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RESEARCH ARTICLE

Robust Iterative Learning Control for 2-D Linear Nonrepetitive Discrete Systems With Iteration-Dependent Trajectory

KAI WAN [1], HENG XIE [1], AND QING-YUAN XU [2]
[1] School of Electronic Information and Electrical Engineering, Huizhou University, Huizhou 516007, China
[2] School of Electronic and Information, Guangdong Polytechnic Normal University, Guangzhou 510665, China

Corresponding author: Kai Wan (wankai606815@163.com)

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ABSTRACT In the existing robust iterative learning control (ILC) for 2-D discrete systems, they typicallly require to satisfy a core hypothesis that the strict repetitiveness of tracking reference trajectory and system model should be satisfied. This paper first investigates the robustness and convergence of a P-type ILC law and a high-order ILC law for 2-D linear nonrepetitive discrete systems (LNDS) with arbitrarily bounded reference trajectory and iteration-dependent reference trajectory described by a high order internal model (HOIM) operator in iteration domain, respectively. It is theoretically proved by using the 2-D linear nonrepetitive inequalities that the ILC tracking error and the control input robustly converge to a bounded range, the bound of which depends continuously on the bounds of all the nonrepetitive uncertainties. If these uncertainties are progressively convergent along the iteration domain, a precise tracking on the 2-D reference trajectory can be achieved. Two illustrative examples are provided to demonstrate the validity of the presented ILC law. Additionally, some comparative result on the practical dynamical processes is given.

INDEX TERMS 2-D linear discrete nonrepetitive systems (LNDS), 2-D linear nonrepetitive inequalities, iteration-dependent trajectory, robust iterative learning control.

I. INTRODUCTION

Iterative learning control (ILC), as an effective and unsupervised control method, has been extensively used in addressing the finite-time-based trajectory tracking problem for systems with nonrepetitive uncertainties, such as linear systems [1], [2], [3], [4], stochastic systems [5], [6], multi-agent systems [7], [8], and nonrepetitive systems [9], [10], [11]. In [9] and [10], the ILC tracking problem for 1-D nonrepetitive discrete systems with nonrepetitive uncertainties in initial states, external disturbances, plant model matrices and desired reference trajectories was investigated. With an extended contraction mapping approach, robustly convergent results have been established. These aforementioned ILC results are designed for 1-D discrete systems.

However, ILC results on 2-D discrete systems with nonrepetitive uncertainties are rarely reported. Among the few exceptions are [12], [13], [14], [15], [16], and [17], which mainly investigate the nonrepetitive uncertainties in boundary states, reference trajectory, trial lengths, and external disturbances. In [12] and [13], the robust ILC tracking on nonrepetitive uncertainties from reference trajectories described by a known high-order internal model (HOIM) operator and boundary states was investigated. By using the HOIM-based inequalities theory, the ultimate ILC tracking error can only converge to a bounded range. In [14], a P-type ILC law with compensation technique is presented to deal with the robust tracking for 2-D repetitive discrete systems with

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nonrepetitive uncertainties arising from reference trajectory, boundary states, and disturbances. With 2-D linear equalities theory, the final ILC tracking error is robustly convergent to be a bounded region. To this end, an adaptive ILC algorithm is proposed in [15] to achieve the precise tracking for 2-D repetitive discrete systems with nonrepetitive uncertainties from boundary states and reference trajectory. Afterwards, adaptive ILC algorithm applied to 2-D linear and repetitive discrete systems are studied in [16] and [17]. Unfortunately, in [15], [16], and [17], they require that the input matrix is assumed to be positive-definite and 2-D discrete systems considered is repetitive.

In practical applications, there have some 2-D linear nonrepetitive systems (LNDS) required to execute tracking control tasks with a repetitive mode, such as 2-D distributed grid sensor networks [18], target echoes collected by a radar [19], thermal process [20], and 2-D multi-functional robotic manipulators [21]. In the 2-D distributed grid sensor networks, the vehicle path under surveillance equipped with regularly spaced sensor nodes, the sensor number is denoted by i. The sensor node signals are sampled in time for discrete processing, and j denotes the sample number, which is a 2-D discrete spatio-temporal system. Concretely, the 2-D multi-functional robotic manipulator needs to pick up and put down different loads for each repetitive operation such that the parameter values of the 2-D robot manipulator may be affected due to various loads at different iterations. Also, to control the temperature of heater exchanger in 2-D thermal process is to reach the desired temperature by repetitively injecting the liquid in the inlet. Influenced by outside temperature at each repetitive operation, the parameter values of heater exchanger may be changed. Therefore, it is essential and meaningful to investigate the robust ILC techniques for 2-D LNDS with iteration-dependent reference trajectory.

This paper aims to investigate the robust ILC tracking problem of a P-type ILC law and a high-order ILC law for 2-D LNDS with arbitrarily bounded reference trajectory and HOIM-based reference trajectory, respectively. With the help of 2-D linear nonrepetitive inequalities approach, it is theoretically proved that the bounds of ILC tracking error and the control input are shown to depend continuously on the bounds of nonrepetitive uncertainties. Particularly, if all the uncertainties converge with increasing iteration, the actual tracking output can precisely track 2-D reference trajectory. The main contributions of this paper are summarized in the following.

1) Compared with the existing ILC algorithms for 2-D repetitive discrete systems with iteration-dependent reference trajectory, this paper first investigates the ILC designs to 2-D LNDS with iteration-dependent reference trajectory.

2) Different from the adaptive ILC algorithm in [15], [16], and [17], the proposed P-type ILC law and high-order ILC law in this paper have no restrictions on the numbers of control inputs and outputs.

3) 2-D linear nonrepetitive inequalities is first proposed to analysis the robust ILC tracking for 2-D LNDS. Additionally, it is verified that the tracking performance of the high-order ILC law outperforms the lower-order ILC law in dealing with the tracking problem on HOIM-based reference trajectory.

The remainder of this paper is arranged as follows: Problem statement for 2-D LNDS is introduced in section II. Section III-IV show robust analysis of the P-type ILC law (3) and the high-order ILC law (39) under Assumption 1. Two simulation examples are provided in section V. Finally, section VI gives some conclusions of this paper.

II. PROBLEM STATEMENT

Consider the ILC issue for 2-D linear nonrepetitive discrete systems (LNDS)[12], [20], operating over a fixed region $i = 0, 1, 2, \cdots, T_1 - 1$ and $j = 0, 1, 2, \cdots, T_2 - 1$:

$$\begin{aligned} x_k(i+1, j+1) &= A_{1,k}(i+1, j)x_k(i+1, j) + A_{2,k}(i, j)x_k(i, j) \\ &\quad + A_{3,k}(i, j+1)x_k(i, j+1) + B_k(i, j)u_k(i, j), \quad (1) \\ y_k(i, j) &= C_k(i, j)x_k(i, j), \quad (2) \end{aligned}$$

where i and j are discrete indexes along the horizontal direction and the vertical direction; $k = 0, 1, 2, \cdots$ represents the iteration number; $u_k(i, j) \in \mathbb{R}$, $x_k(i, j) \in \mathbb{R}^n$, and $y_k(i, j) \in \mathbb{R}$ denote the control input, system state, and system output, respectively; $A_{1,k}(i+1, j) \in \mathbb{R}^{n\times n}$, $A_{2,k}(i, j) \in \mathbb{R}^{n\times n}$, $A_{3,k}(i, j+1) \in \mathbb{R}^{n\times n}$, $B_k(i, j) \in \mathbb{R}^{n\times 1}$, and $C_k(i, j) \in \mathbb{R}^{1\times n}$ are nonrepetitive parameter matrices. In practical 2-D LNDS (1)-(2), i.e., chemical reactors, heater exchangers, and pipe furnaces, the independent indexes i and j usually represent space locations and time instants, respectively [25].

A. ROBUST ILC OBJECTIVE

For 2-D LNDS (1)-(2), the objective of robust ILC is to design an updating learning algorithm on $u_k(i, j)$, such that the ILC tracking error and the control input can robustly converge to a bounded range, the bound of which depends on the boundedness parameters on all the uncertainties, i.e.,

$$\limsup_{k\to+\infty} \|e_k(i, j)\| \le b_e, \ \limsup_{k\to+\infty} \|u_k(i, j)\| \le b_u, \quad (3)$$

where $b_e > 0$ and $b_u > 0$ are two constants. The used norm $\|\cdot\|$ of this paper is Euclidean norm. When all the nonrepetitive uncertainties converges along the iteration direction, the system output tracks the desired reference trajectory perfectly, i.e.,

$$\lim_{k\to+\infty} [y_k(i, j) - y_{d,k}(i, j)] = 0. \quad (4)$$

To investigate the robust ILC problem for 2-D LNDS (1)-(2), the following Assumptions 1-3 and Lemmas 1-2 are made correspondingly.

Assumption 1: For the 2-D LNDS (1)-(2), let

$$\begin{aligned} &\|y_{d,k}(i, j)\| \le b_d, \ i = 0, 1, \cdots, T_1, \ j = 0, 1, \cdots, T_2, \\ &\|A_{1,k}(i+1, j)\| \le b_{A1}, i = 0, 1, \cdots, T_1 - 1, \\ &\qquad j = 0, 1, \cdots, T_2 - 1, \\ &\|A_{2,k}(i, j)\| \le b_{A2}, i = 0, 1, \cdots, T_1 - 1, \end{aligned}$$

$$
\begin{aligned}
& j=0,1,\cdots,T_2-1,\\
\|A_{3,k}(i,j+1)\| &\le b_{A3}, i=0,1,\cdots,T_1-1,\\
& j=0,1,\cdots,T_2-1,\\
\|B_k(i,j)\| &\le b_B, i=0,1,\cdots,T_1-1,\\
& j=0,1,\cdots,T_2-1,\\
\|C_k(i,j)\| &\le b_C, i=0,1,\cdots,T_1,\\
& j=0,1,\cdots,T_2,\\
\|x_k(i,0)\| &\le b_{x_{i0}},\ i=0,1,\cdots,T_1-1,\\
\|x_k(0,j)\| &\le b_{x_{0j}},\ j=0,1,\cdots,T_2,
\end{aligned}
$$

where $b_d>0$, $b_{A1}>0$, $b_{A2}>0$, $b_{A3}>0$, $b_B>0$, $b_C>0$, $b_{x_{i0}}>0$, and $b_{x_{0j}}>0$ are some finite bounds.

Remark 1: In the ILC field of 2-D systems, Assumption 1 is essential and reasonable on the boundedness of 2-D reference trajectory, parameter matrices, and boundary states in iteration domain, which is key relaxation on the strictly repetitive requirement provided in traditional ILC for 2-D systems.

Assumption 2: For any given 2-D reference trajectory $y_d(i,j)$, under boundary states $x_d(0,j)=x_0(0,j)$ and $x_d(i,0)=x_0(i,0)$, there exists a unique input $u_d(i,j)$, $i=0,1,2,\cdots,T_1-1, j=0,1,2,\cdots,T_2-1$ to make

$$
\begin{aligned}
& x_d(i+1,j+1)\\
&\quad = A_1(i+1,j)x_d(i+1,j)+A_2(i,j)x_d(i,j)\\
&\qquad +A_3(i,j+1)x_d(i,j+1)+B(i,j)u_d(i,j),\\
& y_d(i,j)=C(i,j)x_d(i,j),
\end{aligned}
$$

where $x_0(0,j)$ and $x_0(i,0)$ are fixed functions with respective to j and i.

Remark 2: Assumption 2 is a reasonable and basic assumption that it can guarantee our tracking task be achievable, which is popular in [22]. However, it is worth pointing out that Assumption 2 may be difficult to achieve in some cases, because it is usually hard to determine the unique desired input $u_d(i,j)$ for the 2-D reference trajectory. To avoid using the Assumption 2, an alternative analysis approach for ILC is to directly consider the tracking errors $e_k(i+1,j+1)$ and $e_{k+1}(i+1,j+1)$ in [9] and [24].

Assumption 3: Let the reference trajectory $y_{d,k}(i,j)$, system parameters $A_{1,k}(i+1,j)$, $A_{2,k}(i,j)$, $A_{3,k}(i,j+1)$, $B_k(i,j)$ and $C_k(i,j)$, boundary states $x_k(i,0)$ and $x_k(0,j)$ be the progressively convergent, i.e.,

$$
\begin{aligned}
&\lim_{k\to+\infty} y_{d,k}(i,j)=y_d(i,j),\\
&\qquad i=0,1,\cdots,T_1,\ j=0,1,\cdots,T_2,\\
&\lim_{k\to+\infty} A_{1,k}(i+1,j)=A_1(i+1,j),\quad i=0,1,\cdots,T_1-1,\\
&\qquad j=0,1,\cdots,T_2-1,\\
&\lim_{k\to+\infty} A_{2,k}(i,j)=A_2(i,j), i=0,1,\cdots,T_1-1,\\
&\qquad j=0,1,\cdots,T_2-1,\\
&\lim_{k\to+\infty} A_{3,k}(i,j+1)=A_3(i,j+1),\quad i=0,1,\cdots,T_1-1,\\
&\qquad j=0,1,\cdots,T_2-1,\\
&\lim_{k\to+\infty} B_k(i,j)=B(i,j),\quad i=0,1,\cdots,T_1-1,\\
&\qquad j=0,1,\cdots,T_2-1,\\
&\lim_{k\to+\infty} C_k(i,j)=C(i,j),\\
&\qquad i=0,1,\cdots,T_1,\ j=0,1,\cdots,T_2,\\
&\lim_{k\to+\infty} x_k(i,0)=x_0(i,0),\ i=0,1,\cdots,T_1-1,\\
&\lim_{k\to+\infty} x_k(0,j)=x_0(0,j),\ j=0,1,\cdots,T_2,
\end{aligned}
$$

for some iteration-invariant matrices/vectors $y_d(i,j)$, $A_1(i+1,j)$, $A_2(i,j)$, $A_3(i,j+1)$, $B(i,j)$, and $C(i,j)$.

Remark 3: Assumption 3 is an extension to the progressively convergent condition deriving from ILC for 1-D discrete systems and has been used in ILC for 2-D repetitive systems [12], [14]. It is worth noting that Assumption 3 is not essential in achieving the robust ILC objective (3) but necessary in ensuring the perfect ILC tracking on 2-D reference trajectory, which is the same as those for robust ILC of 2-D repetitive systems.

Lemma 1: Let nonnegative real sequences $\{d_k\}$ and $\{w_k\}$ satisfy the following inequality:

$$
d_{k+1}\le\sum_{h=0}^{N-1}\rho_{k-h}d_{k-h}+w_k,\quad k=N-1,N,N+1,\ldots,
$$

with $\rho_{k-h}\ge 0$. If $\rho_k,\rho_{k-1},\ldots,\rho_{k-N+1}$ make $\rho=\sum_{h=0}^{N-1}\rho_{k-h}<1$, then, the condition $\limsup_{k\to+\infty} w_k\le w_\infty$ implies

$$
\limsup_{k\to+\infty} d_k\le\frac{w_\infty}{1-\rho}.
$$

The proof of Lemma 1 is similar with that of Lemma 2.1 in [23], and is thus omitted.

Lemma 2: For the following 2-D linear nonrepetitive inequalities over $j=0,1,2,\cdots,T_2-1$:

$$
\begin{aligned}
\eta_k(j+1) &\le \sum_{h=0}^{N_1-1}\mathbb{D}_{1,k-h}(j)\eta_{k-h}(j)+\sum_{h=0}^{N_2-1}\mathbb{D}_{2,k-h}(j)\\
&\quad\times\psi_{k-h}(j)+\alpha_k(j),\\
\psi_{k+1}(j) &\le \sum_{h=0}^{N_1-1}\mathbb{D}_{3,k-h}(j)\eta_{k-h}(j)+\sum_{h=0}^{N_2-1}\mathbb{D}_{4,k-h}(j)\\
&\quad\times\psi_{k-h}(j)+\beta_k(j),
\end{aligned}
$$

where $\mathbb{D}_{1,k-h}(j)$, $\mathbb{D}_{2,k-h}(j)$, $\mathbb{D}_{3,k-h}(j)$ and $\mathbb{D}_{4,k-h}(j)$ are nonnegative and bounded functions. $\alpha_k(j)$ and $\beta_k(j)$ are nonnegative and bounded functions, i.e., $\|\alpha_k(j)\|\le b_\alpha$, $\|\beta_k(j)\|\le b_\beta$. Let boundary states $\eta_k(0)$ and $\psi_k(j)$, $k=0,1,2,\cdots,N_2-1$ be bounded, if $\sum_{h=0}^{N_2-1}\mathbb{D}_{4,k-h}(j)<1$ is satisfied, then, we have

$$
\begin{aligned}
&\limsup_{k\to+\infty}\|\eta_k(j)\|\le\beta_\eta,\quad j=1,2,\cdots,T_2,\\
&\limsup_{k\to+\infty}\|\psi_k(j)\|\le\beta_\psi,\quad j=0,1,2,\cdots,T_2-1.
\end{aligned}
$$

In particular, as $\lim_{k\to+\infty}\alpha_k(j)=\lim_{k\to+\infty}\beta_k(j)=0$, $j=0,1,2,\cdots,T_2-1$, and $\lim_{k\to+\infty}\eta_k(0)=0$, there is

$$
\lim_{k\to+\infty}\psi_k(j)=0,\quad j=0,1,2,\cdots,T_2-1.
$$

The proof of Lemma 2 can follow that of Lemma 3 in [12], so it is omitted.

Remark 4: Lemma 2 is a 2-D linear nonrepetitive inequalities and can be used to analysis the robustness and convergence of high-order ILC law [13] for 2-D repetitive discrete systems. In traditional ILC results for 2-D repetitive discrete systems, the used analysis approach can not be applied to the nonrepetitive case. To this end, the 2-D linear nonrepetitive inequalities approach in this paper is necessary to be developed.

In this paper, the following P-type ILC law for 2-D LNDS (1)-(2) is used for $i = 0, 1, 2, \cdots, T_1 - 1$ and $j = 0, 1, 2, \cdots, T_2 - 1$:

$$u_{k+1}(i,j) = u_k(i,j) + L_k(i,j)e_k(i+1,j+1), \tag{5}$$

where the learning gain $L_k(i,j)$ is to be designed.

III. ROBUST ANALYSIS OF THE P-TYPE ILC LAW (5) UNDER ASSUMPTION 1

In this section, we will investigate the robustness property of the ILC law (5) for 2-D LNDS (1)-(2), and the following Theorem 1 is presented.

Theorem 1: Consider the 2-D LNDS (1)-(2) under Assumption 1, and let the ILC law (5) be applied. If the learning gain $L_k(i,j)$ is chosen to make

$$|1 - L_k(i,j)C_k(i+1,j+1)B_k(i,j)| < 1, \tag{6}$$

then, the robust tracking objective (3) can be achieved.

Proof: Using the P-type ILC law (5) and considering (1)-(2), it yields

$$\begin{aligned} u_{k+1}&(i,j) - u_k(i,j) \\ &= L_k(i,j)\big[y_{d,k}(i+1,j+1) - C_k(i+1,j+1) \\ &\quad \times x_k(i+1,j+1)\big] \\ &= L_k(i,j)y_{d,k}(i+1,j+1) - L_k(i,j)C_k(i+1,j+1) \\ &\quad \times\big[A_{1,k}(i+1,j)x_k(i+1,j) + A_{2,k}(i,j)x_k(i,j) \\ &\quad + A_{3,k}(i,j+1)x_k(i,j+1) + B_k(i,j)u_k(i,j)\big], \end{aligned} \tag{7}$$

where $i = 0, 1, 2, \cdots, T_1 - 1$ and $j = 0, 1, 2, \cdots, T_2 - 1$. Taking the arrangement on (7), it becomes

$$\begin{aligned} u_{k+1}(i,j) = [1 - L_k(i,j)C_k(i+1,j+1)B_k(i,j)]u_k(i,j) \\ + \phi_k(i,j), \end{aligned} \tag{8}$$

where

$$\begin{aligned} \phi_k&(i,j) \\ &= L_k(i,j)y_{d,k}(i+1,j+1) - L_k(i,j)C_k(i+1,j+1) \\ &\quad \times\big[A_{1,k}(i+1,j)x_k(i+1,j) + A_{2,k}(i,j)x_k(i,j) \\ &\quad + A_{3,k}(i,j+1)x_k(i,j+1)\big]. \end{aligned} \tag{9}$$

Taking the norm on both sides of (8), we obtain

$$\begin{aligned} \|u_{k+1}(i,j)\| \le |1 - L_k(i,j)C_k(i+1,j+1)B_k(i,j)| \\ \times\|u_k(i,j)\| + \|\phi_k(i,j)\|. \end{aligned} \tag{10}$$

With regard to $\phi_k(i,j)$ given in (9), from Assumption 1, there is

$$\begin{aligned} \|\phi_k&(i,j)\| \\ &\le b_d\|L_k(i,j)\| + b_C\|L_k(i,j)\|\big(b_{A1}\|x_k(i+1,j)\| \\ &\quad + b_{A2}\|x_k(i,j)\| + b_{A3}\|x_k(i,j+1)\|\big). \end{aligned} \tag{11}$$

Substituting (11) into (10), it generates

$$\begin{aligned} \|u_{k+1}&(i,j)\| \\ &\le |1 - L_k(i,j)C_k(i+1,j+1)B_k(i,j)|\|u_k(i,j)\| \\ &\quad + b_d\|L_k(i,j)\| + b_C\|L_k(i,j)\|\big(b_{A1}\|x_k(i+1,j)\| \\ &\quad + b_{A2}\|x_k(i,j)\| + b_{A3}\|x_k(i,j+1)\|\big). \end{aligned} \tag{12}$$

Let

$$U_k(j) = \big[\|u_k(0,j)\|, \|u_k(1,j)\|, \cdots, \|u_k(T_1-1,j)\|\big]^T, \tag{13}$$

$$X_k(j) = \big[\|x_k(1,j)\|, \|x_k(2,j)\|, \cdots, \|x_k(T_1,j)\|\big]^T. \tag{14}$$

Then, (12) can be rewritten as

$$\begin{aligned} U_{k+1}&(j) \\ &\le \Phi_{1,k}(j)U_k(j) + \Phi_{2,k}(j)X_k(j+1) + \Phi_{3,k}(j)X_k(j) \\ &\quad + \Phi_{4,k}(j)\|x_k(0,j+1)\| + \Phi_{5,k}(j)\|x_k(0,j)\| \\ &\quad + \Phi_{6,k}(j), \end{aligned} \tag{15}$$

where

$$\Phi_{1,k}(j) = \begin{bmatrix} a_k^0(0,j) & 0 & 0 & \cdots & 0 \\ 0 & a_k^1(0,j) & 0 & \cdots & 0 \\ 0 & 0 & a_k^2(0,j) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & a_k^0(T_1-1,j) \end{bmatrix},$$

$$a_k^0(i,j) = |1 - L_k(i,j)C_k(i+1,j+1)B_k(i,j)|,$$

$$\Phi_{2,k}(j) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ a_k^1(1,j) & 0 & 0 & \cdots & 0 \\ 0 & a_k^1(2,j) & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_k^1(T_1-1,j) & 0 \end{bmatrix},$$

$$a_k^1(i,j) = b_{A3}b_C\|L_k(i,j)\|,$$

$$\Phi_{3,k}(j) = \begin{bmatrix} a_k^2(0,j) & 0 & 0 \\ a_k^3(1,j) & a_k^2(1,j) & 0 \\ 0 & a_k^3(2,j) & a_k^2(2,j) \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \end{bmatrix}$$

$$\begin{bmatrix} \cdots & 0 \\ \cdots & 0 \\ \ddots & \vdots \\ \ddots & 0 \\ a_k^3(T_1-1,j) & a_k^2(T_1-1,j) \end{bmatrix},$$

$a_k^2(i,j) = b_C b_{A1}\|L_k(i,j)\|,\ a_k^3(i,j) = b_C b_{A2}\|L_k(i,j)\|,$

$$\Phi_{4,k}(j) = \begin{bmatrix} b_C b_{A3}\|L_k(0,j)\| \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\Phi_{5,k}(j) = \begin{bmatrix} b_C b_{A2}\|L_k(0,j)\| \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\Phi_{6,k}(j) = \begin{bmatrix} b_L b_d & 0 & 0 & \cdots & 0 \\ 0 & b_L b_d & 0 & \cdots & 0 \\ 0 & 0 & b_L b_d & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & b_L b_d \end{bmatrix}. \tag{16}$$

On the other hand, taking the norm on both sides of (1), there is

$$\begin{aligned} &\|x_k(i+1,j+1)\| \\ &\quad \le b_{A1}\|x_k(i+1,j)\| + b_{A2}\|x_k(i,j)\| + b_{A3}\|x_k(i,j+1)\| \\ &\qquad +\|B_k(i,j)\|\|u_k(i,j)\|. \end{aligned} \tag{17}$$

From (13)-(14), (17) can be formulated as

$$\begin{aligned} &\Psi_{1,k}(j)X_k(j+1) \\ &\quad \le \Psi_{2,k}(j)X_k(j) + \Psi_{3,k}(j)U_k(j) + \Psi_{4,k}(j)\|x_k(0,j+1)\| \\ &\qquad +\Psi_{5,k}(j)\|x_k(0,j)\|, \end{aligned} \tag{18}$$

where

$$\Psi_{1,k}(j) = \begin{bmatrix} I_n & 0 & 0 & \cdots & 0 \\ -b_{A3} & I_n & 0 & \cdots & 0 \\ 0 & -b_{A3} & I_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -b_{A3} & I_n \end{bmatrix},$$

$$\Psi_{2,k}(j) = \begin{bmatrix} b_{A1} & 0 & 0 & \cdots & 0 \\ b_{A2} & b_{A1} & 0 & \cdots & 0 \\ 0 & b_{A2} & b_{A1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & b_{A2} & b_{A1} \end{bmatrix},$$

$$\Psi_{3,k}(j) = \begin{bmatrix} b_B & 0 & 0 & \cdots & 0 \\ 0 & b_B & 0 & \cdots & 0 \\ 0 & 0 & b_B & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & b_B \end{bmatrix},$$

$$\Psi_{4,k}(j) = \begin{bmatrix} b_{A3} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ \Psi_{5,k}(j) = \begin{bmatrix} b_{A2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since $\Psi_{1,k}(j)$ is a nonsingular matrix, multiplying by $\Psi_{1,k}^{-1}(j)$ on both sides of (18), it obtains

$$\begin{aligned} X_k(j+1) \le\ & \Psi_{1,k}^{-1}(j)\Psi_{2,k}(j)X_k(j) + \Psi_{1,k}^{-1}(j)\Psi_{3,k}(j)U_k(j) \\ & +\Psi_{1,k}^{-1}(j)\Psi_{4,k}(j)\|x_k(0,j+1)\| \\ & +\Psi_{1,k}^{-1}(j)\Psi_{5,k}(j)\|x_k(0,j)\|. \end{aligned} \tag{19}$$

Inserting (19) into (15), it yields

$$\begin{aligned} &U_{k+1}(j) \\ &\quad \le \left(\Phi_{1,k}(j) + \Phi_{2,k}(j)\Psi_{1,k}^{-1}(j)\Psi_{3,k}(j)\right)U_k(j) \\ &\qquad +\left(\Phi_{3,k}(j) + \Phi_{2,k}(j)\Psi_{1,k}^{-1}(j)\Psi_{2,k}(j)\right)X_k(j) \\ &\qquad +\left(\Phi_{4,k}(j) + \Phi_{2,k}(j)\Psi_{1,k}^{-1}(j)\Psi_{4,k}(j)\right)\|x_k(0,j+1)\| \\ &\qquad +\left(\Phi_{5,k}(j) + \Phi_{2,k}(j)\Psi_{1,k}^{-1}(j)\Psi_{5,k}(j)\right)\|x_k(0,j)\| \\ &\qquad +\Phi_{6,k}(j). \end{aligned} \tag{20}$$

For (19) and (20), according to Lemma 2, if $\|\Phi_{1,k}(j) + \Phi_{2,k}(j)\Psi_{1,k}^{-1}(j)\Psi_{3,k}(j)\| < 1$ (equivalently, $|1 - L_k(i,j)C_k(i+1,j+1)B_k(i,j)| < 1$), we have

$$\begin{cases} \limsup_{k\to+\infty} X_k(j) \le \beta_X, & j = 1,2,\cdots,T_2, \\ \limsup_{k\to+\infty} U_k(j) \le \beta_U, & j = 0,1,2,\cdots,T_2-1, \end{cases} \tag{21}$$

where $\beta_X > 0$ and $\beta_U > 0$ are two positive constants. From (13) and (14), we further obtain

$$\limsup_{k\to+\infty} \|x_k(i,j)\| \le \beta_X, \tag{22}$$

$$\limsup_{k\to+\infty} \|u_k(i,j)\| \le \beta_U. \tag{23}$$

Using Assumption 1 and (22), we have

$$\begin{aligned} &\limsup_{k\to+\infty} \|e_k(i,j)\| \\ &\quad = \limsup_{k\to+\infty} \left\|y_{d,k}(i,j) - C_k(i,j)x_k(i,j)\right\| \\ &\quad \le b_d + b_C \limsup_{k\to+\infty} \|x_k(i,j)\| \le b_d + b_C\beta_X \triangleq \beta_e, \end{aligned}$$

where β_e is associated with the boundedness parameters b_d, b_{A1}, b_{A2}, b_{A3}, b_C, b_B, $b_{x_{i0}}$, and $b_{x_{0j}}$ presented in Assumption 1. The proof of Theorem 1 is completed.

From Theorem 1, it provides the boundedness result on ILC tracking error and the control input for 2-D LNDS

(1)-(2) under Assumption 1. While a progressively convergent condition on the reference trajectory, system parameters, and boundary states is imposed on the 2-D LNDS (1)-(2), a perfectly convergent result is given. There is the following Theorem 2.

Theorem 2: Consider the 2-D LNDS (1)-(2) under Assumptions 1-3, and let the ILC law (5) be used. If the learning gain $L_k(i,j)$ is chosen to make (6) be satisfied, then, the perfect tracking objective (4) can be accomplished.

Proof: For $i = 0, 1, \cdots, T_1-1$ and $j = 0, 1, \cdots, T_2-1$, define

$$\delta u_k(i,j) = u_d(i,j) - u_k(i,j), \tag{24}$$

and for $i = 1, 2, \cdots, T_1$ and $j = 1, 2, \cdots, T_2$,

$$\delta x_k(i,j) = x_d(i,j) - x_k(i,j). \tag{25}$$

Using (25) and inserting (1) with Assumption 2, there is

$$\begin{aligned}
&\|\delta x_k(i+1,j+1)\| \\
&\quad = \|x_d(i+1,j+1) - x_k(i+1,j+1)\| \\
&\quad = \|A_1(i+1,j)x_d(i+1,j) + A_2(i,j)x_d(i,j) \\
&\qquad + A_3(i,j+1)x_d(i,j+1) + B(i,j)u_d(i,j) \\
&\qquad - A_{1,k}(i+1,j)x_k(i+1,j) - A_{2,k}(i,j)x_k(i,j) \\
&\qquad - A_{3,k}(i,j+1)x_k(i,j+1) - B_k(i,j)u_k(i,j)\| \\
&\quad \le \|A_{1,k}(i+1,j)\|\|\delta x_k(i+1,j)\| + \|A_{1,k}(i+1,j) \\
&\qquad - A_1(i+1,j)\|\|x_d(i+1,j)\| + \|A_{2,k}(i,j)\|\|\delta x_k(i,j)\| \\
&\qquad + \|A_{2,k}(i,j) - A_2(i,j)\|\|x_d(i,j)\| + \|A_{3,k}(i,j+1)\| \\
&\qquad \times \|\delta x_k(i,j+1)\| + \|A_{3,k}(i,j+1) - A_3(i,j+1)\| \\
&\qquad \times \|x_d(i,j+1)\| + \|B_k(i,j)\|\|\delta u_k(i,j)\| \\
&\qquad + \|B(i,j) - B_k(i,j)\|\|u_d(i,j)\|,
\end{aligned} \tag{26}$$

where $\|x_d(i,j)\| \le b_{xd}$ and $\|u_d(i,j)\| \le b_{ud}$.

Subsequently, applying (5) and considering (1), we deduce

$$\begin{aligned}
&\delta u_{k+1}(i,j) \\
&\quad = \delta u_k(i,j) - L_k(i,j)\big[y_{d,k}(i+1,j+1) - y_d(i+1,j+1) \\
&\qquad + C(i+1,j+1)x_d(i+1,j+1) \\
&\qquad - C_k(i+1,j+1)x_k(i+1,j+1)\big] \\
&\quad = \delta u_k(i,j) - L_k(i,j)[y_{d,k}(i+1,j+1) - y_d(i+1,j+1)] \\
&\qquad - L_k(i,j)[C(i+1,j+1) - C_k(i+1,j+1)] \\
&\qquad \times x_d(i+1,j+1) - L_k(i,j)C_k(i+1,j+1) \\
&\qquad \times \delta x_k(i+1,j+1) \\
&\quad = \big[1 - L_k(i,j)C_k(i+1,j+1)B_k(i,j)\big]\delta u_k(i,j) \\
&\qquad + \Psi_k(i,j),
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
&\Psi_k(i,j) \\
&\quad = -L_k(i,j)[y_{d,k}(i+1,j+1) - y_d(i+1,j+1)] \\
&\qquad - L_k(i,j)[C(i+1,j+1) - C_k(i+1,j+1)] \\
&\qquad \times x_d(i+1,j+1) - L_k(i,j)C_k(i+1,j+1) \\
&\qquad \times \Big[A_1(i+1,j)x_d(i+1,j) + A_2(i,j)x_d(i,j) \\
&\qquad + A_3(i,j+1)x_d(i,j+1) - A_{1,k}(i+1,j)x_k(i+1,j) \\
&\qquad - A_{2,k}(i,j)x_k(i,j) - A_{3,k}(i,j+1)x_k(i,j+1) \\
&\qquad + [B(i,j) - B_k(i,j)]u_d(i,j)\Big].
\end{aligned}$$

Taking the norm on both sides of (27), and using (18), we have

$$\begin{aligned}
&\|\delta u_{k+1}(i,j)\| \\
&\quad \le |1 - L_k(i,j)C_k(i+1,j+1)B_k(i,j)|\|\delta u_k(i,j)\| \\
&\qquad + \|\Psi_k(i,j)\|,
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
&\|\Psi_k(i,j)\| \\
&\quad \le \|L_k(i,j)\|\big[\|y_{d,k}(i+1,j+1) - y_d(i+1,j+1)\| \\
&\qquad + b_d\|C(i+1,j+1) - C_k(i+1,j+1)\|\big] \\
&\qquad + \|L_k(i,j)\|b_C\Big[b_{A1}\|\delta x_k(i+1,j)\| \\
&\qquad + \|A_{1,k}(i+1,j) - A_1(i+1,j)\|b_{xd} + b_{A2}\|\delta x_k(i,j)\| \\
&\qquad + \|A_{2,k}(i,j) - A_2(i,j)\|b_{xd} \\
&\qquad + b_{A3}\|\delta x_k(i,j+1)\| + \|A_{3,k}(i,j+1) - A_3(i,j+1)\| \\
&\qquad \times b_{xd} + \|B(i,j) - B_k(i,j)\|b_{ud}\Big].
\end{aligned} \tag{29}$$

Additionally, Using Assumption 1 and (1), we have

$$\begin{aligned}
&\|e_k(i,j)\| \\
&\quad = \big\|y_{d,k}(i,j) - C_k(i,j)x_k(i,j)\big\| \\
&\quad = \big\|y_{d,k}(i,j) - y_d(i,j) + C(i,j)x_d(i,j) - C_k(i,j)x_k(i,j)\big\| \\
&\quad = \|y_{d,k}(i,j) - y_d(i,j) + [C(i,j) - C_k(i,j)]x_d(i,j) \\
&\qquad + C_k(i,j)[x_d(i,j) - x_k(i,j)]\| \\
&\quad \le \|y_{d,k}(i,j) - y_d(i,j)\| + \|C(i,j) - C_k(i,j)\|b_d \\
&\qquad + b_C\|\delta x_k(i,j)\|.
\end{aligned} \tag{30}$$

Let

$$\begin{aligned}
&\delta\bar{X}_k(j) \\
&\quad = \big[\|\delta x_k(1,j)\|, \|\delta x_k(2,j)\|, \cdots, \|\delta x_k(T_1,j)\|\big]^T,
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\delta\bar{U}_k(j) \\
&\quad = \big[\|\delta u_k(0,j)\|, \|\delta u_k(1,j)\|, \cdots, \|\delta u_k(T_1-1,j)\|\big]^T.
\end{aligned} \tag{32}$$

Similar to the proof of Theorem 1, the form of (28) with (29) is similar with that of (10) with (11), and so is (26) to (12). Then, using Lemma 2, if the condition (6) is satisfied, we obtain

$$\lim_{k\to+\infty} \delta\bar{X}_k(j) = 0, \quad j = 1, 2, \cdots, T_2, \tag{33}$$

$$\lim_{k\to+\infty} \delta\bar{U}_k(j) = 0, \quad j = 0, 1, 2, \cdots, T_2-1. \tag{34}$$

From (31) and (32), it yields

$$\lim_{k\to+\infty} \|\delta x_k(i,j)\| = 0, \tag{35}$$

$$\lim_{k\to+\infty} \|\delta u_k(i,j)\| = 0. \tag{36}$$

From Assumption 3, using (30) and (35), there is

$$\lim_{k\to+\infty} \|e_k(i,j)\| = 0,\ i = 1, 2, \cdots, T_1,\ j = 1, 2, \cdots, T_2. \tag{37}$$

The proof of Theorem 2 is completed.

Remark 5: For practical 2-D LNDS (1)-(2), we may identify or estimate the nominal information on $C_k(i,j)$ and $B_k(i,j)$ to determine the learning gain $L_k(i,j)$ in the P-type ILC law (5). Let

$$B_k(i,j) = B(i,j) + \delta B_k(i,j),\ C_k(i,j) = C(i,j) + \delta C_k(i,j),$$

where $B(i,j)$ and $C(i,j)$ are the nominal system matrices, and $\delta B_k(i,j)$ and $\delta C_k(i,j)$ are bounded nonrepetitive uncertainties of them such that $\|\delta B_k(i,j)\| \le \beta_{\delta B}(i,j)$, $\|\delta C_k(i,j)\| \le \beta_{\delta C}(i,j)$. Correspondingly, we employ iteration-invariant learning gain $L_k(i,j) = L(i,j)$. The convergence condition (6) becomes as

$$|1 - L(i,j)C(i+1,j+1)B(i,j)| + \beta_{CB}(i,j)|L(i,j)| < 1,$$

where $\beta_{CB}(i,j) = \beta_{\delta B}(i,j)\|C(i+1,j+1)\| + \beta_{\delta B}(i,j)\beta_{\delta C}(i+1,j+1) + \beta_{\delta C}(i+1,j+1)\|B(i,j)\|$. The identified nominal matrices play a dominant role in implementing P-type ILC law. Therefore, we may reasonably obtain $|C(i+1,j+1)B(i,j)| > \beta_{CB}(i,j)$, which guarantees (6) with the selection of $L(i,j) = \frac{1}{C(i+1,j+1)B(i,j)}$.

Remark 6: In Theorems 1 and 2, the proposed P-type ILC law may be difficult to address the robust ILC problem of the iteration-dependent reference trajectory generated by a HOIM strategy, which is illustrated by simulation example. To this end, a high-order ILC law is designed to deal with this HOIM-based reference trajectory, which is denoted as

$$y_{d,k+1}(i,j) = \sum_{m=0}^{M} h_m y_{d,k-m}(i,j), \quad k \ge M, \tag{38}$$

where $y_{d,k}(i,j)$, $k = 0, 1, 2, \cdots, M$ are the initial reference trajectories; h_m, $m = 0, 1, 2, \cdots, M$ are the coefficients designed to describe the variation of the 2-D reference trajectory in iteration domain, such that all roots of the stable characteristic polynomial $S(z) = z^{M+1} - h_0 z^M - h_1 z^{M-1} - \cdots - h_M = 0$ lie in the unit circle except at least one simple root on the unit circle [12].

In this section, a high-order ILC law for 2-D LNDS (1)-(2) is given as follows:

$$u_{k+1}(i,j) = \sum_{m=0}^{M} h_m u_{k-m}(i,j) + \sum_{m=0}^{M} L_{m,k}(i,j) e_{k-m}(i+1,j+1), \tag{39}$$

where $L_{m,k}(i,j)$, $m = 0, 1, 2, \cdots, M$ are the learning gains to be designed.

IV. ROBUST ANALYSIS OF THE HIGH-ORDER ILC LAW (39) UNDER ASSUMPTION 1

To analysis the robustness and convergence of the proposed high-order ILC law (39) for 2-D LNDS (1)-(2) under Assumption 1 and (38), the following Theorem 3 is presented.

Theorem 3: Consider the 2-D LNDS (1)-(2) under Assumption 1 and HOIM-based reference trajectory (38), and let the high-order ILC law (39) be applied. If the learning gain $L_{m,k}(i,j)$ is chosen to make

$$\sum_{m=0}^{M} |h_m - L_{m,k}(i,j)C_{k-m}(i+1,j+1)B_{k-m}(i,j)| < 1, \tag{40}$$

then, the robust tracking objective (3) can be achieved.

Proof. Using (39) and considering (1)-(2), it yields

$$\begin{aligned} u_{k+1}(i,j) - \sum_{m=0}^{M} h_m u_{k-m}(i,j) &= \sum_{m=0}^{M} L_{m,k}(i,j)[y_{d,k-m}(i+1,j+1) \\ &\quad - C_{k-m}(i+1,j+1)x_{k-m}(i+1,j+1)] \\ &= \sum_{m=0}^{M} L_{m,k}(i,j) y_{d,k-m}(i+1,j+1) \\ &\quad - \sum_{m=0}^{M} L_{m,k}(i,j)C_{k-m}(i+1,j+1) \\ &\quad \times \Big[A_{1,k-m}(i+1,j)x_{k-m}(i+1,j) \\ &\quad + A_{2,k-m}(i,j)x_{k-m}(i,j) \\ &\quad + A_{3,k-m}(i,j+1)x_{k-m}(i,j+1) \\ &\quad + B_{k-m}(i,j)u_{k-m}(i,j)\Big], \end{aligned} \tag{41}$$

where $i = 0, 1, 2, \cdots, T_1 - 1$ and $j = 0, 1, 2, \cdots, T_2 - 1$. Arranging (41), it becomes

$$\begin{aligned} u_{k+1}(i,j) &= \sum_{m=0}^{M} [h_m - L_{m,k}(i,j)C_{k-m}(i+1,j+1)B_{k-m}(i,j)] \\ &\quad \times u_{k-m}(i,j) + \bar{\phi}_k(i,j), \end{aligned} \tag{42}$$

where

$$\begin{aligned} \bar{\phi}_k(i,j) &= \sum_{m=0}^{M} L_{m,k}(i,j) y_{d,k-m}(i+1,j+1) \\ &\quad - \sum_{m=0}^{M} L_{m,k}(i,j)C_{k-m}(i+1,j+1) \\ &\quad \times \Big[A_{1,k-m}(i+1,j)x_{k-m}(i+1,j) \\ &\quad + A_{2,k-m}(i,j)x_{k-m}(i,j) \\ &\quad + A_{3,k-m}(i,j+1)x_{k-m}(i,j+1)\Big]. \end{aligned} \tag{43}$$

Taking the norm on both sides of (42), we obtain

$$\begin{aligned}&\|u_{k+1}(i,j)\|\\&\quad\leq\sum_{m=0}^{M}|h_m-L_{m,k}(i,j)C_{k-m}(i+1,j+1)B_{k-m}(i,j)|\\&\qquad\times\|u_{k-m}(i,j)\|+\|\bar{\phi}_k(i,j)\|.\end{aligned}\tag{44}$$

Regarding to $\bar{\phi}_k(i,j)$ given in (43), from Assumption 1, there is

$$\begin{aligned}&\|\bar{\phi}_k(i,j)\|\\&\quad\leq b_db_L(M+1)+b_Cb_L\sum_{m=0}^{M}\Big(b_{A1}\|x_{k-m}(i+1,j)\|\\&\qquad+b_{A2}\|x_{k-m}(i,j)\|+b_{A3}\|x_{k-m}(i,j+1)\|\Big).\end{aligned}\tag{45}$$

Substituting (45) into (44), it becomes

$$\begin{aligned}&\|u_{k+1}(i,j)\|\\&\quad\leq\sum_{m=0}^{M}|h_m-L_{m,k}(i,j)C_{k-m}(i+1,j+1)B_{k-m}(i,j)|\\&\qquad\times\|u_{k-m}(i,j)\|+b_db_L(M+1)\\&\qquad+b_Cb_L\sum_{m=0}^{M}\Big(b_{A1}\|x_{k-m}(i+1,j)\|+b_{A2}\|x_{k-m}(i,j)\|\\&\qquad+b_{A3}\|x_{k-m}(i,j+1)\|\Big).\end{aligned}\tag{46}$$

From the definitions on $U_k(j)$ and $X_k(j)$ in (13) and (14), we obtain

$$\begin{aligned}&U_{k+1}(j)\\&\quad\leq\sum_{m=0}^{M}\Phi_{1,k-m}(j)U_{k-m}(j)+\sum_{m=0}^{M}\Phi_{2,k-m}(j)X_{k-m}(j+1)\\&\qquad+\sum_{m=0}^{M}\Phi_{3,k-m}(j)X_{k-m}(j)+\sum_{m=0}^{M}\Phi_{4,k-m}(j)\\&\qquad\times\|x_{k-m}(0,j+1)\|+\sum_{m=0}^{M}\Phi_{5,k-m}(j)\|x_{k-m}(0,j)\|\\&\qquad+\sum_{m=0}^{M}\Phi_{6,k-m}(j),\end{aligned}\tag{47}$$

where $\Phi_{1,k}$, $\Phi_{2,k}$, $\Phi_{3,k}$, $\Phi_{4,k}$, $\Phi_{5,k}$, and $\Phi_{6,k}$ are given in (15). Similarly, we still can get (19). Inserting (19) into (47), there is

$$\begin{aligned}&U_{k+1}(j)\\&\quad\leq\sum_{m=0}^{M}\big(\Phi_{1,k-m}(j)+\Phi_{2,k-m}(j)\Psi_{1,k}^{-1}(j)\Psi_{3,k}(j)\big)U_{k-m}(j)\\&\qquad+\sum_{m=0}^{M}\big(\Phi_{3,k-m}(j)+\Phi_{2,k-m}(j)\Psi_{1,k}^{-1}(j)\Psi_{2,k}(j)\big)\\&\qquad\times X_{k-m}(j)\\&\qquad+\sum_{m=0}^{M}\big(\Phi_{4,k-m}(j)+\Phi_{2,k-m}(j)\Psi_{1,k}^{-1}(j)\Psi_{4,k}(j)\big)\\&\qquad\times\|x_{k-m}(0,j+1)\|\\&\qquad+\sum_{m=0}^{M}\big(\Phi_{5,k-m}(j)+\Phi_{2,k-m}(j)\Psi_{1,k}^{-1}(j)\Psi_{5,k}(j)\big)\\&\qquad\times\|x_{k-m}(0,j)\|+\sum_{m=0}^{M}\Phi_{6,k-m}(j),\end{aligned}\tag{48}$$

where $\Psi_{1,k}(j)$, $\Psi_{2,k}(j)$, $\Psi_{3,k}(j)$, $\Psi_{4,k}(j)$, and $\Psi_{5,k}(j)$ are given in (18). For (19) and (48), based on Lemma 2, if $\sum_{m=0}^{M}\|\Phi_{1,k-m}(j)+\Phi_{2,k-m}(j)\Psi_{1,k}^{-1}(j)\Psi_{3,k}(j)\|<1$ (equivalently, $\sum_{m=0}^{M}|h_m-L_{m,k}(i,j)C_{k-m}(i+1,j+1)B_{k-m}(i,j)|<1$), the robust ILC tracking objective (3) can be finished. The proof of Theorem 3 is completed.

Remark 7: Similar to Remark 5, we use the estimated information on $C_k(i,j)$ and $B_k(i,j)$ to select the learning gains $L_{m,k}(i,j), m=0,1,2,\cdots,M$ in the high-order ILC law (39). The convergence condition (40) is reformulated as

$$\begin{aligned}&\sum_{m=0}^{M}\big[|h_m-L_m(i,j)C(i+1,j+1)B(i,j)|\\&\qquad\qquad+\beta_{CB}(i,j)|L_m(i,j)|\big]<1,\end{aligned}$$

where $\beta_{CB}(i,j)$ is defined in Remark 5. Thus, we may reasonably get $|C(i+1,j+1)B(i,j)|>\beta_{CB}(i,j)\sum_{m=0}^{M}h_m$, which guarantees (40) with $L_m(i,j)=\frac{h_m}{C(i+1,j+1)B(i,j)}$.

As the high order ILC law (39) is applied to the 2-D LNDS (1)-(2) with $x_k(0,j)$ and $x_k(i,0)$ satisfying the HOIM-based iterative boundary conditions, which is given as the following Assumption 4.

Assumption 4: Let the system parameters $A_{1,k}(i+1,j)$, $A_{2,k}(i,j)$, $A_{3,k}(i,j+1)$, $B_k(i,j)$ and $C_k(i,j)$ be progressively convergent, i.e.,

$$\begin{aligned}\lim_{k\to+\infty}A_{1,k}(i+1,j)&=A_1(i+1,j),\\\lim_{k\to+\infty}A_{2,k}(i,j)&=A_2(i,j),\\\lim_{k\to+\infty}A_{3,k}(i,j+1)&=A_3(i,j+1),\\\lim_{k\to+\infty}B_k(i,j)&=B(i,j),\\\lim_{k\to+\infty}C_k(i,j)&=C(i,j),\end{aligned}$$

and meanwhile, the boundary states $x_k(i,0)$ and $x_k(0,j)$ satisfy the HOIM asymptotically, i.e.,

$$\begin{aligned}\lim_{k\to\infty}\left[x_{k+1}(i,0)-\sum_{m=0}^{M}h_mx_{k-m}(i,0)\right]&=0,\\\lim_{k\to\infty}\left[x_{k+1}(0,j)-\sum_{m=0}^{M}h_mx_{k-m}(0,j)\right]&=0.\end{aligned}$$

With Assumption 4, we present the following Theorem 4 to provide the perfect tracking result on HOIM-based reference trajectory.

Theorem 4: Consider the 2-D LNDS (1)-(2) under Assumption 4 and HOIM-based reference trajectory (38), and let the high-order ILC law (39) be used. If the learning gains $L_{m,k}(i,j)$ are chosen to make (40) and

$$\sum_{m=0}^{M} |h_m - C_k(i+1,j+1)B_k(i,j)L_{m,k}(i,j)| < 1, \quad (49)$$

be satisfied, then, the result of perfect tracking objective (4) can be obtained.

Proof. Using $e_k(i,j) = y_{d,k}(i,j) - y_k(i,j)$, considering (38), (1) and (2), there is

$$\begin{aligned}
&e_{k+1}(i+1,j+1)\\
&= \sum_{m=0}^{M} h_m y_{d,k-m}(i+1,j+1) - y_{k+1}(i+1,j+1)\\
&= \sum_{m=0}^{M} h_m[e_{k-m}(i+1,j+1) + y_{k-m}(i+1,j+1)]\\
&\quad -C_{k+1}(i+1,j+1)x_{k+1}(i+1,j+1)\\
&= \sum_{m=0}^{M} h_m e_{k-m}(i+1,j+1)\\
&\quad + \sum_{m=0}^{M} h_m C_{k-m}(i+1,j+1)\Big[A_{1,k-m}(i+1,j)\\
&\quad \times x_{k-m}(i+1,j) + A_{2,k-m}(i,j)x_{k-m}(i,j)\\
&\quad +A_{3,k-m}(i,j+1)x_{k-m}(i,j+1) + B_{k-m}(i,j)\\
&\quad \times u_{k-m}(i,j)\Big] - C_{k+1}(i+1,j+1)\Big[A_{1,k+1}(i+1,j)\\
&\quad \times x_{k+1}(i+1,j) + A_{2,k+1}(i,j)x_{k+1}(i,j)\\
&\quad +A_{3,k+1}(i,j+1)x_{k+1}(i,j+1) + B_{k+1}(i,j)u_{k+1}(i,j)\Big]\\
&= \sum_{m=0}^{M} h_m e_{k-m}(i+1,j+1) - C_k(i+1,j+1)B_k(i,j)\\
&\quad \times u_{k+1}(i,j) + \sum_{m=0}^{M} h_m C_{k-m}(i+1,j+1)\\
&\quad \times\Big[A_{1,k-m}(i+1,j)x_{k-m}(i+1,j) + A_{2,k-m}(i,j)\\
&\quad \times x_{k-m}(i,j) + A_{3,k-m}(i,j+1)x_{k-m}(i,j+1)\\
&\quad +B_{k-m}(i,j)u_{k-m}(i,j)\Big] - C_{k+1}(i+1,j+1)\\
&\quad \times\Big[A_{1,k+1}(i+1,j)x_{k+1}(i+1,j) + A_{2,k+1}(i,j)\\
&\quad \times x_{k+1}(i,j) + A_{3,k+1}(i,j+1)x_{k+1}(i,j+1)\Big]\\
&\quad +[C_k(i+1,j+1)B_k(i,j) - C_{k+1}(i+1,j+1)\\
&\quad \times B_{k+1}(i,j)]u_{k+1}(i,j). \qquad (50)
\end{aligned}$$

Using the high-order ILC law (39), there is

$$\begin{aligned}
&e_{k+1}(i+1,j+1)\\
&\quad = \sum_{m=0}^{M}[h_m - C_k(i+1,j+1)B_k(i,j)L_{m,k}(i,j)]\\
&\qquad \times e_{k-m}(i+1,j+1) + \theta_k(i,j), \qquad (51)
\end{aligned}$$

where

$$\begin{aligned}
&\theta_k(i,j)\\
&\quad = \sum_{m=0}^{M} h_m C_{k-m}(i+1,j+1)\Big[A_{1,k-m}(i+1,j)\\
&\qquad \times x_{k-m}(i+1,j) + A_{2,k-m}(i,j)x_{k-m}(i,j)\\
&\qquad +A_{3,k-m}(i,j+1)x_{k-m}(i,j+1)\Big]\\
&\qquad + \sum_{m=0}^{M} h_m[C_{k-m}(i+1,j+1)B_{k-m}(i,j)\\
&\qquad -C_k(i+1,j+1)B_k(i,j)]u_{k-m}(i,j)\\
&\qquad -C_{k+1}(i+1,j+1)\Big[A_{1,k+1}(i+1,j)x_{k+1}(i+1,j)\\
&\qquad +A_{2,k+1}(i,j)x_{k+1}(i,j) + A_{3,k+1}(i,j+1)\\
&\qquad \times x_{k+1}(i,j+1)\Big] + [C_k(i+1,j+1)B_k(i,j)\\
&\qquad -C_{k+1}(i+1,j+1)B_{k+1}(i,j)]u_{k+1}(i,j). \qquad (52)
\end{aligned}$$

Similar with Theorem 3, according to (41) and (51), if the learning gain $L_{m,k}(i,j)$ is selected to satisfy (40) and (49), then, the perfect tracking objective is obtained. The proof of Theorem 4 is complete.

V. TWO ILLUSTRATIVE EXAMPLES

In this section, to illustrate the effectiveness and feasibility of the P-type ILC law (5) and high-order ILC law (39) for 2-D LNDS (1)-(2) under Assumptions 1-4, two simulation examples are presented.

Example 1: We consider the 2-D LNDS (1)-(2) with

$$\begin{aligned}
&A_{1,k}(i+1,j)\\
&\quad = \begin{bmatrix} 0.0139 + 0.1\sin(2\pi(i+j)/10) & 0.01 + m_{1,k} \\ 0.01 & 0.03 \end{bmatrix},\\
&A_{2,k}(i,j)\\
&\quad = \begin{bmatrix} 0.0139 & 0.02 - 0.1\cos(2\pi(i+j)/10) \\ 0.01 & 0.04 + m_{2,k} \end{bmatrix},\\
&A_{3,k}(i,j+1)\\
&\quad = \begin{bmatrix} 0.01 + m_{3,k} & 0.02 \\ 0.0139 & 0.3 + 0.1\sin(2\pi(i+j)/10) \end{bmatrix},\\
&B_k(i,j)\\
&\quad = \begin{bmatrix} 2 \\ 0.1\cos(2\pi(i+j)/10) + m_{4,k} \end{bmatrix},\\
&C_k(i,j)\\
&\quad = \begin{bmatrix} -1 & -0.5\sin(2\pi(i+j)/10) + m_{5,k} \end{bmatrix},
\end{aligned}$$

where $m_1(k)$, $m_2(k)$, $m_3(k)$, $m_4(k)$, and $m_5(k)$ are randomly varying at the intervals $[-0.1, 0.1]$, $[0, 0.2]$, $[-0.1, 0.1]$, $[0, 0.1]$, and $[-0.1, 0.1]$, respectively. In the P-type ILC law (5) with the initial control input $u_0(i,j) = 0$, $i = 0, 1, 2, \cdots, 19$, $j = 0, 1, 2, \cdots, 19$, let the learning gain $L_k(i,j)$ be selected as $L_k(i,j) = -0.1$, which satisfies the convergence condition (6) in Theorems 1 and 2. The following indexes EE_k and SS_k are used to evaluate the accuracy of ILC

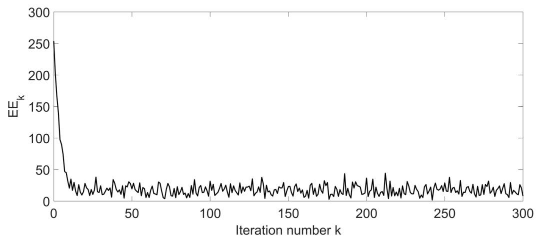

FIGURE 1. The profile of the tracking error index EE_k with k under the P-type ILC law (5).

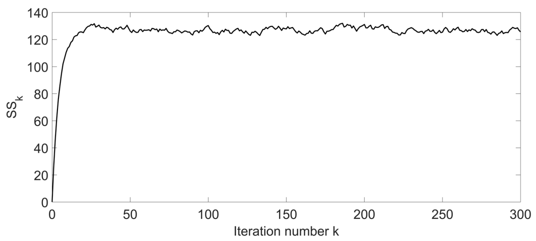

FIGURE 2. The profile of the control input index SS_k with k under the P-type ILC law (5).

tracking error and the control input,

$$EE_k = \sum_{i=1}^{20}\sum_{j=1}^{20}\left|y_{d,k}(i,j) - y_k(i,j)\right|,$$

$$SS_k = \sum_{i=0}^{19}\sum_{j=0}^{19}\left|u_k(i,j)\right|.$$

Case 1: Under Assumption 1, let the 2-D reference trajectory and boundary states be given as

$$y_{d,k}(i,j) = \sin(2\pi(i+j)/10) + m_6(k),$$
$$i = 0, 1, \cdots, 20,\ j = 0, 1, \cdots, 20,$$
$$x_k(0,j) = \begin{bmatrix} 0.5 \\ \sin(j) + m_7(k) \end{bmatrix},\ j = 0, 1, \cdots, 20,$$
$$x_k(i,0) = \begin{bmatrix} -\sin(0.2\pi i) \\ m_8(k) \end{bmatrix},\ i = 1, 2, \cdots, 20,$$

where $m_6(k)$, $m_7(k)$, and $m_8(k)$ are randomly varying at the intervals $(-1, 1)$, $(-1, 1)$ and $(0, 2)$. As a result, the profile of the tracking error index EE_k with k is presented in Figure 1, and the variation situation of the control input index SS_k with k is shown in Figure 2. Obviously, it is observed from Figures 1-2 that the ultimate ILC tracking error and the control input converge to a bounded range. Consequently, the robustness of the P-type ILC law (5) under Assumption 1 is validated.

Case 2: Under Assumption 3, let $m_1(k) = (0.1)^{k+1}$, $m_2(k) = (0.2)^{k+1}$, $m_3(k) = (0.1)^{k+1}$, $m_4(k) = (0.5)^{k+1}$, $m_6(k) = (0.2)^k$, $m_7(k) = (0.3)^k$, and $m_8(k) = (0.2)^k$. The 2-D reference trajectory $y_{d,30}(i,j)$, $i = 0, 1, \cdots, 20$, $j = 0, 1, \cdots, 20$ is shown in Figure 3. Consequently, Figure 4 shows the profiles of ILC tracking error $e_k(i,j)$ at $k = 3, 5, 7, 30$. And the profiles of the tracking error index EE_k with k and the control input index SS_k with k by using the P-type ILC law (5) are presented in Figures 5 and 6. It is

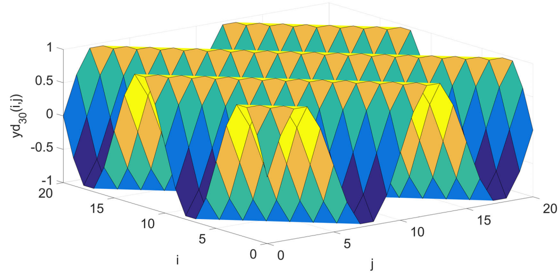

FIGURE 3. The 2-D reference trajectory $y_{d,30}(i,j)$ for $i = 0, 1, \cdots, 20$ and $j = 0, 1, \cdots, 20$.

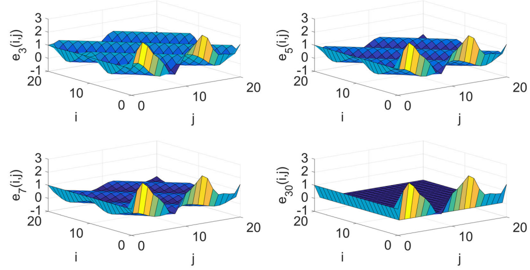

FIGURE 4. The profiles of ILC tracking error $e_k(i,j)$ at $k = 3, 5, 7, 30$.

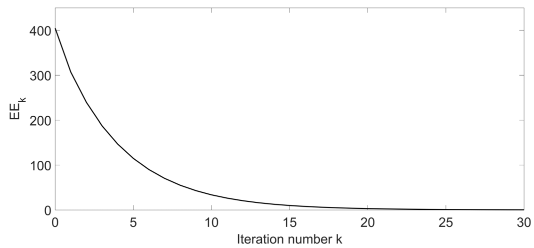

FIGURE 5. The profile of the tracking error index EE_k with k under the P-type ILC law (5).

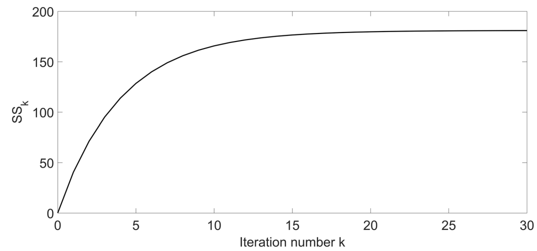

FIGURE 6. The profile of the control input index SS_k with k under the P-type ILC law (5).

observed from Figures 4-6 that the system output $y_k(i,j)$ can precisely track the 2-D reference trajectory. Therefore, the convergence of the P-type ILC law (5) is verified.

Example 2. Consider some practical dynamical linear processes with repetitive operation [20], [22], which is given as

$$\frac{\partial^2 \mathtt{T}_k(w,t)}{\partial w \partial t} = a_{1,k}(w,t)\frac{\partial \mathtt{T}_k(w,t)}{\partial t} + a_{2,k}(w,t)\frac{\partial \mathtt{T}_k(w,t)}{\partial w} + a_{0,k}(w,t)\mathtt{T}_k(w,t) + b_k(w,t)f_k(w,t), \quad (53)$$

where $f_k(w,t)$ is the control input; $\mathtt{T}_k(w,t)$ denotes the temperature; $a_{0,k}(w,t)$, $a_{1,k}(w,t)$, $a_{2,k}(w,t)$, and $b_k(w,t)$ are coefficients. Let

$$x_k(i,j) = \mathbb{T}_k(i\Delta w, j\Delta t),\ u_k(i,j) = f_k(i\Delta w, j\Delta t),$$

where Δw and Δt are sampling step sizes for space variable w and time variable t. Then, (53) can be discretized into the following 2-D LNDS

$$\begin{aligned} x_k(i+1,j+1) &= [1+a_{2,k}(i,j)\Delta t]x_k(i+1,j) \\ &\quad +[a_{0,k}(i,j)\Delta w\Delta t - a_{1,k}(i,j)\Delta w - a_{2,k}(i,j)\Delta t - 1] \\ &\quad \times x_k(i,j) + [1+a_{1,k}(i,j)\Delta w]x_k(i,j+1) \\ &\quad +b_k(i,j)\Delta w\Delta t u_k(i,j). \end{aligned} \tag{54}$$

Let $\Delta w = 0.1$, $\Delta t = 0.1$, $a_{0,k}(i,j) = -145.5 + 110\sin[2\pi(i+j)/10)] + 100m_{2,k}$, $a_{1,k}(i,j) = -9.5 + \sin[2\pi(i+j)/10)] + 10m_{3,k}$, $a_{2,k}(i,j) = -9 + 10\sin[2\pi(i+j)/10)] + 10m_{1,k}$, $b_k(i,j) = 23.5 + 10\cos[2\pi(i+j)/10)] + 10m_{4,k}$, and $y_k(i,j) = 2x_k(i,j)$, where $m_1(k)$, $m_2(k)$, $m_3(k)$, and $m_4(k)$ are randomly varying at the intervals $(-0.1, 0.1)$, $(0, 0.2)$, $(-0.1, 0.1)$, and $(-0.1, 0.1)$, respectively. The following 2-D HOIM-based reference trajectory $y_{d,k}(i,j)$ is given as:

$$\begin{aligned} y_{d,k+1}(i,j) &= \sqrt{2}y_{d,k}(i,j) - y_{d,k-1}(i,j), \\ y_{d,0}(i,j) &= 2\cos[2\pi(i-j)/10], \\ y_{d,1}(i,j) &= \cos[0.2\pi(i+j)]. \end{aligned}$$

Under the high-order ILC law (39) with $u_0(i,j) = u_1(i,j) = 0$, $i = 0,1,2,\cdots,19$, $j = 0,1,2,\cdots,19$, let the learning gains $\Gamma_{1,k}(i,j)$ and $\Gamma_{2,k}(i,j)$ be selected as $\Gamma_{1,k}(i,j) = 0.6 + 0.01i + 0.01j$ and $\Gamma_{2,k}(i,j) = -0.9 + 0.01i + 0.01j$, which satisfy the convergent condition in Theorems 3 and 4. We use the maximum absolute tracking error index MM_k to evaluate the tracking performance, i.e.,

$$\mathrm{MM}_k = \max_{i=1,2,\cdots,20}\max_{j=1,2,\cdots,20}\left|y_{d,k}(i,j) - y_k(i,j)\right|.$$

Case 1: Under Assumption 1, let the boundary states be given as

$$\begin{aligned} x_{k+1}(0,j) &= \sqrt{2}x_k(0,j) - x_{k-1}(0,j) + n_{1,k}, \\ x_0(0,j) &= 2\cos(0.2\pi j),\ x_1(0,j) = \cos(0.2\pi j), \\ x_{k+1}(i,0) &= \sqrt{2}x_k(i,0) - x_{k-1}(i,0) + n_{2,k}, \\ x_0(i,0) &= 2\cos(0.2\pi i),\ x_1(i,0) = \cos(0.2\pi i), \end{aligned}$$

where $n_{1,k}$ and $n_{2,k}$ are uniformly varying at $(0, 0.2)$. The profiles of MM_k with k under the high-order ILC law (39) and P-type ILC law (5) is shown in Figure 7. Apparently, we can be seen from Figure 7 that compared with P-type ILC law (5), the high-order ILC law (39) can well address the robustness ILC tracking on HOIM-based reference trajectory.

Case 2: In Case 1, let $m_1(k) = (0.1)^{k+1}$, $m_2(k) = (0.2)^{k+1}$, $m_3(k) = (0.2)^{k+1}$, and $m_4(k)$ $n_{h,k} = (0.1)^{k+1}$, $n_1(k) = (0.1)^{k+1}$, $n_2(k) = (0.2)^{k+1}$. The ILC tracking errors $e_k(i,j)$ at $k = 5, 10, 20, 70$ and the 2-D reference trajectory $y_{d,70}(i,j)$, $i = 0,1,\cdots,20$, $j = 0,1,\cdots,20$ are presented in Figures 8 and 9, respectively. Obviously, the effectiveness of

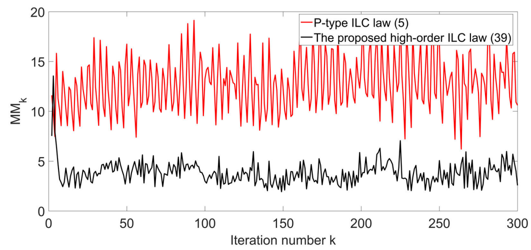

FIGURE 7. The profiles of the tracking error index MM_k with k under the high-order ILC law (39) and P-type ILC law (5).

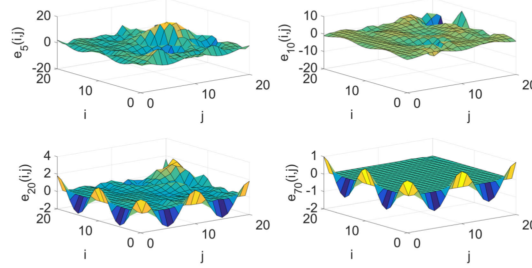

FIGURE 8. The ILC tracking error $e_k(i,j)$ at $k = 5, 10, 20, 70$ under the high-order ILC law (39).

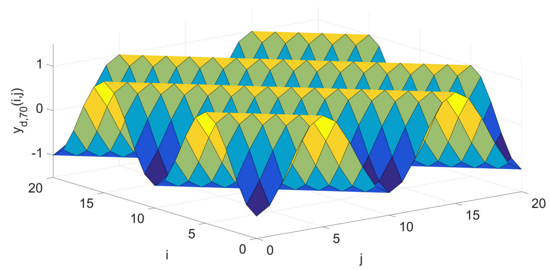

FIGURE 9. The 2-D reference trajectory $y_{d,70}(i,j)$ for $i = 0,1,\cdots,20$ and $j = 0,1,\cdots,20$.

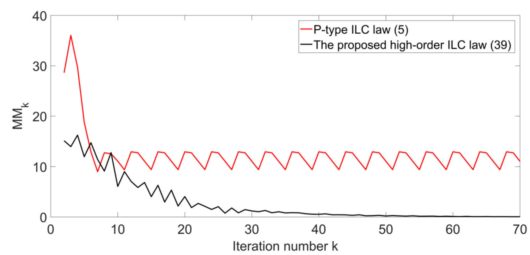

FIGURE 10. The profiles of the tracking error index MM_k with k under the high-order ILC law (39) and P-type ILC law (5).

Theorem 4 is validated. Additionally, we give a comparison result on the proposed high-order ILC law (39) and the P-type ILC law (5) $u_{k+1}(i,j) = u_k(i,j) + (0.6+0.01i+0.01j)e_k(i+1,j+1)$, $i = 0,1,2,\cdots,19$, $j = 0,1,2,\cdots,19$, which is shown in Figure 10. It is verified that the tracking performance of the high-order ILC law outperforms the lower-order ILC law.

VI. CONCLUSION

In this article, the robust and convergent properties of the proposed P-type ILC law and the high-order ILC law for 2-D LNDS with iteration-dependent reference trajectory have been proved. With 2-D linear nonrepetitive inequalities, the ultimate ILC tracking error robustly converges to a bounded range. Certainly, the proposed ILC analysis approach, as a new tool, is applicable to some nonrepetitive systems, i.e., spatially distributed or interconnected systems. Additionally, this paper discloses that high-order ILC law not only is adapted to the variation of 2-D HOIM-based reference trajectories but also can accommodate the nonrepetitiveness from boundary states and system parameters.

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KAI WAN received the Ph.D. degree intelligent control from the School of Electronic and Information Technology, Sun Yat-sen University, Guangzhou, China, in 2019. His research interests include two-dimensional systems, robust control, and iterative learning control.

HENG XIE received the Ph.D. degree in material processing engineering from the South China University of Technology, Guangzhou, China, in 2015. Her research interests include intelligent manufacturing, electrical materials, and iterative learning control.

QING-YUAN XU received the B.S. degree from the School of Information and Electrical Engineering, China University of Mining and Technology, Xuzhou, China, in 2008, and the M.Phil. and Ph.D. degrees from Sun Yat-sen University, Guangzhou, China, in 2010 and 2018, respectively. Her current research interests include iterative learning control and adaptive tracking control.