine the two DPCs through the free edges.

Claim 5: When G_r, $r \ge 2$, contains a terminal, Procedure Find-UDPC-E builds an unpaired k-DPC in $H - F$ unless $k = 2m-3$, $S \subseteq V(G_0)$ and $T \subseteq V(G_{r-1}) \cup V(G_r)$.

Proof. The $k'_{r-1,r}$ free edges of Step 1 exist for the same reason as the free edges exist in Step 1 of Procedure Find-UDPC-A. The unpaired $(k_{0,r-1}+k'_{r-1,r})$-DPC of Step 2 exists unless $k = 2m-3$, $S \subseteq V(G_0)$ and $T \subseteq V(G_{r-1}) \cup V(G_r)$, by the induction hypothesis. In addition, the unpaired $(k_r + k'_{r-1,r})$-DPC of Step 3 exists because $f_r + (k_r + k'_{r-1,r}) \le f + k - 1 = 2m-4$. (Supposing $f_r + (k_r + k'_{r-1,r}) \ge f + k$, i.e., $f_r = f$ and $k_r + k'_{r-1,r} = k$, leads to $k_0 = k_r = 0$ and $k'_{r-1,r} = k$ by the assumption (1), meaning $f_0 = f_r = 0$ and $f = 0$. It follows from the assumptions (1), (2) and (3) that $k = 2m-3$, $S \subseteq V(G_0)$ and $T \subseteq V(G_r)$, which is the exceptional case of this claim.) Thus, this claim is proven. $\square$

It remains to consider the exceptional case of Claim 5, where $k = 2m-3$, $S \subseteq V(G_0)$, and $T \subseteq V(G_{r-1}) \cup V(G_r)$. Firstly, suppose $T \nsubseteq V(G_r)$. In the same way as the proof of

Case 2a for the case when $k = 2m - 3$, $S \subseteq V(G_0)$ and $T \subseteq V(G_{r-1})$, we can build $2m-3$ paths each of which runs from a source to a vertex in G_{r-1}, denoted by s_1–s'_1, ..., s_{2m-3}–s'_{2m-3} paths for some set $S' := \{s'_1, \ldots, s'_{2m-3}\} \subseteq V(G_{r-1})$ such that $|S' \cap T_{r-1}| \leq 1$. It is sufficient to build a generalized $(2m-3)$-DPC joining S' and T in $G_{r-1} \oplus G_r$ by recycling Procedure Find-UDPC-C for the fault set $F' := S' \cap T_{r-1}$ and terminal sets $S' \setminus F'$ and $T \setminus F'$.

Finally, suppose $T \subseteq V(G_r)$, that is, $k = 2m - 3$, $S \subseteq V(G_0)$ and $T \subseteq V(G_r)$, which forms the exceptional configuration of this theorem. Similar to Case 1c, a required DPC can be built as follows: (i) Pick up $2m-4$ free edges bridging G_0 and G_1. Let X_i denote the set of endvertices of the free edges in G_i, $i \in \{0, 1\}$. (ii) Pick up $2m-4$ free edges bridging G_{r-1} and G_r. Let Y_i denote the set of endvertices of the free edges in G_i, $i \in \{r-1, r\}$. It can be assumed that $X_1 \neq Y_{r-1}$ if $r = 2$. (iii) Build a generalized $(2m-4)$-DPC in $G_{1,r-1}$ joining X_1 and Y_{r-1}, which exists by Lemma 7 (if $r = 2$) and by the induction hypothesis (if $r \geq 3$). (iv) Build a generalized $(2m-3)$-DPC in G_0 joining S and $X_0 \cup \{u_i\}$ for each u_i in some subset $\{u_1, u_2\}$. (v) Build a generalized $(2m-3)$-DPC in G_r joining $X_1 \cup \{v_j\}$ and T for each v_j in some subset $\{v_1, v_2\}$. (vi) Merge the three DPCs, the DPC of $G_{1,r-1}$, the DPC of G_0, and the DPC of G_r, through the free edges. This completes the entire proof of Theorem 1. ■

Corollary 1: Let H be a chain $G_0 \oplus \cdots \oplus G_r$ of $(m-1)$-dimensional torus-like graphs, where $m \geq 3$ and $r \geq 1$, such that each G_i is f-fault unpaired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq 2m - 4$ and moreover, G_i is $(2m-5)$-fault Hamiltonian-connected and $(2m-4)$-fault Hamiltonian. Given distinct sets S, T of sources and sinks and a fault set F in the chain H such that $k \geq 2$ and $f + k \leq 2m - 3$, there exists a generalized k-DPC joining S and T in $H - F$ except for the case when $k = 2m-3$, $S \subseteq V(G_0)$ and $T \subseteq V(G_r)$.

Proof: For $F' = S \cap T$ and $f' = |F'|$, a generalized k-DPC joining S and T in $H - F$ can be easily built from an unpaired $(k - f')$-DPC joining $S \setminus F'$ and $T \setminus F'$ in $H - (F \cup F')$. ■

Remark 1: In the exceptional configuration of Theorem 1 where $k = 2m-3$, $S \subseteq V(G_0)$ and $T \subseteq V(G_r)$, it is not known whether or not an unpaired $(2m-3)$-DPC joining S and T exists in H.

IV. UNPAIRED DISJOINT PATH COVERS IN TORUS-LIKE GRAPHS

Let G be an m-dimensional nonbipartite torus-like graph, $m \geq 3$, composed of d components $G_0, \ldots, G_{d-1}$, where each component G_i is an $(m-1)$-dimensional torus-like graph. In this section, we demonstrate that the torus-like graph G has a good disjoint-path-cover property if every component G_i has good Hamiltonian and disjoint-path-cover properties. Specifically, we provide a constructive proof of the theorem presented below, according to which we can design an algorithm for building an unpaired k-disjoint path cover in a torus-like graph with faulty elements.

Theorem 2: Let G be an m-dimensional nonbipartite torus-like graph, $m \geq 3$, composed of d components $G_0, \ldots, G_{d-1}$ such that each G_i is f-fault unpaired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq 2m - 4$ and moreover, G_i is $(2m-5)$-fault Hamiltonian-connected and $(2m-4)$-fault Hamiltonian. Then, G is f-fault unpaired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq 2m - 2$.

Proof: For the proof, assume that we are given disjoint terminal sets $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ in G, along with a fault set F such that

$$f + k = 2m - 2. \tag{4}$$

If the fault set F contains an inter-component edge, say an edge bridging G_{d-1} and G_0, then our problem of building an unpaired k-DPC joining S and T in $G - F$ is reduced to a problem of building an unpaired k-DPC joining S and T in the chain $G_0 \oplus \cdots \oplus G_{d-1}$, the spanning subgraph of G with all the edges between G_0 and G_{d-1} being deleted. The required k-DPC in the chain exists, by Theorem 1, unless $k = 2m-3$, the sources are all contained in one of two components G_0 and G_{d-1} and the sinks are all contained in the other component, because the chain contains $f - 1$ or less faulty elements. For the exceptional case where $k = 2m-3$, $S \subseteq V(G_0)$ and $T \subseteq V(G_{d-1})$, we first pick up $2m-4$ fault-free edges between G_0 and G_{d-1}, and let X_i denote the set of endvertices of the picked edges in G_i, where $i \in \{0, d-1\}$. Then, there is a generalized $(2m-3)$-DPC in G_0 joining S and $X_0 \cup \{u\}$ for some vertex u by Lemma 8(a); also, there is a generalized $(2m-3)$-DPC in G_{d-1} joining $X_{d-1} \cup \{v\}$ and T for some vertex v with $v^- \neq u^+$ again by Lemma 8(a). It suffices to merge the two DPCs through the $2m-4$ edges, and then connect u^+ and v^- via a Hamiltonian path of a chain $G_{1,d-2}$, possibly made of a single component.

Hereafter in this proof, we assume that there is no inter-component edge fault, leading to $f = f_0 + \cdots + f_{d-1}$. Also, we assume that for each $i \in \{1, \ldots, d-1\}$,

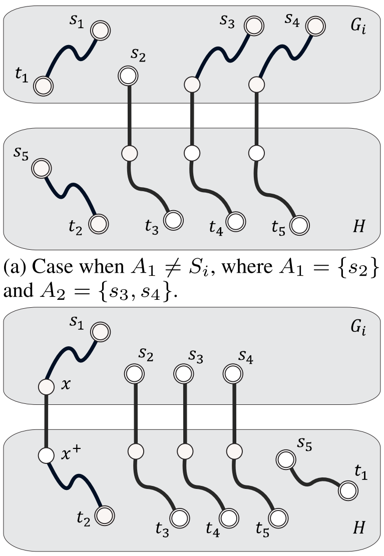
$$\begin{aligned} &f_0 > f_i, \\ &\text{or } f_0 = f_i \text{ and } k_0 > k_i, \\ &\text{or } f_0 = f_i \text{ and } k_0 = k_i \text{ and } |S_0 \cup T_0| \geq |S_i \cup T_i|. \end{aligned} \tag{5}$$

We can further assume

$$|S_0| \geq |T_0|, \tag{6}$$

because the source and sink sets are interchangeable. There are four cases according to the distribution of faults and terminals.

Case 1: *$f_i + k_i + k'_i \leq f + k - 2$ and $f_i + k_i \geq 1$ for some $i \in \{0, \ldots, d-1\}$.* We first present a basic procedure for building an unpaired k-DPC in G that relies on a generalized $(k_i + k'_i)$-DPC of G_i and a generalized $(k - k_i)$-DPC of $H := G - G_i$. The procedure is applicable in most cases, leaving two exceptional cases that will be dealt with separately. For the sake of simplicity, let us assume $|S_i| \geq |T_i|$; we further assume w.l.o.g. $S - S_i \not\subseteq V(G_{i+1})$ or $T - T_i \not\subseteq V(G_{i-1})$ in



(a) Case when $A_1 \neq S_i$, where $A_1 = \{s_2\}$ and $A_2 = \{s_3, s_4\}$.

(b) Case when $A_1 = S_i$, where $A'_1 = \{s_2, s_3, s_4\}$.

FIGURE 8. **Illustrations of procedure Find-UDPC-F.**

order to block the possibility of the exceptional configuration of Theorem 1 in H.

Procedure Find-UDPC-F(S, T, F, G)
/* $f_i + k_i + k'_i \le f + k - 2$ and $f_i + k_i \ge 1$. See Figure 8. */

1: For a set A of k'_i sources in G_i, let $A_1 = \{s_j \in A : s_j^+$ is a nonterminal and $s_j^+, (s_j, s_j^+)$ are both fault-free} and $A_2 = A \setminus A_1$.
2: Case when $A_1 \neq S_i$, i.e., $k_i + |A_2| \ge 1$:
   a: Pick up $|A_2|$ free edges between G_i and G_{i+1}. Let X_j denote the set of endvertices of the free edges in G_j, $j \in \{i, i+1\}$, and let $B_1 = \{s_j^+ : s_j \in A_1\}$.
   b: Build a generalized $(k_i + k'_i)$-DPC in $G_i - F_i$ joining S_i and $T_i \cup A_1 \cup X_i$.
3: Case when $A_1 = S_i$, i.e., $k_i = 0$ and $A_2 = \emptyset$:
   a: Pick up a single free edge (x, x^+) between G_i and G_{i+1}. For a source $s_1 \in A_1$, let $A'_1 = A_1 \setminus \{s_1\}$ and $B_1 = \{s_j^+ : s_j \in A'_1\}$.
   b: Build a generalized $(k_i + k'_i)$-DPC in $G_i - F_i$ joining S_i and $A'_1 \cup X_i$.
4: Build an unpaired $(k - k_i)$-DPC in $H - F_{i+1,i-1}$ joining $S_{i+1,i-1} \cup (B_1 \cup X_{i+1})$ and $T_{i+1,i-1}$.
5: Combine the two DPCs through the free edges and $\{(s_j, s_j^+) : s_j^+ \in B_1\}$.

Claim 6: When $f_i + k_i + k'_i \le f + k - 2$ and $f_i + k_i \ge 1$, Procedure Find-UDPC-F builds an unpaired k-DPC in $G - F$ unless (i) $k_i + k'_i = 0$ or (ii) $k_i = k$.

Proof. Suppose that $k_i + k'_i \ge 1$ and $k_i < k$. The $|A_2|$ free edges of Step 2(a) exist because $|V(G_i)| - (f + 2k - |A_2|) \ge 3^{m-1} - (f+k) - k + |A_2| \ge 3^{m-1} - (2m-2) - (2m-2) + |A_2| > |A_2|$ for $m \ge 3$. The free edge of Step 3(a) also exists for a similar reason that $|V(G_i)| - (f + 2k) \ge 1$. The generalized $(k_i + k'_i)$-DPCs of Steps 2(b) and 3(b) exist by Lemma 7, because $f_i + (k_i + k'_i) \le f + k - 2 = 2m - 4$. Finally, the $(k - k_i)$-DPC of Step 4 exists by Theorem 1, because $f_{i+1,i-1} + (k - k_i) = (f - f_i) + (k - k_i) = f + k - (f_i + k_i) \le f + k - 1 = 2m - 3$. Therefore, the claim is proven. □

We now address the exceptional case (i) of Claim 6 where $k_i + k'_i = 0$ (and $f_i \ge 1$). If $k \ge 3$ or $f_i < f$, meaning $f_i \le 2m - 5$, then it suffices to build an unpaired k-DPC joining S and T in $H - F_{i+1,i-1}$, which exists according to Theorem 1, and then replace an edge $(x, y) \in E(G_{i+1})$ on a path in the DPC such that $x^-, y^- \notin F$ with a Hamiltonian x^-–y^- path of $G_i - F_i$. Now, suppose that $k = 2$ and $f_i = f$ ($= 2m - 4$). If G_{i+1} or G_{i-1}, say G_{i+1}, contains no terminal, we can build a required 2-DPC analogously from an unpaired 2-DPC of $G_{i+2,i-1}$ and a Hamiltonian path of $G_{i,i+1} - F_i$. Note that $G_{i,i+1} - F_i$ is Hamiltonian-connected by Lemma 9. If G_{i+1} contains a single terminal, say s_1, then for some free edge (x, x^+) bridging G_{i+1} and G_{i+2}, it suffices to build a Hamiltonian s_1–x path in $G_{i,i+1} - F_i$ and an unpaired 2-DPC joining $\{x^+, s_2\}$ and $\{t_1, t_2\}$ in $G_{i+2,i-1}$, and then merge them into a required 2-DPC. Finally, if G_1 contains two terminals, say s_1 and s_2, then for some edge $(x, y) \in E(G_{i+1})$ on a Hamiltonian s_1–s_2 path of $G_{i,i+1} - F_i$ such that both x^+ and y^+ are nonterminals, it suffices to combine an unpaired 2-DPC of $G_{i+2,i-1}$ with the s_1–x and s_2–y paths properly. For the exceptional case (ii) of Claim 6 where $k_i = k$, it suffices to build an unpaired k-DPC joining S and T in $G_i - F_i$, which exists because $f_i + k = f_i + k_i + k'_i \le f + k - 2 = 2m - 4$, and replace an edge (x, y) on a path in the DPC such that $x^+, y^+ \notin F$ with a Hamiltonian x^+–y^+ path of $H - F_{i+1,i-1}$.

Case 2: $f_0 \le f - 1$. If $f_0 \le f - 2$, then $f \ge 2$ and $f_0 \ge 1$ by the assumption (5); hence, the case $f_0 \le f - 2$ is reduced to Case 1 for $i = 0$. Similarly, the case $f_0 = f - 1$ is also reduced to Case 1 for $i = 0$ unless $k_0 + k'_0 = k$. So, it remains to consider the case where $f_0 = f - 1$ and $k_0 + k'_0 = k$. Let G_p be the component other than G_0 that contains a fault, that is, $f_p = 1$ and $f_j = 0$ for all $j \neq 0, p$. If $f_p \le f - 2$ or if $f_p = f - 1$ and $k_p + k'_p < k$, then the remaining case is also reduced to Case 1 for $i = p$. Thus, we further assume that $f_p = f - 1$ and $k_p + k'_p = k$, meaning $f_0 = f_p = 1, f = 2, k = 2m-4, S \subseteq V(G_0)$ and $T \subseteq V(G_p)$. Assuming w.l.o.g. $p \neq 1$, it suffices to extract the s_j–x_j paths for $j \in \{1, \ldots, k\}$ from a Hamiltonian cycle $(s_1, \ldots, x_1, s_2, \ldots, x_2, \ldots, s_k, \ldots, x_k)$ of $G_0 - F_0$, and then combine them with a unpaired k-DPC joining $\{x_1^+, \ldots, x_k^+\}$ and T in $G_{1,d-1} - F_p$ through the edges (x_j, x_j^+) for $j \in \{1, \ldots, k\}$.

Case 3: *$f_0 = f \ge 1$, or $f_0 = f = 0$ and $k_0 \ge 1$.* This case is reduced to Case 1 for $i = 0$ if $k_0 + k'_0 \le k - 2$, because $f_0 = f$ and $f_0 + k_0 \ge 1$. Thus, it is sufficient to handle the remaining case where $k_0 + k'_0 \ge k - 1$. Firstly, suppose $k_0 + k'_0 = k$, meaning $S \subseteq V(G_0)$. If $0 \le k_0 \le k - 2$, then assuming w.l.o.g. that not all sinks in $G_{1,d-1}$ belong to G_{d-1}, i.e., $T_{1,d-1} \nsubseteq V(G_{d-1})$ to avoid the exceptional configuration of Theorem 1, we can build a required k-DPC in the following manner: (i) Pick up $(k - 2) - k_0$ free edges between G_0 and G_1. Let X_i denote the set of endvertices of the free edges in G_i for $i \in \{0, 1\}$. (ii) Build a generalized

k-DPC in $G_0 - F_0$ joining S and $T_0 \cup X_0 \cup \{u, v\}$ for some vertices $u, v \in V(G_0)$ such that $\{u^+, v^+\} \neq T_{1,d-1}$, using Lemma 8(b). (iii) Combine the k-DPC of G_0 with a generalized $(k - k_0)$-DPC joining $X_1 \cup \{u^+, v^+\}$ and $T_{1,d-1}$ in $G_{1,d-1}$. The existence of $(k - 2) - k_0$ free edges is due to $|V(G_0)| - (f + 2k) \geq 3^{m-1} - (f + k) - k \geq 3^{m-1} - (2m-2) - (2m-2) \geq 2m - 5 \geq (k-2) - k_0$ for $m \geq 3$. Note that $(k-2) - k_0 \leq \begin{cases} (2m - 3 - 2) - 0 = 2m - 5 & \text{if } f_0 \geq 1, \\ (2m - 2 - 2) - 1 = 2m - 5 & \text{if } k_0 \geq 1. \end{cases}$

If $k_0 = k - 1$, there is a single sink, say t_k, outside G_0. Assuming w.l.o.g. G_1 has no terminals, we first build an unpaired $(k - 1)$-DPC in $G_0 \oplus G_1 - F_0$ joining $S \setminus \{s_1\}$ and T_0. By dividing a path in the $(k-1)$-DPC that passes through s_1 into two path segments, we can build a generalized k-DPC joining S and $T_0 \cup \{u\}$ for some vertex u. The path in the DPC that runs to u is denoted as the s_a–u path. Let u' be the neighbor of u in $G_{2,d-1}$. (The neighbor u' will be u^- if u belongs to G_0; u' will be u^+ if u belongs to G_1.) If $u' \neq t_k$, it suffices to combine the s_a–u path with a Hamiltonian u'–t_k path of $G_{2,d-1}$; if $u' = t_k$, it suffices to extend the s_a–u path to u' by one and then replace an edge (x, y) on a path in the generalized k-DPC such that $x, y \neq u$ with a Hamiltonian x'–y' path of $G_{2,d-1} - \{t_k\}$, where x' and y' are the neighbors of x and y in $G_{2,d-1}$, respectively.

Similar to the case where $k_0 = k - 1$, we can build a required k-DPC when $k_0 = k$. However, instead of building an unpaired $(k - 1)$-DPC joining $S \setminus \{s_1\}$ and T_0, we build an unpaired $(k - 1)$-DPC in $G_0 \oplus G_1 - F_0$ joining $S \setminus \{s_1\}$ and $T \setminus \{t_1\}$, from which we build a generalized $(k+1)$-DPC joining $S \cup \{u\}$ and $T \cup \{v\}$ for some vertices u and v. Let the s_a–v path denote the path in the DPC that runs to v, and let the u–t_b path denote the path that runs from u. The s_a–v and u–t_b paths are merged into an s_a–t_b path; the vertices of $G_{2,d-1}$ are covered by the s_a–t_b path or some other path in the generalized $(k+1)$-DPC depending on whether the neighbors u' and v' of u and v in $G_{2,d-1}$ are distinct or not.

Secondly, suppose $k_0 + k'_0 = k - 1$. Let $S_0 = \{s_1, \ldots, s_{k-1}\}$ and s_k belong to $G_{1,d-1}$. Similar to the previous case where $k_0 + k'_0 = k$, an unpaired k-DPC joining S and T can be built. If $k_0 \leq k - 2$, then assuming w.l.o.g. that not all sinks in $G_{1,d-1}$ belong to G_{d-1}, we build a required k-DPC as described below: (i) Pick up $(k - 2) - k_0$ free edges bridging G_0 and G_1. Let X_i denote the set of endvertices of the free edges in G_i, $i \in \{0, 1\}$. (ii) Build a generalized $(k - 1)$-DPC in $G_0 - F_0$ joining S_0 and $T_0 \cup X_0 \cup \{u\}$ for some vertex u with $u^+ \neq s_k$, using Lemma 8(a). (iii) It suffices to build a generalized $(k - k_0)$-DPC in $G_{1,d-1}$ joining $\{s_k, u^+\} \cup X_1$ and $T_{1,d-1}$ and merge the two DPCs.

Let $k_0 = k - 1$ and $T_0 = \{t_1, \ldots, t_{k-1}\}$ now. If G_1 has no terminals, it suffices to build an unpaired $(k - 1)$-DPC in $G_0 \oplus G_1$ joining S_0 and T_0 and then build a Hamiltonian s_k–t_k path in $G_{2,d-1}$. If G_{d-1} has no terminals, a required k-DPC can be built symmetrically. Finally, we assume that s_k and t_k belong to G_1 and G_{d-1}, respectively. From an unpaired $(k - 1)$-DPC joining S_0 and T_0 in $G_0 \oplus G_1$, we can build a generalized k-DPC joining S and $T_0 \cup \{u\}$ for some vertex $u \in V(G_0) \cup V(G_1)$. For a neighbor u' of u in $G_{2,d-1}$, it suffices to combine a path in the DPC running to u with a Hamiltonian u'–t_k path of $G_{2,d-1}$ if $u' \neq t_k$; if $u' = t_k$, it suffices to extend the path running to u by one to t_k and then replace an edge (x, y) on a path in the DPC such that $x, y \neq u$ with a Hamiltonian x'–y' path of $G_{2,d-1} - \{t_k\}$, where x' and y' are the neighbors of x and y in $G_{2,d-1}$, respectively.

Case 4: *$f = 0$ and $k_0 = 0$.* The assumption (4) leads to $k = 2m - 2$. In particular, the assumption (5) states that no component has both a source and a sink, and that G_0 has no fewer terminals than other components, that is, $k_i = 0$ and $k'_0 \geq k'_i$ for all $i \in \{0, \ldots, d - 1\}$. There are two subcases depending on whether $k'_0 < k$ or not.

Case 4.1: *$k'_0 < k$.* Every component contains fewer than k terminals; therefore, the number of components in G is at least four, i.e., $d \geq 4$ in this subcase. There exists a chain $H := G_{p,q}$ such that (i) one of G_p and G_q contains a source and the other contains a sink, and (ii) every component other than G_p and G_q in chain H contains no terminals. For simplicity, we assume G_p contains a source, G_q contains a sink, and $|S_p| \geq |T_q|$. We further assume $|S_p|$ is the maximum possible over all chains satisfying the above conditions (i) and (ii). Recall that $|S_p| = k_{p,q} + k'_{p,q}$ and $|T_q| = k_{p,q}$.

Let H' be the chain $G - G_{p,q}$, which is made of two or more components. Suppose we can pick up $k'_{p,q}$ free edges between H and H' in such a way that (i) there exists an unpaired $(k_{p,q} + k'_{p,q})$-DPC joining S_p and $T_q \cup X$ and (ii) there exists an unpaired $(k - k_{p,q})$-DPC joining $(S \setminus S_p) \cup Y$ and $T \setminus T_q$, where X and Y are the sets of endvertices of the free edges in H and H', respectively. Then, we can build a required k-DPC just by merging the two DPCs through the free edges. The numbers $k_{p,q} + k'_{p,q}$ and $k - k_{p,q}$ are both nonzero and less than k, so by Theorem 1, it suffices to pick the free edges so that the S_p and $T_q \cup X$ pair in H and the $(S \setminus S_p) \cup Y$ and $T \setminus T_q$ pair in H' both do not form the exceptional configurations of the theorem.

Claim 7: It is possible to pick up $k'_{p,q}$ free edges between H and H' to avoid the exceptional configuration of Theorem 1 unless $k_{p,q} = 2m - 3$, that is, $|S_p| = |T_q| = k - 1$.

Proof. Suppose $k_{p,q} \neq 2m - 3$. If $k'_{p,q} \geq 2$, then it suffices to pick up $\lceil k'_{p,q}/2 \rceil$ free edges between G_p and G_{p-1} and then pick up $\lfloor k'_{p,q}/2 \rfloor$ frees between G_q and G_{q+1}. The free edges exist because $|V(G_0)| - (2k - 2) \geq 3^{m-1} - 2(2m - 2) + 2 \geq 2m - 3 = k - 1$ for $m \geq 3$. If $k'_{p,q} = 1$, it suffices to select a single free edge between G_p and G_{p-1}. The $(S \setminus S_p) \cup Y$ and $T \setminus T_q$ pair in H' does not form an exceptional configuration. (Supposing otherwise means $|S_{p-1}| = k - 2$ and $|T_{q-1}| = k - 1$, leading to $|S_p| = k - 1$ and $|S_p| + |S_{p-1}| = 2k - 3 > k$ for $k = 2m - 2 \geq 4$, which is a contradiction.) If $k'_{p,q} = 0$ and $k_{p,q} < 2m - 3$, the S_p and T_q pair does not form an exceptional configuration; so does the $S \setminus S_p$ and $T \setminus T_q$ pair, as required, thereby proving the claim. □

It remains to consider the case when $|S_p| = |T_q| = k - 1$, where the pair of terminal sets S_p and T_q in H forms the exceptional configuration of Theorem 1. Thus, there exists a generalized k-DPC in H joining $S_p \cup \{v_j\}$ and $T_q \cup \{u_i\}$ for some $u_i \in V(G_p)$ and $v_j \in V(G_q)$ such that u_i^- and v_j^+ are both nonterminals. It suffices to combine the generalized DPC of H with an unpaired 2-DPC of H' joining $(S \setminus S_p) \cup \{u_i^-\}$ and $(T \setminus T_q) \cup \{v_j^+\}$ through edges (u_i, u_i^-) and (v_j, v_j^+).

Case 4.2: $k_0' = k$. Component G_0 contains k sources and $S_0 = S$. We assume w.l.o.g. G_1 has no fewer sinks than G_{d-1}, i.e., $|T_1| \geq |T_{d-1}|$. There are four possibilities depending on the number of sinks in G_1. Firstly, suppose $|T_1| \geq k-1$. A required k-DPC can be built as follows: (i) Pick up $k-2$ edges, not necessarily free edges, between G_0 and G_1. Let X_i denote the set of endvertices of the picked edges in G_i, $i \in \{0, 1\}$. (ii) If $|T_1| = k$, it suffices to build a generalized k-DPC in G_1 joining $X_1 \cup \{u, v\}$ and T_1 for some vertices u and v, build a generalized k-DPC in G_0 joining S_0 and $X_0 \cup \{x, y\}$ for some vertices x, y such that $\{x^-, y^-\} \neq \{u^+, v^+\}$, and then build a generalized 2-DPC in $G_{2,d-1}$ joining $\{x^-, y^-\}$ and $\{u^+, v^+\}$. (iii) If $|T_1| = k-1$, assuming w.l.o.g. $t_k \notin T_1$, it suffices to build a generalized $(k-1)$-DPC in G_1 joining $X_1 \cup \{u\}$ and T_1 for some u with $u^+ \neq t_k$, build a generalized k-DPC in G_0 joining S_0 and $X_0 \cup \{x, y\}$ for some x, y such that $\{x^-, y^-\} \neq \{t_k, u^+\}$, and then build a generalized 2-DPC in $G_{2,d-1}$ joining $\{x^-, y^-\}$ and $\{t_k, u^+\}$. The generalized DPCs exist owing to Lemma 8 and Corollary 1.

Secondly, suppose $2 \leq |T_1| \leq k-2$. We first pick up $(k-2) - |T_1|$ free edges between G_0 and G_{d-1}. The free edges exist because $|V(G_0)| - (2k-2) \geq 3^{m-1} - (4m-6) \geq 2m-4 = k-2$ for $m \geq 3$. Let X_i denote the set of endvertices of Pthe free edges in G_i, $i \in \{0, d-1\}$. In addition, we pick up $|T_1|$ edges, not necessarily free edges, between G_0 and G_1 such that $Y_0 \cap X_0 = \emptyset$ and $Y_1 \neq T_1$, where Y_j is the set of endvertices of the picked edges in G_j, $j \in \{0, 1\}$. It is sufficient to merge the following three DPCs: a generalized k_1'-DPC in G_1 joining Y_1 and T_1, a generalized k-DPC in G_0 joining S and $X_0 \cup Y_0 \cup \{u, v\}$ for some vertices u, v with $\{u^+, v^+\} \neq T_{2,d-1}$, and a generalized $(k-k_1')$-DPC in $G_{2,d-1}$ joining $X_{d-1} \cup \{u^+, v^+\}$ and $T_{2,d-1}$.

Thirdly, suppose $|T_1| = 1$. It follows $|T_{d-1}| \leq 1$, meaning $d \geq 4$ because $k = 2m-2 \geq 4$. We pick up $k-2$ free edges, one edge between G_0 and G_{d-1} and $k-3$ edges between G_1 and G_2. Let X_i denote the set of endvertices of the free edges in G_i, $i \in \{0, 1, 2, d-1\}$. We can build a generalized k-DPC in $G_0 \oplus G_1$ joining S and $T_1 \cup X_0 \cup X_1 \cup \{u\}$ for some vertex u, from an unpaired $(k-1)$-DPC joining $S \setminus \{s_1\}$ and $T_1 \cup X_0 \cup X_1$, which exists by Theorem 1. It suffices to combine the generalized k-DPC with a generalized $(k-1)$-DPC in $G_{2,d-1}$ joining $X_2 \cup X_{d-1} \cup \{u'\}$ and $T_{2,d-1}$, where u' is the neighbor of u contained in $G_{2,d-1}$.

Finally, suppose $|T_1| = 0$, leading to $|T_{d-1}| = 0$ and $d \geq 4$. If $|T_p| = k$ or $k-1$ for some $p \in \{2, \ldots, d-2\}$, assuming w.l.o.g. $|T_i| = 0$ for all $i \in \{1, \ldots, p-1\}$, a required DPC can be built as follows: (i) Pick up $k-2$ free edges, one edge between G_0 and G_{d-1} and $k-3$ edges between G_0 and G_1, such that no two of them are adjacent. Let X_i denote the set of endvertices of the free edges in G_i, $i \in \{0, 1, d-1\}$. (ii) Build a generalized k-DPC in G_0 joining S_0 and $X_0 \cup \{u, v\}$ for some vertices u, v. (iii) Pick up $k-2$ free edges, one edge between G_p and G_{p+1} and $k-3$ edges between G_p and G_{p-1}, such that no two of them are adjacent and moreover, Y_{p-1} and X_1 are disjoint, Y_{p+1} and X_{d-1} are also disjoint, where Y_j denotes the set of endvertices of the free edges in G_j, $j \in \{p-1, p, p+1\}$. (iv) If $|T_p| = k$, there exists a generalized k-DPC in G_p joining $Y_p \cup \{x, y\}$ and T_p for some vertices x and y. Assuming w.l.o.g. $X_1 \cup \{u^+\}$, $Y_{p-1} \cup \{x^-\}$ are distinct and $X_{d-1} \cup \{v^-\}$, $Y_{p+1} \cup \{y^+\}$ are also distinct, it suffices to combine the DPCs of G_0 and G_p with a generalized $(k-2)$-DPC in $G_{1,p-1}$ joining $X_1 \cup \{u^+\}$ and $Y_{p-1} \cup \{x^-\}$ and a generalized 2-DPC in $G_{p+1,d-1}$ joining $X_{d-1} \cup \{v^-\}$ and $Y_{p+1} \cup \{y^+\}$. (v) If $|T_p| = k-1$, there exists a generalized $(k-1)$-DPC in G_p joining $Y_p \cup \{x\}$ and T_p for some vertex x. Assuming w.l.o.g. $t_k \notin T_p$, $X_1 \cup \{u^+\}$ and $Y_{p-1} \cup \{x^-\}$ are distinct, $X_{d-1} \cup \{v^-\}$ and $Y_{p+1} \cup \{t_k\}$ are also distinct, it suffices to combine the DPCs of G_0 and G_p with a generalized $(k-2)$-DPC in $G_{1,p-1}$ joining $X_1 \cup \{u^+\}$ and $Y_{p-1} \cup \{x^-\}$ and a generalized 2-DPC in $G_{p+1,d-1}$ joining $X_{d-1} \cup \{v^-\}$ and $Y_{p+1} \cup \{t_k\}$. Now, we assume $|T_i| \leq k-2$ for all $i \in \{1, \ldots, d-1\}$. The number of components d is 5 or more because G_1 and G_{d-1} contain no sinks and not all sinks are contained in one component. Let p be an index such that $|T_p| \geq 1$ and $|T_i| = 0$ for all $i \in \{1, \ldots, p-1\}$. Then, chains $G_{1,p}$ and $G_{p+1,d-1}$ are both made of two or more components. A required DPC can be built as follows: (i) Pick up $k-2$ free edges, $|T_p|$ edges between G_0 and G_1 and $(k-2) - |T_p|$ edges between G_0 and G_{d-1}, such that no two of them are adjacent. Let X_i denote the set of endvertices of the free edges in G_i, $i \in \{0, 1, d-1\}$. (ii) Build a generalized k-DPC in G_0 joining S_0 and $X_0 \cup \{u, v\}$ for some vertices u, v. (iii) Build an unpaired $|T_p|$-DPC in $G_{1,p}$ joining X_1 and T_p. (iv) Build a generalized $(k - |T_p|)$-DPC in $G_{p+1,d-1}$ joining $X_{d-1} \cup \{u^-, v^-\}$ and $T \setminus T_p$. (v) Merge the three DPCs into a required k-DPC. This completes the entire proof. ■

Corollary 2: Let G be an m-dimensional torus-like graph, $m \geq 3$, that satisfies the preconditions of Theorem 2. Given distinct sets S and T of sources and sinks, and a fault set F in G such that $k \geq 2$ and $f + k \leq 2m - 2$, there exists a generalized k-DPC joining S and T in $G - F$.

Combining Lemmas 2 and 3 with Theorem 2 leads to that:

Theorem 3: Every m-dimensional nonbipartite torus, $m \geq 2$, is f-fault unpaired k-disjoint path coverable for any f and $k \geq 1$ subject to $f + k \leq 2m - 2$.

V. CONCLUSION

In this paper, we have studied the unpaired disjoint path covers of a nonbipartite torus-like graph made of components with good Hamiltonian and disjoint-path-cover properties. Specifically, we proved that an m-dimensional nonbipartite torus-like graph, $m \geq 3$, composed of d components is f-fault unpaired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq 2m - 2$ if each component G_i is f-fault unpaired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq 2m - 4$ and moreover, G_i is $(2m - 5)$-fault Hamiltonian-connected and $(2m-4)$-fault Hamiltonian. As a result, we know that an m-dimensional nonbipartite torus, $m \geq 2$, is f-fault unpaired k-disjoint path coverable for any f and $k \geq 1$ subject to $f + k \leq 2m - 2$. It is open to determine whether a three-dimensional or higher-dimensional nonbipartite torus admits an optimal construction of an unpaired disjoint path cover, that is, whether an m-dimensional nonbipartite torus, $m \geq 3$, is f-fault unpaired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq 2m - 1$.

REFERENCES

[1] B. Parhami, *Introduction to Parallel Processing: Algorithms and Architectures*. Springer, 2006.

[2] L.-H. Hsu and C.-K. Lin, *Graph Theory and Interconnection Networks*. Boca Raton, FL, USA: CRC Press, 2008.

[3] J. A. McHugh, *Algorithmic Graph Theory*. Upper Saddle River, NJ, USA: Prentice-Hall, 1990.

[4] D. Torrieri, "Algorithms for finding an optimal set of short disjoint paths in a communication network," *IEEE Trans. Commun.*, vol. 40, no. 11, pp. 1698–1702, Nov. 1992.

[5] P. D. Seymour, "Disjoint paths in graphs," *Discrete Math.*, vol. 29, no. 3, pp. 293–309, 1980.

[6] S. C. Ntafos and S. L. Hakimi, "On path cover problems in digraphs and applications to program testing," *IEEE Trans. Softw. Eng.*, vol. SE-5, no. 5, pp. 520–529, Sep. 1979.

[7] H. Liu, J. Lu, S. Zhang, and X. Zhong, "On the path cover number of connected quasi-claw-free graphs," *IEEE Access*, vol. 9, pp. 37086–37090, 2021.

[8] H. Ichida and K. Kaneko, "Set-to-set disjoint paths problem in Möbius cubes," *IEEE Access*, vol. 10, pp. 83075–83084, 2022.

[9] K. Asdre and S. D. Nikolopoulos, "The 1-Fixed-Endpoint path cover problem is polynomial on interval graphs," *Algorithmica*, vol. 58, no. 3, pp. 679–710, Nov. 2010.

[10] J.-H. Park, H.-C. Kim, and H.-S. Lim, "Many-to-many disjoint path covers in hypercube-like interconnection networks with faulty elements," *IEEE Trans. Parallel Distrib. Syst.*, vol. 17, no. 3, pp. 227–240, Mar. 2006.

[11] J. H. Park, H. C. Kim, and H. S. Lim, "Many-to-many disjoint path covers in the presence of faulty elements," *IEEE Trans. Comput.*, vol. 58, no. 4, pp. 528–540, Apr. 2009.

[12] J.-H. Park, "Torus-like graphs and their paired many-to-many disjoint path covers," *Discrete Appl. Math.*, vol. 289, pp. 64–77, Jan. 2021.

[13] H.-S. Lim, H.-C. Kim, and J.-H. Park, "Ore-type degree conditions for disjoint path covers in simple graphs," *Discrete Math.*, vol. 339, no. 2, pp. 770–779, Feb. 2016.

[14] J.-H. Park and I. Ihm, "A linear-time algorithm for finding a one-to-many 3-disjoint path cover in the cube of a connected graph," *Inf. Process. Lett.*, vol. 142, pp. 57–63, Feb. 2019.

[15] H. Lü, "Paired many-to-many two-disjoint path cover of balanced hypercubes with faulty edges," *J. Supercomput.*, vol. 75, no. 1, pp. 400–424, Jan. 2019.

[16] H. Lü and T. Wu, "Unpaired many-to-many disjoint path cover of balanced hypercubes," *Int. J. Found. Comput. Sci.*, vol. 32, no. 8, pp. 943–956, 2021.

[17] S.-Y. Hsieh, C.-W. Lee, and C.-H. Huang, "Conditional edge-fault Hamiltonian-connectivity of restricted hypercube-like networks," *Inf. Comput.*, vol. 251, pp. 314–334, Dec. 2016.

[18] R. Niu and M. Xu, "The unpaired many-to-many k-disjoint paths in bipartite hypercube-like networks," *Theor. Comput. Sci.*, vol. 911, pp. 26–40, Apr. 2022.

[19] S.-Y. Kim and J.-H. Park, "Paired many-to-many disjoint path covers in recursive circulants G(2^m,4)," *IEEE Trans. Comput.*, vol. 62, no. 12, pp. 2468–2475, Jun. 2013.

[20] H. Cao, B. Zhang, and Z. Zhou, "One-to-one disjoint path covers in digraphs," *Theor. Comput. Sci.*, vol. 714, pp. 27–35, Mar. 2018.

[21] H.-C. Kim, S.-Y. Cho, and C.-S. Shin, "One-to-One disjoint cover of non-bipartite k-ary n-cube with faulty edges," *J. KIISE*, vol. 48, no. 12, pp. 1251–1273, Dec. 2021.

[22] J. Li, G. Wang, and L. Chen, "Paired 2-disjoint path covers of multi-dimensional torus networks with $2n - 3$ faulty edges," *Theor. Comput. Sci.*, vol. 677, pp. 1–11, May 2017.

[23] J.-H. Park, "Paired 3-Disjoint path covers in bipartite torus-like graphs with edge faults," *Int. J. Found. Comput. Sci.*, vol. 2022, pp. 1–13, Nov. 2022.

[24] J. Li, C. Melekian, S. Zuo, and E. Cheng, "Unpaired many-to-many disjoint path covers on bipartite k-ary n-cube networks with faulty elements," *Int. J. Found. Comput. Sci.*, vol. 31, no. 3, pp. 371–383, Apr. 2020.

[25] H.-C. Kim and J.-H. Park, "Paths and cycles in d-dimensional tori with faults," in *Proc. Workshop Algorithms Comput. (WAAC)*, 2001, pp. 67–74.

[26] B. G. Kronenthal and W. H. T. Wong, "Paired many-to-many disjoint path covers of hypertori," *Discrete Appl. Math.*, vol. 218, pp. 14–20, Feb. 2017.

[27] J.-H. Park, "Many-to-many disjoint path covers in two-dimensional tori," *J. KIISE: Comput. Syst. Theory*, vol. 38, no. 1, pp. 42–48, 2011.

[28] J.-H. Park, "Unpaired many-to-many disjoint path covers in restricted hypercube-like graphs," *Theor. Comput. Sci.*, vol. 617, pp. 45–64, Feb. 2016.

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