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RESEARCH ARTICLE

Consensus of Julia Sets of Potts Models on Diamond-Like Hierarchical Lattice

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ABSTRACT The limit sets of zeros of the partition function for λ-state Potts models on diamond-like hierarchical lattice are the Julia sets of functions in a family of rational functions. In this paper, the consensus problem of Julia sets generated by λ-state Potts models on diamond-like hierarchical lattice is studied. Two types of the consensus problem of Julia sets are considered, one is with a leader and the other is with no leaders. Based on these two types, two different control protocols are proposed respectively to make systems achieve consensus of Julia sets. The simulations confirm the efficacy of control protocols.

INDEX TERMS Julia set, multi-agent system, consensus.

I. INTRODUCTION

In recent years, more and more scholars have participated in the research of multi-agent systems. The consensus problem of multi-agent systems is one of the hottest problems, and a lot of achievements have been made [1], [2], [3], [4], and [5]. Reference [1] investigated the consensus problem for directed networks of agents with external disturbances and model uncertainties on fixed and switching topologies. In [4], a distributed fault-tolerant leader-follower consensus protocol for multi-agent systems is constructed by the proposed adaptive method. In reference [5], a distributed observer type consensus protocol based on relative output measurements is proposed to achieve the consensus of multi-agent systems. The so-called consensus means that a certain state of the multi-agent system tends to be consistent by information exchange among multiple agents connected by a network after designing control protocols. The consensus problem is widely used in artificial intelligence, the coordinated control of unmanned aerial vehicles, and the urban traffic control.

The consensus problem of multi-agent systems is divided into two types, one is with a leader [6], [7] and the other is with no leaders [8], [9]. For the former, agents in the system are divided into two types, namely the follower and the leader. Then control protocols are designed to make some states of followers tend to the corresponding state of the leader. Whereas, for the latter, the state of each agent in the system is finally tended to a certain state by designing a control protocol. Mei et al. [10] studied the distributed containment control problem for networked Lagrangian systems with multiple dynamic leaders in the presence of parametric uncertainties under a directed graph that characterizes the interaction among the leaders and the followers. Li et al. [11] considered two types of the consensus problem of linear multi-agent systems subject to different matching uncertainties with no leaders and with a leader of the bounded unknown control input.

Mandelbrot set and Julia set play an important role in the fractal theory. Famous mathematician Mandelbrot first constructs the Mandelbrot and Julia sets from the complex mapping $z \leftarrow z^{\alpha} + c$ $(\alpha = 2)$ utilizing computer graphics technologies [12]. Subsequently, their properties and images have been studied extensively [13], [14]. The c-plane fractal images from the complex mapping $z \leftarrow z^{\alpha} + c$ $(\alpha = 2)$ were studied in [14]. Dhurandhar et al. [15] discussed the fractal feature of the generalized Julia sets from the generalized transformation function $z \leftarrow z^{\alpha} + c, \alpha < 0$. Kumari et al. [16] proposed a novel approach to visualize Mandelbrot and Julia sets for complex polynomials $w(z) = z^n + mz + r$ $(n \geq 2; m, r \in \mathbb{C})$ through a viscosity

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approximation method. To present the algorithms of generation of Mandelbrot and Julia sets for general complex polynomial $f(x) = \sum_{i=0}^{p} a_i x^i$ ($p \geq 2$, $a_i \in \mathbb{C}$ for $i = 0, 1, \cdots, p$.), a general escape criterion is proposed via extended Jungck-Noor iteration with s-convexity in [17]. Prajapati et al. [18] studied the Julia sets linked to the entire transcendental function $f(z) = ae^{z^n} + bz + c$ ($a, b, c \in \mathbb{C}; n \geq 2$) by using the Mann iterative scheme. The physical significance of the Mandelbrot and Julia sets has also been studied by some scholars [19], [20]. In [19], from the figures of the distribution of zeros of the partition function of an Ising and a q-state Potts model on a diamond hierarchical lattice, we can know that those zeros are just the Julia set corresponding to the renormalization group transformation. The Julia set in nonlinear complex dynamical systems describes a special character, which portrays the degree of the variation of the stable region. Due to the influence of noise or other factors, sometimes it is necessary to restrict the size of the stability region of the system, and sometimes it is required that multiple and different systems show the same or similar behavior and performance according to objective requirements. By controlling the stability of fixed points or periodic orbits, the stability region of the system can be controlled. Recently, some scholars have studied the control and synchronization of several classes of Julia sets [21], [22]. Feedback control and synchronization of Julia sets of the discrete version of the Volterra system are studied in [23].

Since the 1980s, with the development of computer graphics and the hard work of many scholars, the research on complex analytic dynamical systems has gained new vitality and attracted extensive attention from mathematicians all over the world. Inspired by the Newton iteration method and Möbius transformation group, Fatou [24] and Julia [25] began to study the stability of iterative sequences of complex analytic mappings and eventually established the Fatou-Julia theory. Sullivan [26] obtained the final periodicity theorem around 1980, which is a milestone achievement in modern research on complex dynamical systems. In reference [27], Zhou Weimin introduces a stochastic iterative system generated by rational functions on the Ricmaoo sphere. In the 1980s, Sullivan proposed the definition of conformal measure, which provided a new research field for the properties of Julia sets of rational functions and made the dynamic systems generated by rational functions closely related to fractal and other disciplines. It has now been shown that the singularities in many examples are distributed on the Julia sets corresponding to the renormalization transformations, that is, on the smallest closure containing all unstable periodic points [28]. The partition function of Hamiltonian of λ-state Potts models on diamond-like hierarchical lattice is

$$Z = \sum_{\sigma_i} \exp[K \sum_{<i,j>} \delta(\sigma_i, \sigma_j)], \tag{1}$$

where $K = \beta J$, $\beta = \frac{1}{k_1 t}$, k_1 is the Boltzmann constant, and t is the temperature. By using the method of Migdal-Kadanoff renormalization group, Qiao [29] proved that the limit sets of zeros of the partition function (1) are Julia sets of the following rational functions

$$f_\lambda^n = (\frac{z^2 + \lambda - 1}{2z + \lambda - 2})^n, \quad n = 1, 2, \cdots \tag{2}$$

where λ is a complex parameter. Based on the previous consideration of the properties of Julia sets and the consensus of multi-agent systems, we explore such interesting question: Considering a multi-agent dynamic system generated by some functions in the family (2), can control protocols be designed by using the information exchange between agents to make Julia sets corresponding to agents tend to be consistent? That is, when there is a leader, the Julia set corresponding to each follower tends to the Julia set of the leader, or the Julia set corresponding to each agent tends to some set when there are no leaders.

Inspired by the above discussion, the consensus problem of Julia sets of λ-state Potts models on diamond-like hierarchical lattice is discussed in this paper. Section 2 introduces some preliminary knowledge about the consensus problem of Julia sets for multi-agent systems. Section 3 presents two different control protocols for multi-agent systems with a leader to achieve consensus. Section 4 also introduces two different control protocols for multi-agent systems without leaders.

II. PRELIMINARIES

Some basic knowledge about the consensus problem of Julia sets will be introduced here. The graph provides a unified approach to represent various real networks with abstract points and lines [30]. Topological properties of concrete networks can be obtained by studying abstract graphs. Graph $g = \langle v, \epsilon \rangle$ is used to represent the network connecting multiple agents, where $v = \{1, 2, \ldots, N\}$ is a set of nodes or agents, and $\epsilon \subset v \times v$ represents communication links between agents, that is, $(i, j) \in \epsilon$ represents that there is a communication link between the i-th agent and the j-th agent. The graphs in this paper are undirected, then $(i, j) \in \epsilon$ is equivalent to $(j, i) \in \epsilon$. The structure of the graph can be represented by an adjacency matrix $A = (a_{ij})_{N \times N}$, where $a_{ii} = 0$, $a_{ij} = 1$ if and only if $(i, j) \in \epsilon$. Otherwise, $a_{ij} = 0$.

Definition 1: If $(i, j) \in \epsilon$, then node j is called a neighbor of node i. The set of neighbors of node i is denoted as $N_i = \{j | (i, j) \in \epsilon\}$.

The discrete fractal system corresponding to the function in family (2) is shown below:

$$z_i(n + 1) = f_\lambda^i(z_i(n)) \tag{3}$$

Definition 2: The trajectory of system (3) is denoted as $\{z_i(n)\}_{n=0}^{\infty}$. The set

$$K = \{z_i(0) | \{z_i(n)\}_{n=0}^{\infty} \text{is bounded}\}$$

is called the filled Julia set of system (3), and the boundary of K is called the Julia set of system (3).

III. CONTROLLERS OF CONSENSUS OF JULIA SETS WITH A LEADER

Let J_m be the Julia set corresponding to the leader, which is generated by the discrete fractal system

$$z_m(n+1) = f_\lambda^m(z_m(n)), \tag{4}$$

where m is some integer. The Julia set corresponding to the i-th follower is denoted as J_i, which is generated by the following discrete system

$$z_i(n+1) = f_\lambda^i(z_i(n)). \tag{5}$$

f_λ^m and f_λ^i ($i \in \{1, 2, \ldots, N\}$ and $i \neq m$) are both functions in family (2).

Definition 3: The consensus problem of Julia sets with a leader is to design a control protocol $u_i(z_m(n), z_1(n), z_2(n), \cdots, z_N(n); k)$ to join the system (5) corresponding to the i-th follower, and the Julia set $J_i(k)$ generated by the system

$$z_i(n+1) = f_\lambda^i(z_i(n)) + u_i(z_m(n), z_1(n), z_2(n), \cdots, z_N(n); k), \tag{6}$$

satisfies $\lim_{k \to k_0} J_i(k) = J_m$, where $i = 1, 2, \cdots, N$ and k is a control parameter, and k_0 is some constant or ∞.

A. THE FIRST METHOD

Take

$$\begin{aligned} &u_i(z_m(n), z_1(n), z_2(n), \cdots, z_N(n); k) \\ &\quad = k[f_\lambda^i(z_i(n)) - f_\lambda^m(z_m(n))] \\ &\qquad + (k+1)\sum_{j \in N_i} a_{j0}[f_\lambda^j(z_j(n)) - f_\lambda^m(z_m(n))], \end{aligned} \tag{7}$$

where $a_{j0} = 1$ if and only if there is a communication link between the j-th agent and the leader. Otherwise, $a_{j0} = 0$. Obviously, it is impossible that $a_{j0} = 0$ for all $j \in N_i$, because there needs a communication link between some followers in N_i and the leader to achieve consensus. Then

$$\begin{aligned} &z_i(n+1) - z_m(n+1) \\ &\quad = (k+1)[f_\lambda^i(z_i(n)) - f_\lambda^m(z_m(n))] \\ &\qquad + (k+1)\sum_{j \in N_i} a_{j0}[f_\lambda^j(z_j(n)) - f_\lambda^m(z_m(n))]. \end{aligned} \tag{8}$$

Julia set is the boundary of the set of initial points with bounded trajectories, so only the points remaining bounded after iteration are considered. Then there exists $M > 0$ such that $|f_\lambda^i(z_i(n))| < M$ for any $i \in \{1, 2, \cdots, N, m\}$, thus

$$\begin{aligned} &|z_i(n+1) - z_m(n+1)| \\ &\quad \leq 2M|k+1| + 2M|N_i||k+1| \\ &\quad = 2M(|N_i|+1)|k+1|. \end{aligned}$$

Let $k \to -1$, then $|z_i(n+1) - z_m(n+1)| \to 0$ for any $i \in \{1, 2, \cdots, N\}$. So the consensus of trajectories between followers and the leader leads to the consensus of Julia sets.

Next, take rational function $f_2^i = (\frac{z^2+1}{2z})^i$ in family (2) to show the consensus of Julia sets with a leader. Fig.1(a) is the Julia set of the leader when $i = 2$, Fig.1(b), (c) and (d) are Julia sets of followers when $i = 3$, $i = 4$ and $i = 5$ respectively. Denote the leader system as node 1, and the three followers as node 2, node 3 and node 4. Assume that the adjacency matrix of the system composed of these four nodes is

$$A = (a_{ij})_{4\times4} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \tag{9}$$

where $a_{ij} = 1$ if and only if there is a communication link between the i-th node and the j-th node. Otherwise, $a_{ij} = 0$.

Synchronization of Julia sets between two systems is to design control protocols such that the Julia set of one system eventually tends to the Julia set of another. If $a_{j0} = 0$ for all j in (7), then the consensus problem is transformed into the synchronization problem. Take the control parameter k as -0.5 in (7), then the Julia sets corresponding to followers after control are shown in Fig.2. Fig.2 illustrates that although the Julia set corresponding to any follower eventually tends to the Julia set of the leader, the process is not consistent. Conversely, suppose that $a_{j0} = 1$ for all $j \in N_i$ in (7) and the adjacency matrix of the multi-agent system is matrix (9), so the systems of followers after control are

$$\begin{aligned} z_3(n+1) &= f_2^3(z_3(n)) + k[f_2^3(z_3(n)) - f_2^2(z_2(n))] \\ &\quad + (k+1)[f_2^4(z_4(n)) - f_2^2(z_2(n))], \\ z_4(n+1) &= f_2^4(z_4(n)) + k[f_2^4(z_4(n)) - f_2^2(z_2(n))] \\ &\quad + (k+1)[f_2^3(z_3(n)) - f_2^2(z_2(n)) \\ &\quad + f_2^5(z_5(n)) - f_2^2(z_2(n))], \\ z_5(n+1) &= f_2^5(z_5(n)) + k[f_2^5(z_5(n)) - f_2^2(z_2(n))] \\ &\quad + (k+1)[f_2^4(z_4(n)) - f_2^2(z_2(n))]. \end{aligned}$$

When the control parameter k takes different values, the Julia sets corresponding to the above discrete systems are shown in Fig.3. It can be seen from Fig.3 that they eventually tend to the Julia set of the leader as k tends to -1, and the process is also consistent.

B. THE SECOND METHOD

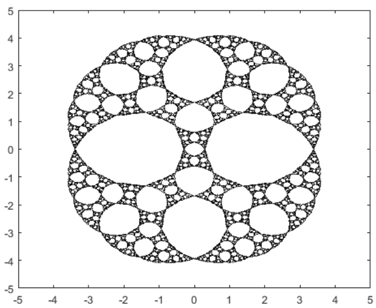
Gradient control methodology [31] is a general technique for controlling nonlinear systems and it is used to achieve the control of Julia sets in [32]. Based on the gradient control method, the control protocol in the consensus of Julia sets of Potts model is designed as

$$\begin{aligned} &u_i(z_m(n), z_1(n), z_2(n), \cdots, z_N(n); k) \\ &\quad = -\frac{k}{1+k}[f_\lambda^i(z_i(n)) - f_\lambda^m(z_m(n))] \\ &\qquad + \frac{1}{1+k}\sum_{j \in N_i} a_{j0}[f_\lambda^j(z_j(n)) - f_\lambda^m(z_m(n))], \end{aligned} \tag{10}$$

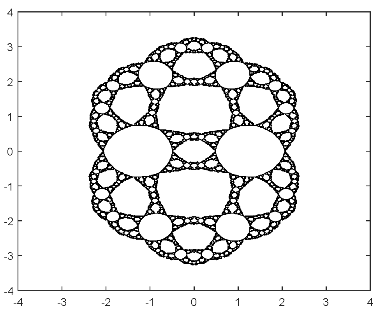
where a_{j0} has the same meaning as it is in the first method. Then

$$\begin{aligned} &z_i(n+1) - z_m(n+1) \\ &\quad = \frac{1}{1+k}[f_\lambda^i(z_i(n)) - f_\lambda^m z_m(n)] \\ &\qquad + \frac{1}{1+k}\sum_{j \in N_i} a_{j0}[f_\lambda^j(z_j(n)) - f_\lambda^m(z_m(n))]. \end{aligned} \tag{11}$$

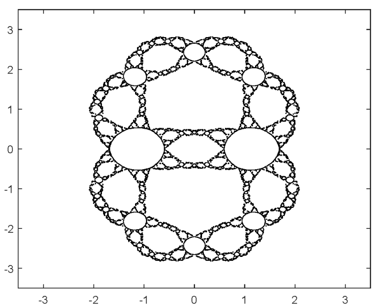
Similarly, we only consider the points remaining bounded after iteration, so suppose there exists $T > 0$ satisfying



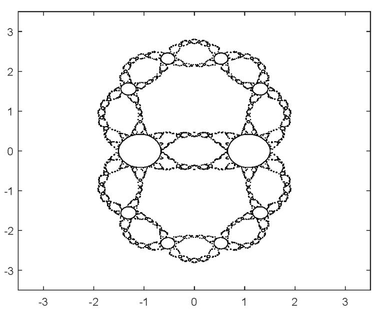
(a) $i = 2$



(b) $i = 3$



(c) $i = 4$

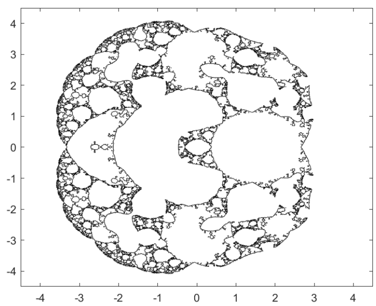


(d) $i = 5$

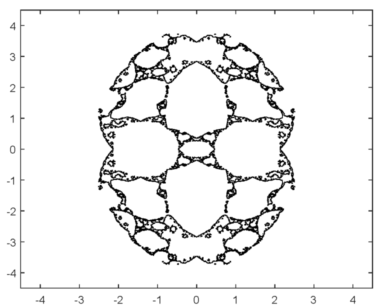
FIGURE 1. Julia sets of the discrete fractal system (3) with different values of *i* when λ = 2.

$|f_\lambda^i(z_i(n))| < T$ for any $i \in \{1, 2, \cdots, N, m\}$, thus

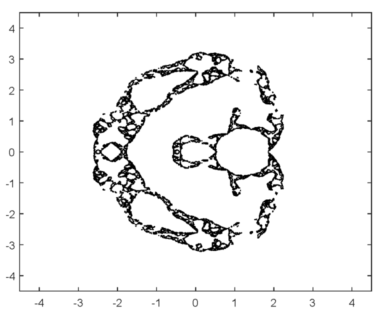
$$\begin{aligned}|z_i(n+1) - z_m(n+1)| \\ &\leq 2T|\frac{1}{1+k}| + 2|N_i|T|\frac{1}{1+k}| \\ &= 2T(|N_i|+1)|\frac{1}{1+k}|.\end{aligned}$$



(a) $i = 3$



(b) $i = 4$



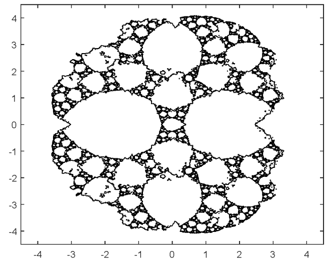
(c) $i = 5$

FIGURE 2. Julia sets of followers when *k* = −0.5.

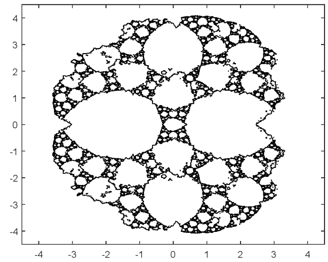
Let $k \to \infty$, then $|z_i(n+1) - z_m(n+1)| \to 0$ for any $i \in \{1, 2, \cdots, N\}$. Finally, the consensus of trajectories between followers and the leader leads to the consensus of Julia sets.

Next, the example in the first method is still used to show the consensus of Julia sets with the leader. If it is true that $a_{j0} = 0$ for all $j \in N_i$ in (10), then the consensus problem is transformed into the synchronization problem. When k is 2, the Julia sets corresponding to followers are shown in Fig.4. It can be seen from Fig.4 that they all tend to the Julia set corresponding to the leader in the end. That is, they achieve consensus, but the process is inconsistent. If $a_{j0} = 1$ for all $j \in N_i$ in (10) and the adjacency matrix of the multi-agent system is still matrix (9), then the systems of followers after control are

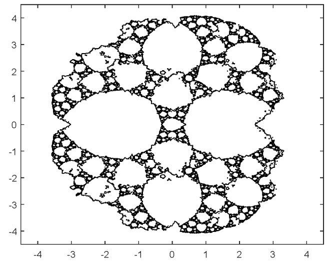
$$\begin{aligned}z_3(n+1) = f_2^3(z_3(n)) - \frac{k}{1+k}[f_2^3(z_3(n)) - f_2^2(z_2(n))] \\ + \frac{1}{1+k}[f_2^4(z_4(n)) - f_2^2(z_2(n))],\end{aligned}$$



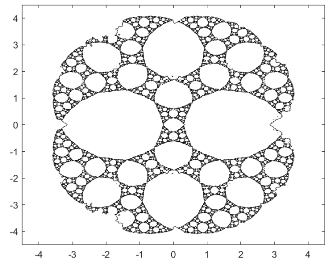
(a) $i = 3, k = -0.9$



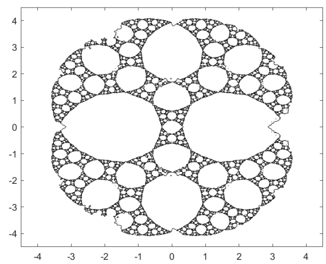
(b) $i = 4, k = -0.9$



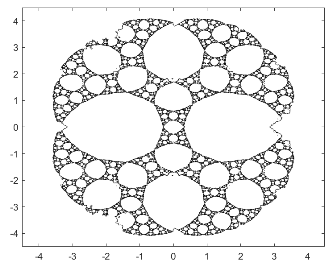
(c) $i = 5, k = -0.9$



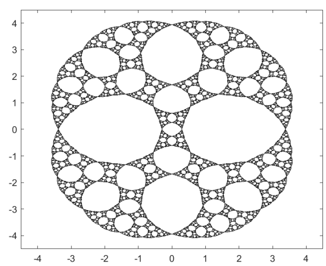
(d) $i = 3, k = -0.95$



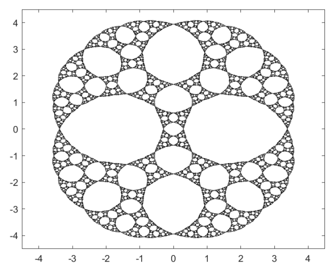
(e) $i = 4, k = -0.95$



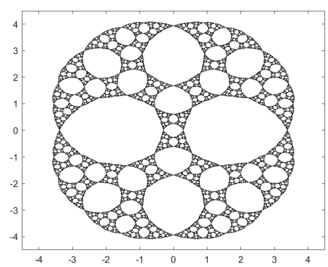
(f) $i = 5, k = -0.95$



(g) $i = 3, k = -0.99$

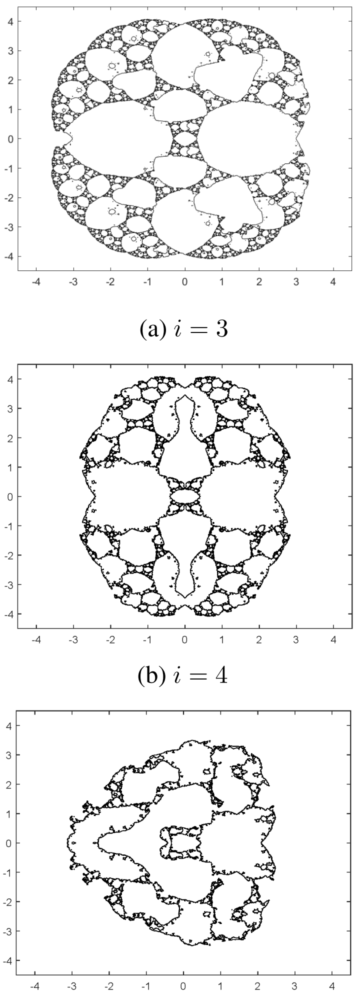


(h) $i = 4, k = -0.99$



(i) $i = 5, k = -0.99$

FIGURE 3. **Julia sets of followers with different values of parameter *k*.**



(a) $i = 3$

(b) $i = 4$

(c) $i = 5$

FIGURE 4. Julia sets of followers when $k = 2$.

$$\begin{aligned} z_4(n+1) &= f_2^4(z_4(n)) - \frac{k}{1+k}[f_2^4(z_4(n)) - f_2^2(z_2(n))] \\ &\quad + \frac{1}{1+k}[f_2^3(z_3(n)) - f_2^2(z_2(n)) \\ &\quad + f_2^5(z_5(n)) - f_2^2(z_2(n))], \\ z_5(n+1) &= f_2^5(z_5(n)) - \frac{k}{1+k}[f_2^5(z_5(n)) - f_2^2(z_2(n))] \\ &\quad + \frac{1}{1+k}[f_2^4(z_4(n)) - f_2^2(z_2(n))]. \end{aligned}$$

When the control parameter k takes different values, the Julia sets corresponding to the above discrete fractal systems are shown in Fig.5. It can be seen from Fig.5 that they eventually tend to the Julia set corresponding to the leader, and the process is also consistent.

IV. CONTROLLERS OF CONSENSUS OF JULIA SETS WITH NO LEADERS

It is assumed that the multi-agent system is composed of discrete systems

$$z_i(n+1) = f_\lambda^i(z_i(n)), \quad i = 1, 2, \cdots, N, \tag{12}$$

where f_λ^i are functions in family (2).

Definition 4: The consensus problem of Julia sets without leaders is to design a controller $u_i(z_1(n), z_2(n), \cdots, z_N(n); t)$ added to the i-th agent, and the Julia set $J_i(t)$ generated by the system

$$z_i(n+1) = f_\lambda^i(z_i(n)) + u_i(z_1(n), z_2(n), \cdots, z_N(n); t), \quad i = 1, 2, \cdots, N, \tag{13}$$

satisfies $\lim_{t \to t_0} J_i(t) = J$, where t is a control parameter, t_0 is some constant or ∞, and J is a certain set.

A. THE FIRST METHOD

Take

$$\begin{aligned} &u_i(z_1(n), z_2(n), \cdots, z_N(n); k, l) \\ &\quad = kf_\lambda^i(z_i(n)) + l \sum_{j \in N_i} a_{ij}[f_\lambda^j(z_j(n)) - f_\lambda^i(z_i(n))], \end{aligned} \tag{14}$$

where $a_{ij} = 1$ if and only if there is a communication link between the i-th agent and the j-th agent. Otherwise, $a_{ij} = 0$. Thus

$$\begin{aligned} &z_i(n+1) - z_j(n+1) \\ &\quad = (k+1)[f_\lambda^i(z_i(n)) - f_\lambda^j(z_j(n))] \\ &\qquad + l\{\sum_{p \in N_i} a_{ip}[f_\lambda^p(z_p(n)) - f_\lambda^i(z_i(n))] \\ &\qquad - \sum_{q \in N_j} a_{jq}[f_\lambda^q(z_q(n)) - f_\lambda^j(z_j(n))]\}. \end{aligned} \tag{15}$$

By the definition of Julia set, we only consider the initial points remaining bounded after iteration, then there exists $M > 0$ such that $|f_\lambda^i(z_i(n))| < M$ for any $i \in \{1, 2, \cdots, N\}$. Therefore

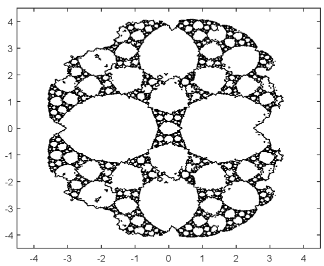
$$|z_i(n+1) - z_j(n+1)| \leq 2M|k+1| + 2Ml(|N_i| + |N_j|).$$

Let $k \to -1$ and $l \to 0$, then $|z_i(n+1) - z_j(n+1)| \to 0$ for any i and j. Finally, the consensus of trajectories leads to the consensus of Julia sets.

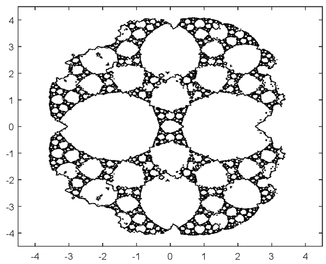
Next, a multi-agent system composed of discrete fractal system $z_3(n+1) = f_6^3(z_3(n))$, $z_4(n+1) = f_6^4(z_4(n))$ and $z_5(n+1) = f_6^5(z_5(n))$ will be taken to achieve consensus of Julia sets without leaders. Denote these three systems or agents as node 1, node 2 and node 3. Suppose that the adjacency matrix corresponding to the multi-agent system is

$$A = (a_{ij})_{3\times3} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{16}$$

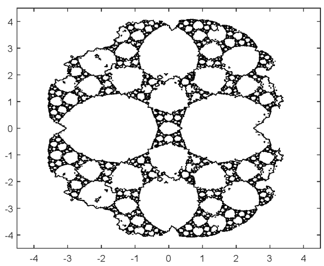
where $a_{ij} = 1$ represents the existence of a communication link between the i-th node and the j-th node. Otherwise, $a_{ij} = 0$. Original Julia sets of the above three discrete fractal systems are shown in Fig.6. Controllers (14)



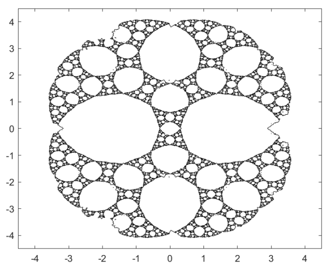
(a) $i = 3, k = 10$



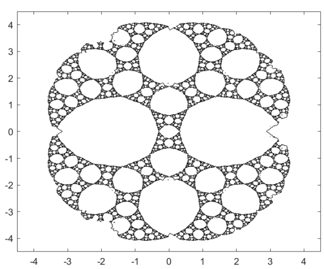
(b) $i = 4, k = 10$



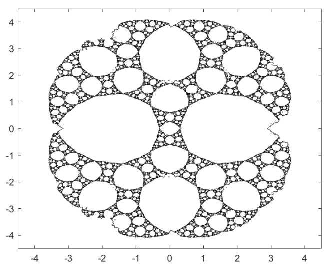
(c) $i = 5, k = 10$



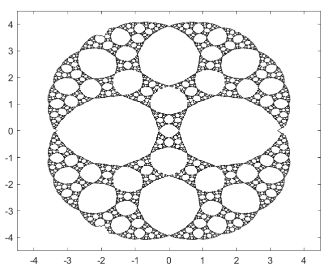
(d) $i = 3, k = 20$



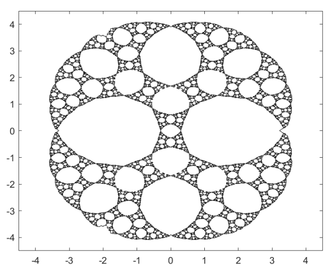
(e) $i = 4, k = 20$



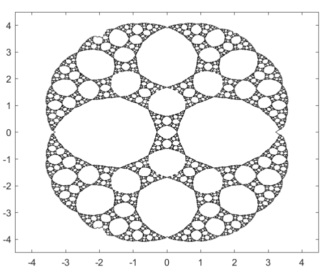
(f) $i = 5, k = 20$



(g) $i = 3, k = 40$



(h) $i = 4, k = 40$



(i) $i = 5, k = 40$

FIGURE 5. **Julia sets of followers with different values of the parameter *k*.**

are added to (12) respectively, then the controlled systems are

$$\begin{aligned} z_3(n+1) &= f_6^3(z_3(n)) + k[f_6^3(z_3(n))] \\ &\quad + l[f_6^4(z_4(n)) - f_6^3(z_3(n))], \\ z_4(n+1) &= f_6^4(z_4(n)) + k[f_6^4(z_4(n))] \\ &\quad + l[f_6^3(z_3(n)) - f_6^4(z_4(n)) \\ &\quad + f_6^5(z_5(n)) - f_6^4(z_4(n))], \\ z_5(n+1) &= f_6^5(z_5(n)) + k[f_6^5(z_5(n))] \\ &\quad + l[f_6^4(z_4(n)) - f_6^5(z_5(n))]. \end{aligned}$$

When the control parameter l takes 0.2 and k takes different values, the Julia set of each system is shown in Fig.7. When the control parameter l takes 0.1 and k takes different values, the Julia set of each system is shown in Fig.8. It can be seen from figures that when l is fixed, as k gets closer and closer to -1, the Julia set of each system becomes more and more similar, and consistently tends to a certain set in the end.

B. THE SECOND METHOD

Take

$$u_i(z_1(n), z_2(n), \cdots, z_N(n); k, l) = -\frac{k}{1+k} f_\lambda^i(z_i(n)) + \frac{1}{k+1} \sum_{j \in N_i} a_{ij}[f_\lambda^j(z_j(n)) - f_\lambda^i(z_i(n))], \tag{17}$$

where $a_{ij} = 1$ if and only if there is a communication link between the i-th agent and the j-th agent. Otherwise, $a_{ij} = 0$. Then

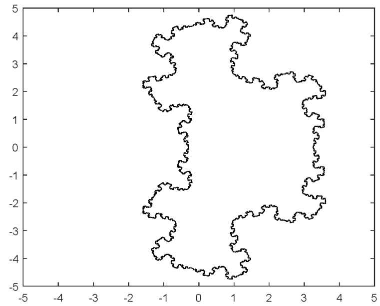
$$\begin{aligned} z_i(n+1) - z_j(n+1) &= \frac{1}{k+1}[f_\lambda^i(z_i(n)) - f_\lambda^j(z_j(n))] \\ &\quad + \frac{1}{k+1}\{\sum_{p \in N_i} a_{ip}[f_\lambda^p(z_p(n)) - f_\lambda^i(z_i(n))] \\ &\quad - \sum_{q \in N_j} a_{jq}[f_\lambda^q(z_q(n)) - f_\lambda^j(z_j(n))]\}. \end{aligned} \tag{18}$$

In the same way, we only consider the points where the corresponding trajectories are bounded, so there exists $T > 0$ satisfying $|f_\lambda^i(z_i(n))| < T$ for any $i \in \{1, 2, \cdots, N\}$. Thus

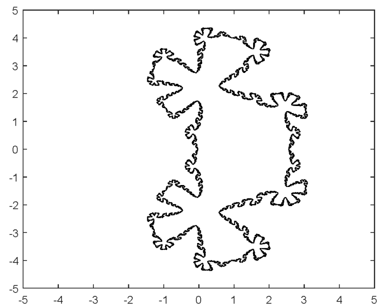
$$\begin{aligned} &|z_i(n+1) - z_j(n+1)| \\ &\quad \le 2T|\frac{1}{k+1}| + 2T|\frac{1}{k+1}|(|N_i| + |N_j|) \\ &\quad = 2T|\frac{1}{k+1}|(1 + |N_i| + |N_j|). \end{aligned}$$

Let $k \to \infty$, then $|z_i(n+1) - z_j(n+1)| \to 0$ for any i and j. Finally, the consensus of trajectories between agents leads to the consensus of Julia sets.

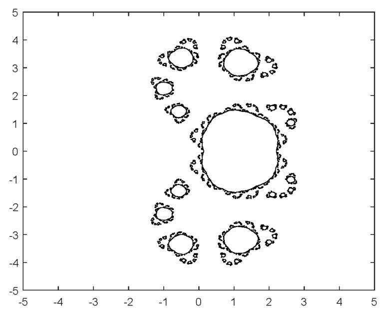
Still use the previous example in the first method to analyze the consensus of Julia sets without leaders. Suppose the adjacency matrix of the multi-agent system is matrix (16), then



(a) $i = 3$



(b) $i = 4$



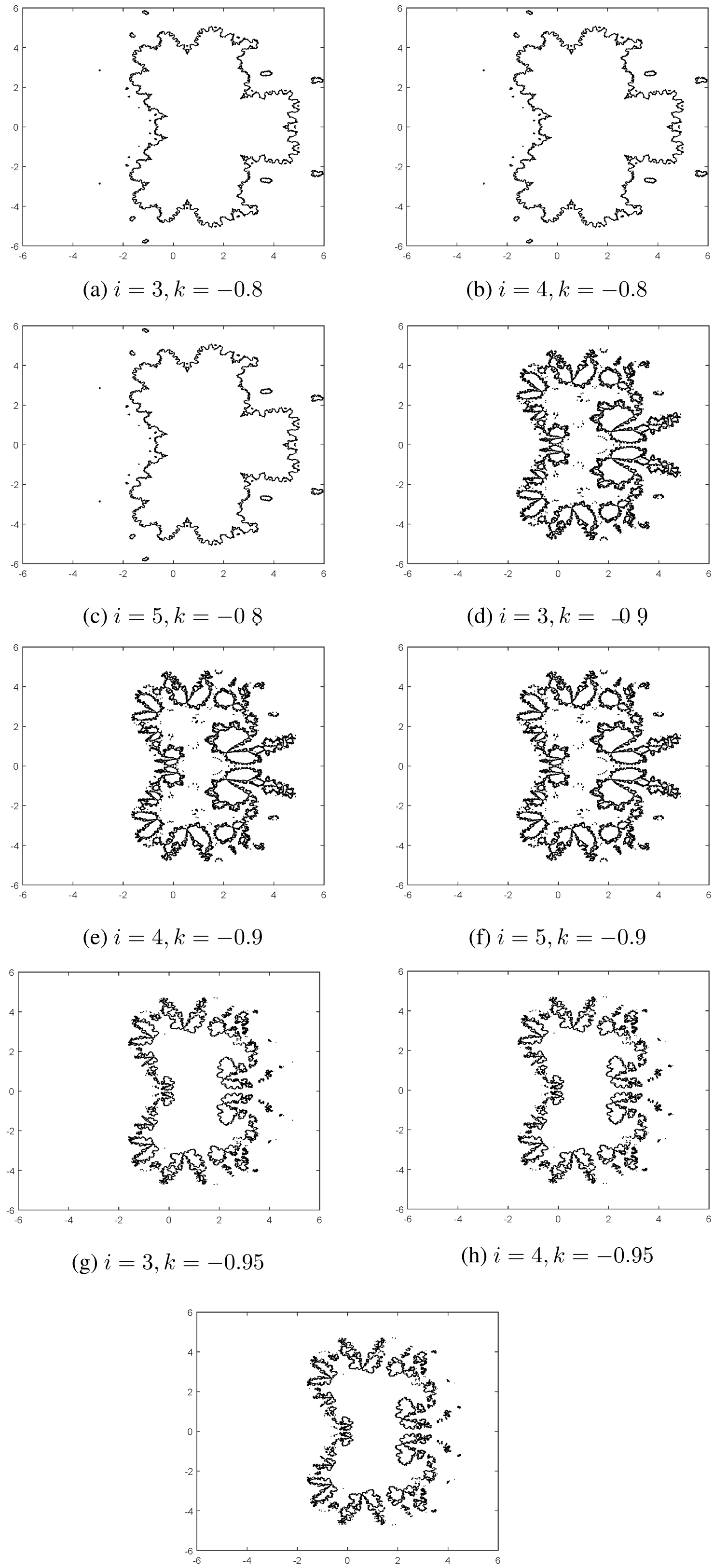
(c) $i = 5$

FIGURE 6. Julia sets of discrete fractal system (12) with different values of i when λ takes 6.

the systems after adding controller (17) are

$$\begin{aligned} z_3(n+1) &= f_6^3(z_3(n)) - \frac{k}{k+1}[f_6^3(z_3(n))] \\ &\quad + \frac{1}{k+1}[f_6^4(z_4(n)) - f_6^3(z_3(n))], \\ z_4(n+1) &= f_6^4(z_4(n)) - \frac{k}{k+1}[f_6^4(z_4(n))] \\ &\quad + \frac{1}{k+1}[f_6^3(z_3(n)) - f_6^4(z_4(n)) \\ &\quad + f_6^5(z_5(n)) - f_6^4(z_4(n))], \\ z_5(n+1) &= f_6^5(z_5(n)) - \frac{k}{k+1}[f_6^5(z_5(n))] \\ &\quad + \frac{1}{k+1}[f_6^4(z_4(n)) - f_6^5(z_5(n))]. \end{aligned}$$

When the parameter k takes different values, the Julia set corresponding to each system is shown in Fig.9. It can be



(a) $i = 3, k = -0.8$ (b) $i = 4, k = -0.8$

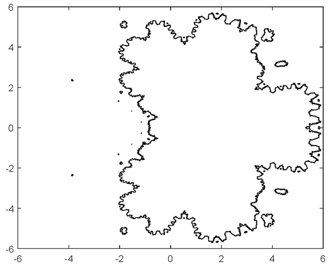
(c) $i = 5, k = -0.8$ (d) $i = 3, k = -0.9$

(e) $i = 4, k = -0.9$ (f) $i = 5, k = -0.9$

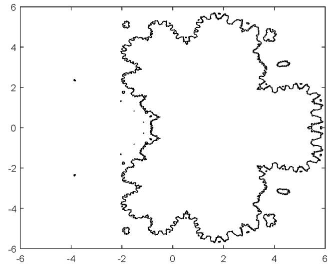
(g) $i = 3, k = -0.95$ (h) $i = 4, k = -0.95$

(i) $i = 5, k = -0.95$

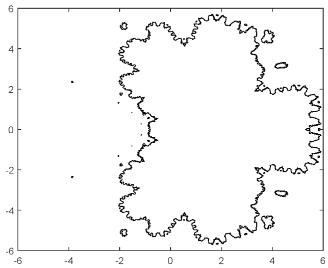
FIGURE 7. Julia sets of discrete fractal system (13) with different values of *k* when *l* takes 0.2.



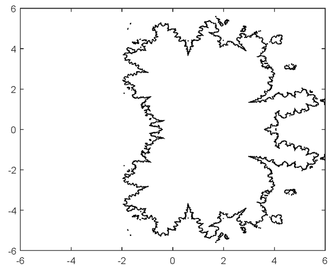
(a) $i = 3, k = -0.9$



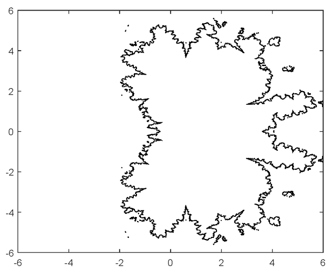
(b) $i = 4, k = -0.9$



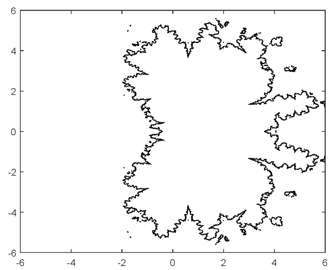
(c) $i = 5, k = -0.9$



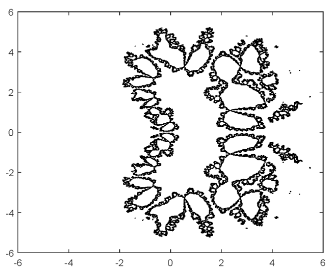
(d) $i = 3, k = -0\ 95$



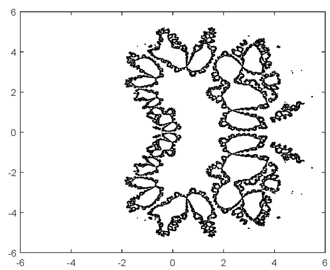
(e) $i = 4, k = -0.95$



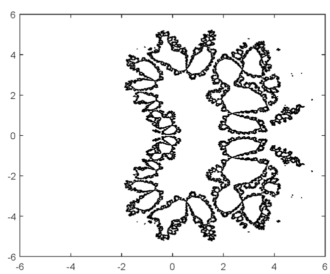
(f) $i = 5, k = -0.95$



(g) $i = 3, k = -0.99$

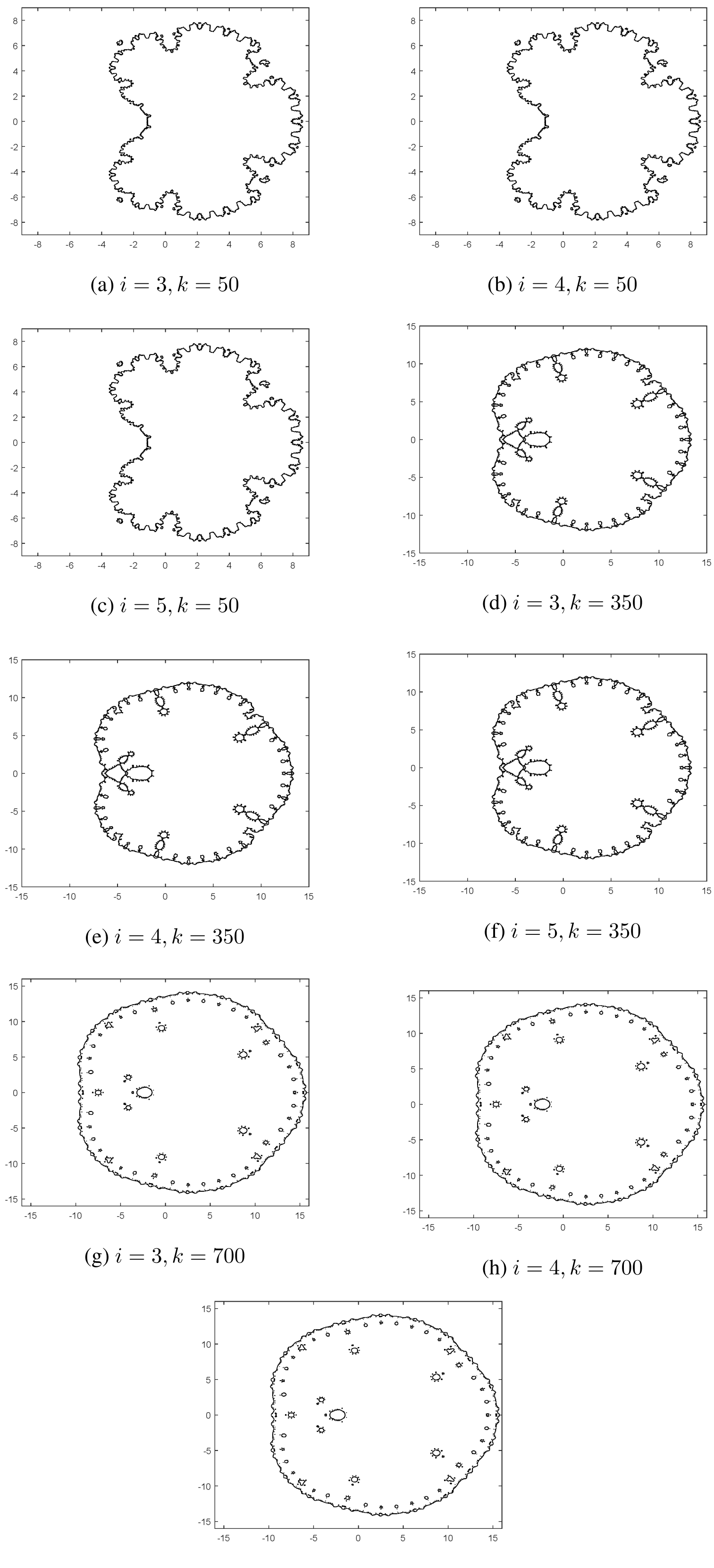


(h) $i = 4, k = -0.99$



(i) $i = 5, k = -0.99$

FIGURE 8. **Julia sets of discrete fractal system (13) with different values of k when l takes 0.1.**



(a) $i = 3, k = 50$
(b) $i = 4, k = 50$
(c) $i = 5, k = 50$
(d) $i = 3, k = 350$
(e) $i = 4, k = 350$
(f) $i = 5, k = 350$
(g) $i = 3, k = 700$
(h) $i = 4, k = 700$
(i) $i = 5, k = 700$

FIGURE 9. Julia sets of discrete fractal system (13) with different values of *k*.

seen from Fig.9 that the Julia set corresponding to each agent consistently tends to a certain set as the absolute value of k becomes larger and larger.

V. CONCLUSION

In this paper, the consensus problem of Julia sets of λ-state Potts models on diamond-like hierarchical lattice is studied. Two types of consensus problems are considered, one is with a leader, and the other is with no leaders. Two different control protocols are designed for Julia sets of λ-state Potts models with a leader and with no leaders, respectively. For consensus of Julia sets with a leader, the Julia sets corresponding to followers after control consistently tend to the Julia set of the leader in the end. For consensus of Julia sets with no leaders, the Julia set corresponding to each agent consistently tends to a certain set.

Although only the consensus problem of Julia sets of λ-state Potts models on diamond-like hierarchical lattice is discussed, the methods employed in this paper are also applicable to models of rational functions in general. In the future, we will try to study control and consensus of Julia sets generated by other types of iterations, such as Mann iteration scheme [18], Halpern method [16], viscosity approximation method [16] and so on.

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