Received 22 May 2022, accepted 9 June 2022, date of publication 4 November 2022, date of current version 17 November 2022.
Digital Object Identifier 10.1109/ACCESS.2022.3219498

On Hamming Distance Distributions of Repeated-Root Cyclic Codes of Length $5p^s$ Over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$

HAI Q. DINH[1], BAC T. NGUYEN[2,3], HIEP L. THI[4], AND WORAPHON YAMAKA[5]
[1]Department of Mathematical Sciences, Kent State University, Kent, OH 44240, USA
[2]Institute of Fundamental and Applied Sciences, Duy Tan University, Ho Chi Minh City, Da Nang 550000, Vietnam
[3]Faculty of Natural Sciences, Duy Tan University, Da Nang 550000, Vietnam
[4]Faculty of Education, Thu Dau Mot University, Binh Duong 75100, Vietnam
[5]Centre of Excellence in Econometrics, Chiang Mai University, Chiang Mai 50200, Thailand

Corresponding author: Hiep L. Thi (lthiep@tdmu.edu.vn)

This research is funded by Thu Dau Mot University, Binh Duong Province, Vietnam under grant number DT.21.2-018; and in part by the Research Administration Centre, Chiang Mai University. The work of Hai Q. Dinh and Woraphon Yamaka was supported in part by the Centre of Excellence in Econometrics, Faculty of Economics, Chiang Mai University.

ABSTRACT Let $p \neq 5$ be any odd prime. Using the algebraic structures of all cyclic codes of length $5p^s$ over the finite commutative chain ring $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, in this paper, the exact values of Hamming distances of all cyclic codes of length $5p^s$ over $\mathcal{R}$ are established. As an application, we identify all maximum distance separable cyclic codes of length $5p^s$.

INDEX TERMS Constacyclic codes, cyclic codes, dual codes, chain rings, hamming distance, singleton bound, MDS codes.

I. INTRODUCTION

The class of constacyclic codes is an important class of linear codes in coding theory. Many optimal linear codes are directly derived from constacyclic codes. Constacyclic codes have practical applications as they are effective for encoding and decoding with shift registers.

For a unit λ of $\mathbb{F}_{p^m}$, λ-constacyclic codes of length n over $\mathbb{F}_{p^m}$ are ideals of the ring $R_\lambda = \frac{\mathbb{F}_{p^m}[x]}{\langle x^n-\lambda\rangle}$. The constacyclic codes of length n over $\mathbb{F}_{p^m}$ are said to be simple-root constacyclic codes if $\gcd(n,p) = 1$. Otherwise, the constacyclic codes are said to be repeated-root constacyclic codes. In 1967, [4] initiated the study of repeated-root constacyclic codes. After that, many researchers studied repeated-root constacyclic codes over finite fields [9], [35], [40], [43].

In 1994, [38] showed that Kerdock and Preparata codes can be constructed from linear codes over $\mathbb{Z}_4$ via the Gray map. After that, codes over finite chain rings received attention because of their new role in algebraic coding theory and their successful applications. Since 2003, special classes of codes over certain classes of finite chain rings have been studied by numerous other authors (see, for example, [1], [5], [26], [45], and [47]).

Linear and cyclic codes over the finite commutative chain ring $\mathbb{F}_2 + u\mathbb{F}_2$, where $u^2 = 0$ are studied in [3]. In general, the class of finite rings of the form $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ has been widely used as alphabets of certain constacyclic codes. The classification of codes plays an important role in studying their structures, but in general, it is very difficult. In 2010, [12] classified all constacyclic codes of length p^s over $\mathcal{R}$. In addition, in 2015, the authors of [30] studied negacyclic codes of length $2p^s$ over $\mathcal{R}$. Moreover, the algebraic structures of all λ-constacyclic codes of length $2p^s$ over $\mathcal{R}$ are determined in [10] and provided the number of codewords and the dual of every λ-constacyclic code. In 2018, all negacyclic and constacyclic codes of length $4p^s$ over $\mathcal{R}$ are established successfully in [21], [22], [23], and [24]. In 2020, [20] studied all λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$. After that, some authors extended these problems to many more general lengths and alphabets (see, e.g., [6] and [7]).

The associate editor coordinating the review of this manuscript and approving it for publication was Luca Barletta.

However, till now very little amount of works on computation of the Hamming distances have been done due to computational complexity. In [14], Dinh obtained the Hamming distances of all the cyclic codes of prime power lengths over $\mathbb{F}_{p^m}$. Later, in [25] and [34], Dinh et al. computed the Hamming distances of all constacyclic codes of lengths $3p^s$, $5p^s$ over $\mathbb{F}_{p^m}$. In addition, Dinh [12] provided Hamming distances of all $(\alpha + u\beta)$-constacyclic codes of prime power lengths over $\mathcal{R}$. Moreover, Dinh et al. [27] determined the Hamming distances of all γ-constacyclic codes of prime power lengths over $\mathcal{R}$. In 2020, the Hamming distance of λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$ is given in $\mathcal{R}$ [20], where $\lambda = \alpha + u\beta$ is not a cube.

Motivated from all these works, we compute Hamming distance distribution for all cyclic codes of length $5p^s$ over $\mathcal{R}$. As an application, we identify all the MDS codes among such codes.

The rest of this paper is organized as follows. Section 2 contains some basic definitions and preliminary results about constacyclic codes of length $5p^s$ over $\mathcal{R}$. In Section 3, we obtain the Hamming distances of cyclic codes of length $5p^s$ over $\mathcal{R}$ and identify all MDS cyclic codes of length $5p^s$ over $\mathcal{R}$ when $p \equiv 2, 3 \pmod 5$. In Section 4, we determine the Hamming distances and provide all MDS codes for all cyclic codes of length $5p^s$ over $\mathcal{R}$, where $p \equiv 1 \pmod 5$. In Section 5, we study the Hamming distances for all cyclic codes of length $5p^s$ over $\mathcal{R}$ when $p \equiv 4 \pmod 5$. We also give all MDS cyclic codes among such codes. In Section 6, we conclude the paper.

II. PRELIMINARIES

Let $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}(u^2 = 0)$ be a finite chain ring. A code C of length n over $\mathcal{R}$ is a non-empty subset of $\mathcal{R}^n$. The code C is said to be linear over $\mathcal{R}$ if it is an $\mathcal{R}$-submodule of $\mathcal{R}^n$. Let $\lambda \in \mathcal{R}$ be a unit element and ν_λ be a map from $\mathcal{R}^n$ to $\mathcal{R}^n$ defined by

$$\nu_\lambda(v_0, v_1, v_2, \dots, v_{n-1}) = (\lambda v_{n-1}, v_0, v_1, \dots, v_{n-2}).$$

A linear code C is said to be a λ-constacyclic code over $\mathcal{R}$ if $\nu_\lambda(C) = C$. If $\lambda = 1$, then C is cyclic, and if $\lambda = -1$, then C is negacyclic code over $\mathcal{R}$.

Consider a code C of length n over $\mathcal{R}$ and let $v = (v_0, v_1, v_2, \dots, v_{n-1}) \in C$ be a codeword, then it can be represented as the polynomial $v(x) = v_0 + v_1x + v_2x^2 + \cdots + v_{n-1}x^{n-1}$ of the ring $\mathcal{R}_\lambda = \frac{\mathcal{R}[x]}{\langle x^n - \lambda\rangle}$. Denote the Hamming weight of v is denoted by $wt_H(v)$. Then $wt_H(v)$ is given by the total number of nonzero components of v, i.e.,

$$wt_H(v) = |\{j : v_j \neq 0\}|.$$

The smallest weight among all its nonzero codewords is the minimum weight of the code C and is denoted by $wt_H(C)$. The Hamming distance of the code C is denoted by $d_H(C)$ and is defined as $d_H(C) = min\{wt_H(v)|\ v \neq 0, v \in C\}$. The following result is one of the most important fact about λ-constacyclic codes.

Proposition 2.1: [8] *Let C be a linear code C of length n over $\mathcal{R}$. Then C is a λ-constacyclic code of length n over $\mathcal{R}$ if and only if C is an ideal of the ring $\mathcal{R}_\lambda = \frac{\mathcal{R}[x]}{\langle x^n-\lambda\rangle}$.*

The ring $\mathcal{R}$ can be expressed as $\mathcal{R} = \frac{\mathbb{F}_{p^m}[u]}{\langle u^2\rangle} = \{a + ub \mid a, b \in \mathbb{F}_{p^m}\}$. Over the last few years, in a series of papers, Dinh et al. ([20], [21], [22], [23], [24]) have done the job of classifying classes of constacyclic codes of certain lengths over $\mathcal{R}$. In 2010, [12] gave the construction of all constacyclic codes of p^s length over $\mathcal{R}$.

Theorem 2.2 (cf. [12]): Let λ be a unit of the ring $\mathcal{R}$, i.e., λ is of the form $\alpha + u\beta$ or γ, where $0 \neq \alpha, \beta, \gamma \in \mathbb{F}_{p^m}$.

1) *If $\lambda = \alpha + u\beta$, there exists $0 \neq \alpha_0 \in \mathbb{F}_{p^m}$ such that $\alpha_0^{p^s} = \alpha$. Then the ring $\frac{\mathcal{R}[x]}{\langle x^{p^s}-(\alpha+u\beta)\rangle}$ is a finite chain ring with maximal ideal $\langle \alpha_0 x - 1\rangle$, and $\langle(\alpha_0x-1)^{p^s}\rangle = \langle u\rangle$. The $(\alpha+u\beta)$-constacyclic codes of p^s length over $\mathcal{R}$ are the ideals $\langle(\alpha_0x-1)^j\rangle$, $0 \le j \le 2p^s$, of the finite chain ring $\frac{\mathcal{R}[x]}{\langle x^{p^s}-(\alpha+u\beta)\rangle}$.*
2) *If $\lambda = \gamma \in \mathbb{F}_{p^m} \setminus \{0\}$, there exists $0 \neq \gamma_0 \in \mathbb{F}_{p^m}$ such that $\gamma_0^{p^s} = \gamma$. Then the ring $\frac{\mathcal{R}[x]}{\langle x^{p^s}-\gamma\rangle}$ is a local ring with the maximal ideal $\langle u, x-\gamma_0\rangle$, but it is not a chain ring. The γ-constacyclic codes of p^s length over $\mathcal{R}$, i.e., ideals of the ring $\frac{\mathcal{R}[x]}{\langle x^{p^s}-\gamma\rangle}$, are given by four types.*
   - *Type 1 are the trivial ideals, i.e., $C = \langle 0\rangle$, $C = \langle 1\rangle$. Number of codewords in theses codes are 1 and p^{2mp^s} respectively.*
   - *Type 2 are the principal ideals generated by nonmonic polynomials, i.e., $C_j = \langle u(x-\gamma_0)^j\rangle$, where $0 \le j \le p^s - 1$. In this case, $|C_j| = p^{m(p^s-j)}$.*
   - *Type 3 are the principal ideals generated by monic polynomials, i.e., $C_j = \langle (x-\gamma_0)^j + u(x-\gamma_0)^t h(x)\ \rangle$, where $1 \le j \le p^s - 1$, $0 \le t < j$, and either $h(x)$ is 0 or $h(x)$ is a unit in $\frac{\mathbb{F}[x]}{\langle x^{p^s}-\gamma\rangle}$. In this case,*
   $$|C_j| = \begin{cases} \bullet\, p^{2m(p^s-j)}, \text{ if } 1 \le j \le p^{s-1} + \lfloor \frac{t}{2} \rfloor \\ \bullet\, p^{m(p^s-t)}, \\ \text{if } p^{s-1} + \lfloor \frac{t}{2} \rfloor < j \le p^s - 1. \end{cases}$$
   - *Type 4 are the nonprincipal ideals, i.e., $\langle (x-\gamma_0)^j + u(x-\gamma_0)^t h(x), u(x-\gamma_0)^\kappa\rangle$, with $h(x)$ as in Type 3, $\deg h(x) \le \kappa - t - 1$, and $\kappa < T$, where T is the smallest integer such that $u(x-\gamma_0)^T \in \langle (x-\gamma_0)^j + u(x-\gamma_0)^t h(x)\ \rangle$; or equivalently, $T = j$, if $h(x) = 0$, otherwise $T = \min\{j, p^s - j + t\}$. The cardinality of C is given by $|C| = p^{m(2p^s-j-\kappa)}$.*

Theorem 2.3: (cf. [10, Section 3])

(i) If λ is a square in $\mathcal{R}$ and $\lambda = \delta^2$, then it follows from the Chinese Reminder Theorem that

$$\Re(2p^s, \lambda) = \frac{\mathcal{R}[x]}{\langle x^{2p^s} - \lambda\rangle} \cong \frac{R[x]}{\langle x^{p^s} + \delta\rangle} \oplus \frac{R[x]}{\langle x^{p^s} - \delta\rangle} = \Re(p^s, -\delta) \oplus \Re(p^s, \delta).$$

It means that any λ-constacyclic code of length $2p^s$ over $\mathcal{R}$, i.e., an ideal C of $\Re(2p^s, \lambda)$, is represented as a direct sum of C_+ and C_-: $C = C_+ \oplus C_-$, where C_+ and C_- are ideals of $\Re(p^s, -\delta)$ and $\Re(p^s, \delta)$, respectively. Thus, the classification, detailed structure, and number of codewords of constacyclic codes C of length $2p^s$ over $\mathcal{R}$ can be obtained from that of the direct summands C_+ and C_-.

(*ii*) If λ is not a square in R, and $\lambda = \alpha + u\beta$, $\alpha, \beta \in \mathbb{F}_{p^m}^*$, let $\alpha_0^{p^s} = \alpha$, then the ring $\Re(2p^s, \alpha + u\beta) = \frac{\mathcal{R}[x]}{\langle x^{2p^s} - (\alpha + u\beta)\rangle}$ is a chain ring whose ideals are

$$\langle 1\rangle \supsetneq \langle x^2 - \alpha_0\rangle \supsetneq \cdots \supsetneq \langle (x^2 - \alpha_0)^{2p^s-1}\rangle \supsetneq\rangle = \langle 0\rangle.$$

In other words, $(\alpha + u\beta)$-constacyclic codes of length $2p^s$ over $\mathcal{R}$ are precisely the ideals $\langle (x^2 - \alpha_0)^i\rangle \subseteq \Re(2p^s, \alpha + u\beta)$, where $0 \le i \le 2p^s$. Each $(\alpha + u\beta)$-constacyclic code $C = \langle (x^2 - \alpha_0)^i\rangle$ has $p^{2m(2p^s-i)}$ codewords, its dual $C^\perp$ is the $(\alpha^{-1} - u\alpha^{-2}\beta)$-constacyclic code

$$C^\perp = \left\langle (x^2 - \alpha_0^{-1})^{2p^s-i}\right\rangle \subseteq \Re(2p^s, \alpha^{-1} - u\alpha^{-2}\beta),$$

which contains p^{2mi} codewords. Moreover, the ideal $\langle u\rangle_{\Re(2p^s,\alpha+u\beta)}$ is the unique self-dual $(\alpha + u\beta)$-constacyclic code of length $2p^s$ over $\mathcal{R}$.

(*iii*) If λ is not a square in $\mathcal{R}$ and $\lambda = \gamma \in \mathbb{F}_{p^m}^*$, then γ-constacyclic codes are classified by categorizing the ideals of the local ring $\Re(2p^s, \gamma) = \frac{\mathcal{R}[x]}{\langle x^{2p^s} - \gamma\rangle}$ into 4 distinct types, where $\gamma_0^{p^s} = \gamma$.

- Type 1 (trivial ideals): $\langle 0\rangle$, $\langle 1\rangle$.
- Type 2 (principal ideals with nonmonic polynomial generators): $\left\langle u(x^2 - \gamma_0)^i\right\rangle$, where $0 \le i \le p^s - 1$.
- Type 3 (principal ideals with monic polynomial generators):

$$\left\langle (x^2 - \gamma_0)^i + u(x^2 - \gamma_0)^t h(x)\right\rangle,$$

where $1 \le i \le p^s - 1$, $0 \le t < i$, and either $h(x)$ is 0 or $h(x)$ is a unit in $\Re(2p^s, \gamma)$ which can be represented as $h(x) = \sum_j (h_{0j}x + h_{1j})(x^2 - \gamma_0)^j$, with $h_{0j}, h_{1j} \in \mathbb{F}_{p^m}$, and $h_{00}x + h_{10} \ne 0$.
- Type 4 (nonprincipal ideals): $\left\langle (v(x))^i + u(v(x))^t h(x), u(v(x))^\omega\right\rangle$, with $v(x) = x^2 - \gamma_0$, and $h(x)$ as in Type 3, $\deg h(x) \le \omega - t - 1$, and $\omega < T$, where T is the smallest integer such that $u(x^2 - \gamma_0)^T \in \langle (x^2 - \gamma_0)^i + u(x^2 - \gamma_0)^t h(x)\rangle$; i.e., such T can be determined as

$$T = \begin{cases} i, & \text{if } h(x) = 0 \\ \min\{i, p^s - i + t\}, & \text{if } h(x) \ne 0\,. \end{cases}$$

Furthermore, the number of distinct γ-constacyclic codes of length $2p^s$ over $\mathcal{R}$, i.e., distinct ideals of the ring $\Re(2p^s, \gamma)$, is $\frac{2(p^{2m}+1)p^{m(p^s-1)} - 2p^{4m} - 2}{(p^{2m}-1)^2} + \frac{(2p^{2m}+3)p^{m(p^s-1)} - 2p^s - 1}{p^{2m}-1} + p^{m(p^s-1)} + 2$.

When $\gamma \in \mathbb{F}_{p^m}^*$, it is easy to see that for any γ-constacyclic code C of length $2p^s$ over $\mathcal{R}$, its residue code $\mathrm{Res}(C)$ and torsion code $\mathrm{Tor}(C)$ are γ-constacyclic codes of length $2p^s$ over $\mathbb{F}_{p^m}$, respectively. By [16], each γ-constacyclic code of length $2p^s$ over $\mathbb{F}_{p^m}$ is an ideal of the form $\langle (x^2 - \gamma_0)^i\rangle$ of the finite chain ring $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{2p^s} - \gamma\rangle}$, where $0 \le i \le p^s$, and each code $\langle (x^2 - \gamma_0)^i\rangle$ contains $p^{2m(p^s-i)}$ codewords. Therefore, we can determine the size of all γ-constacyclic codes of length $2p^s$ over $\mathcal{R}$ in Theorem 2.3 by multiplying the sizes of $\mathrm{Res}(C)$ and $\mathrm{Tor}(C)$ in each case.

Theorem 2.4 (cf. [10, Section 3]): Let $\gamma \in \mathbb{F}_{p^m}^*$ and C be a γ-constacyclic code of length $2p^s$ over $\mathcal{R}$ in Theorem 2.3, then the number of codewords of C, denoted by n_C, is determined as follows.

- If $C = \langle 0\rangle$, then $n_C = 1$.
- If $C = \langle 1\rangle$, then $n_C = p^{4mp^s}$.
- If $C = \langle u(x^2 - \gamma_0)^i\rangle$, where $0 \le i \le p^s - 1$, then $n_C = p^{2m(p^s-i)}$.
- If $C = \langle (x^2 - \gamma_0)^i\rangle$, where $1 \le i \le p^s - 1$, then $n_C = p^{4m(p^s-i)}$.
- If $C = \langle (x^2 - \gamma_0)^i + u(x^2 - \gamma_0)^t h(x)\rangle$, where $1 \le i \le p^s - 1$, $0 \le t < i$, and $h(x)$ is a unit, then

$$n_C = \begin{cases} p^{4m(p^s-i)}, & \text{if } 1 \le i \le p^{s-1} + \frac{t}{2} \\ p^{2m(p^s-i)}, & \text{if } p^{s-1} + \frac{t}{2} < i \le p^s - 1 \end{cases}.$$

- If $C = \langle (x^2 - \gamma_0)^i + u(x^2 - \gamma_0)^t h(x), u(x^2 - \gamma_0)^\kappa\rangle$, where $1 \le i \le p^s - 1$, $0 \le t < i$, either $h(x)$ is 0 or $h(x)$ is a unit, and

$$\kappa < T = \begin{cases} i, & \text{if } h(x) = 0 \\ \min\{i, p^s - i + t\}, & \text{if } h(x) \ne 0\,, \end{cases}$$

then $n_C = p^{2m(2p^s-i-\kappa)}$.

It is well-known from Proposition 2.2 that cyclic codes of length $5p^s$ over $\mathcal{R}$ are ideals of the ring $\mathcal{R}_1 = \frac{\mathcal{R}[x]}{\langle x^{5p^s} - 1\rangle}$. We see that $x^5 - 1 = (x - 1)\left(x^4 + x^3 + x^2 + x + 1\right)$. Let ξ be a primitive $(p^m - 1)$th root of unity, so that $\mathbb{F}_{p^m} = \{0, \xi, \xi^2, \ldots, \xi^{p^m-2}, \xi^{p^m-1} = 1\}$. Assume that $p^m \equiv 1 \pmod 5$, where m is a positive integer. This means that $p^m \equiv 1 \pmod{10}$ and $p^m \equiv 1 \pmod 2$. Hence, $\xi^{\frac{p^m-1}{2}} = -1$. We see that $(-\xi^{\frac{(p^m-1)}{10}})^5 = 1$, i.e., $-\xi^{\frac{(p^m-1)}{10}}$ is a root of the equation $x^5 - 1 = 0$. Similar to $-\xi^{\frac{(p^m-1)}{10}}$, it is easy to see that $-\xi^{\frac{3(p^m-1)}{10}}$, $-\xi^{\frac{7(p^m-1)}{10}}$, $-\xi^{\frac{9(p^m-1)}{10}}$ are also roots of the equation $x^5 - 1 = 0$. Then the equation $x^5 - 1 = 0$ has five distinct roots in $\mathcal{R}$. They are $1, -\xi^{\frac{(p^m-1)}{10}}, -\xi^{\frac{3(p^m-1)}{10}}, -\xi^{\frac{7(p^m-1)}{10}}, -\xi^{\frac{9(p^m-1)}{10}}$. Put $\gamma_2 = -\xi^{\frac{(p^m-1)}{10}}$, $\gamma_3 = -\xi^{\frac{3(p^m-1)}{10}}$, $\gamma_4 = -\xi^{\frac{7(p^m-1)}{10}}$, and $\gamma_5 = -\xi^{\frac{9(p^m-1)}{10}}$. Then $(x^4 + x^3 + x^2 + x + 1)^{p^s}$ can express as follows:

$$(x^4+x^3+x^2+x+1)^{p^s} = \left(x^{p^s}-\gamma_2^{p^s}\right)\left(x^{p^s}-\gamma_3^{p^s}\right) \times \left(x^{p^s}-\gamma_4^{p^s}\right)\left(x^{p^s}-\gamma_5^{p^s}\right).$$

This implies that

$$\begin{aligned} x^{5p^s}-1 &= \left(x^5-1\right)^{p^s} \\ &= \left(x^{p^s}-1\right)\left(x^{p^s}-\gamma_2^{p^s}\right)\left(x^{p^s}-\gamma_3^{p^s}\right) \\ &\times \left(x^{p^s}-\gamma_4^{p^s}\right)\left(x^{p^s}-\gamma_5^{p^s}\right). \end{aligned}$$

By Chinese Reimainder Theorem, we have

$$\begin{aligned} \mathcal{R}_1 &= \frac{\mathcal{R}[x]}{\langle x^{5p^s}-1\rangle} \\ &\cong \frac{\mathcal{R}[x]}{\langle (x^{p^s}-1)\rangle} \bigoplus \frac{\mathcal{R}[x]}{\langle (x^{p^s}-\gamma_2^{p^s})\rangle} \bigoplus \frac{\mathcal{R}[x]}{\langle (x^{p^s}-\gamma_3^{p^s})\rangle} \\ &\oplus \frac{\mathcal{R}[x]}{\langle (x^{p^s}-\gamma_4^{p^s})\rangle} \bigoplus \frac{\mathcal{R}[x]}{\langle (x^{p^s}-\gamma_5^{p^s})\rangle} \\ &\cong \mathcal{R}_+ \bigoplus \mathcal{R}_{\gamma_2} \bigoplus \mathcal{R}_{\gamma_3} \bigoplus \mathcal{R}_{\gamma_4} \bigoplus \mathcal{R}_{\gamma_5}, \end{aligned}$$

where $\mathcal{R}_+ = \frac{\mathcal{R}[x]}{\langle (x^{p^s}-1)\rangle}$ and $\mathcal{R}_{\gamma_i} = \frac{\mathcal{R}[x]}{\langle (x^{p^s}-\gamma_i^{p^s})\rangle}$ ($i = 2, 3, 4, 5$). Hence, ideals of $\mathcal{R}_1$ are of the form $C_1 \bigoplus C_2 \bigoplus C_3 \bigoplus C_4 \bigoplus C_5$, where C_1 is a cyclic code of length p^s over $\mathcal{R}$ and C_i is a γ_i-constacyclic code of length p^s over $\mathcal{R}$ ($i = 2, 3, 4, 5$). Then the algebraic structures of all constacyclic codes of length p^s over $\mathcal{R}$ studied in [12] allow us to determine the algebraic structure of all cyclic codes of length $5p^s$ over $\mathcal{R}$ when $p \equiv 1 \pmod 5$. In [12], Dinh determined the number of codewords in each constacyclic code of length p^s over $\mathcal{R}$. Therefore, the number of codewords in each cyclic code of length $5p^s$ over $\mathcal{R}$ can be obtained. Then we have the following theorem.

Theorem 2.5: Let C be a cyclic code of length $5p^s$ over $\mathcal{R}$. Then

$$C = C_1 \bigoplus C_2 \bigoplus C_3 \bigoplus C_4 \bigoplus C_5,$$

where C_1 is a cyclic code, C_2 is a γ_2-constacyclic code, C_3 is a γ_3-constacyclic code, C_4 is a γ_4-constacyclic code, C_5 is a γ_5-constacyclic code of length p^s over $\mathcal{R}$. Moreover, $|C| = |C_1||C_2||C_3||C_4||C_5|$ and $C^\perp = C_1^\perp \bigoplus C_2^\perp \bigoplus C_3^\perp \bigoplus C_4^\perp \bigoplus C_5^\perp$.

We see that x^5-1 can be expressed as $x^5-1 = (x-1)(x^4+x^3+x^2+x+1)$. Assume that $(x^4+x^3+x^2+x+1)$ is reducible over $\mathbb{F}_{p^m}$. Then there exists $\alpha \in \mathbb{F}_{p^m}$ such that $\alpha^4+\alpha^3+\alpha^2+\alpha+1 = 0$. This implies that $\alpha^5-1 = 0$, i.e., $\alpha^5 = 1$. From $p \neq 5$, we have $\alpha \neq 1$. Since $p \equiv 2 \pmod 5$ or $p \equiv 3 \pmod 5$ ($p^m \not\equiv 1 \pmod 5$), the order of the multiplicative group of $\mathbb{F}_{p^m}$ is not divisible by 5. It follows that $\alpha \notin \mathbb{F}_{p^m}$, which is a contradiction. Therefore, $(x^4+x^3+x^2+x+1)$ is irreducible over $\mathbb{F}_{p^m}$. Assume that $(x^4+x^3+x^2+x+1)$ is reducible over $\mathcal{R}$. Then there exists $\eta \in \mathcal{R}$ satisfying $\eta^4+\eta^3+\eta^2+\eta+1 = 0$, where $\eta = \lambda + u\beta$ and $\lambda, \beta \in \mathbb{F}_{p^m}$. Since $\eta^4+\eta^3+\eta^2+\eta+1 = 0$, we have $\eta^5 = 1$. Hence, we have $\beta = 0$. This implies that $\eta = \lambda \in \mathbb{F}_{p^m}$. Hence, $\lambda^4+\lambda^3+\lambda^2+\lambda+1 = 0$, which is a contradiction because $(x^4+x^3+x^2+x+1)$ is irreducible over $\mathbb{F}_{p^m}$. It means that the polynomial $(x^4+x^3+x^2+x+1)$ is irreducible over $\mathcal{R}$. We have the following result.

Theorem 2.6: Let C be a cyclic code of length $5p^s$ over $\mathcal{R}$. Then cyclic codes of length $5p^s$ over $\mathcal{R}$ can be represented as $C = C_1 \bigoplus C_2$, where C_1 is an ideal of the ring $\frac{\mathcal{R}[x]}{\langle x^{p^s}-1\rangle}$ which is determined in [12] and C_2 is an ideal of the ring $\frac{\mathcal{R}[x]}{\langle (t(x))^{p^s}\rangle}$, where $t(x) = x^4+x^3+x^2+x+1$. Ideals of $\frac{\mathcal{R}[x]}{\langle (t(x))^{p^s}\rangle}$ are

- *Type 1: (trivial ideals)*

$$\langle 0\rangle, \ \langle 1\rangle.$$

- *Type 2: (principal ideals with nonmonic polynomial generators)*

$$\left\langle u(t(x))^i\right\rangle,$$

where $0 \le i \le p^s-1$.

- *Type 3: (principal ideals with monic polynomial generators)*

$$\left\langle (t(x))^i + u(t(x))^t h(x)\right\rangle,$$

where $1 \le i \le p^s-1, 0 \le t < i$, and either $h(x)$ is 0 or $h(x)$ is a unit which can be represented as $h(x) = \sum_j (h_{3j}x^3+h_{2j}x^2+h_{1j}x+h_{0j})(t(x))^j$, with $h_{3j}, h_{2j}, h_{1j}, h_{0j} \in \mathbb{F}_{p^m}$, and $h_{30}x^3+h_{20}x^2+h_{10}x+h_{00} \neq 0$.

- *Type 4: (nonprincipal ideals)*

$$\left\langle (t(x))^i + u\sum_{j=0}^{\omega-1}(v(x))(t(x))^j, u(t(x))^\omega\right\rangle,$$

where $v(x) = a_jx^3+b_jx^2+c_jx+d_j, 1 \le i \le p^s-1, a_j, b_j, c_j, d_j \in \mathbb{F}_{p^m}$, and $\omega < T$, where T is the smallest integer such that

$$u(t(x))^T \in \langle (t(x))^i + u\sum_{j=0}^{i-1}(v(x))(t(x))^j\rangle;$$

or equivalently,

$$\left\langle (t(x))^i + u(t(x))^t h(x), u(t(x))^\omega\right\rangle,$$

with $h(x)$ as in Type 3, and $\deg h(x) \le \omega - t - 1$.

In addition, the enumeration of elements in each ideal of the ring $\frac{\mathcal{R}[x]}{\langle (t(x))^{p^s}\rangle}$ is given as follows. Let I be an ideal of the ring $\frac{\mathcal{R}[x]}{\langle (t(x))^{p^s}\rangle}$. Then the numbers of elements of I, denoted by n_I is determined as follows.

- *If $I = \langle 0\rangle$, then $n_I = 1$.*
- *If $I = \langle 1\rangle$, then $n_I = p^{8mp^s}$.*
- *If $I = \langle u(t(x))^i\rangle$, where $0 \le i \le p^s-1$, then $n_I = p^{4m(p^s-i)}$.*
- *If $I = \langle (t(x))^i\rangle$, where $1 \le i \le p^s-1$, then $n_I = p^{8m(p^s-i)}$.*

• *If* $I = \langle (t(x))^i + u(t(x))^t h(x)\rangle$, *where* $1 \le i \le p^s - 1, 0 \le t < i$, *and* $h(x)$ *is a unit, then*

$$n_I = \begin{cases} p^{8m(p^s-i)}, & \text{if } 1 \le i \le p^{s-1} + \frac{t}{2} \\ p^{4m(2p^s-i-T)}, & \text{if } p^{s-1} + \frac{t}{2} < i \le p^s - 1 \end{cases}.$$

• *If* $I = \langle (t(x))^i + u(t(x))^t h(x), u(t(x))^\kappa\rangle$, *where* $1 \le i \le p^s - 1, 0 \le t < i$, *either* $h(x)$ *is 0 or* $h(x)$ *is a unit, and*

$$\kappa < T = \begin{cases} i, & \text{if } h(x) = 0 \\ \min\{i, p^s - i + t\}, & \text{if } h(x) \ne 0, \end{cases}$$

then $n_I = p^{4m(2p^s-i-\kappa)}$.

When $p \equiv 4 \pmod 5$, we consider the map $\Theta_1 : \frac{\mathcal{R}[x]}{\langle (x^2+(1-\gamma)2^{-1}x+1)^{p^s}\rangle} \to \frac{\mathcal{R}[x]}{\langle (x^2+(5+\gamma)2^{-3})^{p^s}\rangle}$ defined by $f(x) \to f(x-(1-\gamma)2^{-2})$. For polynomials $f(x), g(x) \in \mathcal{R}[x]$, then $f(x) \equiv g(x) \pmod{\left(x^2 + (1-\gamma)2^{-1}x + 1\right)^{p^s}}$ if and only if there exists $q(x) \in R[x]$ such that $f(x) - g(x) = q(x)\left(\left(x^2 + (1-\gamma)2^{-1}x + 1\right)^{p^s}\right)$. Put $\alpha = 1-\gamma$. Then we have

$$\begin{aligned} & f(x-(\alpha)2^{-2}) - g(x-(\alpha)2^{-2}) \\ &= q\left(x-(\alpha)2^{-2}\right)\left[\left(x-(\alpha)2^{-2}\right)^2 \right. \\ &\quad \left. + (\alpha)2^{-1}\left(x-(\alpha)2^{-2}\right) + 1\right]^{p^s} \\ &= q\left(x-(\alpha)2^{-2}\right)\left[x^2 - (\alpha)^2 2^{-4} + 1\right]^{p^s} \\ &= q\left(x-(1-\gamma)2^{-2}\right)\left[x^2 - (6-2\gamma)2^{-4} + 1\right]^{p^s} \\ &= q\left(x-(1-\gamma)2^{-2}\right)\left(x^2 + (5+\gamma)2^{-3}\right)^{p^s}. \end{aligned}$$

This implies that $f(x-(1-\gamma)2^{-2}) \equiv g(x-(1-\gamma)2^{-2} \pmod{(x^2+(5+\gamma)2^{-3})^{p^s}})$. Hence, $\Theta_1(f(x)) = \Theta_1(g(x))$ in $\frac{\mathcal{R}[x]}{\langle (x^2+(5+\gamma)2^{-3})^{p^s}\rangle}$ if and only if $f(x) \equiv g(x)$ in $\frac{\mathcal{R}[x]}{\langle (x^2+(1-\gamma)2^{-1}x+1)^{p^s}\rangle}$. Therefore, Θ_1 is well-defined and one-to-one. It is easy to see that Θ_1 is onto and Θ_1 is a ring homomorphism. It means that Θ_1 is a ring isomorphism. Similar to the map Θ_1, we consider the map $\Theta_2 : \frac{\mathcal{R}[x]}{\langle (x^2+(1+\gamma)2^{-1}x+1)^{p^s}\rangle} \to \frac{\mathcal{R}[x]}{\langle (x^2-(\gamma-5)2^{-3})^{p^s}\rangle}$ defined by $f(x) \to f(x-(1+\gamma)2^{-2})$. Then we can prove that Θ_2 is a ring isomorphism. The algebraic structures of all constacyclic codes of lengths p^s, $2p^s$ over $\mathcal{R}$ studied in [10] and [12] allow us to determine the algebraic structure of all cyclic codes of length $5p^s$ over $\mathcal{R}$. Moreover, [12] and [10] determined the number of codewords in each constacyclic code of lengths p^s, $2p^s$ over $\mathcal{R}$. Therefore, the number of codewords in each cyclic code of length $5p^s$ over $\mathcal{R}$ can be obtained in the following theorem.

Theorem 2.7: If C *is a cyclic code of length* $5p^s$ *over* $\mathcal{R}$*, then* C *can be represented as* $C = C_+ \bigoplus C_{\alpha_1} \bigoplus C_{\alpha_2}$*, where* C_+ *is a cyclic code of length* p^s *over* $\mathcal{R}$*,* C_{α_1} *is an* α_1*-constacyclic code and* C_{α_2} *is an* α_2*-constacyclic code of length* $2p^s$ *over* $\mathcal{R}$*. Moreover,* $|C| = |C_+||C_{\alpha_1}||C_{\alpha_2}|$ *and* $C^\perp = C_+^\perp \bigoplus C_{\alpha_1}^\perp \bigoplus C_{\alpha_2}^\perp$*. In particular,* $C = \langle u\rangle$ *is a self-dual cyclic code of length* $5p^s$ *over* $\mathcal{R}$*.*

The Hamming distance of all λ-constacyclic codes of length p^s over $\mathcal{R}$ is given in the following theorem.

Theorem 2.8 ([12], [19], [28]): Let C *be a* λ*-constacyclic code of length* p^s *over* $\mathcal{R}$*. Then Hamming distance of all* λ*-constacyclic codes* C *is determined as follows.*

1) *[12] If* $\lambda = \alpha + u\gamma$*, then* $C = \langle (\alpha_0 x - 1)^i\rangle \subseteq \frac{\mathcal{R}[x]}{\langle x^{p^s} - (\alpha+u\gamma)\rangle}$*, for* $i \in \{0, 1, \ldots, 2p^s\}$*, and the Hamming distance* $d_H(C)$ *is completely determined by*

$$d_H(C) = \begin{cases} \bullet\, 1, \text{ if } 0 \le j \le p^s \\ \bullet\, (\delta+1)p^k, \\ \text{if } 2p^s - p^{s-k} + (\delta-1)p^{s-k-1} + 1 \le j \\ \text{and } j \le 2p^s - p^{s-k} + \delta p^{s-k-1} \\ \text{where } 1 \le \delta \le p-1, \\ \text{and } 0 \le k \le s-1 \\ \bullet\, 0, \text{ if } j = 2p^s. \end{cases}$$

2) *([19, Theorem 3.2] and [28, Appendix]) If* $\lambda \in \mathbb{F}_{p^m} \setminus \{0\}$*, the* λ*-constacyclic codes of length* p^s *over* $\mathcal{R}$*, i.e., ideals of the ring* $\frac{\mathcal{R}[x]}{\langle x^{p^s} - \lambda\rangle}$ *have their Hamming distances completely determined as follows.*

• *Type* 1 *(trivial ideals):* $\langle 0\rangle$*,* $\langle 1\rangle$*;* $d_H(\langle 0\rangle) = 0$*,* $d_H(\langle 1\rangle) = 1$*.*

• *Type* 2 *(principal ideals with nonmonic polynomial generators):*
$C = \langle u(x-\lambda_0)^j\rangle$*, where* $0 \le j \le p^s - 1$*. Then* $d_H(C) = d_H(\langle (x-\lambda_0)^j\rangle_F)$ *and*

$$d_H(C) = \begin{cases} \bullet\, 1, \text{ if } j = 0 \\ \bullet\, (\delta+1)p^k, \\ \text{if } p^s - p^{s-k} + (\delta-1)p^{s-k-1} + 1 \le j \\ \text{and } j \le p^s - p^{s-k} + \delta p^{s-k-1} \\ \text{where } 1 \le \delta \le p-1, \\ \text{and } 0 \le k \le s-1 \end{cases}$$

• *Type* 3 *(principal ideals with monic polynomial generators):* $C_3 = \left\langle (x-\lambda_0)^j + u(x-\lambda_0)^t h(x)\right\rangle$*, where* $1 \le j \le p^s - 1$*,* $0 \le t < j$*, and either* $h(x)$ *is 0 or* $h(x)$ *is a unit in* $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s} - \lambda\rangle}$*, where* $1 \le T \le i \le p^s - 1, 0 \le t < T$*, either* $h(x)$ *is 0 or* $h(x)$ *is a unit and* T *is the smallest integer satisfying* $u(x-\lambda_0)^T \in C_3$*, i.e.,*

$$T = \begin{cases} j, & \text{if } h(x) = 0 \\ \min\{j, p^s - j + t\}, & \text{if } h(x) \ne 0 \end{cases}.$$

Then $d_H(C_3) = d_H(\langle (x-\lambda_0)^T\rangle_F)$*.*
Moreover,
(a) If $h(x)$ *is 0 or* $h(x)$ *is a unit and* $1 \le j \le \frac{p^s+t}{2}$*, then*

$$\begin{aligned} d_H(C_3) &= d_H(\langle (x-\lambda_0)^j\rangle_F) \\ &= (\delta+1)p^k, \end{aligned}$$

where $p^s - pr + (\delta - 1)r + 1 \leq j \leq p^s - pr + \delta r$, $1 \leq \delta \leq p-1$, $r = p^{s-k-1}$, and $0 \leq k \leq s-1$.
(b) If $h(x)$ is a unit and $\frac{p^s+t}{2} < j \leq p^s - 1$, then

$$\begin{aligned} d_H(C_3) &= d_H(\langle (x-\lambda_0)^{p^s-j+t}\rangle_F) \\ &= (\delta+1)p^k, \end{aligned}$$

where $t + pr - \delta r \leq j \leq t + pr - (\delta - 1)r - 1$, $1 \leq \delta \leq p-1$, and $0 \leq k \leq s-1$.
• Type 4 (nonprincipal ideals):
$C = \left\langle (x-\lambda_0)^j + u(x-\lambda_0)^t h(x), u(x-\lambda_0)^\kappa \right\rangle$, with $h(x)$ as in Type 3, $\deg(h) \leq \kappa - t - 1$, and $\kappa < T$, where T is the smallest integer such that $u(x-\lambda_0)^T \in \left\langle (x-\lambda_0)^j + u(x-\lambda_0)^t h(x) \right\rangle$; i.e., such T can be determined as

$$T = \begin{cases} j, & \text{if } h(x) = 0 \\ \min\{i, p^s - j + t\}, & \text{if } h(x) \neq 0. \end{cases}$$

Then

$$d_H(C) = d_H(\langle (x-\lambda_0)^\kappa\rangle_F) = (\delta+1)p^k,$$

where $p^s - p^{s-k} + (\delta-1)p^{s-k-1} + 1 \leq \kappa \leq p^s - p^{s-k} + \delta p^{s-k-1}$, $1 \leq \delta \leq p-1$, and $0 \leq k \leq s-1$.

The Hamming distance of all non-trivial cyclic codes of length $5p^s$ over $\mathbb{F}_{p^m}$ which are of the form $(x-1)^i(x^4+x^3+x^2+x+1)^j$ is given in the following theorem.

Theorem 2.9: [34, Theorem 5.16] *Assume that $0 \leq \beta_0, \beta_1 \leq p-2$, and $0 \leq \tau_1 \leq \tau_0 \leq s-1$. Let $0 \leq i \leq j \leq p^s$. Then the codes $C = \left\langle (x-1)^i(x^4+x^3+x^2+x+1)^j \right\rangle$ have the following Hamming distances*:

$$d_H(C) = \begin{cases} \bullet 1, \ \text{if } i = j = 0, \\ \bullet 2, \ \text{if } i = 0 \text{ and } 0 < j \leq p^{s-1}, \\ \bullet 3, \ \text{if } i = 0 \text{ and } p^{s-1} < j \leq 2p^{s-1}, \\ \bullet 4, \ \text{if } i = 0 \text{ and } 2p^{s-1} < j \leq 3p^{s-1}, \\ \bullet 5, \ \text{if } i = 0 \text{ and } 3p^{s-1} < j \leq p^s, \\ \bullet \min\{(\beta_0+2)p^{\tau_0}, 5(\beta_1+2)p^{\tau_1}\}, \\ \ \text{if } p^s - p^{s-\tau_0} + \beta_0 p^{s-\tau_0-1} + 1 \leq i \\ \ \text{and } i \leq p^s - p^{s-\tau_0} + (\beta_0+1)p^{s-\tau_0-1}, \\ p^s - p^{s-\tau_1} + \beta_1 p^{s-\tau_1-1} + 1 \leq j \\ \text{and } j \leq p^s - p^{s-\tau_1} + (\beta_1+1)p^{s-\tau_1-1}, \\ \bullet 5(\beta_1+2)p^{\tau_1}, \\ \ \text{if } j = p^s, \\ \ p^s - p^{s-\tau_1} + \beta_1 p^{s-\tau_1-1} + 1 \leq i \\ \ \text{and } j \leq p^s - p^{s-\tau_1} + (\beta_1+1)p^{s-\tau_1-1}, \\ \bullet 0, \ \text{if } i = j = p^s. \end{cases}$$

In [42], the Hamming distance of cyclic codes of length $2p^s$ over $\mathbb{F}_{p^m}$ is studied. Using Theorem 2 (Table 1) in [42], the Hamming distance of cyclic codes of length $2p^s$ over $\mathbb{F}_{p^m}$ when $0 \leq i = j \leq p^s$ is determined as follows.

Theorem 2.10: [42, Theorem 2] Let p be an odd prime, and m, s, τ be intergers. The cyclic codes $C_{i,i}$ of length $2p^s$ over $\mathbb{F}_{p^m}$ are of the form $C_{i,i} = \langle (x^2-1)^i\rangle$ for $i = 0, 1, \cdots, p^s$. Then the Hamming distance $d_H(C_{i,i})$ is determined by:

$$d_H(C_{i,i}) = \begin{cases} \bullet 1, \ \text{if } i = 0 \\ \bullet (\gamma+1)p^{e_1}, \\ \ \text{if } p^s - p^{s-e_1} + \gamma p^{s-e_1-1} + 1 \leq i \\ \text{and } i \leq p^s - p^{s-e_1} + (\gamma+1)p^{s-e-1} \\ \text{where } 0 \leq \gamma \leq p-2, \text{ and } 0 \leq e_1 \leq s-1 \\ \bullet 0, \ \text{if } i = p^s. \end{cases}$$

In 1998, the Singleton bound for finite chain ring $\mathcal{R}$ with respect to the Hamming distance $d_H(C)$ is given in [41]. We review it as follows.

Theorem 2.11(Singleton Bound With Respect to Hamming Distance): [41] *Let C be a linear code of length n over $\mathcal{R}$ with Hamming distance $d_H(C)$. Then $|C| \leq p^{2m(n-d_H(C)+1)}$. In addition, C is said to be a maximum distance separable (MDS) code with respect to the Hamming distance if $|C| = |\mathcal{R}|^{n-d_H(C)+1}$.*

In this paper, the Hamming distances of cyclic codes of length $5p^s$ over $\mathcal{R}$ are given in the following table.

Case	Section
$p \equiv 2, 3 \pmod 5$	3
$p \equiv 1 \pmod 5$	4
$p \equiv 4 \pmod 5$	5

III. HAMMING DISTANCES AND MDS CODES OF CYCLIC CODES OF LENGTH $5p^s$ OVER $\mathcal{R}$ WHEN $p \equiv 2, 3 \pmod 5$

It is well-known that $\mathbb{F}_{p^m}$ is a subring of $\mathcal{R}$. From now on, we denote $d_H(C_F)$ as the Hamming distance of the code C over $\mathbb{F}_{p^m}$. For each codeword $c = (c_0, c_1, c_2, \cdots, c_{n-1})$ over $\mathcal{R}$, the polynomial representation of $c(x)$ is given by $c(x) = \tilde{a}(x) + u\tilde{b}(x)$, where $\tilde{a}(x), \tilde{b}(x)$ are two arbitrary polynomials over $\mathbb{F}_{p^m}$, with corresponding codewords $\tilde{a} = (\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_{n-1})$ and $\tilde{b} = (\tilde{b}_0, \tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_{n-1})$ over $\mathbb{F}_{p^m}$, respectively. As $c_i = \tilde{a}_i + u\tilde{b}_i$, $c_i = 0$ if and only if $\tilde{a}_i = \tilde{b}_i = 0$. It implies that $wt_H(c(x)) \geq \max\{wt_H(\tilde{a}(x)), wt_H(\tilde{b}(x))\}$.

Throughout this section, we denote $x^4 + x^3 + x^2 + x + 1 = \alpha(x)$. By Theorem 2.6, the structure of cyclic codes of length $5p^s$ over $\mathcal{R}$ when $p \equiv 2$ *or* $3 \pmod 5$ is provided. In order to compute the Hamming distance of cyclic codes of length $5p^s$ over $\mathcal{R}$ when $p \equiv 2$ *or* $3 \pmod 5$, we need to determine the Hamming distance for each type of ideals of $\frac{\mathcal{R}[x]}{\langle (\alpha(x))^{p^s}\rangle}$ one by one. Obviously, the Hamming distances of the trivial ideals $\langle 0\rangle$, $\langle 1\rangle$ are given by 0 and 1, respectively.

The Hamming distance of ideals of Type 2 of $\frac{\mathcal{R}[x]}{\langle (\alpha(x))^{p^s}\rangle}$ can be determined in the following result.

Theorem 3.1: Let $C = \langle u(\alpha(x))^j\rangle$, $0 \leq j \leq p^s - 1$ be an ideal of Type 2 of $\frac{\mathcal{R}[x]}{\langle (\alpha(x))^{p^s}\rangle}$. Then $d_H(C) = d_H(\langle (\alpha(x))^j\rangle_F)$, and $d_H(C)$ is given by

$$d_H(C) = \begin{cases} \bullet 1, & \text{if } i=j=0, \\ \bullet 2, & \text{if } i=0 \text{ and } 0<j\le p^{s-1}, \\ \bullet 3, & \text{if } i=0 \text{ and } p^{s-1}<j\le 2p^{s-1}, \\ \bullet 4, & \text{if } i=0 \text{ and } 2p^{s-1}<j\le 3p^{s-1}, \\ \bullet 5, & \text{if } i=0 \text{ and } 3p^{s-1}<j\le p^s. \end{cases}$$

Proof: We consider the following two cases:

Case 1: If $j=0$, then $d_H(C)=1$.

Case 2: If $p^s-p^{s-\varsigma}+(\delta-1)p^{s-\varsigma-1}+1 \le j \le p^s-p^{s-\varsigma}+\delta p^{s-\varsigma-1}$, the codewords of the code $C=\langle u(\alpha(x))^j\rangle$ are exactly same as the codewords of the constacyclic codes $\langle(\alpha(x))^j\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle(\alpha(x))^{p^s}\rangle}$ multiplied by u, where $0\le j\le p^s-1$. Hence, we have $d_H(C)=d_H(\langle(\alpha(x))^j\rangle_F)$. Using Theorem 2.9,

$$d_H(C) = \begin{cases} \bullet 1, & \text{if } i=j=0, \\ \bullet 2, & \text{if } i=0 \text{ and } 0<j\le p^{s-1}, \\ \bullet 3, & \text{if } i=0 \text{ and } p^{s-1}<j\le 2p^{s-1}, \\ \bullet 4, & \text{if } i=0 \text{ and } 2p^{s-1}<j\le 3p^{s-1}, \\ \bullet 5, & \text{if } i=0 \text{ and } 3p^{s-1}<j\le p^s, \end{cases}$$

completing our proof. □

Next, we discuss the Hamming distance of ideals of Type 3 of $\frac{\mathcal{R}[x]}{\langle(\alpha(x))^{p^s}\rangle}$.

Theorem 3.2: Let $C=\langle(\alpha(x))^j+u(\alpha(x))^t v(x)\rangle$ be an ideal of Type 3 of $\frac{\mathcal{R}[x]}{\langle(\alpha(x))^{p^s}\rangle}$, where $1\le j\le p^s-1$, $0\le t<j$ and either $v(x)$ is a unit in $\frac{\mathcal{R}[x]}{\langle(\alpha(x))^{p^s}\rangle}$ or 0. Then, we have $d_H(C)=d_H(\langle(\alpha(x))^R\rangle_F)$, where R is the smallest integer satisfying $u(\alpha(x))^R\in\langle(\alpha(x))^j+u(\alpha(x))^t v(x)\rangle$ and

$$R = \begin{cases} j, & \text{if } v(x)=0 \\ \min\{j, p^s-j+t\}, & \text{if } v(x)\ne 0. \end{cases}$$

Moreover,

(1) If $v(x)$ is 0 or $v(x)$ is a unit and $1\le j\le\frac{p^s+t}{2}$, then

$$d_H(C) = \begin{cases} \bullet 1, & \text{if } i=j=0, \\ \bullet 2, & \text{if } i=0 \text{ and } 0<j\le p^{s-1}, \\ \bullet 3, & \text{if } i=0 \text{ and } p^{s-1}<j\le 2p^{s-1}, \\ \bullet 4, & \text{if } i=0 \text{ and } 2p^{s-1}<j\le 3p^{s-1}, \\ \bullet 5, & \text{if } i=0 \text{ and } 3p^{s-1}<j\le p^s. \end{cases}$$

(2) If $v(x)$ is a unit and $\frac{p^s+t}{2}<j\le p^s-1$, then

$$d_H(C) = \begin{cases} \bullet 1, \text{ if } i=j=0, \\ \bullet 2, \text{ if } i=0 \text{ and } p^s-p^{s-1}+t<j\le p^s+t, \\ \bullet 3, \text{ if } i=0 \text{ and } p^s-2p^{s-1}+t<j \\ \text{and } j\le p^s-p^{s-1}+t, \\ \bullet 4, \text{ if } i=0 \text{ and } p^s-3p^{s-1}+t<j \\ \text{and } j\le p^s-2p^{s-1}+t, \\ \bullet 5, \text{ if } i=0 \text{ and } t<j\le p^s-3p^{s-1}+t. \end{cases}$$

Proof. Since R is the smallest integer satisfying $u(\alpha(x))^R\in\langle(\alpha(x))^j+u(\alpha(x))^t v(x)\rangle$, therefore we have,

$$\begin{aligned} d_H(C) &\le d_H(\langle u(\alpha(x))^R\rangle) \\ &= d_H(\langle(\alpha(x))^R\rangle_F). \end{aligned}$$

This implies that $d_H(C_3)\le d_H(\langle(\alpha(x))^R\rangle_F)$. Now, let us take an arbitrary polynomial $c(x)\in C$. So, there exist two polynomials $f_0(x)$ and $f_u(x)$ over $\mathbb{F}_{p^m}$ satisfying

$$\begin{aligned} c(x) &= [t(x)][(\alpha(x))^j+u(\alpha(x))^t v(x)] \\ &= f_0(x)(\alpha(x))^j+u[f_0(x)(\alpha(x))^t v(x)+f_u(x)(\alpha(x))^j], \end{aligned}$$

where $t(x)=f_0(x)+uf_u(x)$. Now, we consider two cases:

Case 1: When $v(x)=0$, then we have

$$\begin{aligned} wt_H(c(x)) &\ge \max\left\{wt_H(f_{0,\lambda_1}(x)), wt_H(f_{u,\lambda_1}(x))\right\} \\ &\ge \max\left\{wt_H(f_{0,\lambda_1}(x)), wt_H(f_{0,\lambda_1}(x))\right\} \\ &\ge d_H(\langle(\alpha(x))^j\rangle_F), \\ &= d_H(\langle(\alpha(x))^R\rangle_F), \end{aligned}$$

where $f_{0,\lambda_1}(x)=f_0(x)(\alpha(x))^j$ and $f_{u,\lambda_1}(x)=f_u(x)(\alpha(x))^j$.

Case 2: When $v(x)\ne 0$, then we have

$$\begin{aligned} wt_H(c(x)) &\ge \max\left\{wt_H(f_{0,\lambda_1}(x)), wt_H(h(x))\right\} \\ &\ge \max\left\{wt_H(f_{0,\lambda_1}(x)), wt_H(f_0(x)(\alpha(x))^{p^s-j+t})\right\} \\ &\ge d_H(\langle(\alpha(x))^{\min\{j,\, p^s-j+t\}}\rangle_F), \\ &= d_H(\langle(\alpha(x))^R\rangle_F), \end{aligned}$$

where $f_{0,\lambda_1}(x)=f_0(x)(\alpha(x))^j$, and $h(x)=f_0(x)(\alpha(x))^t v(x)+f_u(x)(\alpha(x))^j$. Hence, by combining both the cases, we get $d_H(\langle(\alpha(x))^R\rangle_F)\le d_H(C)$, which implies that, $d_H(\langle(\alpha(x))^R\rangle_F)=d_H(C)$.

If $v(x)$ is 0 or $v(x)$ is a unit and $1\le j\le\frac{p^s+t}{2}$, we have

$$d_H(C) = \begin{cases} \bullet 1, & \text{if } i=j=0, \\ \bullet 2, & \text{if } i=0 \text{ and } 0<j\le p^{s-1}, \\ \bullet 3, & \text{if } i=0 \text{ and } p^{s-1}<j\le 2p^{s-1}, \\ \bullet 4, & \text{if } i=0 \text{ and } 2p^{s-1}<j\le 3p^{s-1}, \\ \bullet 5, & \text{if } i=0 \text{ and } 3p^{s-1}<j\le p^s. \end{cases}$$

If $v(x)$ is a unit and $\frac{p^s+t}{2}<j\le p^s-1$, then

$$d_H(C) = \begin{cases} \bullet 1, \text{ if } i=j=0, \\ \bullet 2, \text{ if } i=0 \text{ and } p^s-p^{s-1}+t<j\le p^s+t, \\ \bullet 3, \text{ if } i=0, p^s-2p^{s-1}+t<j \\ \text{and } j\le p^s-p^{s-1}+t, \\ \bullet 4, \text{ if } i=0, p^s-3p^{s-1}+t<j \\ \text{and } j\le p^s-2p^{s-1}+t, \\ \bullet 5, \text{ if } i=0 \text{ and } t<j\le p^s-3p^{s-1}+t. \end{cases}$$

□

We compute the Hamming distance of ideals of Type 4 of $\frac{\mathcal{R}[x]}{\langle(\alpha(x))^{p^s}\rangle}$.

Theorem 3.3: Let $C = \langle(\alpha(x))^j + u(\alpha(x))^t v(x), u(\alpha(x))^\omega\rangle$ be an ideal of Type 4 of $\frac{\mathcal{R}[x]}{\langle(\alpha(x))^{p^s}\rangle}$, where $v(x)$ is same as given in Type 3, $\deg(v) \leq \omega - t - 1$, $\omega < R$, and R is the smallest integer such that $u(\alpha(x))^R \in \langle(\alpha(x))^j + u(\alpha(x))^t v(x)\rangle$, i.e., $R = j$, if $v(x) = 0$ and otherwise $R = \min\{j, p^s - j + t\}$. Then, we have $\mathrm{d_H}(C) = \mathrm{d_H}(\langle(\alpha(x))^\omega\rangle_F)$, and

$$\mathrm{d_H}(C) = \begin{cases} \bullet 1, & \text{if } i = j = 0, \\ \bullet 2, & \text{if } i = 0 \text{ and } 0 < \omega \leq p^{s-1}, \\ \bullet 3, & \text{if } i = 0 \text{ and } p^{s-1} < \omega \leq 2p^{s-1}, \\ \bullet 4, & \text{if } i = 0 \text{ and } 2p^{s-1} < \omega \leq 3p^{s-1}, \\ \bullet 5, & \text{if } i = 0 \text{ and } 3p^{s-1} < \omega \leq p^s. \end{cases}$$

Proof: It is easy to see that $C = \langle(\alpha(x))^j + u(\alpha(x))^t v(x), u(\alpha(x))^\omega\rangle \supseteq \langle u(\alpha(x))^\omega\rangle \supseteq \langle u(\alpha(x))^j\rangle$, since $\omega < R \leq j$. Therefore, $\mathrm{d_H}(C) \leq \mathrm{d_H}(\langle u(\alpha(x))^\omega\rangle) = \mathrm{d_H}(\langle(\alpha(x))^\omega\rangle_F)$. To prove that $\mathrm{d_H}(\langle(\alpha(x))^\omega\rangle_F) \leq \mathrm{d_H}(C)$, we take an arbitrary polynomial $c(x) \in C$ and proceed to show that $\mathrm{wt_H}(c(x)) \geq \mathrm{d_H}(\langle(\alpha(x))^\omega\rangle_F)$. Now, there exist polynomials $f_0(x), f_u(x), g_0(x)$ and $g_u(x)$ over $\mathbb{F}_{p^m}$ such that

$$\begin{aligned} c(x) &= [f_0(x) + uf_u(x)] \\ &\quad \times [(\alpha(x))^j + u(\alpha(x))^t v(x) + u(\alpha(x))^\omega][g_0(x) + ug_u(x)] \\ &= f_0(x)(\alpha(x))^j + u[f_0(x)(\alpha(x))^t v(x)] + uf_u(x)(\alpha(x))^j \\ &\quad + ug_0(x)(\alpha(x))^\omega \\ &= f_0'(x)(\alpha(x))^\omega + u[f_0(x)(\alpha(x))^t v(x) + g_0'(x)(\alpha(x))^\omega], \end{aligned}$$

where $f_0'(x) = f_0(x)(\alpha(x))^{j-\omega} \in \mathbb{F}_{p^m}[x]$, $g_0'(x) = f_u(x)(\alpha(x))^{j-\omega} + g_0(x) \in \mathbb{F}_{p^m}[x]$. Hence,

$$\begin{aligned} \mathrm{wt_H}(c(x)) &\geq \max\left\{\mathrm{wt_H}(a(x)), \mathrm{wt_H}(h'(x))\right\} \\ &\geq \max\left\{\mathrm{wt_H}(a(x)), \mathrm{wt_H}(a(x))\right\} \\ &\geq \mathrm{d_H}(\langle(\alpha(x))^\omega\rangle_F), \end{aligned}$$

where $a(x) = f_0'(x)(\alpha(x))^\omega)$ and $h'(x) = f_0(x)(\alpha(x))^t v(x) + g_0'(x)(\alpha(x))^\omega$. It implies that $\mathrm{d_H}(C) = \mathrm{d_H}(\langle(\alpha(x))^\omega\rangle_F)$. Hence

$$\mathrm{d_H}(C) = \begin{cases} \bullet 1, & \text{if } i = j = 0, \\ \bullet 2, & \text{if } i = 0 \text{ and } 0 < \omega \leq p^{s-1}, \\ \bullet 3, & \text{if } i = 0 \text{ and } p^{s-1} < \omega \leq 2p^{s-1}, \\ \bullet 4, & \text{if } i = 0 \text{ and } 2p^{s-1} < \omega \leq 3p^{s-1}, \\ \bullet 5, & \text{if } i = 0 \text{ and } 3p^{s-1} < \omega \leq p^s. \end{cases}$$

□

Applying Theorems 3.1-3.3, we give the Hamming distance of λ-constacyclic codes of length $5p^s$ over $\mathcal{R}$ when $p \equiv 2, 3 \pmod 5$.

Theorem 3.4: Let $C = C_1 \oplus C_2$ be a non-zero cyclic code of length $5p^s$ over $\mathcal{R}$ as in Theorem 2.4. Then the Hamming distances $\mathrm{d_H}(C) = \min\{\mathrm{d_H}(C_1), \mathrm{d_H}(C_2)\}$, where $C_1 \neq \langle 0\rangle$ and $C_2 \neq \langle 0\rangle$.

Proof: Without loss of generality, assume that $\mathrm{d_H}(C_1) = d_1 = \min\{\mathrm{d_H}(C_1), \mathrm{d_H}(C_2)\}$ (that means $C_1 \neq \langle 0\rangle$). Let d be the Hamming distance of $C = C_1 \oplus C_2$. Let c_1 be a non-zero codeword of minimum weight in C_1, i.e., $d_1 = \mathrm{wt_H}(c_1)$. Since $(c_1, 0) \in C$ and $\mathrm{wt_H}(c_1, 0) = d_1$, we have $d \leq d_1$. Let $z = (z_1, z_2)$ be an arbitrary non-zero codeword in $C_1 \oplus C_2$. If $z_1 \neq 0$, then $\mathrm{wt_H}(z) = \mathrm{wt_H}(z_1) + \mathrm{wt_H}(z_2) \geq \mathrm{wt_H}(z_1) \geq d_1$. Thus, $d \geq d_1$ when $z_1 \neq 0$. If $z_1 = 0$, then $z_2 \neq 0$. Since $\mathrm{d_H}(C_1) = d_1 = \min\{\mathrm{d_H}(C_1), \mathrm{d_H}(C_2)\}$, the Hamming weight of z is $\mathrm{wt_H}(z) = \mathrm{wt_H}(z_1) + \mathrm{wt_H}(z_2) = \mathrm{wt_H}(z_2) \geq \mathrm{d_H}(C_2) \geq d_1$. It implies that $d \geq d_1$ when $z_1 = 0$. Since z is an arbitrary non-zero codeword, we have $d \geq d_1$. Thus, $d = d_1$, completing our proof. □

Let $C = C_1 \oplus C_2$ be a λ-constacyclic code over $\mathcal{R}$, where C_1 is a cyclic code of length p^s over $\mathcal{R}$, C_2 is an ideal of $\frac{\mathcal{R}[x]}{\langle(\alpha(x))^{p^s}\rangle}$. Then the MDS codes of cyclic codes over $\mathcal{R}$ are determined by the following propositions.

Proposition 3.5: Let $C = C_1 \oplus C_2$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where $C_1 = \langle 0\rangle$ is a cyclic code of length p^s over $\mathcal{R}$, and C_2 is any ideal of $\frac{\mathcal{R}[x]}{\langle(\alpha(x))^{p^s}\rangle}$. Then C is not an MDS code.

Proof: Using Theorem 3.4, we have $\mathrm{d_H}(C) = \mathrm{d_H}(C_2) \leq 5$. We see that $|C| = p^\ell \leq p^{8mp^s}$. By Theorem 2.11, C is an MDS code when $\ell = 2m(5p^s - \mathrm{d_H}(C_2) + 1)$. If $\mathrm{d_H}(C_2) = 1$, then $\ell = 10mp^s$, which is a contradiction. If $\mathrm{d_H}(C_2) = 2$, then $\ell = 2m(5p^s - 1)$. This is impossible since $2m(5p^s - 1) > 8mp^s \geq \ell$. Thus, C is not an MDS code when $\mathrm{d_H}(C_2) = 2$. If $\mathrm{d_H}(C_2) = 3$, then $\ell = 2m(5p^s - 2)$. Since $\mathrm{d_H}(C_2) = 3$, we have $C_2 \neq \langle 1\rangle$, i.e., $|C| = p^\ell < p^{8mp^s}$. Since $2m(5p^s - 2) > \ell$, by Theorem 2.11, C is not an MDS code when $\mathrm{d_H}(C_2) = 3$. If $\mathrm{d_H}(C) = 4$, then $\ell = 2m(5p^s - 4)$. Since $\mathrm{d_H}(C_2) = 4$, we have $C_2 \neq \langle 1\rangle$, i.e., $|C| = p^\ell < p^{8mp^s}$. Since $2m(5p^s - 4) > \ell$, by Theorem 2.11, C is not an MDS code when $\mathrm{d_H}(C_2) = 4$. If $\mathrm{d_H}(C_2) = 5$, then $\ell = 2m(5p^s - 5)$. Since $\mathrm{d_H}(C_2) = 5$, we have $C_2 \neq \langle 1\rangle$, i.e., $|C| = p^\ell < p^{8mp^s}$. Since $2m(5p^s - 5) > \ell$, by Theorem 2.11, C is not an MDS code when $\mathrm{d_H}(C_2) = 5$. □

Proposition 3.6: Let $C = C_1 \oplus C_2$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where $C_1 = \langle 1\rangle$ is a cyclic code of length p^s over $\mathcal{R}$, C_2 is any ideal of $\frac{\mathcal{R}[x]}{\langle(\alpha(x))^{p^s}\rangle}$. Then C is an MDS code if $C = \langle 1\rangle$.

Proof: By applying Theorem 3.4, we see that $\mathrm{d_H}(C) = 1$. We have $|C| = p^{2mp^s} \cdot p^{\ell_5}$, where $|C_2| = p^{\ell_5} \leq p^{8mp^s}$. By Theorem 2.11, C is an MDS code when $2mp^s + \ell_5 = 2m(5p^s)$. It implies that $\ell_5 = 8mp^s$. Therefore, C is an MDS code if and only if $C_1 = \langle 1\rangle$ and $C_2 = \langle 1\rangle$, i.e., $C = \langle 1\rangle$. □

Proposition 3.7: Let $C = C_1 \oplus C_2$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where $C_1 = \langle(x - \gamma_0)^{j_1}\rangle$ is a λ_0-constacyclic code of length p^s over $\mathcal{R}$, C_2 is any ideal of $\frac{\mathcal{R}[x]}{\langle(\alpha(x))^{p^s}\rangle}$ (j_1 is defined as in Theorem 2.2). Then C is not an MDS code.

Proof: We have two cases: $C_2 = \langle 0\rangle$ and $C_2 \neq \langle 0\rangle$.

Case 1: $C_2 = \langle 0\rangle$. From Theorem 3.4, we see that $\mathrm{d_H}(C) = \mathrm{d_H}(C_1)$. We have $|C| = p^{m(p^s - j_1)}$. Using Theorem 2.3, $\mathrm{d_H}(C) = \mathrm{d_H}(\langle(x-1)^{j_1}\rangle_F)$ and

$$\mathrm{d_H}(C) = \begin{cases} \bullet 1, \text{ if } j_1 = 0 \\ \bullet (\delta_1 + 1)p^{k_1}, \\ \text{ if } p^s - p^{s-k_1} + (\delta_1 - 1)p^{s-k_1-1} + 1 \leq j_1 \\ \text{and } j_1 \leq p^s - p^{s-k_1} + \delta_1 p^{s-k_1-1} \\ \text{where } 1 \leq \delta_1 \leq p - 1, \text{ and } 0 \leq k_1 \leq s - 1. \end{cases}$$

If $d_H(C) = 1$, then C is an MDS code when $m(p^s) = 2m(5p^s)$, which is a contradiction. Hence, C is not an MDS code when $d_H(C) = 1$. If $d_H(C) = (\delta_1 + 1)p^{k_1}$, then C is an MDS code when $m(p^s - j_1) = 2m(5p^s - (\delta_1 + 1)p^{k_1} + 1)$, i.e., $j_1 = -9p^s + 2(\delta_1+1)p^{k_1} - 2$. Since $-9p^s + 2(\delta_1+1)p^{k_1} - 2 < 0$, we see that $j_1 \neq -9p^s + 2(\delta_1+1)p^{k_1} - 2$. Thus, C is not an MDS code in this case.

Case 2: $C_2 \neq \langle 0 \rangle$. Using Theorem 3.4, $d_H(C) = \min\{d_H(C_1), d_H(C_2)\}$. By Theorems 3.1-3.3, we see that $d_H(C) = 1$ or $d_H(C) = 2$ or $d_H(C) = 3$ or $d_H(C) = 4$ or $d_H(C) = 5$. We have $|C| = p^{m(p^s - j_4)} \cdot p^{\ell_2}$, where $|C_2| = p^{\ell_2} \leq p^{8mp^s}$. If $d_H(C) = 1$, then C is an MDS code when $m(p^s - j_1) + \ell_2 = 2m(5p^s)$, i.e., $\ell_2 = m(9p^s - j_1)$, which is a contradiction since $1 \leq j_1 \leq p^s - 1$ and $\ell_2 \leq 8mp^s$. Hence, C is not an MDS code when $d_H(C) = 1$. If $d_H(C) = 2$, then C is an MDS code when $m(p^s - j_1) + \ell_2 = 2m(5p^s - 1)$, i.e., $\ell_2 = m(9p^s - 2 + j_1) > 8mp^s$, which is a contradiction. Thus, C is not an MDS code when $d_H(C) = 2$. If $d_H(C) = 3$, then $\ell_2 < 8mp^s$. By Theorem 2.11, C is an MDS code when $m(p^s - j_4) + \ell_2 = 2m(5p^s - 2)$, i.e., $\ell_2 = m(9p^s - 4 + j_1)$. It is easy to check that if $j_1 \geq 3$, then $\ell_2 = m(9p^s - 4 + j_1) \geq m(9p^s - 1) > 8mp^s$, which is a contradiction. Thus, we must consider two cases: $j_1 = 1$ and $j_1 = 2$. If $j_1 = 1$, by Theorem 2.3, then $d_H(C_1) = 2 < d_H(C)$. This is a contradiction. Hence, C is not an MDS code when $j_1 = 2$ and $d_H(C) = 3$. If $j_1 = 2$, then $\ell_2 = m(9p^s - 2)$. Since $m(9p^s - 2) \geq 8mp^s$, C is not an MDS code when $j_1 = 2$ and $d_H(C) = 3$. Thus, C is not an MDS code when $d_H(C) = 3$. If $d_H(C) = 4$, then $\ell_2 < 8mp^s$. Hence, C is not an MDS code when $d_H(C) = 4$. If $d_H(C) = 5$, then $\ell_2 < 8mp^s$. Therefore, C is not an MDS code. Combining Cases 1 and 2, we conclude that C is not an MDS code. □

Combining Propositions 3.5-3.7, we have the following theorem.

Theorem 3.8: Let C be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_1 is a cyclic code of length p^s over $\mathcal{R}$, C_2 is an ideal of $\frac{\mathcal{R}[x]}{\langle(\alpha(x))^{p^s}\rangle}$. Then C is an MDS code when $C_1 = \langle 1 \rangle$ and $C_2 = \langle 1 \rangle$.

We give an example to illustrate our work in this section.

Example 3.9: Put $\mathcal{R} = \mathbb{F}_{11} + u\mathbb{F}_{11}$. We consider cyclic codes of length 55 over $\mathcal{R}$. Let C be a cyclic code of length 55 over $\mathcal{R}$. Then $C = C_1 \oplus C_2$, where C_1 is a cyclic code of length 11 over $\mathcal{R}$ and C_2 is an ideal of $\frac{\mathcal{R}[x]}{\langle(\alpha(x))^{11}\rangle}$.

1) Let $C_1 = \langle (x-1)^3 \rangle$ and $C_2 = \langle u(\alpha(x))^6 \rangle$. By using Theorem 2.3, we have $d_H(C_1) = 4$. Applying Theorem 3.2, we have $d_H(C_2) = 5$. From Theorem 3.4, we have $d_H(C) = 4$. Then C has parameters $[55, 11^{18}, 4]$.

2) Let $C_1 = \langle (x-1)^5 \rangle$ and $C_2 = \langle (\alpha(x))^8, u(\alpha(x))^6 \rangle$. By using Theorem 2.3, we have $d_H(C_1) = 6$. Applying Theorem 3.3, we have $d_H(C_2) = 5$. From Theorem 3.4, we have $d_H(C) = 5$. Then C has parameters $[55, 11^{22}, 5]$.

IV. HAMMING DISTANCE AND MDS CODES WHEN $p \equiv 1 \pmod 5$

As in Theorem 2.5, cyclic codes of length $5p^s$ over $\mathcal{R}$ is of the form $C = C_1 \bigoplus C_2 \bigoplus C_3 \bigoplus C_4 \bigoplus C_5$. We compute the Hamming distance of cyclic codes of length $5p^s$ over $\mathcal{R}$ when $p \equiv 1 \pmod 5$ as follows.

Theorem 4.1: Let $C = C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5$ be a non-zero cyclic code of length $5p^s$ over $\mathcal{R}$ as in Theorem 2.5. Then the Hamming distance $d_H(C) = \min\{d_H(C_i) | i \in \{1,2,3,4,5\}, C_i \neq \langle 0 \rangle\}$.

Proof: Without loss of generality, assume that $d_H(C_1) = d_1 = \min\{d_H(C_i)\}$ (that means $C_1 \neq \langle 0 \rangle$). Let d be the Hamming distance of $C = C_1 \oplus C_2 \oplus C_3$. Let c_1 be a non-zero codeword of minimum weight in C_1, i.e., $d_1 = wt_H(c_1)$. Since $(c_1, 0, 0, 0, 0) \in C$ and $wt_H(c_1, 0, 0, 0, 0) = d_1$, we have $d \leq d_1$. Let $z = (z_1, z_2, z_3, z_4, z_5)$ be an arbitrary non-zero codeword in $C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5$. If $z_1 = 0$, then there exists $t \in \{2,3,4,5\}$ such that $z_t \neq 0$. Since $d_H(C_1) = d_1 = \min\{d_H(C_i)\}$, we have $wt_H(z_t) \geq d_H(C_t) \geq d_1$. Hence, the Hamming weight of z is $wt_H(z) = wt_H(z_1) + wt_H(z_2) + wt_H(z_3) + wt_H(z_4) + wt_H(z_5) \geq wt_H(z_t) \geq d_H(C_t) \geq d_1$. It implies that $d \geq d_1$ when $z_1 = 0$. If $z_1 \neq 0$, then the Hamming weight of z is $wt_H(z) = wt_H(z_1) + wt_H(z_2) + wt_H(z_3) + wt_H(z_4) + wt_H(z_5) \geq d_1$. It means that $d \geq d_1$ when $z_1 \neq 0$. Since z is an arbitrary non-zero codeword, we have $d \geq d_1$. Thus, $d = d_1$. □

Since $\gamma_2, \gamma_3, \gamma_4, \gamma_5 \in \mathbb{F}_{p^m}$, there exist $\alpha_2, \alpha_3, \alpha_4, \alpha_5 \in \mathbb{F}_{p^m}$ such that $\alpha_i^{p^s} = \gamma_i$. By Theorem 2.2, we classify all C_i $(i = 2,3,4,5)$ as follows.

- Type 1: (trivial ideals)

$$\langle 0 \rangle, \ \langle 1 \rangle.$$

- Type 2: (principal ideals with nonmonic polynomial generators)

$$\left\langle u(x-\alpha_i)^{j_i} \right\rangle,$$

where $0 \leq j_i \leq p^s - 1$.
- Type 3: (principal ideals with monic polynomial generators)

$$\left\langle (x-\alpha_i)^{j_i} + u(x-\alpha_i)^{t_i} h_i(x) \right\rangle,$$

where $1 \leq j_i \leq p^s - 1$, $0 \leq t_i < j_i$, and either $h_i(x)$ is 0 or $h_i(x)$ is a unit.
- Type 4: (nonprincipal ideals) $\langle (x-\alpha_i)^{j_i} + u(x-\alpha_i)^{t_i}\ h_i(x), u(x-\alpha_0)^{\kappa_i} \rangle$, with $h_i(x)$ as in Type 3, $\deg(h_i) \leq \kappa_i - t_i - 1$, and $\kappa_i < T_i$, where T_i is the smallest integer such that $u(x-\alpha_i)^{T_i} \in \left\langle (x-\alpha_i)^{j_i} + u(x-\alpha_i)^{t_i} h_i(x) \right\rangle$; or equivalently, $T_i = j_i$, if $h_i(x) = 0$, otherwise $T_i = \min\{j_i, p^s - j_i + t_i\}$.

To get MDS codes, we consider the following propositions.

Proposition 4.2: Let $C = C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5$ be a cyclic code of length $3p^s$ over $\mathcal{R}$, where C_1, C_2, C_3, C_4, C_5 are defined as in Theorem 2.5 such that there exists $C_i = \langle 0 \rangle$ for $i \in \{1,2,3,4,5\}$. Then C is not an MDS code.

Proof: Without loss of generality, assume that $C_1 = \langle 0 \rangle$. From Theorem 4.1, $d_H(C) = \min\{d_H(C_j)\} \leq p^s$, where $j = 2,3,4,5$. We have $|C| = |C_2| \times |C_3| \times |C_4| \times |C_5| = p^{\ell_2} \cdot p^{\ell_3} \cdot$

$p^{\ell_4} \cdot p^{\ell_5}$, where $|C_2| = p^{\ell_2}, |C_3| = p^{\ell_3}, |C_4| = p^{\ell_4}, |C_5| = p^{\ell_5}$ and $0 \le \ell_2, \ell_3, \ell_4, \ell_5 \le 2mp^s$. We see that $\ell_2+\ell_3+\ell_4+\ell_5 \le 8mp^s$ and $2m(5p^s - d_H(C)+1) > 8mp^s$. Thus, $\ell_2+\ell_3+\ell_4+\ell_5 < 2m(5p^s - d_H(C)+1)$. Using Theorem 2.11, C is an MDS code when $\ell_2+\ell_3+\ell_4+\ell_5 = 2m(5p^s - d_H(C)+1)$. Since $\ell_2+\ell_3+\ell_4+\ell_5 < 2m(5p^s - d_H(C)+1)$, C is not an MDS code. □

Proposition 4.3: Let $C = C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_1, C_2, C_3, C_4, C_5 are defined as in Theorem 2.5 such that there exists $C_i = \langle 1 \rangle$ for $i \in \{1, 2, 3, 4, 5\}$. Then C is an MDS code if and only if $C = \langle 1 \rangle$.

Proof: Without loss of generality, assume that $C_1 = \langle 1 \rangle$ and $|C_2| = p^{\ell_2}, |C_3| = p^{\ell_3}, |C_4| = p^{\ell_4}$, and $|C_5| = p^{\ell_5}$, where $0 \le \ell_2, \ell_3, \ell_4, \ell_5 \le 2mp^s$. Using Theorem 4.1, $d_H(C) = d_H(C_1) = 1$. By Theorem 2.11, C is an MDS code when $p^{2mp^s} \cdot p^{\ell_2} \cdot p^{\ell_3} \cdot p^{\ell_4} \cdot p^{\ell_5} = p^{2m(5p^s-1+1)}$, where $0 \le \ell_2, \ell_3, \ell_4, \ell_5 \le 2mp^s$. It implies that $2mp^s + \ell_2 + \ell_3 + \ell_4 + \ell_5 = 10mp^s$. Thus, $\ell_2 + \ell_3 + \ell_4 + \ell_5 = 8mp^s$. Hence, $\ell_2 = \ell_3 = \ell_4 = \ell_5 = 2mp^s$. Then C is an MDS code when $C_1 = \langle 1 \rangle, C_2 = \langle 1 \rangle, C_3 = \langle 1 \rangle, C_4 = \langle 1 \rangle$ and $C_5 = \langle 1 \rangle$, i.e., $C = \langle 1 \rangle$. □

By Propositions 4.2 and 4.3, if $C_i = \langle 0 \rangle$ or $C_i = \langle 1 \rangle$ for $i \in \{1, 2, 3, 4, 5\}$, then C is an MDS code when $C = \langle 1 \rangle$. Thus, we consider the case when all $C_i \neq \langle 0 \rangle$ and $C_i \neq \langle 1 \rangle$ for $i \in \{1, 2, 3, 4, 5\}$ in the following propositions.

Proposition 4.4: Let $C = C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_1, C_2, C_3, C_4, C_5 are defined as in Theorem 2.5 such that all C_i are constacyclic codes of Type 2 of length p^s over $\mathcal{R}$ for $i \in \{2, 3, 4, 5\}$. Then C is not an MDS code.

Proof: We have $C_1 = \langle (x-1)^{j_1} \rangle, C_2 = \langle u(x-\gamma_2)^{j_2} \rangle, C_3 = \langle u(x-\gamma_3)^{j_3} \rangle, C_4 = \langle u(x-\gamma_4)^{j_4} \rangle, C_5 = \langle u(x-\gamma_5)^{j_5} \rangle$ (j_2, j_3, j_4, j_5 are defined as in Theorem 2.2). We see that $|C| = p^{m(p^s-j_1)} \cdot p^{m(p^s-j_2)} \cdot p^{m(p^s-j_3)} \cdot p^{m(p^s-j_4)} \cdot p^{m(p^s-j_5)}$. From Theorem 4.1, $d_H(C) \le p^s$. Using Theorem 2.11, C is an MDS code when $m(p^s-j_1) + m(p^s-j_2) + m(p^s-j_3) + m(p^s-j_4) + m(p^s-j_5) = 2m(5p^s - d_H(C)+1)$. Since $m(p^s-j_1) + m(p^s-j_2) + m(p^s-j_3) + m(p^s-j_4) + m(p^s-j_5) < 8mp^s$ and $2m(5p^s - d_H(C)+1) > 8mp^s$, we have $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+m(p^s-j_4)+m(p^s-j_5) < 2m(5p^s - d_H(C)+1)$. Hence, C is not an MDS code. □

Proposition 4.5: Let $C = C_1 \oplus C_i \oplus C_j \oplus C_k \oplus C_t$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_i, C_j, C_k, C_t are defined as in Theorem 2.5 and i, j, k, t are different numbers in $\{2, 3, 4, 5\}$. If C_i, C_j, and C_k are constacyclic codes of Type 2 of length p^s over $\mathcal{R}$, then C is not an MDS code.

Proof: If C_t is a constacyclic code of Type 2 of length p^s over $\mathcal{R}$, then C is not an MDS code by Proposition 4.4. Thus, C_t is a constacyclic code of Type 3 or Type 4 of length p^s over $\mathcal{R}$. Without loss of generality, assume that $C_2 = \langle u(x-\gamma_2)^{j_2} \rangle, C_3 = \langle u(x-\gamma_3)^{j_3} \rangle, C_4 = \langle u(x-\gamma_3)^{j_3} \rangle$ and $C_5 = \langle (x-\gamma_5)^{j_5} + u(x-\gamma_5)^{t_5} h_5(x) \rangle$ or $C_5 = \langle (x-\gamma_5)^{j_5} + u(x-\gamma_5)^{t_5}\, h_5(x), u(x-\gamma_5)^{\kappa_5} \rangle$ ($j_1, j_2, j_3, j_4, t_5, \kappa_5$ are defined as in Theorem 2.2).

Case 1: $C_1 = \langle (x-\gamma_1)^{j_1} \rangle, C_2 = \langle u(x-\gamma_2)^{j_2} \rangle, C_3 = \langle u(x-\gamma_3)^{j_3} \rangle, C_4 = \langle u(x-\gamma_4)^{j_4} \rangle$ and $C_5 = \langle (x-\gamma_5)^{j_5} + u(x-\gamma_5)^{t_5} h_5(x) \rangle$. We have $|C| = p^{m(p^s-j_1)} . p^{m(p^s-j_2)} . p^{m(p^s-j_3)} . p^{m(p^s-j_4)} \cdot p^{2m(p^s-j_5)}$ or $|C| = p^{m(p^s-j_1)} \cdot p^{m(p^s-j_2)} \cdot p^{m(p^s-j_3)} \cdot p^{m(p^s-j_4)} \cdot p^{m(p^s-t_5)}$. By applying Theorem 4.1, $d_H(C) \le p^s$. Using Theorem 2.11, C is an MDS code when $m(p^s-j_1) + m(p^s-j_2) + m(p^s-j_3) + m(p^s-j_4) + 2m(p^s-j_5) = 2m(5p^s - d_H(C)+1)$ or $m(p^s-j_1) + m(p^s-j_2) + m(p^s-j_3) + m(p^s-j_4) + m(p^s-t_5) = 2m(5p^s - d_H(C)+1)$. Since $m(p^s-j_1) + m(p^s-j_2) + m(p^s-j_3) + m(p^s-j_4) + 2m(p^s-j_5) < 8mp^s$, $m(p^s-j_1) + m(p^s-j_2) + m(p^s-j_3) + m(p^s-j_4) + m(p^s-t_5) < 8mp^s$ and $2m(5p^s - d_H(C)+1) > 4mp^s$, we see that $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+m(p^s-j_4)+2m(p^s-j_5) \neq 2m(5p^s - d_H(C)+1)$ and $m(p^s-j_1) + m(p^s-j_2) + m(p^s-j_3) + m(p^s-j_4) + m(p^s-t_5) < 2m(5p^s - d_H(C)+1)$. Hence, C is not an MDS code in this case.

Case 2: $C_1 = \langle (x-\gamma_1)^{j_1} \rangle, C_2 = \langle u(x-\gamma_2)^{j_2} \rangle, C_3 = \langle u(x-\gamma_3)^{j_3} \rangle, C_4 = \langle u(x-\gamma_4)^{j_4} \rangle$ and $C_5 = \langle (x-\gamma_5)^{j_5} + u(x-\gamma_5)^{t_5}$ $h_5(x), u(x-\gamma_5)^{\kappa_5} \rangle$. We see that $|C| = p^{m(p^s-j_1)} \cdot p^{m(p^s-j_2)} \cdot p^{m(p^s-j_3)} \cdot p^{m(p^s-j_4)} \cdot p^{m(2p^s-j_5-\kappa_5)}$. Applying Theorem 4.1, $d_H(C) \le p^s$. By Theorem 2.11, C is an MDS code when $m(p^s-j_1) + m(p^s-j_2) + m(p^s-j_3) + m(p^s-j_4) + m(2p^s - j_5 - \kappa_5) = 2m(5p^s - d_H(C)+1)$. Since $m(p^s-j_1) + m(p^s-j_2) + m(p^s-j_3) + m(p^s-j_4) + m(2p^s-j_5-\kappa_5) < 8mp^s$ and $2m(5p^s - d_H(C)+1) > 8mp^s$, we have $m(p^s-j_1) + m(p^s-j_2) + m(p^s-j_3) + m(p^s-j_4) + m(2p^s-j_5-\kappa_5) < 2m(5p^s - d_H(C)+1)$. Hence, C is not an MDS code. □

Proposition 4.6: Let $C = C_1 \oplus C_i \oplus C_j \oplus C_k \oplus C_t$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_i, C_j, C_k, C_t are defined as in Theorem 2.5 and i, j, k, t are different numbers in $\{2, 3, 4, 5\}$. If C_i, C_j are constacyclic codes of Type 2 of length p^s over $\mathcal{R}$, then C is not an MDS code.

Proof: If C_k, C_t are constacyclic codes of Type 2 of length p^s over $\mathcal{R}$, then C is not an MDS code by Proposition 4.5. Thus, C_k, C_t are constacyclic codes of Type 3 or Type 4 of length p^s over $\mathcal{R}$. Without loss of generality, assume that $C_1 = \langle (x-1)^{j_1} \rangle, C_2 = \langle u(x-\gamma_2)^{j_2} \rangle, C_3 = \langle u(x-\gamma_3)^{j_3} \rangle$ and $C_4 = \langle (x-\gamma_4)^{j_4} + u(x-\gamma_4)^{t_4} h_4(x) \rangle$ or $C_4 = \langle (x-\gamma_4)^{j_4} + u(x-\gamma_4)^{t_4}\, h_4(x), u(x-\gamma_4)^{\kappa_4} \rangle$ and $C_5 = \langle (x-\gamma_5)^{j_5} + u(x-\gamma_5)^{t_5} h_5(x) \rangle$ or $C_5 = \langle (x-\gamma_5)^{j_5} + u(x-\gamma_5)^{t_5}\, h_5(x), u(x-\gamma_5)^{\kappa_5} \rangle$ ($j_1, j_2, j_3, j_4, t_5, \kappa_5$ are defined as in Theorem 2.2).

Case 1: $C_1 = \langle (x-\gamma_1)^{j_1} \rangle, C_2 = \langle u(x-\gamma_2)^{j_2} \rangle, C_3 = \langle u(x-\gamma_3)^{j_3} \rangle, C_4 = \langle (x-\gamma_4)^{j_4} + u(x-\gamma_4)^{t_4} h_4(x) \rangle$ and $C_5 = \langle (x-\gamma_5)^{j_5} + u(x-\gamma_5)^{t_5} h_5(x) \rangle$. Then we have $|C| = p^{m(p^s-j_1)} \cdot p^{m(p^s-j_2)} \cdot p^{m(p^s-j_3)} \cdot p^{2m(p^s-j_4)} \cdot p^{2m(p^s-j_5)}$ or $|C| = p^{m(p^s-j_1)} \cdot p^{m(p^s-j_2)} \cdot p^{m(p^s-j_3)} \cdot p^{2m(p^s-j_4)} \cdot p^{m(p^s-t_5)}$ or $p^{m(p^s-j_1)} \cdot p^{m(p^s-j_2)} \cdot p^{m(p^s-j_3)} \cdot p^{m(p^s-t_4)} \cdot p^{2m(p^s-j_5)}$ or $p^{m(p^s-j_1)} \cdot p^{m(p^s-j_2)} \cdot p^{m(p^s-j_3)} \cdot p^{m(p^s-t_4)} \cdot p^{2m(p^s-j_5)}$. By applying Theorem 4.1, $d_H(C) \le p^s$. Using Theorem 2.11, C is an MDS code when $m(p^s-j_1) + m(p^s-j_2) + m(p^s-j_3) + 2m(p^s-j_4) + 2m(p^s-j_5) = 2m(5p^s - d_H(C)+1)$ or $m(p^s-j_1) + m(p^s-j_2) + m(p^s-j_3) + 2m(p^s-j_4) + m(p^s-t_5) = 2m(5p^s - d_H(C)+1)$ or $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+m(p^s-t_4)+2m(p^s-j_5) = 2m(5p^s - d_H(C)+1)$ or $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+m(p^s-t_4) + m(p^s-t_5) = 2m(5p^s - d_H(C)+1)$. We see that $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+2m(p^s-j_4)+2m(p^s-j_5) <$

$8mp^s$, $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+2m(p^s-j_4)+m(p^s-t_5) < 8mp^s$, $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+m(p^s-t_4)+2m(p^s-j_5) < 8mp^s$, $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+m(p^s-t_4)+m(p^s-t_5) < 8mp^s$. It is easy to see that $2m(5p^s-d_H(C)+1) > 8mp^s$. Then we see that $|C| < p^{2m(5p^s-d_H(C)+1)}$. Hence, C is not an MDS code in this case.

Case 2: $C_1 = \langle (x-\gamma_1)^{j_1}\rangle$, $C_2 = \langle u(x-\gamma_2)^{j_2}\rangle$, $C_3 = \langle u(x-\gamma_3)^{j_3}\rangle$, $C_4 = \langle (x-\gamma_4)^{j_4}+u(x-\gamma_4)^{t_4}h_4(x)\rangle$ *and* $C_5 = \langle (x-\gamma_5)^{j_5}+u(x-\gamma_5)^{t_5}h_5(x), u(x-\gamma_5)^{\kappa_5}\rangle$. We see that $|C| = p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{m(p^s-j_3)}\cdot p^{2m(p^s-j_4)}\cdot p^{m(2p^s-j_5-\kappa_5)}$ or $|C| = p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{m(p^s-j_3)}\cdot p^{m(p^s-t_4)}\cdot p^{m(2p^s-j_5-\kappa_5)}$. Applying Theorem 4.1, $d_H(C) \le p^s$. By Theorem 2.11, C is an MDS code when $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+2m(p^s-j_4)+m(2p^s-j_5-\kappa_5) = 2m(5p^s-d_H(C)+1)$ or $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+m(p^s-t_4)+m(2p^s-j_5-\kappa_5) = 2m(5p^s-d_H(C)+1)$. It is easy to check that $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+2m(p^s-j_4)+m(2p^s-j_5-\kappa_5) < 8mp^s$ and $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+m(p^s-t_4)+m(2p^s-j_5-\kappa_5) < 8mp^s$. We see that $2m(5p^s-d_H(C)+1) > 8mp^s$. Hence, $|C| < p^{2m(5p^s-d_H(C)+1)}$. Hence, C is not an MDS code in this case.

Case 3: $C_1 = \langle (x-\gamma_1)^{j_1}\rangle$, $C_2 = \langle u(x-\gamma_2)^{j_2}\rangle$, $C_3 = \langle u(x-\gamma_3)^{j_3}\rangle$, $C_4 = \langle (x-\gamma_4)^{j_4}+u(x-\gamma_4)^{t_4}h_4(x), u(x-\gamma_4)^{\kappa_4}\rangle$ *and* $C_5 = \langle (x-\gamma_5)^{j_5}+u(x-\gamma_5)^{t_5}h_5(x)\rangle$. Then we have $|C| = p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{m(p^s-j_3)}\cdot p^{m(2p^s-j_4-\kappa_4)}\cdot p^{2m(p^s-j_5)}$ or $|C| = p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{m(p^s-j_3)}\cdot p^{m(2p^s-j_4-\kappa_4)}\cdot p^{m(p^s-t_5)}$. By applying Theorem 4.1, $d_H(C) \le p^s$. Using Theorem 2.11, C is an MDS code when $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+m(2p^s-j_4-\kappa_4)+2m(p^s-j_5) = 2m(5p^s-d_H(C)+1)$ or $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+m(2p^s-j_4-\kappa_4)+m(p^s-t_5) = 2m(5p^s-d_H(C)+1)$. We have $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+m(2p^s-j_4-\kappa_4)+2m(p^s-j_5) < 8mp^s$, $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+m(2p^s-j_4-\kappa_4)+m(p^s-t_5) < 8mp^s$. It is easy to see that $2m(5p^s-d_H(C)+1) > 8mp^s$. Then we see that $|C| < p^{2m(5p^s-d_H(C)+1)}$. Hence, C is not an MDS code in this case.

Case 4: $C_1 = \langle (x-\gamma_1)^{j_1}\rangle$, $C_2 = \langle u(x-\gamma_2)^{j_2}\rangle$, $C_3 = \langle u(x-\gamma_3)^{j_3}\rangle$, $C_4 = \langle (x-\gamma_4)^{j_4}+u(x-\gamma_4)^{t_4}h_4(x), u(x-\gamma_4)^{\kappa_4}\rangle$ *and* $C_5 = \langle (x-\gamma_5)^{j_5}+u(x-\gamma_5)^{t_5}h_5(x), u(x-\gamma_5)^{\kappa_5}\rangle$. We see that $|C| = p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{m(p^s-j_3)}\cdot p^{m(2p^s-j_4-\kappa_4)}\cdot p^{m(2p^s-j_5-\kappa_5)}$. Applying Theorem 4.1, $d_H(C) \le p^s$. By Theorem 2.11, C is an MDS code when $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+m(2p^s-j_4-\kappa_4)+m(2p^s-j_5-\kappa_5) = 2m(5p^s-d_H(C)+1)$. We have $m(p^s-j_1)+m(p^s-j_2)+m(p^s-j_3)+m(2p^s-j_4-\kappa_4)+m(2p^s-j_5-\kappa_5) < 8mp^s$. We see that $2m(5p^s-d_H(C)+1) > 8mp^s$. Hence, $|C| < p^{2m(5p^s-d_H(C)+1)}$. Hence, C is not an MDS code in this case. □

Proposition 4.7: Let $C = C_1 \oplus C_i \oplus C_j \oplus C_k \oplus C_t$ *be a cyclic code of length* $5p^s$ *over* $\mathcal{R}$, *where* C_i, C_j, C_k, C_t *are defined as in Theorem 2.5 and* i, j, k, t *are different numbers in* $\{2,3,4,5\}$. *If* C_i *is a constacyclic code of Type 2 of length* p^s *over* $\mathcal{R}$, *then* C *is not an MDS code.*

Proof: If C_j, C_k, C_t are constacyclic codes of Type 2 of length p^s over $\mathcal{R}$, then C is not an MDS code by Proposition 4.5. Thus, C_j, C_k, C_t are constacyclic codes of Type 3 or Type 4 of length p^s over $\mathcal{R}$. Without loss of generality, assume that $C_1 = \langle (x-1)^{j_1}\rangle$, $C_2 = \langle u(x-\gamma_2)^{j_2}\rangle$, $C_3 = \langle (x-\gamma_3)^{j_3}+u(x-\gamma_3)^{t_3}h_3(x)\rangle$ or $C_3 = \langle (x-\gamma_3)^{j_3}+u(x-\gamma_3)^{t_3}h_3(x), u(x-\gamma_3)^{\kappa_3}\rangle$ and $C_4 = \langle (x-\gamma_4)^{j_4}+u(x-\gamma_4)^{t_4}h_4(x)\rangle$ or $C_4 = \langle (x-\gamma_4)^{j_4}+u(x-\gamma_4)^{t_4}h_4(x), u(x-\gamma_4)^{\kappa_4}\rangle$ and $C_5 = \langle (x-\gamma_5)^{j_5}+u(x-\gamma_5)^{t_5}h_5(x)\rangle$ or $C_5 = \langle (x-\gamma_5)^{j_5}+u(x-\gamma_5)^{t_5}h_5(x), u(x-\gamma_5)^{\kappa_5}\rangle$ ($j_1, j_2, j_3, j_4, t_5, \kappa_5$ are defined as in Theorem 2.2).

Case 1: $C_1 = \langle (x-\gamma_1)^{j_1}\rangle$, $C_2 = \langle u(x-\gamma_2)^{j_2}\rangle$, $C_3 = \langle (x-\gamma_3)^{j_3}+u(x-\gamma_3)^{t_3}h_3(x)\rangle$, $C_4 = \langle (x-\gamma_4)^{j_4}+u(x-\gamma_4)^{t_4}h_4(x)\rangle$ *and* $C_5 = \langle (x-\gamma_5)^{j_5}+u(x-\gamma_5)^{t_5}h_5(x)\rangle$. Then we have $|C| = p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{2m(p^s-j_3)}\cdot p^{2m(p^s-j_4)}\cdot p^{2m(p^s-j_5)}$ or $|C| = p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{2m(p^s-j_3)}\cdot p^{2m(p^s-j_4)}\cdot p^{m(p^s-t_5)}$ or $p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{2m(p^s-j_3)}\cdot p^{m(p^s-t_4)}\cdot p^{2m(p^s-j_5)}$ or $p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{2m(p^s-j_3)}\cdot p^{m(p^s-t_4)}\cdot p^{m(p^s-t_5)}$ or $|C| = p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{m(p^s-t_3)}\cdot p^{2m(p^s-j_4)}\cdot p^{2m(p^s-j_5)}$ or $|C| = p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{m(p^s-t_3)}\cdot p^{2m(p^s-j_4)}\cdot p^{m(p^s-t_5)}$ or $p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{m(p^s-t_3)}\cdot p^{m(p^s-t_4)}\cdot p^{2m(p^s-j_5)}$ or $p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{m(p^s-t_3)}\cdot p^{m(p^s-t_4)}\cdot p^{m(p^s-t_5)}$. By applying Theorem 4.1, $d_H(C) \le p^s$. Using Theorem 2.11, C is an MDS code when $|C| = p^{2m(5p^s-d_H(C)+1)}$. We see that $|C| < 8mp^s$ and $2m(5p^s-d_H(C)+1) > 8mp^s$. Then we see that $|C| < p^{2m(5p^s-d_H(C)+1)}$. Hence, C is not an MDS code in this case.

Case 2: $C_1 = \langle (x-\gamma_1)^{j_1}\rangle$, $C_2 = \langle u(x-\gamma_2)^{j_2}\rangle$, $C_3 = \langle (x-\gamma_3)^{j_3}+u(x-\gamma_3)^{t_3}h_3(x)\rangle$, $C_4 = \langle (x-\gamma_4)^{j_4}+u(x-\gamma_4)^{t_4}h_4(x)\rangle$ *and* $C_5 = \langle (x-\gamma_5)^{j_5}+u(x-\gamma_5)^{t_5}h_5(x), u(x-\gamma_5)^{\kappa_5}\rangle$. We see that $|C| = p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{2m(p^s-j_3)}\cdot p^{2m(p^s-j_4)}\cdot p^{m(2p^s-j_5-\kappa_5)}$ or $|C| = p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{2m(p^s-j_3)}\cdot p^{m(p^s-t_4)}\cdot p^{m(2p^s-j_5-\kappa_5)}$ or $|C| = p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{m(p^s-t_3)}\cdot p^{2m(p^s-j_4)}\cdot p^{m(2p^s-j_5-\kappa_5)}$ or $|C| = p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{m(p^s-t_3)}\cdot p^{m(p^s-t_4)}\cdot p^{m(2p^s-j_5-\kappa_5)}$. Applying Theorem 4.1, $d_H(C) \le p^s$. By Theorem 2.11, C is an MDS code when $m(p^s-j_1)+m(p^s-j_2)+2m(p^s-j_3)+2m(p^s-j_4)+m(2p^s-j_5-\kappa_5) = 2m(5p^s-d_H(C)+1)$ or $m(p^s-j_1)+m(p^s-j_2)+2m(p^s-j_3)+m(p^s-t_4)+m(2p^s-j_5-\kappa_5) = 2m(5p^s-d_H(C)+1)$ or $m(p^s-j_1)+m(p^s-j_2)+m(p^s-t_3)+2m(p^s-j_4)+m(2p^s-j_5-\kappa_5) = 2m(5p^s-d_H(C)+1)$ or $m(p^s-j_1)+m(p^s-j_2)+m(p^s-t_3)+m(p^s-t_4)+m(2p^s-j_5-\kappa_5) = 2m(5p^s-d_H(C)+1)$. It is easy to check that $m(p^s-j_1)+m(p^s-j_2)+2m(p^s-j_3)+2m(p^s-j_4)+m(2p^s-j_5-\kappa_5) < 8mp^s$, $m(p^s-j_1)+m(p^s-j_2)+2m(p^s-j_3)+m(p^s-t_4)+m(2p^s-j_5-\kappa_5) < 8mp^s$, $m(p^s-j_1)+m(p^s-j_2)+m(p^s-t_3)+2m(p^s-j_4)+m(2p^s-j_5-\kappa_5) < 8mp^s$, and $m(p^s-j_1)+m(p^s-j_2)+m(p^s-t_3)+m(p^s-t_4)+m(2p^s-j_5-\kappa_5) < 8mp^s$. We see that $2m(5p^s-d_H(C)+1) > 8mp^s$. Hence, $|C| < p^{2m(5p^s-d_H(C)+1)}$. Thus, C is not an MDS code in this case.

Case 3: $C_1 = \langle (x-\gamma_1)^{j_1}\rangle$, $C_2 = \langle u(x-\gamma_2)^{j_2}\rangle$, $C_3 = \langle (x-\gamma_3)^{j_3}+u(x-\gamma_3)^{t_3}h_3(x)\rangle$, $C_4 = \langle (x-\gamma_4)^{j_4}+u(x-\gamma_4)^{t_4}h_4(x), u(x-\gamma_4)^{\kappa_4}\rangle$ *and* $C_5 = \langle (x-\gamma_5)^{j_5}+u(x-\gamma_5)^{t_5}h_5(x), u(x-\gamma_5)^{\kappa_5}\rangle$. Then we have $|C| = p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{2m(p^s-j_3)}\cdot p^{m(2p^s-j_4-\kappa_4)}\cdot p^{m(2p^s-j_5-\kappa_5)}$ or $|C| = p^{m(p^s-j_1)}\cdot p^{m(p^s-j_2)}\cdot p^{m(p^s-t_3)}\cdot p^{m(2p^s-j_4-\kappa_4)}\cdot p^{m(2p^s-j_5-\kappa_5)}$. By applying Theorem 4.1, $d_H(C) \le p^s$. Using Theorem 2.11, C is an MDS code when $m(p^s-j_1)+m(p^s-j_2)+2m(p^s-j_3)+m(2p^s-j_4-\kappa_4)+m(2p^s-j_5-\kappa_5) = 2m(5p^s-d_H(C)+1)$

or $m(p^s - j_1) + m(p^s - j_2) + m(p^s - t_3) + m(2p^s - j_4 - \kappa_4) + m(2p^s - j_5 - \kappa_5) = 2m(5p^s - d_H(C) + 1)$. We have $m(p^s - j_1) + m(p^s - j_2) + 2m(p^s - j_3) + m(2p^s - j_4 - \kappa_4) + m(2p^s - j_5 - \kappa_5) < 8mp^s$, $m(p^s - j_1) + m(p^s - j_2) + m(p^s - t_3) + m(2p^s - j_4 - \kappa_4) + m(2p^s - j_5 - \kappa_5) < 8mp^s$ and $2m(5p^s - d_H(C) + 1) > 8mp^s$. Then we see that $|C| < p^{2m(5p^s - d_H(C)+1)}$. Hence, C is not an MDS code in this case.

Case 4: $C_1 = \langle (x-\gamma_1)^{j_1}\rangle$, $C_2 = \langle u(x-\gamma_2)^{j_2}\rangle$, $C_3 = \langle (x-\gamma_3)^{j_3} + u(x-\gamma_3)^{t_3} h_3(x), u(x-\gamma_3)^{\kappa_3}\rangle$, $C_4 = \langle (x-\gamma_4)^{j_4} + u(x-\gamma_4)^{t_4} h_4(x)\rangle$ *and* $C_5 = \langle (x-\gamma_5)^{j_5} + u(x-\gamma_5)^{t_5} h_5(x)\rangle$. Similar to Case 2, we can conclude that C is not an MDS code.

Case 5: $C_1 = \langle (x-\gamma_1)^{j_1}\rangle$, $C_2 = \langle u(x-\gamma_2)^{j_2}\rangle$, $C_3 = \langle (x-\gamma_3)^{j_3} + u(x-\gamma_3)^{t_3} h_3(x), u(x-\gamma_3)^{\kappa_3}\rangle$, $C_4 = \langle (x-\gamma_4)^{j_4} + u(x-\gamma_4)^{t_4} h_4(x)\rangle$ *and* $C_5 = \langle (x-\gamma_5)^{j_5} + u(x-\gamma_5)^{t_5} h_5(x), u(x-\gamma_5)^{\kappa_5}\rangle$. Similar to Case 3, C is not an MDS code.

Case 6: $C_1 = \langle (x-\gamma_1)^{j_1}\rangle$, $C_2 = \langle u(x-\gamma_2)^{j_2}\rangle$, $C_3 = \langle (x-\gamma_3)^{j_3} + u(x-\gamma_3)^{t_3} h_3(x), u(x-\gamma_3)^{\kappa_3}\rangle$, $C_4 = \langle (x-\gamma_4)^{j_4} + u(x-\gamma_4)^{t_4} h_4(x), u(x-\gamma_4)^{\kappa_4}\rangle$ *and* $C_5 = \langle (x-\gamma_5)^{j_5} + u(x-\gamma_5)^{t_5} h_5(x), u(x-\gamma_5)^{\kappa_5}\rangle$. Then we have $|C| = p^{m(p^s-j_1)} \cdot p^{m(p^s-j_2)} \cdot p^{m(2p^s-j_3-\kappa_3)} \cdot p^{m(2p^s-j_4-\kappa_4)} \cdot p^{m(2p^s-j_5-\kappa_5)}$. By applying Theorem 4.1, $d_H(C) \le p^s$. Using Theorem 2.11, C is an MDS code when $m(p^s - j_1) + m(p^s - j_2) + m(2p^s - j_3 - \kappa_3) + m(2p^s - j_4 - \kappa_4) + m(2p^s - j_5 - \kappa_5) = 2m(5p^s - d_H(C) + 1)$. We see that $m(p^s - j_1) + m(p^s - j_2) + m(2p^s - j_3 - \kappa_3) + m(2p^s - j_4 - \kappa_4) + m(2p^s - j_5 - \kappa_5) < 8mp^s$ and $2m(5p^s - d_H(C) + 1) > 8mp^s$. Thus, C is not an MDS code. □

Proposition 4.8: Let $C = C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5$ *be a cyclic code of length* $5p^s$ *over* $\mathcal{R}$*, where* C_1, C_2, C_3, C_4, C_5 *are defined as in Theorem 4.1 such that all* C_i *are constacyclic codes of Type 3 of length* p^s *over* $\mathcal{R}$ *for* $i \in \{2, 3, 4, 5\}$*. Then* C *is not an MDS code.*

Proof: We have $C_1 = \langle (x-1)^{j_1}\rangle$, $C_2 = \langle (x-\gamma_2)^{j_2} + u(x-\gamma_2)^{t_2} h_2(x)\rangle$, $C_3 = \langle (x-\gamma_3)^{j_3} + u(x-\gamma_3)^{t_3} h_3(x)\rangle$, $C_4 = \langle (x-\gamma_4)^{j_4} + u(x-\gamma_4)^{t_4} h_4(x)\rangle$, $C_5 = \langle (x-\gamma_5)^{j_5} + u(x-\gamma_5)^{t_5} h_5(x)\rangle$ (j_2, j_3, j_4, j_5 are defined as in Theorem 2.2). If $|C_2| = p^{m(p^s-t_2)}$, then $|C| = p^{m(p^s-j_1)} \cdot p^{m(p^s-t_2)} \cdot |C_3| \cdot |C_4| \cdot |C_5|$. From Theorem 4.1, $d_H(C) \le p^s$. Using Theorem 2.11, C is an MDS code when $p^{m(p^s-j_1)} \cdot p^{m(p^s-t_2)} \cdot |C_3| \cdot |C_4| \cdot |C_5| = p^{2m(5p^s-d_H(C)+1)}$. It is easy to check that $p^{m(p^s-j_1)} \cdot p^{m(p^s-t_2)} \cdot |C_3| \cdot |C_4| \cdot |C_5| < p^{8mp^s}$. Since $2m(5p^s - d_H(C)+1) > 8mp^s$, we have $|C| < p^{2m(5p^s-d_H(C)+1)}$. Hence, C is not an MDS code if $|C_2| = p^{m(p^s-t_2)}$. It is easy to see that if there exists $\ell \in \{2, 3, 4, 5\}$ such that $|C_\ell| = p^{m(p^s-t_\ell)}$, then C is not an MDS code. Thus, we need to consider the case $|C| = p^{m(p^s-j_1)} \cdot p^{2m(p^s-j_2)} \cdot p^{2m(p^s-j_3)} \cdot p^{2m(p^s-j_4)} \cdot p^{2m(p^s-j_5)}$. By Theorem 2.11, C is an MDS code when $m(p^s - j_1) + 2m(p^s - j_2) + 2m(p^s - j_3) + 2m(p^s - j_4) + 2m(p^s - j_5) = 2m(5p^s - d_H(C) + 1)$. Hence, C is an MDS code when $j_1 + 2j_2 + 2j_3 + 2j_4 + 2j_5 = -p^s + 2\,d_H(C) - 2$. Applying Theorem 4.1, we see that $d_H(C) = \min\{d_H(C_i)\}$, where $i = \{1, 2, 3, 4, 5\}$. Then we have $d_H(C) = d_H(C_1)$ or $d_H(C) = d_H(C_2)$ or $d_H(C) = d_H(C_3)$ or $d_H(C) = d_H(C_4)$ or $d_H(C) = d_H(C_5)$.

Case 1: $d_H(C) = d_H(C_1)$. From Theorem 2.3, $d_H(C_1) = 1$ or $d_H(C_1) = (\delta_1 + 1)p^{k_1}$.

Subcase 1.1: $d_H(C_1) = 1$. In this subcase, $2j_2 + 2j_3 + 2j_4 + 2j_5 = -p^s$, which is a contradiction. Hence, C is not an MDS code in this subcase.

Subcase 1.2: $d_H(C_1) = (\delta_1 + 1)p^{k_1}$. Then we see that $2j_2 + 2j_3 + 2j_4 + 2j_5 = -p^s + 2(\delta_1 + 1)p^{k_1} - 2$ (1). From (1), we have $2(\delta_1 + 1)p^{k_1} > p^s$. Since $\delta_1 + 1 \le p$, we have $p^s < 2p \cdot p^{k_1}$. It implies that $p^{s-k_1-1} < 2$. Hence, $s - k_1 - 1 = 0$, i.e., $s = k_1 + 1$. By asumption $p^s - p^{s-k_1} + (\delta_1 - 1)p^{s-k_1-1} + 1 \le j_1 \le p^s - p^{s-k_1} + \delta_1 p^{s-k_1-1}$, we see that $p^s - p + \delta_1 - 1 + 1 \le j_1 \le p^s - p + \delta_1$. It follows that $j_1 = p^s - p + \delta_1$. Put $T_1 = 2j_2 + 2j_3 + 2j_4 + 2j_5$. Then $T_1 = 2j_2 + 2j_3 + 2j_4 + 2j_5 = -p^s + 2(\delta_1 + 1)p^{s-1} - 2 - (p^s - p + \delta_1)$. Hence,

$$\begin{aligned} T_1 &= p^{s-1}[2(\delta_1 + 1) - 2p] - (\delta_1 - p + 1) - 1 \\ &= [2p^{s-1} - 1][\delta_1 - p + 1] - 1. \end{aligned}$$

Since $2p^{s-1} - 1 > 0$ and $\delta_1 - p + 1 \le 0$, we have $[2p^{s-1} - 1][\delta_1 - p + 1] - 1 < 0$. Thus, $j_1 + 2j_2 + 2j_3 + 2j_4 + 2j_5 \ne -p^s + 2(\delta_1 + 1)p^{k_1} - 2$, i.e., C is not an MDS code when $d_H(C) = d_H(C_1)$.

Case 2: $d_H(C) = d_H(C_2)$. When $d_H(C) = d_H(C_3)$, $d_H(C) = d_H(C_4)$ and $d_H(C) = d_H(C_5)$ can be done similarly. Using Theorem 2.3, we have 2 subcases.

Subcase 2.1: $d_H(C_2) = d_H(\langle (x-\gamma_2)^{j_2}\rangle_F)$ *when* $h_2(x)$ *is 0 or* $h_2(x)$ *is a unit and* $1 \le j_2 \le \frac{p^s+t_2}{2}$. Then

$$d_H(C_2) = d_H(\langle (x-\gamma_2)^{j_2}\rangle_F) = (\delta_2 + 1)p^{k_2},$$

where $p^s - pr_2 + (\delta_2 - 1)r_2 + 1 \le j_2 \le p^s - pr_2 + \delta_2 r_2$, $1 \le \delta_2 \le p - 1$, $r_2 = p^{s-k_2-1}$, and $0 \le k_2 \le s - 1$. Then C is an MDS code when $j_1 + 2j_2 + 2j_3 + 2j_4 + 2j_5 = -p^s + 2(\delta_2 + 1)p^{k_2} - 2$. Similar to Subcase 1.2 of Case 1, we can conclude that $j_1 + 2j_2 + 2j_3 + 2j_4 + 2j_5 \ne -p^s + 2(\delta_2 + 1)p^{k_2} - 2$, i.e., C is not an MDS code in this subcase.

Subcase 2.2: $d_H(C_2) = d_H(\langle (x-\gamma_2)^{j_2}\rangle_F)$ *when* $h_2(x)$ *is a unit and* $\frac{p^s+t_2}{2} < j_2 \le p^s - 1$. Then

$$d_H(C_2) = d_H(\langle (x-\gamma_2)^{j_2}\rangle_F) = (\delta_2 + 1)p^{k_2},$$

where $t_2 + pr_2 - \delta_2 r_2 \le j_2 \le t_2 + pr_2 - (\delta_2 - 1)r_2 - 1$, $1 \le \delta_2 \le p - 1$, $r_2 = p^{s-k_2-1}$, and $0 \le k_2 \le s - 1$. Then C is an MDS code when $j_1 + 2j_2 + 2j_3 + 2j_4 + 2j_5 = -p^s + 2(\delta_2 + 1)p^{k_2} - 2$. From $j_2 > \frac{p^s+t_2}{2}$, we have $2j_2 > p^s + t_2$. We see that $j_1 + 2j_2 + 2j_3 + 2j_4 + 2j_5 + p^s - 2(\delta_2 + 1)p^{k_2} + 2 > j_1 + p^s + t_2 + 2j_3 + 2j_4 + 2j_5 + p^s - 2(\delta_2 + 1)p^{k_2} + 2 > 0$, i.e., $j_1 + 2j_2 + 2j_3 + 2j_4 + 2j_5 > -p^s + 2(\delta_2 + 1)p^{k_2} - 2$. Hence, C is not an MDS code in this subcase. Thus, C is not an MDS code. □

Proposition 4.9: Let $C = C_1 \oplus C_i \oplus C_j \oplus C_k \oplus C_t$ *be a cyclic code of length* $5p^s$ *over* $\mathcal{R}$*, where* C_i, C_j, C_k, C_t *are defined as in Theorem 2.5 and* i, j, k, t *are different numbers in* $\{2, 3, 4, 5\}$*. If* C_i*,* C_j*, and* C_k *are constacyclic codes of Type 3 of length* p^s *over* $\mathcal{R}$*, then* C *is not an MDS code.*

Proof: If C_t is a constacyclic code of Type 3 of length p^s over $\mathcal{R}$, then C is not an MDS code by Proposition 4.8.

Thus, C_t is a constacyclic code of Type 4 of length p^s over $\mathcal{R}$. Without loss of generality, assume that $C_2 = \langle (x-\gamma_2)^{j_2} + u(x-\gamma_2)^{t_2}h_2(x)\rangle$, $C_3 = \langle (x-\gamma_3)^{j_3} + u(x-\gamma_3)^{t_3}h_3(x)\rangle$, $C_4 = \langle (x-\gamma_4)^{j_4} + u(x-\gamma_4)^{t_4}h_4(x)\rangle$ and $C_5 = \langle (x-\gamma_5)^{j_5} + u(x-\gamma_5)^{t_5} h_5(x), u(x-\gamma_5)^{\kappa_5}\rangle$ ($j_1, j_2, j_3, j_4, t_5, \kappa_5$ are defined as in Theorem 2.2). If there exists $\ell \in \{2,3,4\}$ such that $|C_\ell| = p^{m(p^s-t_\ell)}$. Without loss of generality, assume that $|C_2| = p^{m(p^s-t_2)}$. Using Theorem 2.11, C is an MDS code when $p^{m(p^s-j_1)} \cdot p^{m(p^s-t_2)} \cdot |C_3| \cdot |C_4| \cdot |C_5| = p^{2m(5p^s-d_H(C)+1)}$. It is easy to check that $p^{m(p^s-j_1)} \cdot p^{m(p^s-t_2)} \cdot |C_3|\cdot|C_4|\cdot|C_5| < p^{8mp^s}$ because $|C_3|, |C_4|, |C_5| < 2mp^s$. Since $2m(5p^s - d_H(C)+1) > 8mp^s$, we have $|C| < p^{2m(5p^s-d_H(C)+1)}$. Hence, C is not an MDS code when $|C_2| = p^{m(p^s-t_2)}$. Thus, C is not an MDS code if there exists $\ell \in \{2,3,4\}$ such that $|C_\ell| = p^{m(p^s-t_\ell)}$. We consider the remaining case: $|C| = p^{m(p^s-j_1)} \cdot p^{2m(p^s-j_2)} \cdot p^{2m(p^s-j_3)} \cdot p^{2m(p^s-j_4)} \cdot p^{m(2p^s-j_5-\kappa_5)}$. By Theorem 2.11, C is an MDS code when $m(p^s-j_1) + 2m(p^s-j_2) + 2m(p^s-j_3) + 2m(p^s-j_4) + m(2p^s-j_5-\kappa_5) = 2m(5p^s - d_H(C)+1)$. Hence, C is not an MDS code when $j_1 + 2j_2 + 2j_3 + 2j_4 + j_5 + \kappa_5 = -p^s + 2(d_H(C) - 2$. Applying Theorem 4.1, we see that $d_H(C) = \min\{d_H(C_i)\}$, where $i = \{1,2,3,4,5\}$. Then we have $d_H(C) = d_H(C_1)$ or $d_H(C) = d_H(C_2)$ or $d_H(C) = d_H(C_3)$ or $d_H(C) = d_H(C_4)$ or $d_H(C) = d_H(C_5)$.

Case 1: $d_H(C) = d_H(C_1)$. From Theorem 2.3, $d_H(C_1) = 1$ or $d_H(C_1) = (\delta_1+1)p^{k_1}$.

Subcase 1.1: $d_H(C_1) = 1$. In this subcase, $2j_2+2j_3+2j_4+j_5+\kappa_5 = -p^s$, which is a contradiction. Hence, C is not an MDS code in this subcase.

Subcase 1.2: $d_H(C_1) = (\delta_1+1)p^{k_1}$. Then we see that $= -p^s+2(\delta_1+1)p^{k_1}-2$ (2). From (2), we have $2(\delta_1+1)p^{k_1} > p^s$. Since $\delta_1+1 \le p$, we have $p^s < 2p\cdot p^{k_1}$. It implies that $p^{s-k_1-1} < 2$. Hence, $s-k_1-1 = 0$, i.e., $s = k_1+1$. By asumption $p^s - p^{s-k_1} + (\delta_1-1)p^{s-k_1-1} + 1 \le j_1 \le p^s - p^{s-k_1} + \delta_1 p^{s-k_1-1}$, we see that $p^s - p + \delta_1 - 1 + 1 \le j_1 \le p^s - p + \delta_1$. It follows that $j_1 = p^s - p + \delta_1$. Put $T_2 = 2j_2+2j_3+2j_4+j_5+\kappa_5$. Then $T_2 = 2j_2+2j_3+2j_4+j_5+\kappa_5 = -p^s + 2(\delta_1+1)p^{s-1} - 2 - (p^s - p + \delta_1)$. Hence,

$$\begin{aligned} T_2 &= p^{s-1}[2(\delta_1+1) - 2p] - (\delta_1 - p + 1) - 1 \\ &= [2p^{s-1} - 1][\delta_1 - p + 1] - 1. \end{aligned}$$

Since $2p^{s-1} - 1 > 0$ and $\delta_1 - p + 1 \le 0$, we have $[2p^{s-1}-1][\delta_1-p+1] - 1 < 0$. Thus, $j_1 + 2j_2 + 2j_3 + 2j_4 + j_5 + \kappa_5 \ne -p^s + 2(\delta_1+1)p^{k_1} - 2$, i.e., C is not an MDS code when $d_H(C) = d_H(C_1)$.

Case 2: $d_H(C) = d_H(C_2)$. We see that $d_H(C) = d_H(C_3)$ and $d_H(C) = d_H(C_4)$ can be done similarly. Using Theorem 2.3, we have 2 subcases.

Subcase 2.1: $d_H(C_2) = d_H(\langle (x-\gamma_2)^{j_2}\rangle_F)$ *when* $h_2(x)$ *is 0 or* $h_2(x)$ *is a unit and* $1 \le j_2 \le \frac{p^s+t_2}{2}$. Then

$$d_H(C_2) = d_H(\langle (x-\gamma_2)^{j_2}\rangle_F) = (\delta_2+1)p^{k_2},$$

where $p^s - pr_2 + (\delta_2-1)r_2 + 1 \le j_2 \le p^s - pr_2 + \delta_2 r_2$, $1 \le \delta_2 \le p-1$, $r_2 = p^{s-k_2-1}$, and $0 \le k_2 \le s-1$. Then C is an MDS code when $j_1 + 2j_2 + 2j_3 + 2j_4 + j_5 + \kappa_5 = -p^s + 2(\delta_2+1)p^{k_2} - 2$. Similar to Subcase 1.2 of Case 1, we can conclude that $j_1 + 2j_2 + 2j_3 + 2j_4 + j_5 + \kappa_5 \ne -p^s + 2(\delta_2+1)p^{k_2} - 2$, i.e., C is not an MDS code in this subcase.

Subcase 2.2: $d_H(C_2) = d_H(\langle (x-\gamma_2)^{j_2}\rangle_F)$ *when* $h_2(x)$ *is a unit and* $\frac{p^s+t_2}{2} < j_2 \le p^s - 1$. Then

$$d_H(C_2) = d_H(\langle (x-\gamma_2)^{j_2}\rangle_F) = (\delta_2+1)p^{k_2},$$

where $t_2 + pr_2 - \delta_2 r_2 \le j_2 \le t_2 + pr_2 - (\delta_2-1)r_2 - 1$, $1 \le \delta_2 \le p-1$, $r_2 = p^{s-k_2-1}$, and $0 \le k_2 \le s-1$. Then C is an MDS code when $j_1 + 2j_2 + 2j_3 + 2j_4 + j_5 + \kappa_5 = -p^s + 2(\delta_2+1)p^{k_2} - 2$. From $j_2 > \frac{p^s+t_2}{2}$, we have $2j_2 > p^s + t_2$. We see that $j_1 + 2j_2 + 2j_3 + 2j_4 + j_5 + \kappa_5 + p^s - 2(\delta_2+1)p^{k_2} + 2 > j_1 + p^s + t_2 + 2j_3 + 2j_4 + j_5 + \kappa_5 + p^s - 2(\delta_2+1)p^{k_2} + 2 > 0$, i.e., $j_1 + 2j_2 + 2j_3 + 2j_4 + j_5 + \kappa_5 > -p^s + 2(\delta_2+1)p^{k_2} - 2$. Hence, C is not an MDS code in this subcase. Thus, C is not an MDS code.

Case 3: $d_H(C) = d_H(C_5) = d_H(\langle (x-\gamma_5)^{\kappa_5}\rangle_F) = (\delta_5+1)p^{k_5}$, *where* $p^s - p^{s-k_5} + (\delta_5-1)p^{s-k_5-1} + 1 \le \kappa_5 \le p^s - p^{s-k_5} + \delta_5 p^{s-k_5-1}$, $1 \le \delta_5 \le p-1$, *and* $0 \le k_5 \le s-1$. Then C is an MDS code when $j_1+2j_2+2j_3+2j_4+j_5+\kappa_5 = -p^s + 2(\delta_5+1)p^{k_5} - 2$. From $j_1+2j_2+2j_3+2j_4+j_5+\kappa_5 = -p^s+2(\delta_5+1)p^{k_5}-2$, we see that $2(\delta_5+1)p^{k_5} > p^s$ (5). Since $\delta_5+1 \le p$, we have $p^s < 2p\cdot p^{k_5}$. It implies that $p^{s-k_5-1} < 2$. Hence, $s = k_5+1$. From $p^s - p^{s-k_5} + (\delta_5-1)p^{s-k_5-1} + 1 \le j_5 \le p^s - p^{s-k_5} + \delta_5 p^{s-k_5-1}$, we see that $j_5 = p^s - p + \delta_5$. From $s = k_5+1$, (5) becomes $2(\delta_5+1)p^{s-1} > p^s$. It implies that $\delta_5 + 1 \le p < 2(\delta_5+1)$. By assumption, $(\delta_5+1)p^{k_5} \le (\delta_5+1)p^{k_5}$. Hence, $(\delta_5+1)p^{s-1} \le p^{k_2+1}$. It follows that $k_2+1 > s-1$. Therefore, $k_2 > s-2$. It shows that $k_2 = s-1$ since $k_2 \le s-1$. Similarly, we see that $k_1 = s-1$, $k_3 = s-1$ and $k_4 = s-1$. Since $(\delta_5+1)p^{k_5} \le (\delta_1+1)p^{k_1} \le (\delta_2+1)p^{k_2} \le (\delta_3+1)p^{k_3} \le (\delta_4+1)p^{k_4}$, we have $\delta_1 \le \delta_2 \le \delta_3 \le \delta_4$. From this, $j_1 = p^s - p + \delta_1$, $j_2 = p^s - p + \delta_2$, $j_3 = p^s - p + \delta_3$ and $j_4 = p^s - p + \delta_4$. From $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_3 = -p^s + 2(\delta_5+1)p^{k_5} - 2$, $j_1 = p^s - p + \delta_1$ and $j_2 = p^s - p + \delta_2$, $j_3 = p^s - p + \delta_3$ and $j_4 = p^s - p + \delta_4$, we see that $(p^s-p+\delta_1) + 2(p^s-p+\delta_2) + 2(p^s-p+\delta_3) + 2(p^s-p+\delta_4) + j_5 + \kappa_5 = -p^s + 2(\delta_5+1)p^{s-1} - 2$. Hence,

$$\begin{aligned} j_5 + \kappa_5 &= -8p^s + 2(\delta_3+1)p^{s-1} - 2 \\ &\quad + 7p - \delta_1 - 2\delta_2 - 2\delta_3 - 2\delta_4 \\ &= [-2p^s + 2(\delta_5+1)p^{s-1}] \\ &\quad + [p - \delta_1 - p^s] + [p - 2\delta_2 - p^s] \\ &\quad + [p - 2\delta_3 - p^s] + [p - 2\delta_4 - p^s] \\ &\quad + [3p - 2p^s - 2]. \end{aligned}$$

If $s \ge 2$, then $j_3 + \kappa_3 < 0$. This is a contradiction. If $s = 1$, then $k_1 = k_2 = k_3 = k_4 = 0$. Hence, $j_1 = \delta_1, j_2 = \delta_2$, $j_3 = \delta_3$ and $j_4 = \delta_4$. It implies that $j_1 + 2j_2 + j_3 + 2j_4 + j_5 + \kappa_5 = 2\delta_5 - p < 2\delta_1$, which is a contradiction. Therefore, $j_1 + 2j_2 + 2j_3 + 2j_4 + j_5 + \kappa_5 \ne -p^s + 2(\delta_5+1)p^{k_5} - 2$, i.e., C is not an MDS code. □

Proposition 4.10: Let $C = C_1 \oplus C_i \oplus C_j \oplus C_k \oplus C_t$ *be a cyclic code of length* $5p^s$ *over* $\mathcal{R}$, *where* C_i, C_j, C_k, C_t

are defined as in Theorem 2.5 and i, j, k, t *are different numbers in* $\{2, 3, 4, 5\}$. *If* C_i, C_j *are constacyclic codes of Type 3 of length* p^s *over* $\mathcal{R}$, *then* C *is not an MDS code.*

Proof: If C_k, C_t are constacyclic codes of Type 3 of length p^s over $\mathcal{R}$, then C is not an MDS code by Proposition 4.8. Thus, C_k, C_t are constacyclic codes of Type 4 of length p^s over $\mathcal{R}$. Without loss of generality, we have $C_2 = \langle (x-\gamma_2)^{j_2} + u(x-\gamma_2)^{t_2} h_2(x)\rangle$, $C_3 = \langle (x-\gamma_3)^{j_3} + u(x-\gamma_3)^{t_3} h_3(x)\rangle$, $C_4 = \langle (x-\gamma_4)^{j_4} + u(x-\gamma_4)^{t_4} h_4(x), u(x-\gamma_4)^{\kappa_4}\rangle$ and $C_5 = \langle (x-\gamma_5)^{j_5} + u(x-\gamma_5)^{t_5} h_5(x), u(x-\gamma_5)^{\kappa_5}\rangle$ ($j_1, j_2, j_3, j_4, t_5, \kappa_5$ are defined as in Theorem 2.2). Assume that there exists $\ell \in \{2, 3\}$ such that $|C_\ell| = p^{m(p^s-t_\ell)}$. If $|C_2| = p^{m(p^s-t_2)}$, using Theorem 2.11, then C is an MDS code when $p^{m(p^s-j_1)} \cdot p^{m(p^s-t_2)} \cdot |C_3| \cdot |C_4| \cdot |C_5| = p^{2m(5p^s-\mathrm{d_H}(C)+1)}$. It is easy to check that $p^{m(p^s-j_1)} \cdot p^{m(p^s-t_2)} \cdot |C_3| \cdot |C_4| \cdot |C_5| < p^{8mp^s}$ because $|C_3|, |C_4|, |C_5| < 2mp^s$. Since $2m(5p^s - \mathrm{d_H}(C) + 1) > 8mp^s$, we have $|C| < p^{2m(5p^s-\mathrm{d_H}(C)+1)}$. Hence, C is not an MDS code when $|C_2| = p^{m(p^s-t_2)}$. Thus, C is not an MDS code if there exists $\ell \in \{2, 3\}$ satisfying $|C_\ell| = p^{m(p^s-t_\ell)}$. We consider the remaining case $|C| = p^{m(p^s-j_1)} \cdot p^{2m(p^s-j_2)} \cdot p^{2m(p^s-j_3)} \cdot p^{m(2p^s-j_4-\kappa_4)} \cdot p^{m(2p^s-j_5-\kappa_5)}$. By Theorem 2.11, C is an MDS code when $m(p^s - j_1) + 2m(p^s - j_2) + 2m(p^s - j_3) + m(2p^s - j_4 - \kappa_4) + m(2p^s - j_5 - \kappa_5) = 2m(5p^s - \mathrm{d_H}(C) + 1)$. Hence, C is not an MDS code when $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2\,\mathrm{d_H}(C) - 2$. Applying Theorem 4.1, we see that $\mathrm{d_H}(C) = \min\{\mathrm{d_H}(C_i)\}$, where $i = \{1, 2, 3, 4, 5\}$. Then we have $\mathrm{d_H}(C) = \mathrm{d_H}(C_1)$ or $\mathrm{d_H}(C) = \mathrm{d_H}(C_2)$ or $\mathrm{d_H}(C) = \mathrm{d_H}(C_3)$ or $\mathrm{d_H}(C) = \mathrm{d_H}(C_4)$ or $\mathrm{d_H}(C) = \mathrm{d_H}(C_5)$.

Case 1: $\mathrm{d_H}(C) = \mathrm{d_H}(C_1)$. From Theorem 2.3, $\mathrm{d_H}(C_1) = 1$ or $\mathrm{d_H}(C_1) = (\delta_1 + 1)p^{k_1}$.

Subcase 1.1: $\mathrm{d_H}(C_1) = 1$. In this subcase, $2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s$, which is a contradiction. Hence, C is not an MDS code.

Subcase 1.2: $\mathrm{d_H}(C_1) = (\delta_1 + 1)p^{k_1}$. Then we see that $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_1 + 1)p^{k_1} - 2$ (4). From (4), we have $2(\delta_1 + 1)p^{k_1} > p^s$. Since $\delta_1 + 1 \le p$, we have $p^s < 2p \cdot p^{k_1}$. It implies that $p^{s-k_1-1} < 2$. Hence, $s - k_1 - 1 = 0$, i.e., $s = k_1 + 1$. By asumption $p^s - p^{s-k_1} + (\delta_1 - 1)p^{s-k_1-1} + 1 \le j_1 \le p^s - p^{s-k_1} + \delta_1 p^{s-k_1-1}$, we see that $p^s - p + \delta_1 - 1 + 1 \le j_1 \le p^s - p + \delta_1$. It follows that $j_1 = p^s - p + \delta_1$. Put $T_4 = 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5$. Then $T_4 = 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_1 + 1)p^{s-1} - 2 - (p^s - p + \delta_1)$. Hence,

$$\begin{aligned} T_4 &= p^{s-1}[2(\delta_1 + 1) - 2p] - (\delta_1 - p + 1) - 1 \\ &= [2p^{s-1} - 1][\delta_1 - p + 1] - 1. \end{aligned}$$

Since $2p^{s-1} - 1 > 0$ and $\delta_1 - p + 1 \le 0$, we have $[2p^{s-1} - 1][\delta_1 - p + 1] - 1 < 0$. Thus, $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 \ne -p^s + 2(\delta_1 + 1)p^{k_1} - 2$, i.e., C is not an MDS code when $\mathrm{d_H}(C) = \mathrm{d_H}(C_1)$.

Case 2: $\mathrm{d_H}(C) = \mathrm{d_H}(C_2)$. We see that $\mathrm{d_H}(C) = \mathrm{d_H}(C_3)$ can be done similarly. Using Theorem 2.3, we have 2 subcases.

Subcase 2.1: $\mathrm{d_H}(C_2) = \mathrm{d_H}(\langle (x-\gamma_2)^{j_2}\rangle_F)$ *when* $h_2(x)$ *is 0 or* $h_2(x)$ *is a unit and* $1 \le j_2 \le \frac{p^s+t_2}{2}$. Then

$$\mathrm{d_H}(C_2) = \mathrm{d_H}(\langle (x-\gamma_2)^{j_2}\rangle_F) = (\delta_2 + 1)p^{k_2},$$

where $p^s - pr_2 + (\delta_2 - 1)r_2 + 1 \le j_2 \le p^s - pr_2 + \delta_2 r_2$, $1 \le \delta_2 \le p - 1$, $r_2 = p^{s-k_2-1}$, and $0 \le k_2 \le s - 1$. Then C is an MDS code when $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_2 + 1)p^{k_2} - 2$. Similar to Subcase 1.2 of Case 1, we can conclude that $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 \ne -p^s + 2(\delta_2 + 1)p^{k_2} - 2$, i.e., C is not an MDS code in this subcase.

Subcase 2.2: $\mathrm{d_H}(C_2) = \mathrm{d_H}(\langle (x-\gamma_2)^{j_2}\rangle_F)$ *when* $h_2(x)$ *is a unit and* $\frac{p^s+t_2}{2} < j_2 \le p^s - 1$. Then

$$\mathrm{d_H}(C_2) = \mathrm{d_H}(\langle (x-\gamma_2)^{j_2}\rangle_F) = (\delta_2 + 1)p^{k_2},$$

where $t_2 + pr_2 - \delta_2 r_2 \le j_2 \le t_2 + pr_2 - (\delta_2 - 1)r_2 - 1$, $1 \le \delta_2 \le p - 1$, $r_2 = p^{s-k_2-1}$, and $0 \le k_2 \le s - 1$. Then C is an MDS code when $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_2 + 1)p^{k_2} - 2$. From $j_2 > \frac{p^s+t_2}{2}$, we have $2j_2 > p^s + t_2$. We see that $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 + p^s - 2(\delta_2 + 1)p^{k_2} + 2 > j_1 + p^s + t_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 + p^s - 2(\delta_2 + 1)p^{k_2} + 2 > 0$, i.e., $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 > -p^s + 2(\delta_2 + 1)p^{k_2} - 2$. Hence, C is not an MDS code in this subcase. Thus, C is not an MDS code.

Case 3: $\mathrm{d_H}(C) = \mathrm{d_H}(C_5) = \mathrm{d_H}(\langle (x-\gamma_5)^{\kappa_5}\rangle_F) = (\delta_5 + 1)p^{k_5}$, *where* $p^s - p^{s-k_5} + (\delta_5 - 1)p^{s-k_5-1} + 1 \le \kappa_5 \le p^s - p^{s-k_5} + \delta_5 p^{s-k_5-1}$, $1 \le \delta_5 \le p - 1$, *and* $0 \le k_5 \le s - 1$. We see that $\mathrm{d_H}(C) = \mathrm{d_H}(C_4)$ can be done similarly. If $\mathrm{d_H}(C) = \mathrm{d_H}(C_5)$, then C is an MDS code when $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_5 + 1)p^{k_5} - 2$. From $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_5 + 1)p^{k_5} - 2$, we see that $2(\delta_5 + 1)p^{k_5} > p^s$ (5). Since $\delta_5 + 1 \le p$, we have $p^s < 2p \cdot p^{k_5}$. It implies that $p^{s-k_5-1} < 2$. Hence, $s = k_5 + 1$. From $p^s - p^{s-k_5} + (\delta_5 - 1)p^{s-k_5-1} + 1 \le j_5 \le p^s - p^{s-k_5} + \delta_5 p^{s-k_5-1}$, we see that $j_5 = p^s - p + \delta_5$. From $s = k_5 + 1$, (3) becomes $2(\delta_5 + 1)p^{s-1} > p^s$. It implies that $\delta_5 + 1 \le p < 2(\delta_5 + 1)$. By assumption, $(\delta_5 + 1)p^{k_5} \le (\delta_5 + 1)p^{k_5}$. Hence, $(\delta_5 + 1)p^{s-1} \le p^{k_2+1}$. It follows that $k_2 + 1 > s - 1$. Therefore, $k_2 > s - 2$. It shows that $k_2 = s - 1$ since $k_2 \le s - 1$. Similarly, we see that $k_1 = s - 1$, $k_3 = s - 1$ and $k_4 = s - 1$. Since $(\delta_5 + 1)p^{k_5} \le (\delta_1 + 1)p^{k_1} \le (\delta_2 + 1)p^{k_2} \le (\delta_3 + 1)p^{k_3} \le (\delta_4 + 1)p^{k_4}$, we have $\delta_1 \le \delta_2 \le \delta_3 \le \delta_4$. From this, $j_1 = p^s - p + \delta_1$, $j_2 = p^s - p + \delta_2$, $j_3 = p^s - p + \delta_3$ and $j_4 = p^s - p + \delta_4$. From $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_3 = -p^s + 2(\delta_5 + 1)p^{k_5} - 2$, $j_1 = p^s - p + \delta_1$ and $j_2 = p^s - p + \delta_2$, $j_3 = p^s - p + \delta_3$ and $j_4 = p^s - p + \delta_4$, we see that $(p^s - p + \delta_1) + 2(p^s - p + \delta_2) + 2(p^s - p + \delta_3) + (p^s - p + \delta_4) + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_5 + 1)p^{s-1} - 2$. Hence,

$$\begin{aligned} j_5 + \kappa_5 + \kappa_4 &= -7p^s + 2(\delta_5 + 1)p^{s-1} - 2 \\ &\quad + 6p - \delta_1 - 2\delta_2 - 2\delta_3 - \delta_4 \\ &= [-2p^s + 2(\delta_5 + 1)p^{s-1}] + [p - \delta_1 - p^s] \\ &\quad + [p - 2\delta_2 - p^s] + [p - 2\delta_3 - p^s] \\ &\quad + [p - \delta_4 - p^s] + [2p - p^s - 2]. \end{aligned}$$

If $s \ge 2$, then $j_3 + \kappa_5 + \kappa_4 < 0$, which is a contradiction. If $s = 1$, then $k_1 = k_2 = k_3 = 0$. Hence, $j_1 = \delta_1, j_2 = \delta_2$,

$j_3 = \delta_3$. It implies that $j_1 + 2j_2 + j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = 2\delta_5 - p < 2\delta_1$. This is a contradiction. Therefore, $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 \neq -p^s + 2(\delta_5 + 1)p^{k_5} - 2$, i.e., C is not an MDS code. □

Proposition 4.11: Let $C = C_1 \oplus C_i \oplus C_j \oplus C_k \oplus C_t$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_i, C_j, C_k, C_t are defined as in Theorem 2.5 and i, j, k, t are different numbers in $\{2, 3, 4, 5\}$. If C_i is a constacyclic code of Type 3 of length p^s over $\mathcal{R}$, then C is not an MDS code.

Proof: If C_j, C_k, C_t are constacyclic codes of Type 3 of length p^s over $\mathcal{R}$, then C is not an MDS code by Proposition 4.10. Thus, C_j, C_k, C_t are constacyclic codes of Type 4 of length p^s over $\mathcal{R}$. Without loss of generality, assume that $C_2 = \langle (x-\gamma_2)^{j_2} + u(x-\gamma_2)^{t_2}h_2(x)\rangle$, $C_3 = \langle (x-\gamma_3)^{j_3} + u(x-\gamma_3)^{t_3}\, h_3(x), u(x-\gamma_3)^{\kappa_3}\rangle$, $C_4 = \langle (x-\gamma_4)^{j_4} + u(x-\gamma_4)^{t_4}\, h_4(x), u(x-\gamma_4)^{\kappa_4}\rangle$ and $C_5 = \langle (x-\gamma_5)^{j_5} + u(x-\gamma_5)^{t_5}\, h_5(x), u(x-\gamma_5)^{\kappa_5}\rangle$ ($j_1, j_2, j_3, j_4, t_5, \kappa_5$ are defined as in Theorem 2.2). If $|C_2| = p^{m(p^s-t_2)}$, by using Theorem 2.11, C is an MDS code when $p^{m(p^s-j_1)} \cdot p^{m(p^s-t_2)} \cdot |C_3| \cdot |C_4| \cdot |C_5| = p^{2m(5p^s-\mathrm{d_H}(C)+1)}$. It is easy to check that $p^{m(p^s-j_1)} \cdot p^{m(p^s-t_2)} \cdot p^{m(2p^s-j_3-\kappa_3)} \cdot p^{m(2p^s-j_4-\kappa_4)} \cdot p^{m(2p^s-j_5-\kappa_5)} < p^{8mp^s}$. Since $2m(5p^s - \mathrm{d_H}(C) + 1) > 8mp^s$, we have $|C| < p^{2m(5p^s-\mathrm{d_H}(C)+1)}$. Hence, C is not an MDS code when $|C_2| = p^{m(p^s-t_2)}$. We consider the remaining case $|C| = p^{m(p^s-j_1)} \cdot p^{2m(p^s-j_2)} \cdot p^{m(2p^s-j_3-\kappa_3)} \cdot p^{m(2p^s-j_4-\kappa_4)} \cdot p^{m(2p^s-j_5-\kappa_5)}$. By Theorem 2.11, C is an MDS code when $m(p^s - j_1) + 2m(p^s - j_2) + m(p^s - j_3 - \kappa_3) + m(2p^s - j_4 - \kappa_4) + m(2p^s - j_5 - \kappa_5) = 2m(5p^s - \mathrm{d_H}(C) + 1)$, which is equivalent to $j_1 + 2j_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2\,\mathrm{d_H}(C) - 2$. Applying Theorem 4.1, we see that $\mathrm{d_H}(C) = \min\{\mathrm{d_H}(C_i)\}$, where $i = \{1, 2, 3, 4, 5\}$. Then we have $\mathrm{d_H}(C) = \mathrm{d_H}(C_1)$ or $\mathrm{d_H}(C) = \mathrm{d_H}(C_2)$ or $\mathrm{d_H}(C) = \mathrm{d_H}(C_3)$ or $\mathrm{d_H}(C) = \mathrm{d_H}(C_4)$ or $\mathrm{d_H}(C) = \mathrm{d_H}(C_5)$.

Case 1: $\mathrm{d_H}(C) = \mathrm{d_H}(C_1)$. From Theorem 2.3, $\mathrm{d_H}(C_1) = 1$ or $\mathrm{d_H}(C_1) = (\delta_1 + 1)p^{k_1}$.

Subcase 1.1: $\mathrm{d_H}(C_1) = 1$. In this subcase, $2j_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s$, which is a contradiction. Hence, C is not an MDS code in this subcase.

Subcase 1.2: $\mathrm{d_H}(C_1) = (\delta_1 + 1)p^{k_1}$. Then we see that $j_1 + j_2 + \kappa_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_1 + 1)p^{k_1} - 2$ (6). From (6), we have $2(\delta_1 + 1)p^{k_1} > p^s$. Since $\delta_1 + 1 \leq p$, we have $p^s < 2p \cdot p^{k_1}$. It implies that $p^{s-k_1-1} < 2$. Hence, $s - k_1 - 1 = 0$, i.e., $s = k_1 + 1$. By asumption $p^s - p^{s-k_1} + (\delta_1 - 1)p^{s-k_1-1} + 1 \leq j_1 \leq p^s - p^{s-k_1} + \delta_1 p^{s-k_1-1}$, we see that $p^s - p + \delta_1 - 1 + 1 \leq j_1 \leq p^s - p + \delta_1$. It follows that $j_1 = p^s - p + \delta_1$. Put $T_4 = 2j_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5$. Then $T_4 = 2j_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_1 + 1)p^{s-1} - 2 - (p^s - p + \delta_1)$. Hence,

$$\begin{aligned} T_4 &= p^{s-1}[2(\delta_1 + 1) - 2p] - (\delta_1 - p + 1) - 1 \\ &= [2p^{s-1} - 1][\delta_1 - p + 1] - 1. \end{aligned}$$

Since $2p^{s-1} - 1 > 0$ and $\delta_1 - p + 1 \leq 0$, we have $[2p^{s-1} - 1][\delta_1 - p + 1] - 1 < 0$. Thus, $j_1 + 2j_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5 \neq -p^s + 2(\delta_1 + 1)p^{k_1} - 2$, i.e., C is not an MDS code when $\mathrm{d_H}(C) = \mathrm{d_H}(C_1)$.

Case 2: $\mathrm{d_H}(C) = \mathrm{d_H}(C_2)$. Using Theorem 2.3, we have 2 subcases.

Subcase 2.1: $\mathrm{d_H}(C_2) = \mathrm{d_H}(\langle (x-\gamma_2)^{j_2}\rangle_F)$ *when $h_2(x)$ is 0 or $h_2(x)$ is a unit and* $1 \leq j_2 \leq \frac{p^s+t_2}{2}$. Then

$$\mathrm{d_H}(C_2) = \mathrm{d_H}(\langle (x-\gamma_2)^{j_2}\rangle_F) = (\delta_2 + 1)p^{k_2},$$

where $p^s - pr_2 + (\delta_2 - 1)r_2 + 1 \leq j_2 \leq p^s - pr_2 + \delta_2 r_2$, $1 \leq \delta_2 \leq p - 1$, $r_2 = p^{s-k_2-1}$, and $0 \leq k_2 \leq s - 1$. Then C is an MDS code when $j_1 + j_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_2 + 1)p^{k_2} - 2$. Similar to Subcase 1.2 of Case 1, we can conclude that $j_1 + j_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5 \neq -p^s + 2(\delta_2 + 1)p^{k_2} - 2$, i.e., C is not an MDS code in this subcase.

Subcase 2.2: $\mathrm{d_H}(C_2) = \mathrm{d_H}(\langle (x-\gamma_2)^{j_2}\rangle_F)$ *when $h_2(x)$ is a unit and* $\frac{p^s+t_2}{2} < j_2 \leq p^s - 1$. Then

$$\mathrm{d_H}(C_2) = \mathrm{d_H}(\langle (x-\gamma_2)^{j_2}\rangle_F) = (\delta_2 + 1)p^{k_2},$$

where $t_2 + pr_2 - \delta_2 r_2 \leq j_2 \leq t_2 + pr_2 - (\delta_2 - 1)r_2 - 1$, $1 \leq \delta_2 \leq p - 1$, $r_2 = p^{s-k_2-1}$, and $0 \leq k_2 \leq s - 1$. Then C is an MDS code when $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_2 + 1)p^{k_2} - 2$. From $j_2 > \frac{p^s+t_2}{2}$, we have $2j_2 > p^s + t_2$. We see that $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 + p^s - 2(\delta_2 + 1)p^{k_2} + 2 > j_1 + p^s + t_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 + p^s - 2(\delta_2 + 1)p^{k_2} + 2 > 0$, i.e., $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 > -p^s + 2(\delta_2 + 1)p^{k_2} - 2$. Hence, C is not an MDS code in this subcase. Thus, C is not an MDS code.

Case 3: $\mathrm{d_H}(C) = \mathrm{d_H}(C_5) = \mathrm{d_H}(\langle (x-\gamma_5)^{\kappa_5}\rangle_F) = (\delta_5 + 1)p^{k_5}$, *where* $p^s - p^{s-k_5} + (\delta_5 - 1)p^{s-k_5-1} + 1 \leq \kappa_5 \leq p^s - p^{s-k_5} + \delta_5 p^{s-k_5-1}$, $1 \leq \delta_5 \leq p - 1$, *and* $0 \leq k_5 \leq s - 1$. We see that $\mathrm{d_H}(C) = \mathrm{d_H}(C_3)$ and $\mathrm{d_H}(C) = \mathrm{d_H}(C_4)$ can be done similarly. If $\mathrm{d_H}(C) = \mathrm{d_H}(C_5)$, then C is an MDS code when $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_5 + 1)p^{k_5} - 2$. From $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_5 + 1)p^{k_5} - 2$, we see that $2(\delta_5 + 1)p^{k_5} > p^s$ (7). Since $\delta_5 + 1 \leq p$, we have $p^s < 2p \cdot p^{k_5}$. It implies that $p^{s-k_5-1} < 2$. Hence, $s = k_5 + 1$. From $p^s - p^{s-k_5} + (\delta_5 - 1)p^{s-k_5-1} + 1 \leq j_5 \leq p^s - p^{s-k_5} + \delta_5 p^{s-k_5-1}$, we have $j_5 = p^s - p + \delta_5$. From $s = k_5 + 1$, (7) becomes $2(\delta_5 + 1)p^{s-1} > p^s$. It implies that $\delta_5 + 1 \leq p < 2(\delta_5 + 1)$. By assumption, $(\delta_5 + 1)p^{k_5} \leq (\delta_5 + 1)p^{k_5}$. Hence, $(\delta_5 + 1)p^{s-1} \leq p^{k_2+1}$. It follows that $k_2 + 1 > s - 1$. Therefore, $k_2 > s - 2$. It shows that $k_2 = s - 1$ since $k_2 \leq s - 1$. Similarly, we see that $k_1 = s - 1$, $k_3 = s - 1$ and $k_4 = s - 1$. Since $(\delta_5 + 1)p^{k_5} \leq (\delta_1 + 1)p^{k_1} \leq (\delta_2 + 1)p^{k_2} \leq (\delta_3 + 1)p^{k_3} \leq (\delta_4 + 1)p^{k_4}$, we have $\delta_1 \leq \delta_2 \leq \delta_3 \leq \delta_4$. From this, $j_1 = p^s - p + \delta_1$, $j_2 = p^s - p + \delta_2$, $j_3 = p^s - p + \delta_3$ and $j_4 = p^s - p + \delta_4$. From $j_1 + 2j_2 + 2j_3 + 2j_4 + j_5 + \kappa_3 = -p^s + 2(\delta_5 + 1)p^{k_5} - 2$, $j_1 = p^s - p + \delta_1$ and $j_2 = p^s - p + \delta_2$, $j_3 = p^s - p + \delta_3$ and $j_4 = p^s - p + \delta_4$, we see that $(p^s - p + \delta_1) + 2(p^s - p + \delta_2) + 2(p^s - p + \delta_3) + (p^s - p + \delta_4) + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_5 + 1)p^{s-1} - 2$. Hence,

$$\begin{aligned} j_5 + \kappa_5 + \kappa_4 &= -7p^s + 2(\delta_3 + 1)p^{s-1} - 2 \\ &\quad + 6p - \delta_1 - 2\delta_2 - 2\delta_3 - \delta_4 \\ &= [-2p^s + 2(\delta_5 + 1)p^{s-1}] + [p - \delta_1 - p^s] \\ &\quad + [p - 2\delta_2 - p^s] + [p - 2\delta_3 - p^s] \\ &\quad + [p - \delta_4 - p^s] + [2p - p^s - 2]. \end{aligned}$$

If $s \geq 2$, then $j_3 + \kappa_5 + \kappa_4 < 0$, which is a contradiction. If $s = 1$, then $k_1 = k_2 = k_3 = 0$. Hence, $j_1 = \delta_1, j_2 = \delta_2$, $j_3 = \delta_3$. It implies that $j_1 + 2j_2 + j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = 2\delta_5 - p < 2\delta_1$. This is a contradiction. Therefore, $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 \neq -p^s + 2(\delta_5 + 1)p^{k_5} - 2$, i.e., C is not an MDS code. □

Proposition 4.12: Let $C = C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_1, C_2, C_3, C_4, C_5 are defined as in Theorem 2.5 such that all C_i are constacyclic codes of Type 4 of length p^s over $\mathcal{R}$ for $i \in \{2, 3, 4, 5\}$. Then C is not an MDS code.

Proof: We have $C_1 = \langle (x-1)^{j_1} \rangle$, $C_2 = \langle (x-\gamma_2)^{j_2} + u(x-\gamma_2)^{t_2} h_2(x), u(x-\gamma_2)^{\kappa_2} \rangle$, $C_3 = \langle (x-\gamma_3)^{j_3} + u(x-\gamma_3)^{t_3} h_3(x), u(x-\gamma_3)^{\kappa_3} \rangle$, $C_4 = \langle (x-\gamma_4)^{j_4} + u(x-\gamma_4)^{t_4} h_4(x), u(x-\gamma_4)^{\kappa_4} \rangle$, $C_5 = \langle (x-\gamma_5)^{j_5} + u(x-\gamma_5)^{t_5} h_5(x), u(x-\gamma_5)^{\kappa_5} \rangle$ (j_2, j_3, j_4, j_5 are defined as in Theorem 2.2). By Theorem 2.11, C is an MDS code when $m(p^s - j_1) + m(2p^s - j_2 - \kappa_2) + m(2p^s - j_3 - \kappa_3) + m(2p^s - j_4 - \kappa_4) + m(p^s - j_5 - \kappa_5) = 2m(5p^s - d_H(C) + 1)$. Hence, C is an MDS code when $j_1 + j_2 + \kappa_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2\,d_H(C) - 2$. Applying Theorem 4.1, we see that $d_H(C) = \min\{d_H(C_i)\}$, where $i = \{1, 2, 3, 4, 5\}$. Then we have $d_H(C) = d_H(C_1)$ or $d_H(C) = d_H(C_2)$ or $d_H(C) = d_H(C_3)$ or $d_H(C) = d_H(C_4)$ or $d_H(C) = d_H(C_5)$.

Case 1: $d_H(C) = d_H(C_1)$. From Theorem 2.3, $d_H(C_1) = 1$ or $d_H(C_1) = (\delta_1 + 1)p^{k_1}$.

Subcase 1.1: $d_H(C_1) = 1$. In this subcase, $j_2 + \kappa_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s$, which is a contradiction. Hence, C is not an MDS code in this subcase.

Subcase 1.2: $d_H(C_1) = (\delta_1 + 1)p^{k_1}$. Then we see that $j_1 + j_2 + \kappa_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_1 + 1)p^{k_1} - 2$ (8). From (8), we have $2(\delta_1 + 1)p^{k_1} > p^s$. Since $\delta_1 + 1 \leq p$, we have $p^s < 2p \cdot p^{k_1}$. It implies that $p^{s-k_1-1} < 2$. Hence, $s - k_1 - 1 = 0$, i.e., $s = k_1 + 1$. By asumption $p^s - p^{s-k_1} + (\delta_1 - 1)p^{s-k_1-1} + 1 \leq j_1 \leq p^s - p^{s-k_1} + \delta_1 p^{s-k_1-1}$, we see that $p^s - p + \delta_1 - 1 + 1 \leq j_1 \leq p^s - p + \delta_1$. It follows that $j_1 = p^s - p + \delta_1$. Put $T_6 = j_2 + \kappa_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5$. Then $T_6 = j_2 + \kappa_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_1 + 1)p^{s-1} - 2 - (p^s - p + \delta_1)$. Hence,

$$\begin{aligned} T_6 &= p^{s-1}[2(\delta_1 + 1) - 2p] - (\delta_1 - p + 1) - 1 \\ &= [2p^{s-1} - 1][\delta_1 - p + 1] - 1. \end{aligned}$$

Since $2p^{s-1} - 1 > 0$ and $\delta_1 - p + 1 \leq 0$, we have $[2p^{s-1} - 1][\delta_1 - p + 1] - 1 < 0$. Thus, $j_1 + j_2 + \kappa_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5 \neq -p^s + 2(\delta_1 + 1)p^{k_1} - 2$, i.e., C is not an MDS code when $d_H(C) = d_H(C_1)$.

Case 2: $d_H(C) = d_H(C_2)$.

Subcase 2.1: $d_H(C_2) = d_H(\langle (x - \gamma_2)^{j_2} \rangle_F)$ *when $h_2(x)$ is 0 or $h_2(x)$ is a unit and* $1 \leq j_2 \leq p^s - 1$. We have

$$d_H(C_2) = d_H(\langle (x - \gamma_2)^{j_2} \rangle_F) = (\delta_2 + 1)p^{k_2},$$

where $p^s - pr_2 + (\delta_2 - 1)r_2 + 1 \leq j_2 \leq p^s - pr_2 + \delta_2 r_2$, $1 \leq \delta_2 \leq p - 1$, $r_2 = p^{s-k_2-1}$, and $0 \leq k_2 \leq s - 1$. Then C is an MDS code when $j_1 + 2j_2 + 2j_3 + 2j_4 + 2j_5 = -p^s + 2(\delta_2 + 1)p^{k_2} - 2$. Similar to Subcase 1.2 of Case 1, we can conclude that $j_1 + 2j_2 + 2j_3 + 2j_4 + 2j_5 \neq -p^s + 2(\delta_2 + 1)p^{k_2} - 2$, i.e., C is not an MDS code in this subcase.

Subcase 2.2: $d_H(C_2) = d_H(\langle (x - \gamma_2)^{j_2} \rangle_F)$ *when $h_2(x)$ is a unit and* $\frac{p^s + t_2}{2} < j_2 \leq p^s - 1$. Then

$$d_H(C_2) = d_H(\langle (x - \gamma_2)^{j_2} \rangle_F) = (\delta_2 + 1)p^{k_2},$$

where $t_2 + pr_2 - \delta_2 r_2 \leq j_2 \leq t_2 + pr_2 - (\delta_2 - 1)r_2 - 1$, $1 \leq \delta_2 \leq p - 1$, $r_2 = p^{s-k_2-1}$, and $0 \leq k_2 \leq s - 1$. Then C is an MDS code when $j_1 + 2j_2 + 2j_3 + 2j_4 + 2j_5 = -p^s + 2(\delta_2 + 1)p^{k_2} - 2$. From $j_2 > \frac{p^s + t_2}{2}$, we have $2j_2 > p^s + t_2$. We see that $j_1 + 2j_2 + 2j_3 + 2j_4 + 2j_5 + p^s - 2(\delta_2 + 1)p^{k_2} + 2 > j_1 + p^s + t_2 + 2j_3 + 2j_4 + 2j_5 + p^s - 2(\delta_2 + 1)p^{k_2} + 2 > 0$, i.e., $j_1 + 2j_2 + 2j_3 + 2j_4 + 2j_5 > -p^s + 2(\delta_2 + 1)p^{k_2} - 2$. Hence, C is not an MDS code in this subcase. Thus, C is not an MDS code.

Case 3: $d_H(C) = d_H(C_5) = d_H(\langle (x - \gamma_5)^{\kappa_5} \rangle_F) = (\delta_5 + 1)p^{k_5}$, *where* $p^s - p^{s-k_5} + (\delta_5 - 1)p^{s-k_5-1} + 1 \leq \kappa_5 \leq p^s - p^{s-k_5} + \delta_5 p^{s-k_5-1}$, $1 \leq \delta_5 \leq p - 1$, *and* $0 \leq k_5 \leq s - 1$. We see that $d_H(C) = d_H(C_4)$ and $d_H(C) = d_H(C_3)$ can be done similarly. If $d_H(C) = d_H(C_5)$, then C is an MDS code when $j_1 + j_2 + \kappa_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_5 + 1)p^{k_5} - 2$. From $j_1 + j_2 + \kappa_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_5 + 1)p^{k_5} - 2$, we see that $2(\delta_5 + 1)p^{k_5} > p^s$ (9). Since $\delta_5 + 1 \leq p$, we have $p^s < 2p \cdot p^{k_5}$. It implies that $p^{s-k_5-1} < 2$. Hence, $s = k_5 + 1$. From $p^s - p^{s-k_5} + (\delta_5 - 1)p^{s-k_5-1} + 1 \leq j_5 \leq p^s - p^{s-k_5} + \delta_5 p^{s-k_5-1}$, we have $j_5 = p^s - p + \delta_5$. From $s = k_5 + 1$, (9) becomes $2(\delta_5 + 1)p^{s-1} > p^s$. It implies that $\delta_5 + 1 \leq p < 2(\delta_5 + 1)$. By assumption, $(\delta_5 + 1)p^{k_5} \leq (\delta_5 + 1)p^{k_5}$. Hence, $(\delta_5 + 1)p^{s-1} \leq p^{k_2+1}$. It follows that $k_2 + 1 > s - 1$. Therefore, $k_2 > s - 2$. It shows that $k_2 = s - 1$ since $k_2 \leq s - 1$. Similarly, we see that $k_1 = s - 1$, $k_3 = s - 1$ and $k_4 = s - 1$. Since $(\delta_5 + 1)p^{k_5} \leq (\delta_1 + 1)p^{k_1} \leq (\delta_2 + 1)p^{k_2} \leq (\delta_3 + 1)p^{k_3} \leq (\delta_4 + 1)p^{k_4}$, we have $\delta_1 \leq \delta_2 \leq \delta_3 \leq \delta_4$. From this, $j_1 = p^s - p + \delta_1$, $j_2 = p^s - p + \delta_2$, $j_3 = p^s - p + \delta_3$ and $j_4 = p^s - p + \delta_4$. From $j_1 + j_2 + \kappa_2 + j_3 + \kappa_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_5 + 1)p^{k_5} - 2$, $j_1 = p^s - p + \delta_1$ and $j_2 = p^s - p + \delta_2$, $j_3 = p^s - p + \delta_3$ and $j_4 = p^s - p + \delta_4$, we see that $(p^s - p + \delta_1) + (p^s - p + \delta_2) + \kappa_2 + (p^s - p + \delta_3) + \kappa_3 + (p^s - p + \delta_4) + \kappa_4 + j_5 + \kappa_5 = -p^s + 2(\delta_5 + 1)p^{s-1} - 2$. Hence,

$$\begin{aligned} j_5 + \kappa_5 + \kappa_4 + \kappa_3 + \kappa_2 &= -4p^s + 2(\delta_3 + 1)p^{s-1} - 2 \\ &\quad + 3p - \delta_1 - \delta_2 - \delta_3 - \delta_4 \\ &= [-2p^s + 2(\delta_5 + 1)p^{s-1}] \\ &\quad + [p - \delta_1 - p^s] + [p - 2\delta_2 - p^s] \\ &\quad + [p - 2\delta_3 - p^s] \\ &\quad + [p - \delta_4 - p^s] + [-p - p^s - 2]. \end{aligned}$$

If $s \geq 2$, then $j_3 + \kappa_5 + \kappa_4 < 0$, which is a contradiction. If $s = 1$, then $k_1 = k_2 = k_3 = 0$. Hence, $j_1 = \delta_1, j_2 = \delta_2$, $j_3 = \delta_3$. It implies that $j_1 + 2j_2 + j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 = 2\delta_5 - p < 2\delta_1$. This is a contradiction. Therefore, $j_1 + 2j_2 + 2j_3 + j_4 + \kappa_4 + j_5 + \kappa_5 \neq -p^s + 2(\delta_5 + 1)p^{k_5} - 2$, i.e., C is not an MDS code. □

Combining Propositions 4.2-4.12, we have the following result.

Theorem 4.13: Let $C = C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_1, C_2, C_3, C_4, C_5 are defined as in Theorem 2.5. Then C is an MDS code if and only if $C = \langle 1 \rangle$.

We finish this section by giving an example.

Example 4.14: Put $\mathcal{R} = \mathbb{F}_{11} + u\mathbb{F}_{11}$. We consider cyclic codes of length 55 over $\mathcal{R}$. Then $C = C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5$, where C_1 is a cyclic code of length 11 over $\mathcal{R}$, C_2 is a 9-constacyclic code of length 11 over $\mathcal{R}$, C_3 is a 5-constacyclic code of length 55 over $\mathcal{R}$, C_4 is a 4-constacyclic code of length 55 over $\mathcal{R}$, and C_5 is a 3-constacyclic code of length 55 over $\mathcal{R}$.

(1) Let $C_1 = \langle (x-1)^7 \rangle$, $C_2 = \langle u(x-9)^5 \rangle$, $C_3 = \langle u(x-5)^6 \rangle$, $C_4 = \langle u(x-4)^5 \rangle$ and $C_5 = \langle u(x-3)^5 \rangle$. By Theorem 2.3, we have $d_H(C_1) = 8, d_H(C_2) = 6, d_H(C_3) = 7, d_H(C_4) = 6$ and $d_H(C_5) = 6$. Using Theorem 4.1, we see that $d_H(C) = 6$. Then C has parameters $[55, 11^{27}, 6]$.

(2) Let $C_1 = \langle (x-1)^8 \rangle$, $C_2 = \langle u(x-9)^9 \rangle$, $C_3 = \langle (x-5)^7 + u(x-5)^6 \rangle$, $C_4 = \langle (x-4)^5 + u(x-4)^6 \rangle$ and $C_5 = \langle (x-3)^4 + u(x-3)^6 \rangle$. By Theorem 2.3, we see that $d_H(C_1) = 9, d_H(C_2) = 10, d_H(C_3) = 8, d_H(C_4) = 6$ and $d_H(C_5) = 5$. By applying Theorem 4.1, $d_H(C) = 6$. Then C has parameters $[55, 11^{29}, 5]$.

V. HAMMING DISTANCE AND MDS CODES OF LENGTH $5p^s$ OVER $\mathcal{R}$ WHEN $p \equiv 4 \pmod 5$

As in Theorem 2.7, cyclic codes of length $5p^s$ over $\mathcal{R}$ can be represented as $C = \mathrm{C}_+ \bigoplus \mathrm{C}_{\alpha_1} \bigoplus \mathrm{C}_{\alpha_2}$, where C_+ is a cyclic code of length p^s over $\mathcal{R}$, C_{α_1} is an α_1-constacyclic code and C_{α_2} is an α_2-constacyclic code of length $2p^s$ over $\mathcal{R}$. We determine the Hamming distance of C as follows.

Theorem 5.1: Let $C = \mathrm{C}_+ \bigoplus \mathrm{C}_{\alpha_1} \bigoplus \mathrm{C}_{\alpha_2}$ be a non-zero cyclic code of length $5p^s$ over $\mathcal{R}$. Then the Hamming distance $d_H(C) = \min\{d_H(C_i) | i \in \{1, 2, 3\}, C_i \neq \langle 0 \rangle\}$.

Proof: Without loss of generality, assume that $d_H(C_1) = d_1 = \min\{d_H(C_i)\}$ (that means $C_1 \neq \langle 0 \rangle$). Let d be the Hamming distance of $C = C_1 \oplus C_2 \oplus C_3$. Let c_1 be a non-zero codeword of minimum weight in C_1, i.e., $d_1 = wt_H(c_1)$. Since $(c_1, 0, 0) \in C$ and $wt_H(c_1, 0, 0) = d_1$, we have $d \leq d_1$. Let $z = (z_1, z_2, z_3)$ be an arbitrary non-zero codeword in $C_1 \oplus C_2 \oplus C_3$. If $z_1 = 0$, then there exists $t \in \{2, 3\}$ such that $z_t \neq 0$. Since $d_H(C_1) = d_1 = \min\{d_H(C_i)\}$, we have $wt_H(z_t) \geq d_H(C_t) \geq d_1$. Hence, the Hamming weight of z is $wt_H(z) = wt_H(z_1) + wt_H(z_2) + wt_H(z_3) \geq wt_H(z_t) \geq d_H(C_t) \geq d_1$. It implies that $d \geq d_1$ when $z_1 = 0$. If $z_1 \neq 0$, then the Hamming weight of z is $wt_H(z) = wt_H(z_1) + wt_H(z_2) + wt_H(z_3) \geq d_1$. It means that $d \geq d_1$ when $z_1 \neq 0$. Since z is an arbitrary non-zero codeword, we have $d \geq d_1$. Thus, $d = d_1$. □

We compute the Hamming distance of γ-constacyclic codes of Type 2 in the following theorem.

Theorem 5.2: Let $C = \langle u(x^2 - \alpha_0)^j \rangle$, $0 \leq j \leq p^s - 1$, be a γ-constacyclic code of Type 2 of length $2p^s$ over $\mathcal{R}$. Then $d_H(C) = d_H(\langle (x^2 - \alpha_0)^j \rangle_F)$, and $d_H(C)$ is given by

$$d_H(C) = \begin{cases} \bullet\, 1, \text{ if } j = 0 \\ \bullet\, (\delta+1)p^{\varsigma}, \text{ if } \; p^s - p^{s-\varsigma} + (\delta-1)p^{s-\varsigma-1} + 1 \leq j \\ \text{and } j \leq p^s - p^{s-\varsigma} + \delta p^{s-\varsigma-1} \\ \bullet\, 0, \text{ if } j = p^s \end{cases}$$

where $1 \leq \delta \leq p-1$, $0 \leq \varsigma \leq s-1$.

Proof: We consider the following two cases:

Case 1: If $j = 0$, then $d_H(C) = 1$.

Case 2: If $p^s - p^{s-\varsigma} + (\delta-1)p^{s-\varsigma-1} + 1 \leq j \leq p^s - p^{s-\varsigma} + \delta p^{s-\varsigma-1}$, then for a Type 2 code $C = \langle u(x^2 - \alpha_0)^j \rangle$, $0 \leq j \leq p^s - 1$, the codewords of the code C are exactly same as the codewords of the γ-constacyclic codes $\langle (x^2 - \alpha_0)^j \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{3p^s} - \gamma \rangle}$ multiplied by u. Thus, we get $d_H(C) = d_H(\langle (x^2 - \alpha_0)^j \rangle_F)$. By Theorem 2.3, $d_H(C)$ is given by

$$d_H(C) = \begin{cases} \bullet\, 1, \text{ if } j = 0 \\ \bullet\, (\delta+1)p^{\varsigma}, \text{ if } \; p^s - p^{s-\varsigma} + (\delta-1)p^{s-\varsigma-1} + 1 \leq j \\ \text{and } j \leq p^s - p^{s-\varsigma} + \delta p^{s-\varsigma-1} \\ \bullet\, 0, \text{ if } j = p^s \end{cases}$$

where $1 \leq \delta \leq p-1$, $0 \leq \varsigma \leq s-1$, as required. □

We provide the Hamming distance of γ-constacyclic codes of Type 3 of length $3p^s$ over $\mathcal{R}$ in the following theorem.

Theorem 5.3: Let $C = \langle (x^2 - \alpha_0)^j + u(x^2 - \alpha_0)^r v(x) \rangle$ be a γ-constacyclic code of Type 3 of length $2p^s$ over $\mathcal{R}$, where $1 \leq j \leq p^s - 1$, $0 \leq r < j$ and either $v(x)$ is a unit in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{3p^s} - \gamma \rangle}$ or 0. Then $d_H(C) = d_H(\langle (x^2 - \alpha_0)^R \rangle_F)$, where R is the smallest integer satisfying $u(x^3 - \gamma_0)^R \in \langle (x^3 - \gamma_0)^j + u(x^3 - \gamma_0)^r v(x) \rangle$, which is given by

$$R = \begin{cases} j, & \text{if } v(x) = 0 \\ \min\{j, p^s - j + r\}, & \text{if } v(x) \neq 0 \end{cases}$$

and is determined by

$$d_H(C) = (\delta + 1)p^{\varsigma},$$

where $p^s - p^{s-\varsigma} + (\delta-1)p^{s-\varsigma-1} + 1 \leq R \leq p^s - p^{s-\varsigma} + \delta p^{s-\varsigma-1}$, $1 \leq \delta \leq p-1$ and $0 \leq \varsigma \leq s-1$.

Proof: Since R is the smallest integer such that $u(x^2 - \alpha_0)^R \in \langle (x^2 - \alpha_0)^j + u(x^2 - \alpha_0)^r v(x) \rangle$, therefore we have,

$$d_H(C) \leq d_H(\langle u(x^2 - \alpha_0)^R \rangle) = d_H(\langle (x^2 - \alpha_0)^R \rangle_F).$$

Now, let us take an arbitrary polynomial $c(x) \in C$. So, there exist two polynomials $f_0(x)$ and $f_u(x)$ over $\mathbb{F}_{p^m}$ satisfying

$$\begin{aligned} c(x) &= [f_0(x) + uf_u(x)][(x^2 - \alpha_0)^j + u(x^2 - \alpha_0)^r v(x)] \\ &= f_0(x)(x^2 - \alpha_0)^j \\ &\quad + u[f_0(x)(x^2 - \alpha_0)^r v(x) + f_u(x)(x^2 - \alpha_0)^j]. \end{aligned}$$

Now, we consider two cases:

Case 1: When $v(x) = 0$, then we have

$$\begin{aligned}\mathrm{wt_H}(c(x)) &\geq \max\left\{\mathrm{wt_H}(f_0(x)(x^2-\alpha_0)^j), \mathrm{wt_H}(f_u(x)(x^2-\alpha_0)^j)\right\}\\ &\geq \max\left\{\mathrm{wt_H}(f_0(x)(x^2-\alpha_0)^j), \mathrm{wt_H}(f_0(x)(x^2-\alpha_0)^j)\right\}\\ &\geq \mathrm{d_H}(\langle(x^2-\alpha_0)^j\rangle_F),\\ &= \mathrm{d_H}(\langle(x^2-\alpha_0)^R\rangle_F),\end{aligned}$$

Case 2: When $v(x) \neq 0$, then we have

$$\begin{aligned}&\mathrm{wt_H}(c(x))\\ &\quad\geq \max\left\{\mathrm{wt_H}(f_0(x)(\beta(x))^j), \mathrm{wt_H}(h(x))\right\}\\ &\quad\geq \max\left\{\mathrm{wt_H}(f_0(x)(\beta(x))^j), \mathrm{wt_H}(f_0(x)(\beta(x))^{p^s-j+r})\right\}\\ &\quad\geq \mathrm{d_H}(\langle(\beta(x))^{\min\{j,\, p^s-j+r\}}\rangle_F),\\ &\quad= \mathrm{d_H}(\langle(\beta(x))^R\rangle_F),\end{aligned}$$

where $\beta(x) = x^2 - \alpha_0$ and $f(\lambda_0, x) = f_0(x)(x^2-\alpha_0)^{p^s-j+r}$ and $h(x) = f_0(x)(x^2-\alpha_0)^r v(x) + f_u(x)(x^2-\alpha_0)^j$. Hence, by combining both the cases, we get $\mathrm{d_H}(\langle(x^2-\alpha_0)^R\rangle_F) \leq \mathrm{d_H}(C)$, which implies that, $\mathrm{d_H}(\langle(x^2-\alpha_0)^R\rangle_F) = \mathrm{d_H}(C)$. □

We determine the Hamming distance of γ-constacyclic codes of Type 4 in the following result.

Theorem 5.4: Let $C = \langle(x^2-\alpha_0)^j + u(x^2-\alpha_0)^r v(x), u(x^2-\alpha_0)^\omega\rangle$ be a γ-constacyclic code of Type 4 of length $2p^s$ over $\mathcal{R}$, where $v(x)$ is same as given in Type 3, $1 \leq j \leq p^s - 1$, $\deg(v) \leq \omega - r - 1$, $\omega < R$, and R is the smallest integer such that $u(x^2-\alpha_0)^R \in \langle(x^2-\alpha_0)^j + u(x^2-\alpha_0)^r v(x)\rangle$, i.e., $R = j$, if $v(x) = 0$ and otherwise $R = \min\{j, p^s - j + r\}$. Then $\mathrm{d_H}(C) = \mathrm{d_H}(\langle(x^2-\alpha_0)^\omega\rangle_F)$, and is given by

$$\mathrm{d_H}(C) = (\delta + 1)p^\varsigma,$$

where $p^s - p^{s-\varsigma} + (\delta - 1)p^{s-\varsigma-1} + 1 \leq \omega \leq p^s - p^{s-\varsigma} + \delta p^{s-\varsigma-1}$, $1 \leq \delta \leq p-1$ and $0 \leq \varsigma \leq s-1$.

Proof: Clearly, we have $C = \langle(x^2-\alpha_0)^j + u(x^2-\alpha_0)^r v(x), u(x^2-\alpha_0)^\omega\rangle \supseteq \langle u(x^2-\alpha_0)^\omega\rangle \supseteq \langle u(x^2-\alpha_0)^j\rangle$, since $\omega < R \leq j$. Thus, $\mathrm{d_H}(C) \leq \mathrm{d_H}(\langle u(x-\gamma_0)^\omega\rangle) = \mathrm{d_H}(\langle(x^2-\alpha_0)^\omega\rangle_F)$. To prove that $\mathrm{d_H}(\langle(x^2-\alpha_0)^\omega\rangle_F) \leq \mathrm{d_H}(C)$, we take an arbitrary polynomial $c(x) \in C$ and proceed to show that $\mathrm{wt_H}(c(x)) \geq \mathrm{d_H}(\langle(x^2-\alpha_0)^\omega\rangle_F)$. Now, there exist polynomials $f_0(x), f_u(x), g_0(x)$ and $g_u(x)$ over $\mathbb{F}_{p^m}$ such that

$$\begin{aligned}c(x) &= [f_0(x) + uf_u(x)][(\beta(x))^j + u(\beta(x))^r v(x)]\\ &\quad + u(\beta(x))^\omega[g_0(x) + ug_u(x)]\\ &= f_0(x)(\beta(x))^j\\ &\quad + u[f_0(x)(\beta(x))^r v(x) + f_u(x)(\beta(x))^j + g_0(x)(\beta(x))^\omega]\\ &= f_0'(x)(\beta(x))^\omega + u[f_0(x)(\beta(x))^r v(x) + g_0'(x)(\beta(x))^\omega],\end{aligned}$$

where $\beta(x) = x^2 - \alpha_0$ and $f_0'(x) = f_0(x)(x^2-\alpha_0)^{j-\omega} \in \mathbb{F}_{p^m}[x]$, $g_0'(x) = f_u(x)(x^2-\alpha_0)^{j-\omega} + g_0(x) \in \mathbb{F}_{p^m}[x]$. Hence,

$$\begin{aligned}\mathrm{wt_H}(c(x)) &\geq \max\left\{\mathrm{wt_H}(f_0'(x)(x^2-\alpha_0)^\omega), \mathrm{wt_H}(h'(x))\right\}\\ &\geq \max\left\{\mathrm{wt_H}(f_0'(x)(x^2-\alpha_0)^\omega), \mathrm{wt_H}(a(x))\right\}\\ &\geq \mathrm{d_H}(\langle(x^2-\alpha_0)^\omega\rangle_F),\end{aligned}$$

where $a(x) = f_0'(x)(x^2-\alpha_0)^\omega)$ and $h'(x) = f_0(x)(x^2-\alpha_0)^r v(x) + g_0'(x)(x^2-\alpha_0)^\omega$. □

To get MDS codes, we consider the following propositions.

Proposition 5.5: Let $C = C_1 \oplus C_2 \oplus C_3$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_1, C_2, C_3 are defined as in Theorem 2.7 such that $C_1 = \langle 0\rangle$. Then C is not an MDS code.

Proof: From Theorem 5.1, $\mathrm{d_H}(C) = \min\{\mathrm{d_H}(C_2), \mathrm{d_H}(C_3)\} \leq p^s$. We have $|C| = |C_2| \times |C_3| = p^{\ell_2} \cdot p^{\ell_3}$, where $|C_2| = p^{\ell_2}$, $|C_3| = p^{\ell_3}$ and $0 \leq \ell_2, \ell_3 \leq 2mp^s$. We see that $\ell_2 + \ell_3 \leq 4mp^s$ and $2m(5p^s - \mathrm{d_H}(C) + 1) > 4mp^s$. Thus, $\ell_2 + \ell_3 < 2m(5p^s - \mathrm{d_H}(C) + 1)$. Since $\ell_2 + \ell_3 < 2m(5p^s - \mathrm{d_H}(C) + 1)$, C is not an MDS code by Theorem 2.11. □

Proposition 5.6: Let $C = C_1 \oplus C_2 \oplus C_3$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_1, C_2, C_3 are defined as in Theorem 2.7 such that there exists $C_i = \langle 0\rangle$ for $i \in \{2, 3\}$. Then C is not an MDS code.

Proof: Without loss of generality, assume that $C_2 = \langle 0\rangle$. From Theorem 5.1, $\mathrm{d_H}(C) = \min\{\mathrm{d_H}(C_1), \mathrm{d_H}(C_3)\} \leq p^s$. We have $|C| = |C_1| \times |C_3| = p^{\ell_1} \cdot p^{\ell_3}$, where $|C_1| = p^{\ell_1}$, $|C_3| = p^{\ell_3}$ and $0 \leq \ell_1 \leq 2mp^s$, $0 \leq \ell_3 \leq 4mp^s$. We see that $\ell_1 + \ell_3 \leq 6mp^s$ and $2m(5p^s - \mathrm{d_H}(C) + 1) > 8mp^s$. Thus, $\ell_2 + \ell_3 < 2m(5p^s - \mathrm{d_H}(C) + 1)$. Using Theorem 2.11, C is an MDS code when $\ell_1 + \ell_3 = 2m(5p^s - \mathrm{d_H}(C) + 1)$. Since $\ell_1 + \ell_3 < 2m(5p^s - \mathrm{d_H}(C) + 1)$, C is not an MDS code. □

Proposition 5.7: Let $C = C_1 \oplus C_2 \oplus C_3$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_1, C_2, C_3 are defined as in Theorem 2.7 such that there exists $C_i = \langle 1\rangle$ for $i \in \{1, 2, 3\}$. Then C is an MDS code if and only if $C = \langle 1\rangle$.

Proof: Without loss of generality, assume that $C_1 = \langle 1\rangle$ and $|C_2| = p^{\ell_2}$ and $|C_3| = p^{\ell_3}$, where $0 \leq \ell_2, \ell_3 \leq 4mp^s$. Using Theorem 5.1, $\mathrm{d_H}(C) = \mathrm{d_H}(C_1) = 1$. By Theorem 2.11, C is an MDS code when $p^{2mp^s} \cdot p^{\ell_2} \cdot p^{\ell_3} = p^{2m(5p^s-1+1)}$, where $0 \leq \ell_2, \ell_3 \leq 4mp^s$. It implies that $2mp^s + \ell_2 + \ell_3 = 10mp^s$. Thus, $\ell_2 + \ell_3 = 8mp^s$. Hence, $\ell_2 = \ell_3 = 4mp^s$. Then C is an MDS code when $C_1 = \langle 1\rangle$, $C_2 = \langle 1\rangle$ and $C_3 = \langle 1\rangle$, i.e., $C = \langle 1\rangle$. □

Proposition 5.8: Let $C = C_1 \oplus C_2 \oplus C_3$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_1, C_2, C_3 are defined as in Theorem 2.7 such that there exists $C_i = \langle 1\rangle$ for $i \in \{2, 3\}$. Then C is an MDS code if and only if $C = \langle 1\rangle$.

Proof. Without loss of generality, assume that $|C_2| = p^{\ell_2}$ and $|C_3| = p^{\ell_3}$, where $0 \leq \ell_2, \ell_3 \leq 4mp^s$. Using Theorem 5.1, $\mathrm{d_H}(C) = \mathrm{d_H}(C_1) = 1$. By Theorem 2.11, C is an MDS code when $p^{2mp^s} \cdot p^{\ell_2} \cdot p^{\ell_3} = p^{2m(5p^s-1+1)}$, where $0 \leq \ell_2, \ell_3 \leq 4mp^s$. It implies that $2mp^s + \ell_2 + \ell_3 = 10mp^s$. Thus, $\ell_2 + \ell_3 = 8mp^s$. Hence, $\ell_2 = \ell_3 = 4mp^s$. Then C is an MDS code when $C_1 = \langle 1\rangle$, $C_2 = \langle 1\rangle$ and $C_3 = \langle 1\rangle$, i.e., $C = \langle 1\rangle$. □

Proposition 5.9: Let $C = C_1 \oplus C_j \oplus C_k$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_j, C_k are defined as in Theorem 2.7 and j, k are different numbers in $\{2, 3\}$. If C_j, C_k are constacyclic codes of Type 2 of length $2p^s$ over $\mathcal{R}$, then C is not an MDS code.

Proof: Without loss of generality, assume that $C_1 = \langle(x-1)^{j_1}\rangle$, $C_2 = \langle u(x^2-\gamma_2)^{j_2}$, $C_3 = \langle u(x^2-\gamma_3)^{j_3}\rangle$ (j_1, j_2, j_3 are defined as in Theorem 2.2). By Theorem 2.11, C is an MDS code when $m(p^s - j_1) + m(2p^s - j_2) + m(2p^s - j_3) =$

$2m(5p^s - d_H(C) + 1)$. Hence, C is an MDS code when $j_1 + j_2 + j_3 = -5p^s + 2\,d_H(C) - 2$. Applying Theorem 4.1, we see that $d_H(C) = \min\{d_H(C_i)\}$, where $i = \{1, 2, 3\}$. Then we have $d_H(C) = d_H(C_1)$ or $d_H(C) = d_H(C_2)$ or $d_H(C) = d_H(C_3)$. Since Theorems 2.3, 5.1 and 5.2, $d_H(C) \le p^s$. Thus, $-5p^s + 2\,d_H(C) - 2 < 0$. It implies that $j_1 + j_2 + j_3 \neq -5p^s + 2\,d_H(C) - 2$. Hence, C is not an MDS code. □

Proposition 5.10: Let $C = C_1 \oplus C_j \oplus C_k$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_j, C_k are defined as in Theorem 2.7 and j, k are different numbers in $\{2, 3\}$. If C_j is a constacyclic code of Type 2 of length $2p^s$ over $\mathcal{R}$, then C is not an MDS code.

Proof: Without loss of generality, assume that $C_1 = \langle (x-1)^{j_1} \rangle$, $C_2 = \langle u(x^2 - \gamma_2)^{j_2}$, $C_3 = \langle (x^2 - \gamma_3)^{j_3} + u(x^2 - \gamma_3)^{t_3}\, h_3(x), u(x^2 - \gamma_3)^{\kappa_3} \rangle$, or $C_3 = \langle (x^2 - \gamma_3)^{j_3} + u(x^2 - \gamma_3)^{t_3}\, h_3(x), u(x^2 - \gamma_3)^{\kappa_3} \rangle$. ($j_1, j_2, j_3$ are defined as in Theorem 2.2). By Theorem 2.11, C is an MDS code when $m(p^s - j_1) + m(2p^s - j_2) + \ell_3 = 2m(5p^s - d_H(C) + 1)$, where $p^{\ell_3} = |C_3|$. Hence, C is an MDS code when $j_1 + j_2 + \ell_3 = -7p^s + 2\,d_H(C) - 2$. Since Theorems 2.3, 5.1 and 5.2, $d_H(C) \le p^s$. Thus, $-5p^s + 2\,d_H(C) - 2 < 0$. It implies that $j_1 + j_2 + j_3 \neq -5p^s + 2\,d_H(C) - 2$. Hence, C is not an MDS code. □

Proposition 5.11: Let $C = C_1 \oplus C_j \oplus C_k$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_j, C_k are defined as in Theorem 2.7 and j, k are different numbers in $\{2, 3\}$. If C_j, C_k are constacyclic codes of Type 3 of length $2p^s$ over $\mathcal{R}$, then C is not an MDS code.

Proof: Without loss of generality, assume that $C_1 = \langle (x-1)^{j_1} \rangle$, $C_2 = \langle (x^2 - \gamma_2)^{j_2} + u(x^2 - \gamma_2)^{t_2}\, h_2(x) \rangle$, $C_3 = \langle (x^2 - \gamma_3)^{j_3} + u(x^2 - \gamma_3)^{t_3}\, h_3(x) \rangle$. By Theorem 2.11, C is an MDS code when $m(p^s - j_1) + \ell_2 + \ell_3 = 2m(5p^s - d_H(C) + 1)$. Applying Theorem 4.1, we see that $d_H(C) = \min\{d_H(C_i)\}$, where $i = \{1, 2, 3\}$. Then we have $d_H(C) = d_H(C_1)$ or $d_H(C) = d_H(C_2)$ or $d_H(C) = d_H(C_3)$. Since Theorems 2.3, 5.1 and 5.2, $d_H(C) \le p^s$. Therefore, if $\ell_2 < 4m(p^s - j_2)$ or $\ell_3 < 4m(p^s - j_3)$, then $m(p^s - j_1) + \ell_2 + \ell_3 \neq 2m(5p^s - d_H(C) + 1)$. Thus, C is not an MDS code when $\ell_2 < 4m(p^s - j_2$ or $\ell_3 < 4m(p^s - j_3)$. We consider the case $\ell_2 = 4m(p^s - j_2)$ or $\ell_3 = 4m(p^s - j_3)$. We divide into 3 cases as follows.

Case 1: $d_H(C) = d_H(C_1)$. From Theorem 2.3, $d_H(C_1) = 1$ or $d_H(C_1) = (\delta_1 + 1)p^{k_1}$.

Subcase 1.1: $d_H(C_1) = 1$. In this subcase, $j_1 + \ell_2 + \ell_3 = -p^s$, which is a contradiction. Hence, C is not an MDS code in this subcase.

Subcase 1.2: $d_H(C_1) = (\delta_1 + 1)p^{k_1}$. Then we see that $j_1 + 4j_2 + 4j_3 = -p^s + 2(\delta_1 + 1)p^{k_1} - 2$ (10). From (10), we have $2(\delta_1 + 1)p^{k_1} > p^s$. Since $\delta_1 + 1 \le p$, we have $p^s < 2p \cdot p^{k_1}$. It implies that $p^{s-k_1-1} < 2$. Hence, $s - k_1 - 1 = 0$, i.e., $s = k_1 + 1$. By asumption $p^s - p^{s-k_1} + (\delta_1 - 1)p^{s-k_1-1} + 1 \le j_1 \le p^s - p^{s-k_1} + \delta_1 p^{s-k_1-1}$, we see that $p^s - p + \delta_1 - 1 + 1 \le j_1 \le p^s - p + \delta_1$. It follows that $j_1 = p^s - p + \delta_1$. Put $T_6 = 4j_2 + 4j_3$. Then $T_6 = 4j_2 + 4j_3 = -p^s + 2(\delta_1 + 1)p^{s-1} - 2 - (p^s - p + \delta_1)$. Hence,

$$\begin{aligned} T_6 &= p^{s-1}[2(\delta_1 + 1) - 2p] - (\delta_1 - p + 1) - 1 \\ &= [2p^{s-1} - 1][\delta_1 - p + 1] - 1. \end{aligned}$$

Since $2p^{s-1} - 1 > 0$ and $\delta_1 - p + 1 \le 0$, we have $[2p^{s-1} - 1][\delta_1 - p + 1] - 1 < 0$. Thus, $j_1 + 4j_2 + 4j_3 \neq -p^s + 2(\delta_1 + 1)p^{k_1} - 2$, i.e., C is not an MDS code when $d_H(C) = d_H(C_1)$.

Case 2: $d_H(C) = d_H(C_2)$. We see that $d_H(C) = d_H(C_3)$ can be done similarly. We have $d_H(C) = (\delta_2 + 1)p^{k_2}$ where $p^s - p^{s-k_2} + (\delta_2 - 1)p^{s-k_2-1} + 1 \le R \le p^s - p^{s-k_2} + \delta_2 p^{s-k_2-1}$, $1 \le \delta_2 \le p - 1$ and $0 \le k_2 \le s - 1$. By Theorem 2.11, C is an MDS code when $m(p^s - j_1) + 4m(p^s - j_2) + 4m(p^s - j_3) = 2m(5p^s - d_H(C) + 1)$, where $p^{\ell_2} = |C_2|$. Hence, C is an MDS code when $j_1 + 4j_2 + 4j_3 = -p^s + 2\,d_H(C) - 2 = -p^s + 2(\delta_2 + 1)p^{k_2} - 2$ (11). From (11), we have $2(\delta_2 + 1)p^{k_2} > p^s$. Since $\delta_2 + 1 \le p$, we have $p^s < 2p \cdot p^{k_2}$. It implies that $p^{s-k_2-1} < 2$. Hence, $s - k_2 - 1 = 0$, i.e., $s = k_2 + 1$. By asumption $p^s - p^{s-k_2} + (\delta_2 - 1)p^{s-k_2-1} + 1 \le j_2 \le p^s - p^{s-k_2} + \delta_3 p^{s-k_2-1}$, we see that $p^s - p + \delta_2 - 1 + 1 \le j_2 \le p^s - p + \delta_2$. It follows that $j_2 = p^s - p + \delta_2$. Put $T_6 = 4j_2 + 4j_3$. Then $T_6 = 4j_2 + 4j_3 = -p^s + 2(\delta_2 + 1)p^{s-1} - 2 - (p^s - p + \delta_2)$. Hence,

$$\begin{aligned} T_6 &= p^{s-1}[2(\delta_2 + 1) - 2p] - (\delta_2 - p + 1) - 1 \\ &= [2p^{s-1} - 1][\delta_2 - p + 1] - 1. \end{aligned}$$

Since $2p^{s-1} - 1 > 0$ and $\delta_2 - p + 1 \le 0$, we have $[2p^{s-1} - 1][\delta_2 - p + 1] - 1 < 0$. Thus, $j_1 + 4j_2 + 4j_3 \neq -p^s + 2(\delta_2 + 1)p^{k_2} - 2$, i.e., C is not an MDS code. □

Proposition 5.12: Let $C = C_1 \oplus C_j \oplus C_k$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_j, C_k are defined as in Theorem 2.11 and j, k are different numbers in $\{2, 3\}$. If C_j is a constacyclic code of Type 3 of length $2p^s$ over $\mathcal{R}$ and C_k is a constacyclic code of Type 4 of length $2p^s$ over $\mathcal{R}$, then C is not an MDS code.

Proof: Without loss of generality, assume that $C_1 = \langle (x-1)^{j_1} \rangle$, $C_2 = \langle (x^2 - \gamma_2)^{j_2} + u(x^2 - \gamma_2)^{t_2}\, h_2(x) \rangle$, $C_3 = \langle (x^2 - \gamma_3)^{j_3} + u(x^2 - \gamma_3)^{t_3}\, h_3(x), u(x^2 - \gamma_3)^{\kappa_3} \rangle$. Applying Theorem 4.1, we see that $d_H(C) = \min\{d_H(C_i)\}$, where $i = \{1, 2, 3\}$. Then we have $d_H(C) = d_H(C_1)$ or $d_H(C) = d_H(C_2)$ or $d_H(C) = d_H(C_3)$. By Theorem 2.11, C is an MDS code when $m(p^s - j_1) + \ell_2 + \ell_3 = 2m(5p^s - d_H(C) + 1)$. It is easy to see that if $\ell_2 = p^{2m(p^s - j_2)}$, then $m(p^s - j_1) + \ell_2 + \ell_3 \neq 2m(5p^s - d_H(C) + 1)$. Thus, C is not an MDS code when $\ell_2 = p^{2m(p^s - j_2)}$. We consider the case when $\ell_2 = p^{4m(p^s - j_2)}$. We divide into 3 cases, namely, $d_H(C) = d_H(C_1)$, $d_H(C) = d_H(C_2)$ and $d_H(C) = d_H(C_3)$.

Case 1: $d_H(C) = d_H(C_1)$. From Theorem 2.3, $d_H(C_1) = 1$ or $d_H(C_1) = (\delta_1 + 1)p^{k_1}$.

Subcase 1.1: $d_H(C_1) = 1$. In this subcase, $j_1 + \ell_2 + \ell_3 = -9p^s$, which is a contradiction. Hence, C is not an MDS code in this subcase.

Subcase 1.2: $d_H(C_1) = (\delta_1 + 1)p^{k_1}$. Then we see that $j_1 + 4j_2 + 2j_3 + 2\kappa_2 = -p^s + 2(\delta_1 + 1)p^{k_1} - 2$ (12). From (12), we have $2(\delta_1 + 1)p^{k_1} > p^s$. Since $\delta_1 + 1 \le p$, we have $p^s < 2p \cdot p^{k_1}$. It implies that $p^{s-k_1-1} < 2$. Hence, $s - k_1 - 1 = 0$, i.e., $s = k_1 + 1$. By asumption $p^s - p^{s-k_1} + (\delta_1 - 1)p^{s-k_1-1} + 1 \le j_1 \le p^s - p^{s-k_1} + \delta_1 p^{s-k_1-1}$, we see that $p^s - p + \delta_1 - 1 + 1 \le j_1 \le p^s - p + \delta_1$. It follows that $j_1 = p^s - p + \delta_1$. Put $T_7 = 4j_2 + 2j_3 + 2\kappa_2$. Then $T_7 = 4j_2 + 2j_3 + 2\kappa_2 =$

$-p^s+2(\delta_1+1)p^{s-1}-2-(p^s-p+\delta_1)$. Hence,

$$\begin{aligned}T_7 &= p^{s-1}[2(\delta_1+1)-2p]-(\delta_1-p+1)-1\\ &= [2p^{s-1}-1][\delta_1-p+1]-1.\end{aligned}$$

Since $2p^{s-1}-1>0$ and $\delta_1-p+1\leq 0$, we have $[2p^{s-1}-1][\delta_1-p+1]-1<0$. Thus, $j_1+4j_2+2j_3+2\kappa_2\neq -p^s+2(\delta_1+1)p^{k_1}-2$, i.e., C is not an MDS code when $\mathrm{d_H}(C)=\mathrm{d_H}(C_1)$.

Case 2: $\mathrm{d_H}(C)=\mathrm{d_H}(C_2)$. We see that $\mathrm{d_H}(C)=(\delta_2+1)p^{k_2}$ where $p^s-p^{s-k_2}+(\delta_2-1)p^{s-k_2-1}+1\leq R\leq p^s-p^{s-k_2}+\delta_2p^{s-k_2-1}$, $1\leq\delta_2\leq p-1$ and $0\leq k_2\leq s-1$. By Theorem 2.11, C is an MDS code when $m(p^s-j_1)+m(4p^s-j_2)+2m(2p^s-j_3-\kappa_3)=2m(5p^s-\mathrm{d_H}(C)+1)$. Hence, C is an MDS code when $j_1+j_2+2j_3+2\kappa_3=-p^s+2\,\mathrm{d_H}(C)-2=-p^s+2(\delta_2+1)p^{k_2}-2$ (13). From (13), we have $2(\delta_2+1)p^{k_2}>p^s$. Since $\delta_2+1\leq p$, we have $p^s<2p\cdot p^{k_2}$. It implies that $p^{s-k_2-1}<2$. Hence, $s-k_2-1=0$, i.e., $s=k_2+1$. By asumption $p^s-p^{s-k_2}+(\delta_2-1)p^{s-k_2-1}+1\leq j_2\leq p^s-p^{s-k_2}+\delta_3p^{s-k_2-1}$, we see that $p^s-p+\delta_2-1+1\leq j_2\leq p^s-p+\delta_2$. It follows that $j_2=p^s-p+\delta_2$. Put $T_8=j_1+j_2+2j_3+2\kappa_3$. Then $T_8=j_1+j_2+2j_3+2\kappa_3=-p^s+2(\delta_2+1)p^{s-1}-2-(p^s-p+\delta_2)$. Hence,

$$\begin{aligned}T_8 &= p^{s-1}[2(\delta_2+1)-2p]-(\delta_2-p+1)-1\\ &= [2p^{s-1}-1][\delta_2-p+1]-1.\end{aligned}$$

Since $2p^{s-1}-1>0$ and $\delta_2-p+1\leq 0$, we have $[2p^{s-1}-1][\delta_2-p+1]-1<0$. Thus, $j_1+j_2+2j_3+2\kappa_3\neq -p^s+2(\delta_2+1)p^{k_2}-2$, i.e., C is not an MDS code when $C_2=\langle(x^2-\alpha_2)^j+u(x^2-\alpha_2)^r v(x)\rangle$.

Case 3: $\mathrm{d_H}(C)=\mathrm{d_H}(C_3)$. We see that $\mathrm{d_H}(C)=(\delta_3+1)p^{k_3}$, where $p^s-p^{s-k_3}+(\delta_3-1)p^{s-k_3-1}+1\leq R\leq p^s-p^{s-k_3}+\delta_3p^{s-k_3-1}$, $1\leq\delta_3\leq p-1$ and $0\leq k_3\leq s-1$. By Theorem 2.11, C is an MDS code when $m(p^s-j_1)+4m(p^s-j_2)+2m(2p^s-j_3-\kappa_3)=2m(5p^s-\mathrm{d_H}(C)+1)$. Hence, C is an MDS code when $j_1+4j_2+2j_3+2\kappa_3=-p^s+2\,\mathrm{d_H}(C)-2=-p^s+2(\delta_3+1)p^{k_3}-2$ (14). From (14), we have $2(\delta_3+1)p^{k_3}>p^s$. Since $\delta_3+1\leq p$, we have $p^s<2p\cdot p^{k_3}$. It implies that $p^{s-k_3-1}<3$. Hence, $s-k_3-1=0$, i.e., $s=k_3+1$. By asumption $p^s-p^{s-k_3}+(\delta_3-1)p^{s-k_3-1}+1\leq j_3\leq p^s-p^{s-k_3}+\delta_3p^{s-k_3-1}$, we see that $p^s-p+\delta_3-1+1\leq j_3\leq p^s-p+\delta_3$. It follows that $j_3=p^s-p+\delta_3$. Put $T_9=j_1+4j_2+2j_3+2\kappa_3$. Then $T_9=j_1+4j_2+2j_3+2\kappa_3=-p^s+3(\delta_3+1)p^{s-1}-3-(p^s-p+\delta_3)$. Hence,

$$\begin{aligned}T_9 &= p^{s-1}[3(\delta_3+1)-2p]-(\delta_3-p+1)-1\\ &= [2p^{s-1}-1][\delta_3-p+1]-1.\end{aligned}$$

Since $2p^{s-1}-1>0$ and $\delta_3-p+1\leq 0$, we have $[2p^{s-1}-1][\delta_3-p+1]-1<0$. Thus, $j_1+4j_2+2j_3+2\kappa_3\neq -p^s+3(\delta_3+1)p^{k_3}-3$, i.e., C is not an MDS code.

Proposition 5.13: Let $C=C_1\oplus C_j\oplus C_k$ be a cyclic code of length $5p^s$ over $\mathcal{R}$, where C_j, C_k are defined as in Theorem 2.7 and j, k are different numbers in $\{2,3\}$. If C_j, C_k are constacyclic codes of Type 4 of length $2p^s$ over $\mathcal{R}$, then C is not an MDS code.

Proof: Without loss of generality, assume that $C_1=\langle(x-1)^{j_1}\rangle$, $C_2=\langle(x^2-\gamma_2)^{j_2}+u(x^2-\gamma_2)^{t_2}\,h_2(x)\rangle$, $C_3=\langle(x^2-\gamma_3)^{j_3}+u(x^2-\gamma_3)^{t_3}\,h_3(x), u(x^2-\gamma_3)^{\kappa_3}\rangle$. Applying Theorem 4.1, we see that $\mathrm{d_H}(C)=\min\{\mathrm{d_H}(C_i)\}$, where $i=\{1,2,3\}$. Then we have $\mathrm{d_H}(C)=\mathrm{d_H}(C_1)$ or $\mathrm{d_H}(C)=\mathrm{d_H}(C_2)$ or $\mathrm{d_H}(C)=\mathrm{d_H}(C_3)$. Using Theorems 2.3, 5.1-5.3, $\mathrm{d_H}(C)\leq p^s$. By Theorem 2.11, C is an MDS code when $m(p^s-j_1)+2m(2p^s-j_2-\kappa_2)+2m(2p^s-j_3-\kappa_3)=2m(5p^s-\mathrm{d_H}(C)+1)$. We divide into 3 cases, namely, $\mathrm{d_H}(C)=\mathrm{d_H}(C_1)$, $\mathrm{d_H}(C)=\mathrm{d_H}(C_2)$ and $\mathrm{d_H}(C)=\mathrm{d_H}(C_3)$.

Case 1: $\mathrm{d_H}(C)=\mathrm{d_H}(C_1)$. From Theorem 2.3, $\mathrm{d_H}(C_1)=1$ or $\mathrm{d_H}(C_1)=(\delta_1+1)p^{k_1}$.

Subcase 1.1: $\mathrm{d_H}(C_1)=1$. In this subcase, $j_1+2j_2+2\kappa_2+2j_3+2\kappa_3=-p^s$, which is a contradiction. Hence, C is not an MDS code in this subcase.

Subcase 1.2: $\mathrm{d_H}(C_1)=(\delta_1+1)p^{k_1}$. Then we see that $j_1+2j_2+2\kappa_2+2j_3+2\kappa_2=-p^s+2(\delta_1+1)p^{k_1}-2$ (15). From (15), we have $2(\delta_1+1)p^{k_1}>p^s$. Since $\delta_1+1\leq p$, we have $p^s<2p\cdot p^{k_1}$. It implies that $p^{s-k_1-1}<2$. Hence, $s-k_1-1=0$, i.e., $s=k_1+1$. By asumption $p^s-p^{s-k_1}+(\delta_1-1)p^{s-k_1-1}+1\leq j_1\leq p^s-p^{s-k_1}+\delta_1p^{s-k_1-1}$, we see that $p^s-p+\delta_1-1+1\leq j_1\leq p^s-p+\delta_1$. It follows that $j_1=p^s-p+\delta_1$. Put $T_{10}=2j_2+2\kappa_2+2j_3+2\kappa_3$. Then $T_{10}=2j_2+2\kappa_2+2j_3+2\kappa_3=-p^s+2(\delta_1+1)p^{s-1}-2-(p^s-p+\delta_1)$. Hence,

$$\begin{aligned}T_{10} &= p^{s-1}[2(\delta_1+1)-2p]-(\delta_1-p+1)-1\\ &= [2p^{s-1}-1][\delta_1-p+1]-1.\end{aligned}$$

Since $2p^{s-1}-1>0$ and $\delta_1-p+1\leq 0$, we have $[2p^{s-1}-1][\delta_1-p+1]-1<0$. Thus, $j_1+2j_2+2\kappa_2+2j_3+2\kappa_3\neq -p^s+2(\delta_1+1)p^{k_1}-2$, i.e., C is not an MDS code when $\mathrm{d_H}(C)=\mathrm{d_H}(C_1)$.

Case 2: $\mathrm{d_H}(C)=\mathrm{d_H}(C_2)$. We see that $\mathrm{d_H}(C)=\mathrm{d_H}(C_3)$ can be done similarly. We have $\mathrm{d_H}(C)=(\delta_2+1)p^{k_2}$, where $p^s-p^{s-k_2}+(\delta_2-1)p^{s-k_2-1}+1\leq R\leq p^s-p^{s-k_2}+\delta_2p^{s-k_2-1}$, $1\leq\delta_2\leq p-1$ and $0\leq k_2\leq s-1$. By Theorem 2.4, C is an MDS code when $m(p^s-j_1)+2m(2p^s-j_2-\kappa_2)+2m(2p^s-j_3-\kappa_3)=2m(5p^s-\mathrm{d_H}(C)+1)$. Hence, C is an MDS code when $j_1+2j_2+2\kappa_2+2j_3+2\kappa_3=-p^s+2\,\mathrm{d_H}(C)-2=-p^s+2(\delta_2+1)p^{k_2}-2$ (16). From (16), we have $2(\delta_2+1)p^{k_2}>p^s$. Since $\delta_2+1\leq p$, we have $p^s<2p\cdot p^{k_2}$. It implies that $p^{s-k_2-1}<2$. Hence, $s-k_2-1=0$, i.e., $s=k_2+1$. By asumption $p^s-p^{s-k_2}+(\delta_2-1)p^{s-k_2-1}+1\leq j_2\leq p^s-p^{s-k_2}+\delta_3p^{s-k_2-1}$, we see that $p^s-p+\delta_2-1+1\leq j_2\leq p^s-p+\delta_2$. It follows that $j_2=p^s-p+\delta_2$. Put $T_{11}=j_1+2j_2+2\kappa_2+2j_3+2\kappa_3$. Then $T_{11}=j_1+2j_2+2\kappa_2+2j_3+2\kappa_3=-p^s+2(\delta_2+1)p^{s-1}-2-(p^s-p+\delta_2)$. Hence,

$$\begin{aligned}T_{11} &= p^{s-1}[2(\delta_2+1)-2p]-(\delta_2-p+1)-1\\ &= [2p^{s-1}-1][\delta_2-p+1]-1.\end{aligned}$$

Since $2p^{s-1}-1>0$ and $\delta_2-p+1\leq 0$, we have $[2p^{s-1}-1][\delta_2-p+1]-1<0$. Thus, $j_1+2j_2+2\kappa_2+2j_3+2\kappa_3\neq -p^s+2(\delta_2+1)p^{k_2}-2$, i.e., C is not an MDS code. ☐

Combining Propositions 5.5-5.13, we have the following theorem.

Theorem 5.14: Let $C = C_1 \oplus C_2 \oplus C_3$ be a cyclic code of length $5p^s$ over $\mathcal{R}$. Then C is an MDS code when $C = \langle 1 \rangle$.

We finish this section by the following examples.

Example 5.15: Put $\mathcal{R} = \mathbb{F}_{19} + u\mathbb{F}_{19}$. Then cyclic codes of length 95 over $\mathcal{R}$ are ideals of $\frac{\mathcal{R}[x]}{\langle x^{95}-1 \rangle}$. We have $x^{95} - 1 = (x-1)^{19}(x^2+5x+1)^{38}(x^2+15x+1)^{38}$. It is easy to see that $\gamma = 10$. By Theorem 2.7, $C = C_1 \oplus C_2 \oplus C_3$, where C_1 is a cyclic code of length 19 over $\mathcal{R}$, C_2 is a 11-constacyclic code of length 38 over $\mathcal{R}$, C_3 is a 10-constacyclic code of length 38 over $\mathcal{R}$. Let $C_1 = \langle (x-1)^5 \rangle$, $C_2 = \langle u(x^2-11)^5 \rangle$, $C_3 = \langle u(x^2-10)^4 \rangle$. By applying Theorem 2.3, $\mathrm{d_H}(C_1) = 6$. Using Theorem 5.4, we have $\mathrm{d_H}(C_2) = 6$ and $\mathrm{d_H}(C_3) = 5$. From Theorem 5.1, $\mathrm{d_H}(C) = 5$. Then C has the parameters $[95, 19^{70}, 5]$.

Example 5.16: Put $\mathcal{R} = \mathbb{F}_{29} + u\mathbb{F}_{29}$. Then cyclic codes of length 145 over $\mathcal{R}$ are ideals of $\frac{\mathcal{R}[x]}{\langle x^{145}-1 \rangle}$. We have $x^{145} - 1 = (x-1)^{29}(x^2+6x+1)^{58}(x^2+24x+1)^{58}$. It is easy to see that $\gamma = 18$. By Theorem 2.7, $C = C_1 \oplus C_2 \oplus C_3$, where C_1 is a cyclic code of length 29 over $\mathcal{R}$, C_2 is a 27-constacyclic code of length 58 over $\mathcal{R}$, C_3 is a 15-constacyclic code of length 58 over $\mathcal{R}$. Let $C_1 = \langle (x-1)^5 \rangle$, $C_2 = \langle u(x^2-27)^5 \rangle$, $C_3 = \langle u(x^2-15)^4 \rangle$. By using Theorem 2.3, $\mathrm{d_H}(C_1) = 6$. Applying Theorem 5.4, we have $\mathrm{d_H}(C_2) = 6$ and $\mathrm{d_H}(C_3) = 5$. By Theorem 5.1, $\mathrm{d_H}(C) = 5$. Then C has the parameters $[145, 29^{131}, 5]$.

VI. CONCLUSION

In this paper, the Hamming distances of all cyclic codes of length $5p^s$ over $\mathcal{R}$ are studied. When $p \equiv 2, 3 \pmod 5$, we provided the Hamming distance for cyclic codes of length $5p^s$ over $\mathcal{R}$ in Theorems 3.1-3.4. In Theorems 3.5-3.8, we gave all MDS cyclic codes of length $5p^s$ over $\mathcal{R}$. In Section 4, the Hamming distance of all cyclic codes of length $5p^s$ over $\mathcal{R}$ is given in Theorem 4.1 when $p \equiv 1 \pmod 5$. Using Propositions 4.2-4.13, we determined all MDS cyclic codes of length $5p^s$ over $\mathcal{R}$. In Section 5, we determined Hamming distance of all cyclic codes of length $5p^s$ over $\mathcal{R}$ when $p \equiv 4 \pmod 5$ (Theorem 5.1). Theorems 5.2-5.4 provided the Hamming distance of ideals of Types 2,3,4 of $\frac{\mathcal{R}[x]}{\langle (x^2-\alpha_1)^{p^s} \rangle}$. Applying Propositions 5.5-5.13, Theorem 5.14 gave all MDS cyclic codes of length $5p^s$ over $\mathcal{R}$ when $p \equiv 4 \pmod 5$. We gave three examples to illustrate our work in Sections 3, 4, and 5 (Examples 3.11, 4.14, 5.15 and 5.16).

MDS b-symbol codes can be considered as a generalization of MDS codes and MDS symbol-pair codes. In 2018, [11] established the Singleton bound for b-symbol codes as follows: Let q be a prime power and $b \leq \mathrm{d_b} \leq n$, for any b-symbol code C of length n with size M and minimum b-distance $\mathrm{d_b}$ over $\mathbb{F}_q$, $M \leq q^{n-\mathrm{d_b}+b}$. If the equality holds, the b-symbol code C is called an optimal code with respect to the Singleton bound, or an MDS b-symbol code. It will be very interesting to discuss the b-symbol metrics for all constacyclic codes of length $5p^s$ over $\mathcal{R}$ and then we will identify all MDS constacyclic codes of length $5p^s$ with respect to b-symbol distances.

Since classical error-correcting codes can not be used in quantum computation, quantum error-correcting (briefly, QEC) codes are proposed to protect quantum information from errors due to the decoherence and other quantum noise. For future work, it will be interesting to apply these distances in constructing quantum error-correcting codes from the class of λ-constacyclic codes of length $5p^s$ over $\mathcal{R}$.

REFERENCES

[1] T. Abualrub and R. Oehmke, "On the generators of $\mathbb{Z}_4$ cyclic codes of length 2^e," *IEEE Trans. Inf. Theory*, vol. 49, no. 9, pp. 2126–2133, Sep. 2003.
[2] T. Abualrub and I. Siap, "Constacyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2$," *J. Franklin Inst.*, vol. 346, pp. 520–529, Jun. 2009.
[3] M. M. Al-Ashker, "Simplex codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2$," *Arab. J. Sci. Eng. A, Sci*, vol. 30, no. 2, pp. 277–285, 2005.
[4] S. Berman, "Semisimple cyclic and Abelian codes. II," *Cybernetics*, vol. 3, pp. 21–30, May 1967.
[5] T. Blackford, "Cyclic codes over $\mathbb{Z}_4$ of oddly even length," in *Proc. Int. Workshop Coding Cryptogr. (WCC)*, vol. 128, Paris, France, 2003, pp. 27–46.
[6] Y. Cao, Y. Cao, H. Q. Dinh, F. Fu, J. Gao, and S. Sriboonchitta, "Constacyclic codes of length np^s over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$," *Adv. Math. Commun.*, vol. 12, no. 2, pp. 231–262, 2018.
[7] Y. Cao, Y. Cao, H. Q. Dinh, F. Fu, J. Gao, and S. Sriboonchitta, "A class of repeated-root constacyclic codes over $\mathbb{F}_{p^m}[u]/\langle u^e \rangle$ of type 2," *Finite Fields Appl.*, vol. 55, pp. 238–267, Jan. 2019.
[8] Y. Cassuto and M. Blaum, "Codes for symbol-pair read channels," *IEEE Trans. Inf. Theory*, vol. 57, no. 12, pp. 8011–8020, Dec. 2011.
[9] G. Castagnoli, J. L. Massey, P. A. Schoeller, and N. von Seemann, "On repeated-root cyclic codes," *IEEE Trans. Inf. Theory*, vol. 37, no. 2, pp. 337–342, Mar. 1991.
[10] B. Chen, H. Q. Dinh, H. Liu, and L. Wang, "Constacyclic codes of length p^s over $F^{p^m} + uF^{p^m}$," *Finite Fields Appl.*, vol. 37, pp. 108–130, Jan. 2016.
[11] B. Ding, T. Zhang, and G. Ge, "Maximum distance separable codes for b-symbol read channels," *Finite Fields Their Appl.*, vol. 49, pp. 180–197, Jan. 2018.
[12] H. Q. Dinh, "Constacyclic codes of length p^s over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$," *J. Algebra*, vol. 324, pp. 940–950, Sep. 2010.
[13] H. Q. Dinh, S. Dhompongsa, and S. Sriboonchitta, "On constacyclic codes of length $3p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$," *Discrete Math.*, vol. 340, pp. 832–849, Apr. 2017.
[14] H. Q. Dinh, "On the linear ordering of some classes of negacyclic and cyclic codes and their distance distributions," *Finite Fields Appl.*, vol. 14, no. 1, pp. 22–40, 2008.
[15] H. Q. Dinh, "Constacyclic codes of length 2^s over Galois extension rings of $\mathbb{F}_2 + u\mathbb{F}_2$," *IEEE Trans. Inf. Theory*, vol. 55, no. 4, pp. 1730–1740, Apr. 2009.
[16] H. Q. Dinh, "Repeated-root constacyclic codes of length $2p^s$," *Finite Fields Appl.*, vol. 18, no. 1, pp. 133–143, 2012.
[17] H. Q. Dinh, "Structure of repeated-root constacyclic codes of length $3p^s$ and their duals," *Discrete Math.*, vol. 313, no. 9, pp. 983–991, May 2013.
[18] H. Q. Dinh, "Structure of repeated-root cyclic and negacyclic codes of length $6p^s$ and their duals," *AMS Contemp. Math.*, vol. 339, pp. 69–87, Feb. 2014.
[19] H. Q. Dinh, B. T. Nguyen, A. K. Singh, and S. Sriboonchitta, "On the symbol-pair distance of repeated-root constacyclic codes of prime power lengths," *IEEE Trans. Inf. Theory*, vol. 64, no. 4, pp. 2417–2430, Apr. 2018.
[20] H. Q. Dinh, B. T. Nguyen, and W. Yamaka, "Constacyclic codes of length $3p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ and their application in various distance distributions," *IEEE Access*, vol. 8, pp. 204031–204056, 2020.
[21] H. Q. Dinh, B. T. Nguyen, and S. Sriboonchitta, "Negacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$," *Discrete Math.*, vol. 341, pp. 1055–1071, Apr. 2018.
[22] H. Q. Dinh, S. Dhompongsa, and S. Sriboonchitta, "On constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$," *Discrete Math.*, vol. 340, pp. 832–849, Apr. 2017.

[23] H. Q. Dinh, B. T. Nguyen, S. Sriboonchitta, and T. M. Vo, ''On a class of constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$,'' *J. Algebra Appl.*, vol. 18, no. 8, 2019, Art. no. 1950022.
[24] H. Q. Dinh, B. T. Nguyen, S. Sriboonchitta, and T. M. Vo, ''On $(\alpha + u\beta)$-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$,'' *J. Algebra Appl.*, vol. 18, Feb. 2019, Art. no. 1950023.
[25] H. Q. Dinh, X. Wang, H. Liu, and S. Sriboonchitta, ''On the Hamming distances of repeated-root constacyclic codes of length $4p^s$,'' *Discrete Math.*, vol. 342, pp. 1456–1470, May 2019.
[26] H. Q. Dinh and S. R. López-Permouth, ''Cyclic and negacyclic codes over finite chain rings,'' *IEEE Trans. Inf. Theory*, vol. 50, no. 8, pp. 1728–1744, Aug. 2004.
[27] H. Q. Dinh, B. T. Nguyen, A. K. Singh, and S. Sriboonchitta, ''Hamming and symbol-pair distances of repeated-root constacyclic codes of prime power lengths over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$,'' *IEEE Commun. Lett.*, vol. 22, no. 12, pp. 2400–2403, Dec. 2018.
[28] H. Q. Dinh, B. T. Nguyen, A. K. Singh, and W. Yamaka, ''MDS constacyclic codes and MDS symbol-pair constacyclic codes,'' *IEEE Access*, vol. 9, pp. 137970–137990, 2021.
[29] H. Q. Dinh, X. Wang, and J. Sirisrisakulchai, ''On the Hamming distances of constacyclic codes of length 5pS,'' *IEEE Access*, vol. 8, pp. 46242–46254, 2020.
[30] H. Q. Dinh, L. Wang, and S. Zhu, ''Negacyclic codes of length $2p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$,'' *Finite Fields Appl.*, vol. 31, pp. 178–201, Jan. 2015.
[31] H. Q. Dinh, S. Dhompongsa, and S. Sriboonchitta, ''Repeated-root constacyclic codes of prime power length over $\frac{\mathbb{F}_{p^m}[u]}{\langle u^a \rangle}$ and their duals,'' *Discrete Math.*, vol. 339, pp. 1706–1715, Jun. 2016.
[32] H. Q. Dinh, S. Satpati, A. K. Singh, and W. Yamaka, ''Symbol-triple distance of repeated-root constacyclic codes of prime power lengths,'' *J. Algebra Appl.*, vol. 19, no. 11, 2019, Art. no. 2050209.
[33] H. Q. Dinh, P. Kunam, P. Kumar, S. Satpati, A. K. Singh, and W. Yamaka, ''MDS symbol-pair repeated-root constacyclic codes of prime power lengths over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$,'' *IEEE Access*, vol. 7, pp. 145039–145048, 2019.
[34] H. Q. Dinh, X. Wang, and S. Jirakom, ''On the Hamming distance of constacyclic codes of length $5p^s$,'' *IEEE Access*, vol. 8, pp. 46242–46254, 2020.
[35] J. H. van Lint, ''Repeated-root cyclic codes,'' *IEEE Trans. Inf. Theory*, vol. 37, no. 2, pp. 343–345, Mar. 1991.
[36] M. Grassl. *Bounds on the Minimum Distance of Linear Codes and Quantum Codes*. Accessed: Oct. 15, 2021. [Online]. Available: http://www.codetables.de
[37] K. Guenda and A. Gulliver, ''Repeated-root constacyclic codes of length mp^s over $\mathbb{F}_{p^r} + u\mathbb{F}_{p^r} + \cdots + u^{e-1}\mathbb{F}_{p^r}$,'' *J. Algebra Appl.*, vol. 14, no. 1, 2015, Art. no. 1450081.
[38] A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, and P. Solé, ''The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals, and related codes,'' *IEEE Trans. Inf. Theory*, vol. 40, no. 2, pp. 301–319, Feb. 1994.
[39] L. Liu, L. Li, X. Kai, and S. Zhu, ''Repeated-root constacyclic codes of length $3lp^s$ and their dual codes,'' *Finite Fields Appl.*, vol. 42, pp. 269–295, Nov. 2016.
[40] Y. Liu, M. Shi, H. Q. Dinh, and S. Sriboonchitta, ''Repeated-root constacyclic codes of length $3l^mp^s$,'' *Adv. Math. Commun.*, vol. 14, pp. 359–378, Jan. 2020.
[41] G. H. Norton and A. Salagean, ''On the Hamming distance of linear codes over a finite chain ring,'' *IEEE Trans. Inf. Theory*, vol. 46, no. 3, pp. 1060–1067, May 2000.
[42] H. Ozadam and F. Ozbudak, ''The minimum Hamming distance of cyclic codes of length $2p^s$,'' in *Proc. Int. Symp. Appl. Algebra, Algebr. Algorithms, Error-Correcting Codes*, 2009, pp. 92–100.
[43] R. Roth and G. Seroussi, ''On cyclic MDS codes of length q over $GF(q)$,'' *IEEE Trans. Inf. Theory*, vol. IT-32, no. 2, pp. 284–285, Mar. 1986.
[44] M. Shi, S. Zhu, and S. Yang, ''A class of optimal p-ary codes from one-weight codes over $\frac{\mathbb{F}_p[u]}{\langle u^m \rangle}$,'' *J. Franklin Inst.*, vol. 350, pp. 929–937, Jun. 2013.
[45] M. Shi, L. Qian, and P. Solé, ''On self-dual negacirculant codes of index two and four,'' *Des., Codes Cryptogr.*, vol. 86, no. 11, pp. 2485–2494, Nov. 2018.
[46] M. Shi, R. Wu, Y. Liu, and P. Sole, ''Two and three weight codes over $\mathbb{F}_p + u\mathbb{F}_p$,'' *Cryptogr. Commun.*, vol. 9, no. 5, pp. 637–646, 2017.
[47] E. Yaakobi, J. Bruck, and P. H. Siegel, ''Constructions and decoding of cyclic codes over b-symbol read channels,'' *IEEE Trans. Inf. Theory*, vol. 62, no. 4, pp. 1541–1551, Apr. 2016.