IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 29 September 2022, accepted 21 October 2022, date of publication 1 November 2022, date of current version 8 November 2022.

Digital Object Identifier 10.1109/ACCESS.2022.3218695

RESEARCH ARTICLE

A Study on Soft Multi-Granulation Rough Sets and Their Applications

SABA AYUB[1], WAQAS MAHMOOD[1], MUHAMMAD SHABIR[1], ALI N. A. KOAM[2], AND RIZWAN GUL[1]

[1]Department of Mathematics, Quaid-i-Azam University, Islamabad 45320, Pakistan
[2]Department of Mathematics, College of Science, Jazan University, Jazan 2097, Saudi Arabia

Corresponding author: Saba Ayub (sabamianayub@gmail.com)

ABSTRACT Rough set (RS) and soft set (SS) theories are two successful mathematical approaches to dealing with uncertainty in data analysis. The classical soft rough set (SRS) theory proposed by Feng et al. (2011) offers a formal theoretical framework for solving the uncertainty under a single granulation environment. However, it is essential to note that the SRS theory cannot be applied in the context of multi-granulation in the real world. To address this issue, in this paper, we introduce the idea of soft multi-granulation RS (SMGRS) model based on two soft binary relations (S-BRs). Axiomatic operations, lower soft rough approximation space (lower SRA-space) and upper soft rough approximation space (upper SRA-space), are defined through after sets of soft relations. After that, the concept of SMGRSs is applied to a significant part of commutative algebra, group theory. In this respect, the primitive notions of SRA-spaces are defined with the help of two normal soft groups (NSGs). In groups, several important structural properties related to SMGRS are investigated in detail with illustrative examples. It is shown that SMGRS in groups may be influential in decision-making (DM) by some numerical examples. To demonstrate the flexibility, superiority, and effectiveness of the suggested technique, some comparative examples are given with some existing methods.

INDEX TERMS Multi-granulation rough sets, soft sets, soft binary relations, normal soft groups, decision-making.

I. INTRODUCTION

In the present era of technology, the intricacy of modeling real-world problems such as in medical sciences, social sciences, environmental sciences, and engineering, the complexity of human DM is increasing. To deal with real-world problems, this pursuit has given rise to many resourceful techniques such as fuzzy set (FS) [1], RS [2], and SS [3].

The concept of SS theory was introduced by a Russian Scholar Molodtsov [3] in 1999, as a parameterized family of subsets of the universal set. SS theory is an extension of set theory as it contains a family of parameters to describe the membership of objects rather than the set $\{0, 1\}$. The gospel truth of no further constraints on the parameters in case of limited information gramatically simplifies the DM process.

The associate editor coordinating the review of this manuscript and approving it for publication was Abdullah Iliyasu.

SS theory has ample utilizations in probability theory, game theory, measurement theory, operational research, Riemann integration, and the smoothness of functions [3].

Due to the flexibility of this computing technique, research on SS has been significantly advanced since the theory was born. Maji et al. [4] initially studied the elementary operations of SSs and successfully utilized them in a DM problem [5]. For the detailed study of the central notions of SSs, we refer the reader to study the initial work presented by Ali et al. [6] and Aktaş and Çağman [7]. Moreover, SS theory has a significant influence on algebraic structures. For instance, Aktaş and Çağman [7] initiated the conceptualization of S groups (SGs). Further, they have shown that the theories of RS and SS are closely related. A similar relationship between SS and RS was established by Feng et al. [8]. Alkhazaleh and Marei [9] put forward new SRS approximations. Feng et al. [10] proposed generalized uni–int DM schemes based on choice value SSs.

This work is licensed under a Creative Commons Attribution 4.0 License. For more information, see https://creativecommons.org/licenses/by/4.0/

In 2017, Fan et al. [11] applied the notion of Feng et al. [8] to MGRS and established the concept of MGSRS. Sezğin and Atagün [12] and Aslam and Qurashi [13] pursued the study of SGs and defined the concept of NSGs. Furthermore, Çağman et al. [14] have investigated the notion of S-intersection groups, which was further studied by Kaygisiz in [15] and [16]. In 2011, Shabir and Naz [17] independently worked on the topological structure of SSs. To generalize the idea of fuzzy relations (FRs), Feng et al. [18] initiated the idea of S-BRs with applications in semigroups. In [19], Ali defined the notions of lower and upper A-spaces by employing the idea of soft equivalence relations.

In literature, researchers have tackled the problems of uncertain or incomplete information in different ways. In 2017, under the attack on complex networks, Shang [20] has contributed to robust statistics and established the robustness of a system. The concept of RS theory was initiated by Pawlak [2] in 1982 as a systematic approach to the classification of objects. RS theory is also an extension of set theory. In RS theory, the membership of objects is defined through a pair of sets, namely the lower and upper A-spaces. These A-spaces are defined using equivalence classes induced by equivalence relations (ERs). By using these spaces, RS theory characterizes the objects. Let us illustrate this concept: Consider a group of some patients who have diabetes. To diagnose diabetes, one must see various symptoms, like feeling very thirsty, hungry, urinating often, and weight loss. The patients revealing the same symptoms are indiscernible concerning the available information and form elementary classes (granules) of knowledge. So, they are part of RS. Similarly, two acids with PH levels of 4.12 and 4.53 will, in many contexts, be perceived as so equally weak that they are similar concerning this attribute. They are a part of RS "weak acids" as compared to "strong" or "medium". Alternatively, whatsoever other categories are relevant in this context of classification.

In practical life, we often need to describe the concept via multiple relations over the universe based on user requirements or the target of tackling the problem. Hence, Qian et al. [21] extended the single granulation RS model to the multi-granulation RS (MGRS) model, which has recently emerged as a prominent topic in artificial intelligence, attracting a wide range of research from both theoretical and application perspectives.

In 1994, Biswas and Nanda [22] put forward the roughness of groups based on only the lower approximation. So, in 1996, Kuroki and Wang [23] introduced the notion of lower and upper A-space in groups by using a normal subgroup. Recently, Mahmood et al. [24] established a connection between the A-spaces of two different groups by utilizing a group homomorphism. The same scholars investigated the concept of roughness in quotient groups in [25]. Furthermore, Davvaz [26] and in [27], Davvaz and Mahdavipour introduced the notions of roughness in rings and modules, respectively, and studied some related properties. Inspiring by the mentioned studies, recently Chen et al. [28] introduced the concept of roughness in modules of fractions and established a connection between the A-spaces of two different modules of fractions by utilizing a module homomorphism. Ayub et al. [29] and [30] have also implemented the idea of RSs to S-intersection groups and groups by utilizing the concept of normal subgroups and NSGs, respectively. In [29] and [30], the authors have also created relationships among the SA-spaces of two different groups by employing the group homomorphisms. In Ayub et al. [31] combined the theories of SS, FS, and RS and introduced the notion of fuzzy modules of fractions in terms of multi-granulation. In [32], Wang and Garg give an algorithm for multiple-attribute DM. Further study on FSs and algebraic structures such as groups and fields can be found in [33], [34], [35], [36], and [37].

Although the theories of SS and RS are distinct but can be joined together in constructive manners (see [38], [39], [40], [41], [42], [43], and [44]). Ma et al. [45] presented various techniques of hybrid models with applications in DM. In [46], [47], [48], and [49], the authors have employed the notions of SRS [43] to groups and modified soft rough set (MSRS) [44] to groups, semigroups and fuzzy hemirings, respectively. In [50], Zhan and Alcantud, have also investigated the notion of SR covering and an application to multi-criteria group DM (MCDM).

BRs, particularly ERs, are significant in mathematics, artificial intelligence, computer science, DM, and classification. However, the condition of ER for Pawlak's RS [2] is too restrictive for many practical applications. Therefore, different generalizations of Pawlak's RSs [2] have introduced by compensating ERs with a BR [51], [52], a set-valued map [53], [54], a tolerance relation [55], a similarity relation [56], multi ERs [21], multi S-BRs [57], NSGs [30]. In Pawlak's RSs [2], each equivalence class may be regarded as a granule containing similar components concerning the attributes. The partition induced by an ER is a granulation structure. Zadeh first explored the concept of granular computing in 1997 and later studied by Qian et al. [58]. Thus, more general granulation structures have been acquired by weakening the condition of an ER in view of extensions, as mentioned earlier.

A. RESEARCH GAP AND MOTIVATION

All through the above analysis, our leading motivations and research gaps are summarized below:

(1) MGRS proposed by Qian et al. [21] is a universality of Pawlak's RS. MGRS theory is a practical approach to solving the problems in the context of multi-granulations which extended the application areas of Pawlak's single granulation RS model. However, the authors have found fewer studies on weakening the condition of ERs on MGRS (see [57], [59], and [60]).

(2) Since the innovation of MGRSs, many scholars have extended Qian et a.'s MGRSs in various directions (see [61], [62], [63], and [64]). However, there is a lack of investigation on the SRSs in the context of multi-granulation using S-BRs. This is the primary

motivation to introduce the idea of the SMGRS model by means of S-BRs.

(3) To search for the applications of new mathematical models in different algebraic structures are essential and fascinating artistry to find the model's influences. Unfortunately, we have not found a single study on the applications of MGRSs or their universality mentioned above, on algebraic structures.

(4) Because of these research motivations and to fill up the research gap mentioned above, a very interesting universality of MGRS model to SSs is proposed with some applications of this model in group theory and DM.

B. AIM OF THE PROPOSED STUDY

The primary goal of this study is to propose another interesting and novel version of SMGRS by utilizing two S-BRs.

We highlight the article by the following pioneering work:

- A novel concept known as SMGRS is proposed, which is a hybridization of MGRS theory and SS via S-BRs.
- Some important structural properties of SMGRS are investigated in detail with some concrete examples.
- To apply the proposed hybrid model in group theory and explore their related structural properties in groups.
- A detailed comparative analysis with other existing methods is carried out in show the advantages of the proposed methodology.

C. ORGANIZATION OF THE PAPER

Remaining of the paper is organized in the following manner: Section 2 consists of some preliminary concepts related to RS, MGRS, SS, S-BR and NSG. The primary idea of this paper is SMGRS which is introduced based on two SBRs in section 3. The constitutive operations of SMGRS, which are SRA-spaces, defined by using the after sets of both BRs. It is shown that some intrinsic properties in the absence of ERs may hold with some weaker conditions. In section 4, the applications of SMGRSs in group theory are discussed. In this respect, the concepts of lower and upper SRA-spaces in groups are defined by employing two NSGs [13]. In the end, an application of SMGRSs over groups are given in DM and an algorithm in section 5. It also presents a comparative analysis with some existing techniques of Pan and Zhan's [46], which is based on the SRA-spaces regarding only one normal subgroup. Finally, section 6 contains some conclusions of the paper.

II. PRELIMINARIES

This section presents some indispensable concepts of RS, MGRS, SSs, SBRs and NSGs to apprehend the advanced part of this paper.

Definition 1 [2]: An object of the form $\Delta = (\mathscr{X}, \pi)$ is termed as *an approximation space* (A_s), where $\mathscr{X}$ is a finite non-empty universe and π is an ER over $\mathscr{X}$.

If $\varnothing \neq \mathcal{M} \subseteq \mathscr{X}$, then the lower and upper approximations of $\mathcal{M}$ w.r.t Δ are respectively defined as:

$$\underline{apr}_{\pi}(\mathcal{M}) = \{x \in \mathscr{X} : [x]_{\pi} \subseteq \mathcal{M}\}, \tag{1}$$

$$\widetilde{apr}_{\pi}(\mathcal{M}) = \{x \in \mathscr{X} : [x]_{\pi} \cap \mathcal{M} \neq \varnothing\}, \tag{2}$$

where,

$$[x]_{\pi} = \{y \in \mathscr{X} \mid (x, y) \in \pi\}. \tag{3}$$

Moreover, the boundary region of RS is defined as:

$$\mathcal{B}nd_{\pi}(\mathcal{M}) = \widetilde{apr}_{\pi}(\mathcal{M}) - \underline{apr}_{\pi}(\mathcal{M}). \tag{4}$$

Thus, the set $\mathcal{M}$ is called *definable* if $\underline{apr}_{\pi}(\mathcal{M}) = \widetilde{apr}_{\pi}(\mathcal{M})$; otherwise, it is called *RS*.

Pawlak's RS theory uses a single ER. Qian et al. [21] proposed MGRS by using more than one ER, as stated below.

Definition 2 [21]: Let π_1 and π_2 be two independent ERs over a universe $\mathscr{X}$ and $\mathcal{M} \subseteq \mathscr{X}$. Then, we define:

$$\underset{\sim}{\pi_1 + \pi_2}(\mathcal{M}) = \{x \in \mathscr{X} : [x]_{\pi_1} \subseteq \mathcal{M} \text{ or } [x]_{\pi_2} \subseteq \mathcal{M}\}, \tag{5}$$

$$\widetilde{\pi_1 + \pi_2}(\mathcal{M}) = \left(\underset{\sim}{\pi_1 + \pi_2}(\mathcal{M}^c)\right)^c, \tag{6}$$

as the lower and upper approximation of $\mathcal{M}$ w.r.t π_1, π_2. Moreover, if $\pi_1 = \pi_2$ then MGRS model degenerate into the Pawlak RS. The boundary region of $\mathcal{M} \subseteq \mathscr{X}$ under MGRS environment is defined as:

$$Bnd_{(\pi_1+\pi_2)}(X) = \widetilde{\pi_1 + \pi_2}(\mathcal{M}) - \underset{\sim}{\pi_1 + \pi_2}(\mathcal{M}). \tag{7}$$

Definition 3 [3]: Let $\mathscr{X}$ be an initial universe and $\mathscr{E}$ be a set of parameters or attributes of the objects in $\mathscr{X}$. Then a pair $(\eta, \mathscr{A})$ is called a SS over $\mathscr{X}$, where $\eta : \mathscr{A} \longrightarrow \mathcal{P}(\mathscr{X})$ is a set-valued map, $\mathscr{A} \subseteq \mathscr{E}$ and $\mathcal{P}(\mathscr{X})$ represents the power set of $\mathscr{X}$.

In other words, a SS over the universe $\mathscr{X}$ offers a parameterized family of subsets of the universe $\mathscr{X}$. For $\mathfrak{e} \in \mathscr{A}$, $\eta(\mathfrak{e})$ could also be considered as the set of $\mathfrak{e}$-approximate elements of $\mathscr{X}$ by the SS $(\eta, \mathscr{A})$.

Numerous researchers studied fundamental operations of SSs (see [4], [6], [7], and [14]). In 2009, Ali et al. [6] defined some new operations of SSs.

Definition 4 [6]: Let $(\eta_1, \mathscr{A}_1)$ and $(\eta_2, \mathscr{A}_2)$ be two SSs over $\mathscr{X}$. Then, $(\eta_1, \mathscr{A}_1)$ is called a S subset of $(\eta_2, \mathscr{A}_2)$, denoted by $(\eta_1, \mathscr{A}_1)\tilde{\subseteq}(\eta_2, \mathscr{A}_2)$, if:

(1) $\mathscr{A}_1 \subseteq \mathscr{A}_2$, and
(2) $\eta_1(\mathfrak{e}) \subseteq \eta_2(\mathfrak{e})$, for all $\mathfrak{e} \in \mathscr{A}_1$.

Two SSs $(\eta_1, \mathscr{A}_1)$ and $(\eta_2, \mathscr{A}_2)$ are said to be *equal*, if $(\eta_1, \mathscr{A}_1)\tilde{\subseteq}(\eta_2, \mathscr{A}_2)$ and $(\eta_1, \mathscr{A}_1)\tilde{\supseteq}(\eta_2, \mathscr{A}_2)$, represented by $(\eta_1, \mathscr{A}_1)\tilde{=}(\eta_2, \mathscr{A}_2)$.

Definition 5 [6]: Let $(\eta_1, \mathscr{A}_1)$ and $(\eta_2, \mathscr{A}_2)$ be two SSs over $\mathscr{X}$. Then, their restricted intersection $(\eta_1, \mathscr{A}_1) \Cap (\eta_2, \mathscr{A}_2) = (\gamma_1, \mathscr{A}_3)$ and restricted union $(\eta_1, \mathscr{A}_1) \cup_R (\eta_2, \mathscr{A}_2) = (\gamma_2, \mathscr{A}_3)$ is defined as follows:

(1) $\gamma_1(\mathfrak{e}) = \eta_1(\mathfrak{e}) \cap \eta_2(\mathfrak{e})$, for all $\mathfrak{e} \in \mathscr{A}_3$.
(2) $\gamma_2(\mathfrak{e}) = \eta_1(\mathfrak{e}) \cup \eta_2(\mathfrak{e})$, for all $\mathfrak{e} \in \mathscr{A}_3$.

where $\mathscr{A}_3 = \mathscr{A}_1 \cap \mathscr{A}_2 \neq \emptyset$.

Feng et al. [43] put forth a link between SS and RS theory, and laid the foundation of SRS, which is stated as:

Definition 6: Let $(\eta, \mathscr{A})$ be a SSs over $\mathscr{X}$. Then $P = \big(\mathscr{X}, (\eta, \mathscr{A})\big)$ is called soft approximation space. Based on P, the subsequent two operators are defined for any $T \subseteq \mathscr{X}$ as follows:

$$\underline{\mathcal{S}}_P(T) = \big\{x \in \mathscr{X} : \exists\, e \in \mathscr{A}, \big[x \in \eta(e) \subseteq T\big]\big\}, \quad (8)$$
$$\overline{\mathcal{S}}_P(T) = \big\{x \in \mathscr{X} : \exists\, e \in \mathscr{A}, \big[x \in \eta(e), \eta(e) \cap T \neq \emptyset\big]\big\}, \quad (9)$$

are regarded as *soft P-lower* and *soft P-upper approximations* of T.

In 2007, Aktas and Cagman [7] initiated some soft algebraic structures. They established a relationship among the theories of SSs, RSs and FSs and proved that FGs are special instances of SGs.

Definition 7 [7]: A SS $(\eta, \mathscr{A}_1)$ over a group $\mathscr{G}$ is called a soft group (SG) over $\mathscr{G}$, if $\eta(\mathfrak{e})$ is a subgroup of $\mathscr{G}$, for each parameter $\mathfrak{e} \in \mathscr{A}_1$.
In [12], [13], Aslam and Qureshi and Sezgin and Atagun have further investigated the idea of SGs and some associated properties.

Definition 8 [7], [12], [13]: A SG $(\eta, \mathscr{A}_1)$ over $\mathscr{G}$ is called a NSG over $\mathscr{G}$, if $\eta(\mathfrak{e})$ is a normal subgroup of $\mathscr{G}$, for each parameter $\mathfrak{e} \in \mathscr{A}_1$.

Definition 9 [13]: Let $(\eta_1, \mathscr{A}_1)$ and $(\eta_2, \mathscr{A}_2)$ be two SGs over $\mathcal{G}$. Then, their restricted soft product written as $(\eta_1, \mathscr{A}_1)\hat{\circ}(\eta_2, \mathscr{A}_2) = (\zeta, \mathscr{A}_3)$ and defined as $\zeta(\mathfrak{e}) = \eta_1(\mathfrak{e}) \cdot \eta_2(\mathfrak{e})$, for all $\mathfrak{e} \in \mathscr{A}_3 = \mathscr{A}_1 \cap \mathscr{A}_2 \neq \emptyset$.

Theorem 1 [13]: Let $(\eta_1, \mathscr{A}_1)$ and $(\eta_2, \mathscr{A}_2)$ be two SGs over $\mathscr{G}$. Then, $(\zeta, \mathscr{A}_3) = (\eta_1, \mathscr{A}_1)\hat{\circ}(\eta_2, \mathscr{A}_2)$ given in Definition 9, is a SG over $\mathscr{G}$ if and only if $\eta_1(\mathfrak{e})\cdot\eta_2(\mathfrak{e}) = \eta_2(\mathfrak{e})\cdot\eta_1(\mathfrak{e})$, for all $e \in \mathscr{A}_3$, where $\mathscr{A}_3 = \mathscr{A}_1 \cap \mathscr{A}_2 \neq \emptyset$. Moreover, $(\eta_1, \mathcal{A}_1)\hat{\circ}(\eta_1, \mathscr{A}_1)\widetilde{=}(\eta_1, \mathscr{A}_1)$.

In 2013, Feng et al. [18] have introduced the concept of S-BRs and then applied it to the notion of semigroups.

Definition 10 [18]: A SS $(\varrho, \mathcal{E})$ over $\mathscr{X} \times \mathscr{X}$ is called a S-BR. In other words, a S-BR is a parameterized family of the binary relations on $\mathscr{X}$.

Definition 11 [18]: Let $(\varrho, \mathscr{E})$ be a S-BR over $\mathscr{X}$. Then, $(\varrho, \mathscr{E})$ is called:

(1) soft reflexive, if $\varrho(\mathfrak{e})$ is reflexive, for all $\mathfrak{e} \in \mathscr{E}$.
(2) soft symmetric, if $\varrho(\mathfrak{e})$ is symmetric, for all $\mathfrak{e} \in \mathscr{E}$.
(3) soft transitive, if $\varrho(\mathfrak{e})$ is transitive, for all $\mathfrak{e} \in \mathscr{E}$.
(4) soft ER over $\mathscr{X}$, if $\varrho(e)$ is an ER on $\mathscr{X}$, for each $e \in \mathscr{E}$.

III. SOFT MULTI-GRANULATION ROUGH sets(SMGRSs)

The concept of SRSs have studied by many authors, such as [11], [19], [29], [30], [38], [39], [42], [43], [44], [45], [48], [49], [50], and [57]. Since BRs play a fundamental role in both pure and applied sciences. In this section, the concept of SRS is enhanced to a productive approach of SMGRSs employing two S-BRs over a common universe $\mathscr{U}$. In this regard, a pair of SRA-spaces are defined by exerting the after sets of both S-BRs. We signify $\mathscr{U}$ as a universal set and $\mathscr{E}$ a set of parameters all over this segment.

Definition 12: Let $(\varrho_1, \mathscr{E})$ and $(\varrho_2, \mathscr{E})$ be two S-BRs over $\mathscr{U}$. Then, $(\mathscr{U}, \varrho_1, \varrho_2, \mathscr{E})$ is called a SMGRA-space. Define a pair of SSs, namely the lower SRA-space $(\underline{\mathscr{X}}_{(\varrho_1,\varrho_2)}, \mathscr{E})$ and the upper SRA-space $(\overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}, \mathscr{E})$ for any subset $\mathscr{X}$ of $\mathscr{U}$ as follows:

$$\underline{\mathscr{X}}_{(\varrho_1,\varrho_2)}(\mathfrak{e}) = \left\{\begin{matrix} \mathfrak{u} \in \mathscr{U} : \emptyset \neq \mathfrak{u}\varrho_1(\mathfrak{e}) \subseteq \mathscr{X} \\ \text{or } \emptyset \neq \mathfrak{u}\varrho_2(\mathfrak{e}) \subseteq \mathscr{X} \end{matrix}\right\} \quad (10)$$
$$\overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{e}) = \left\{\begin{matrix} \mathfrak{u} \in \mathscr{U} : \mathfrak{u}\varrho_1(\mathfrak{e}) \cap \mathscr{X} \neq \emptyset \\ \text{and } \mathfrak{u}\varrho_2(\mathfrak{e}) \cap \mathscr{X} \neq \emptyset \end{matrix}\right\} \quad (11)$$

for all $\mathfrak{e} \in \mathscr{E}$, where

$$u\varrho_i(\mathfrak{e}) = \big\{\mathfrak{v} \in \mathscr{U} : (\mathfrak{u}, \mathfrak{v}) \in \varrho_i(\mathfrak{e})\big\} \quad (12)$$

called the after sets of $\varrho_i(\mathfrak{e})$.

Remark 1: (1) The notion of MGSRS proposed by Fan et al. [11] was defined by employing SS as an extension of SRSs proposed by Feng et al. [8]. Whereas our proposed model of SMGRS is defined by using two S-BRs as a generalization of the work of Kanwal and Shabir [42] and Li et al. [38] to MGRS. Thus, our concept of SMGRS is different and independent from [11]. There is no connection between the model proposed by Fan et al. [11] and our model of SMGRS.
(2) Moreover, the notion of MGRS based on multi soft relations defined by Shabir et al. [57] depend on two different universes, while our proposed SMGRS model is dependent on a single universe.

Remark 2: The Definition 12 can be extended to n S-BRs in the similar manners.

To illustrate the notion of SMGRSs over $\mathscr{U}$, let us consider an example.

Example 1: Let $\mathscr{U} = \{1, w, w^2, w^3\}$ and $\mathscr{E} = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. Define two S-BRs $(\varrho_1, \mathscr{E})$ and $(\varrho_2, \mathscr{E})$ over $\mathscr{U}$ as follows:

$$\varrho_1(\mathfrak{e}) = \begin{cases} \big\{(1, 1), (w, w^3), (w^3, w)\big\}, & \text{if } \mathfrak{e} = \mathfrak{p}_1 \\ \mathscr{I}, & \text{if } \mathfrak{e} = \mathfrak{p}_2, \end{cases}$$
$$\varrho_2(\mathfrak{e}) = \begin{cases} \big\{(w, w), (w^3, w^3)\big\}, & \text{if } \mathfrak{e} = \mathfrak{p}_1 \\ \big\{(1, w), (1, w^2)\big\}, & \text{if } \mathfrak{e} = \mathfrak{p}_2 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{E}$, where $\mathscr{I}$ denotes the identity relation on $\mathscr{U}$. From Definition 12, we obtain:

$$\underline{\emptyset}_{(\varrho_1,\varrho_2)}(\mathfrak{p}_1) = \{1, w^2\} \neq \emptyset,$$
$$\overline{\emptyset}^{(\varrho_1,\varrho_2)}(\mathfrak{p}_1) = \{w^2\} \neq \emptyset,$$
$$\overline{\mathscr{U}}^{(\varrho_1,\varrho_2)}(\mathfrak{p}_1) = \{w, w^3\} \neq \mathscr{U}.$$

Further, if we assume $\mathscr{X} = \{1, w, w^2\}$, then

$$\underline{\mathscr{X}}_{(\varrho_1,\varrho_2)}(\mathfrak{p}_1) = \{1, w, w^3\} \nsubseteq \mathscr{X} \text{ and}$$
$$\overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{p}_1) = \emptyset \nsupseteq \mathscr{X}.$$

In above example, we note that $\underline{\mathscr{X}}_{(\varrho_1,\varrho_2)}(\mathfrak{p}_1) \nsubseteq \mathscr{X} \nsubseteq \overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{p}_1)$, (see the next result).

Lemma 1: Let $(\varrho_i, \mathscr{E})$ be soft reflexive, for all $1 \leq i \leq 2$. Then,

$$\underline{\mathscr{X}}_{(\varrho_1,\varrho_2)}(\mathfrak{e}) \subseteq \mathscr{X} \subseteq \overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{e}), \text{ for all } \mathfrak{e} \in \mathscr{E}.$$

Proof: The proof is evident in the view of the Definition 12 and the given assumptions. ■

The following result can be easily deduced from the Definition 12, so its proof is omitted.

Proposition 1: Let $(\varrho_i, \mathscr{E})$ be S-BRs, $1 \leq i \leq 2$ and $\mathscr{X}$ be a non-empty subset of $\mathscr{U}$. Then,

(1) $(\underline{\mathscr{X}}_{(\varrho_1,\varrho_2)}, \mathscr{E}) \widetilde{=} (\underline{\mathscr{X}}_{\varrho_1}, \mathscr{E}) \cup_R (\underline{\mathscr{X}}_{\varrho_2}, \mathscr{E})$.

(2) $(\overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}, \mathscr{E}) \widetilde{=} (\overline{\mathscr{X}}^{\varrho_1}, \mathscr{E}) \Cap (\overline{\mathscr{X}}^{\varrho_2}, \mathscr{E})$.

Proof: Straightforward. ■

Some essential properties which may not hold for any arbitrary S-BR are proved in the following result.

Theorem 2: Let $(\varrho_i, \mathscr{E})$, $1 \leq i \leq 2$ be two S-BRs over $\mathscr{U}$ and $\mathscr{X}$ be a non-empty subset of $\mathscr{U}$. Then the following properties hold:

(1) If $(\varrho_i, \mathscr{E})$ are soft reflexive, $1 \leq i \leq 2$, then $\underline{\overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{e})}_{(\varrho_1,\varrho_2)}(\mathfrak{e}) \subseteq \overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{e})$, for all $\mathfrak{e} \in \mathscr{E}$.

(2) If $(\varrho_i, \mathscr{E})$ are soft symmetric and transitive, $1 \leq i \leq 2$, then the reverse inclusion of (1) is true.

(3) If $(\varrho_i, \mathscr{E})$ are soft reflexive, for all $1 \leq i \leq 2$, then $\overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{e}) \subseteq \overline{\overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{e})}^{(\varrho_1,\varrho_2)}(\mathfrak{e})$, for all $\mathfrak{e} \in \mathscr{E}$.

(4) If $(\varrho_i, \mathscr{E})$ are soft transitive, $1 \leq i \leq 2$, then the reverse inclusion of (3) is true.

(5) If $(\varrho_i, \mathscr{E})$ are soft reflexive, $1 \leq i \leq 2$, then $\underline{\underline{\mathscr{X}}_{(\varrho_1,\varrho_2)}(\mathfrak{e})}_{(\varrho_1,\varrho_2)}(\mathfrak{e}) \subseteq \underline{\mathscr{X}}_{(\varrho_1,\varrho_2)}(\mathfrak{e})$, for all $\mathfrak{e} \in \mathscr{E}$.

(6) If $(\varrho_i, \mathscr{E})$ are soft transitive, $1 \leq i \leq 2$, then the reverse inclusion of (5) is true.

(7) If $(\varrho_i, \mathscr{E})$ are soft reflexive, $1 \leq i \leq 2$, then $\underline{\mathscr{X}}_{(\varrho_1,\varrho_2)}(\mathfrak{e}) \subseteq \overline{\underline{\mathscr{X}}_{(\varrho_1,\varrho_2)}(\mathfrak{e})}^{(\varrho_1,\varrho_2)}(\mathfrak{e})$, for all $\mathfrak{e} \in \mathscr{E}$.

Proof: Since $(\varrho_i, \mathscr{E})$ are soft reflexive for $1 \leq i \leq 2$. Thus the proofs of the part (1), (3), (5), and (7) are direct consequences of Lemma 1.

(2) Suppose that $\mathfrak{t} \in \overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{e})$, where $\mathfrak{e} \in \mathscr{E}$. Then from Definition 12, we have $\mathfrak{a} \in \mathfrak{t}\varrho_1(\mathfrak{e}) \cap \mathscr{X}$ and $\mathfrak{b} \in \mathfrak{t}\varrho_2(\mathfrak{e}) \cap \mathscr{X}$, for some $\mathfrak{a}, \mathfrak{b} \in \mathscr{U}$. We claim that $\mathfrak{t}\varrho_1(\mathfrak{e}) \subseteq \overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{e})$ or $\mathfrak{t}\varrho_2(\mathfrak{e}) \subseteq \overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{e})$. So, let $\mathfrak{u} \in \mathfrak{t}\varrho_1(\mathfrak{e})$. Since $\mathfrak{a} \in \mathfrak{t}\varrho_1(\mathfrak{e})$. Using the assumptions on $\varrho_1(\mathfrak{e})$, we have $\mathfrak{a} \in \mathfrak{u}\varrho_1(\mathfrak{e})$. But $\mathfrak{a} \in \mathscr{X}$. Therefore, $\mathfrak{a} \in \mathfrak{u}\varrho_1(\mathfrak{e}) \cap \mathscr{X}$ or $\mathfrak{u} \in \overline{\mathscr{X}}^{\varrho_1}(\mathfrak{e})$. Similarly, $\mathfrak{u} \in \overline{\mathscr{X}}^{\varrho_2}(\mathfrak{e})$. Therefore, $\mathfrak{u} \in \overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{e})$ (see Proposition 1 (2)). This proves that $\mathfrak{t}\varrho_1(\mathfrak{e}) \subseteq \overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{e})$. Hence, $\mathfrak{t} \in \underline{\overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}}_{(\varrho_1,\varrho_2)}(\mathfrak{e})$.

(4) Assume that $\mathfrak{t} \in \overline{\overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}}^{(\varrho_1,\varrho_2)}(\mathfrak{e})$, $\mathfrak{e} \in \mathscr{E}$. Then by Definition 12, there exists $\mathfrak{u}, \mathfrak{v} \in \mathscr{U}$ such that $\mathfrak{u} \in \mathfrak{t}\varrho_1(\mathfrak{e}) \cap \overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{e})$ and $\mathfrak{v} \in \mathfrak{t}\varrho_2(\mathfrak{e}) \cap \overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{e})$. Therefore, $\mathfrak{a}_1 \in \mathfrak{u}\varrho_1(\mathfrak{e}) \cap \mathscr{X}$ and $\mathfrak{b}_2 \in \mathfrak{v}\varrho_2(\mathfrak{e}) \cap \mathscr{X}$ for some $\mathfrak{a}_1, \mathfrak{b}_2 \in \mathscr{U}$. Because $(\varrho_i, \mathscr{E})$, $1 = 1, 2$ are soft transitive, $\mathfrak{a}_1 \in \mathfrak{t}\varrho_1(\mathfrak{e})$ and $\mathfrak{b}_2 \in \mathfrak{t}\varrho_2(\mathfrak{e})$. Hence, $\mathfrak{a}_1 \in \mathfrak{t}\varrho_1(\mathfrak{e}) \cap \mathscr{X}$ and $\mathfrak{b}_2 \in \mathfrak{t}\varrho_2(\mathfrak{e}) \cap \mathscr{X}$. This proves that $\mathfrak{t} \in \overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{e})$.

(6) Let $\mathfrak{t} \in \underline{\mathscr{X}}_{(\varrho_1,\varrho_2)}(\mathfrak{e})$, where $\mathfrak{e} \in \mathscr{E}$. Then, $\mathfrak{t}\varrho_1(\mathfrak{e}) \subseteq \mathscr{X}$ or $\mathfrak{t}\varrho_2(\mathfrak{e}) \subseteq \mathscr{X}$. Assume that $\mathfrak{t}\varrho_1(\mathfrak{e}) \subseteq \mathscr{X}$. We claim that $\mathfrak{t}\varrho_1(\mathfrak{e}) \subseteq \underline{\mathscr{X}}_{(\varrho_1,\varrho_2)}(\mathfrak{e})$. Letting $\mathfrak{u} \in \mathfrak{t}\varrho_1(\mathfrak{e})$ and $\mathfrak{v} \in \mathfrak{u}\varrho_1(\mathfrak{e})$. By using transitivity of $(\varrho_1, \mathscr{E})$, we have $\mathfrak{v} \in \mathfrak{t}\varrho_1(\mathfrak{e}) \subseteq \mathscr{X}$. Hence, $\mathfrak{u}\varrho_1(\mathfrak{e}) \subseteq \mathscr{X}$. Therefore, $\mathfrak{u} \in \underline{\mathscr{X}}_{(\varrho_1,\varrho_2)}(\mathfrak{e})$. Thus, $\mathfrak{t}\varrho_1(\mathfrak{e}) \subseteq \underline{\mathscr{X}}_{(\varrho_1,\varrho_2)}(\mathfrak{e})$. This proves that $\mathfrak{t} \in \underline{\underline{\mathscr{X}}_{(\varrho_1,\varrho_2)}}_{(\varrho_1,\varrho_2)}(\mathfrak{e})$.

This completes the proof. ■

As an illustration of Theorem 2, we construct the following example.

Example 2: Let $\mathscr{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3\}$ and $\mathscr{E} = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. Define two S-BRs $(\varrho_1, \mathscr{E})$ and $(\varrho_2, \mathscr{E})$ over $\mathscr{U}$ as follows:

$$\varrho_1(\mathfrak{e}) = \begin{cases} \left\{ \begin{array}{l} (\mathfrak{u}_1,\mathfrak{u}_1),(\mathfrak{u}_2,\mathfrak{u}_2),(\mathfrak{u}_3,\mathfrak{u}_3), \\ (\mathfrak{u}_1,\mathfrak{u}_2),(\mathfrak{u}_1,\mathfrak{u}_3),(\mathfrak{u}_2,\mathfrak{u}_3) \end{array} \right\}, & \text{if } \mathfrak{e} = \mathfrak{p}_1 \\ \left\{ \begin{array}{l} (\mathfrak{u}_1,\mathfrak{u}_1),(\mathfrak{u}_2,\mathfrak{u}_2),(\mathfrak{u}_3,\mathfrak{u}_3), \\ (\mathfrak{u}_1,\mathfrak{u}_2),(\mathfrak{u}_1,\mathfrak{u}_3),(\mathfrak{u}_2,\mathfrak{u}_1) \end{array} \right\}, & \text{if } \mathfrak{e} = \mathfrak{p}_2 \end{cases}$$

$$\varrho_2(\mathfrak{e}) = \begin{cases} \mathscr{U} \times \mathscr{U}, & \text{if } \mathfrak{e} = \mathfrak{p}_1 \\ \left\{ \begin{array}{c} (\mathfrak{u}_1,\mathfrak{u}_1),(\mathfrak{u}_2,\mathfrak{u}_2),(\mathfrak{u}_3,\mathfrak{u}_3), \\ (\mathfrak{u}_1,\mathfrak{u}_3),(\mathfrak{u}_3,\mathfrak{u}_1) \end{array} \right\}, & \text{if } \mathfrak{e} = \mathfrak{p}_2 \end{cases}$$

for all $\mathfrak{e} \in \mathcal{E}$. It can be easily checked that $(\varrho_1, \mathcal{E})$ is not soft transitive. Let $\mathscr{X} = \{\mathfrak{u}_1, \mathfrak{u}_3\}$. By simple computations, we get:

$$\overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{p}_2) = \{\mathfrak{u}_1, \mathfrak{u}_3\},$$
$$\overline{\overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{p}_2)}^{(\varrho_1,\varrho_2)}(\mathfrak{p}_1) = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3\},$$

which shows that $\overline{\overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{p}_2)}^{(\varrho_1,\varrho_2)}(\mathfrak{p}_1) \nsubseteq \overline{\mathscr{X}}^{(\varrho_1,\varrho_2)}(\mathfrak{p}_2)$. Hence, the inclusion in part of (4) of Theorem 2 does not generally hold. Similar calculations can be done for other assertions.

IV. APPLICATIONS OF SMGRSs IN GROUPS

In literature, roughness in algebraic structures, especially in groups, has been studied by various scholars (see [22], [23], [24], [29], [30], and [46]). In 2020, Ayub et al. [30] applied the notion of SRSs on groups using a NSG and established a relationship among lower and upper SRA-spaces. In this Segment, the idea of SMGRSs is implemented in group theory using two NSGs. Some fundamental properties of lower and upper SRA-spaces over groups are studied in detail with corroborative examples. In the sequel of this paper, we assume that $\mathscr{G}$ is a multiplicative group with an identity element $1_{\mathscr{G}}$, as a universal set and $\mathscr{E}$ as a set of its attributes.

Definition 13: Let $(\eta_1, \mathscr{E})$ and $(\eta_2, \mathscr{E})$ be two NSGs over $\mathscr{G}$. Then, SMGRA-space for $\mathscr{G}$ is an object of the form $(\mathscr{G}, \eta_1, \eta_2, \mathscr{E})$. For any $\mathscr{X} \subseteq \mathscr{G}$, define the lower SRA-space $(\underline{\mathscr{X}}_{(\eta_1,\eta_2)}, \mathscr{E})$ and the upper SRA-space $(\overline{\mathscr{X}}^{(\eta_1,\eta_2)}, \mathscr{E})$

as follows:

$$\underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e}) = \left\{\begin{matrix} \mathfrak{u} \in \mathscr{G} : \mathfrak{u}\eta_1(\mathfrak{e}) \subseteq \mathscr{X} \\ \text{or } \mathfrak{u}\eta_2(\mathfrak{e}) \subseteq \mathscr{X} \end{matrix}\right\} \quad (13)$$

$$\overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}) = \left\{\begin{matrix} \mathfrak{u} \in \mathscr{G} : \mathfrak{u}\eta_1(\mathfrak{e}) \cap \mathscr{X} \neq \emptyset \\ \text{and } \mathfrak{u}\eta_2(\mathfrak{e}) \cap \mathscr{X} \neq \emptyset \end{matrix}\right\} \quad (14)$$

where $\mathfrak{u}\eta_i(\mathfrak{e})$ represents the cosets of the normal subgroups $\eta_i(\mathfrak{e})$, for each parameter $\mathfrak{e} \in \mathscr{E}$.

Remark 3: It is important to note that if $(\eta_1, \mathscr{E}) \widetilde{=} (\eta_2, \mathscr{E})$, then the Definition 13 degenerates into the Definition 3.1 given in Ayub et al. [30].

Theorem 3: Let $(\eta_1, \mathscr{E})$ and $(\eta_2, \mathscr{E})$ be two NSGs over $\mathscr{G}$. Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are any two subsets of $\mathscr{G}$. Then the following properties hold:

(1) $\big(\underline{(\mathscr{X} \cap \mathscr{Y})}_{(\eta_1,\eta_2)}, \mathscr{E}\big) \widetilde{=} \big[(\underline{\mathscr{X}}_{\eta_1}, \mathscr{E}) \Cap (\underline{\mathscr{Y}}_{\eta_1}, \mathscr{E})\big] \cup_R \big[(\underline{\mathscr{X}}_{\eta_2}, \mathscr{E}) \Cap (\underline{\mathscr{Y}}_{\eta_2}, \mathscr{E})\big]$;

(2) $\big(\overline{(\mathscr{X} \cup \mathscr{Y})}^{(\eta_1,\eta_2)}, \mathscr{E}\big) \widetilde{=} \big[(\overline{\mathscr{X}}^{\eta_1}, \mathscr{E}) \ cup_R (\overline{\mathscr{Y}}^{\eta_1}, \mathscr{E})\big] \Cap \big[(\overline{\mathscr{X}}^{\eta_2}, \mathscr{E}) \cup_R (\overline{\mathscr{Y}}^{\eta_2}, \mathscr{E})\big]$;

(3) $\big(\underline{(\mathscr{X} \cap \mathscr{Y})}_{(\eta_1,\eta_2)}, \mathscr{E}\big) \widetilde{\subseteq} (\underline{\mathscr{X}}_{(\eta_1,\eta_2)}, \mathscr{E}) \Cap (\underline{\mathscr{Y}}_{(\eta_1,\eta_2)}, \mathscr{E})$;

(4) $\big(\overline{(\mathscr{X} \cup \mathscr{Y})}^{(\eta_1,\eta_2)}, \mathscr{E}\big) \widetilde{\supseteq} (\overline{\mathscr{X}}^{(\eta_1,\eta_2)}, \mathscr{E}) \cup_R (\overline{\mathscr{Y}}^{(\eta_1,\eta_2)}, \mathscr{E})$;

(5) $\mathscr{X} \subseteq \mathscr{Y} \Longrightarrow (\underline{\mathscr{X}}_{(\eta_1,\eta_2)}, \mathscr{E}) \widetilde{\subseteq} (\underline{\mathscr{Y}}_{(\eta_1,\eta_2)}, \mathscr{E})$;

(6) $\mathscr{X} \subseteq \mathscr{Y} \Longrightarrow (\overline{\mathscr{X}}^{(\eta_1,\eta_2)}, \mathscr{E}) \widetilde{\subseteq} (\overline{\mathscr{Y}}^{(\eta_1,\eta_2)}, \mathscr{E})$;

(7) $\big(\underline{(\mathscr{X} \cup \mathscr{Y})}_{(\eta_1,\eta_2)}, \mathscr{E}\big) \widetilde{\supseteq} (\underline{\mathscr{X}}_{(\eta_1,\eta_2)}, \mathscr{E}) \cup_R (\underline{\mathscr{Y}}_{(\eta_1,\eta_2)}, \mathscr{E})$;

(8) $\big(\overline{(\mathscr{X} \cap \mathscr{Y})}^{(\eta_1,\eta_2)}, \mathscr{E}\big) \widetilde{\subseteq} (\overline{\mathscr{X}}^{(\eta_1,\eta_2)}, \mathscr{E}) \Cap (\overline{\mathscr{Y}}^{(\eta_1,\eta_2)}, \mathscr{E})$;

where $\Cap$ and $\cup_R$ represents the restricted intersection and restricted union of two SSs as given in Definition 5.

Proof: It can be directly obtained by Definition 12, Proposition 1, Definition 3.1 of [30] and Theorem 4.15 (5) of [30]. ■

Theorem 4: With the same assumptions as in Theorem 3, the following statement hold:

$$(\overline{\mathscr{X}}^{(\eta_1,\eta_2)}, \mathscr{E})\hat{\circ}(\overline{\mathscr{Y}}^{(\eta_1,\eta_2)}, \mathscr{E}) \widetilde{\subseteq} (\overline{\mathscr{X}\mathscr{Y}}^{(\eta_1,\eta_2)}, \mathscr{E}).$$

Proof: Let $\mathfrak{x} = \mathfrak{y}\mathfrak{z}$, for some $\mathfrak{y} \in \overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e})$ and $\mathfrak{z} \in \overline{\mathscr{Y}}^{(\eta_1,\eta_2)}(\mathfrak{e})$, where $\mathfrak{e} \in \mathscr{E}$. Then, there exists $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2 \in \mathscr{G}$ such that $\mathfrak{a}_1 \in \mathfrak{y}\eta_1(\mathfrak{e}) \cap \mathscr{X}$ and $\mathfrak{a}_2 \in \mathfrak{y}\eta_2(\mathfrak{e}) \cap \mathscr{X}$, $\mathfrak{b}_1 \in \mathfrak{z}\eta_1(\mathfrak{e}) \cap \mathscr{Y}$ and $\mathfrak{b}_2 \in \mathfrak{z}\eta_2(\mathfrak{e}) \cap \mathscr{Y}$. It implies that $\mathfrak{a}_1\mathfrak{b}_1 \in \mathfrak{y}\eta_1(\mathfrak{e}) \cdot \mathfrak{z}\eta_1(\mathfrak{e}) \cap \mathscr{X}\mathscr{Y}$ and $\mathfrak{a}_2\mathfrak{b}_2 \in \mathfrak{y}\eta_2(\mathfrak{e}) \cdot \mathfrak{z}\eta_2(\mathfrak{e}) \cap \mathscr{X}\mathscr{Y}$. Since $(\eta_i, \mathscr{E})$ are NSGs, for all $i = 1, 2$, we have $\mathfrak{a}_1\mathfrak{b}_1 \in (\mathfrak{y}\mathfrak{z})\eta_1(\mathfrak{e}) \cap \mathscr{X}\mathscr{Y}$ and $\mathfrak{a}_2\mathfrak{b}_2 \in (\mathfrak{y}\mathfrak{z})\eta_2(\mathfrak{e}) \cap \mathscr{X}\mathscr{Y}$. This proves that $\mathfrak{x} = \mathfrak{y}\mathfrak{z} \in \overline{\mathscr{X}\mathscr{Y}}^{(\eta_1,\eta_2)}(\mathfrak{e})$. ■

It is important to note that Theorem 4 is inconsistent with the part (1) of Proposition 3.5 given in [30], which is illustrated by the following example.

Example 3: Let $\mathscr{G} = \mathscr{S}_3$ and $\mathscr{E} = \{\mathfrak{e}_1, \mathfrak{e}_2\}$. Define two NSGs $(\eta_i, \mathscr{E})$, $i = 1, 2$ by the maps $\eta_i : \mathscr{E} \to \mathcal{P}(\mathscr{G})$ defined as follows:

$$\eta_1(\mathfrak{e}) = \begin{cases} \mathscr{A}_3, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{1_{\mathscr{G}}, (12)\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2, \end{cases}$$

$$\eta_2(\mathfrak{e}) = \begin{cases} \{1_{\mathscr{G}}, (23)\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{1_{\mathscr{G}}, (13)\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{E}$. Let $\mathscr{X} = \{1_{\mathscr{G}}, (12), (13)\}$ and $\mathscr{Y} = \{(123)\}$, then $\mathscr{X}\mathscr{Y} = \{(12), (23), (123)\}$. In the light of Definition 13, we obtain:

$$\overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}_2) = \{1_{\mathscr{G}}, (12), (13)\},$$
$$\overline{\mathscr{Y}}^{(\eta_1,\eta_2)}(\mathfrak{e}_2) = \{(123)\},$$
$$\overline{\mathscr{X}\mathscr{Y}}^{(\eta_1,\eta_2)}(\mathfrak{e}_2) = \{(12), (23), (123), (132)\},$$
$$\overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}_2) \cdot \overline{\mathscr{Y}}^{(\eta_1,\eta_2)}(\mathfrak{e}_2) = \{(12), (23), (123)\}.$$

This demonsrates that $\overline{\mathscr{X}\mathscr{Y}}^{(\eta_1,\eta_2)}(\mathfrak{e}_2) \nsubseteq \overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}_2) \cdot \overline{\mathscr{Y}}^{(\eta_1,\eta_2)}(\mathfrak{e}_2)$.

Theorem 5: With the same assumptions as in Theorem 4, the following statement hold:

$$(\underline{\mathscr{X}}_{(\eta_1,\eta_2)}, \mathscr{E})\hat{\circ}(\underline{\mathscr{Y}}_{(\eta_1,\eta_2)}, \mathscr{E}) \widetilde{\subseteq} (\underline{\mathscr{X}\mathscr{Y}}_{(\eta_1,\eta_2)}, \mathscr{E}).$$

Proof: Let $e \in \mathscr{E}$. Then,

$$\begin{aligned}
&\underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e}) \cdot \underline{\mathscr{Y}}_{(\eta_1,\eta_2)}(\mathfrak{e}) \\
&= \big(\underline{\mathscr{X}}_{\eta_1}(\mathfrak{e}) \cup \underline{\mathscr{X}}_{\eta_2}(\mathfrak{e})\big) \cdot \big(\underline{\mathscr{Y}}_{\eta_1}(\mathfrak{e}) \cup \underline{\mathscr{Y}}_{\eta_2}(\mathfrak{e})\big), \\
&\qquad \text{by part(1) Theorem 3} \\
&= \big(\underline{\mathscr{X}}_{\eta_1}(\mathfrak{e}) \cdot \underline{\mathscr{Y}}_{\eta_1}(\mathfrak{e})\big) \cup \big(\underline{\mathscr{X}}_{\eta_1}(\mathfrak{e}) \cdot \underline{\mathscr{Y}}_{\eta_2}(\mathfrak{e})\big) \cup \big(\underline{\mathscr{X}}_{\eta_2}(\mathfrak{e}) \cdot \underline{\mathscr{Y}}_{\eta_1}(\mathfrak{e})\big) \cup \big(\underline{\mathscr{X}}_{\eta_2}(\mathfrak{e}) \cdot \underline{\mathscr{Y}}_{\eta_2}(\mathfrak{e})\big) \\
&\supseteq \big(\underline{\mathscr{X}}_{\eta_1}(\mathfrak{e}) \cdot \underline{\mathscr{Y}}_{\eta_1}(\mathfrak{e})\big) \cup \big(\underline{\mathscr{X}}_{\eta_2}(\mathfrak{e}) \cdot \underline{\mathscr{Y}}_{\eta_2}(\mathfrak{e})\big) \\
&\supseteq \underline{\mathscr{X}\mathscr{Y}}_{\eta_1}(\mathfrak{e}) \cup \underline{\mathscr{X}\mathscr{Y}}_{\eta_2}(\mathfrak{e}), \\
&\qquad \text{by part (2) Proposition 3.8 of [30]} \\
&= \underline{\mathscr{X}\mathscr{Y}}_{(\eta_1,\eta_2)}(\mathfrak{e}).
\end{aligned}$$

This completes the proof. ■

As an illustration of the above theorem, we consider the following example.

Example 4: Let $\mathscr{G} = \mathcal{S}_3$ and $\mathscr{E} = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. Define two NSGs $(\eta_i, \mathscr{E})$, $i = 1, 2$ by the following maps $\eta_i : \mathscr{E} \to \mathcal{P}(\mathscr{G})$, where

$$\eta_1(\mathfrak{e}) = \begin{cases} \{1_{\mathscr{G}}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \mathscr{A}_3, & \text{if } \mathfrak{e} = \mathfrak{e}_2, \end{cases}$$

$$\eta_2(\mathfrak{e}) = \begin{cases} \mathscr{A}_3, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \mathscr{S}_3, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{E}$. Suppose that $\mathscr{X} = \{(12), (13), (23)\}$ and $\mathscr{Y} = \{(123), (132)\}$. Then, $\mathscr{X}\mathscr{Y} = \mathscr{X}$. According to Definition 13, we can easily get the following:

$$\underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e}_2) = \mathscr{X} = \underline{\mathscr{X}\mathscr{Y}}_{(\eta_1,\eta_2)}(\mathfrak{e}_2),$$
$$\underline{\mathscr{Y}}_{(\eta_1,\eta_2)}(\mathfrak{e}_2) = \emptyset,$$
$$\underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e}_2) \cdot \underline{\mathscr{Y}}_{(\eta_1,\eta_2)}(\mathfrak{e}_2) = \emptyset.$$

Clearly, $\underline{\mathscr{X}\mathscr{Y}}_{(\eta_1,\eta_2)}(\mathfrak{e}_2) \nsubseteq \underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e}_2) \cdot \underline{\mathscr{Y}}_{(\eta_1,\eta_2)}(\mathfrak{e}_2)$.

The following result is significant and infers us that the SRA-spaces of a subgroup do not yield any other information.

Theorem 6: Let $(\eta_1, \mathscr{E})$ and $(\eta_2, \mathscr{E})$ be two NSGs over $\mathscr{G}$. If $\mathscr{X}$ is a subgroup of $\mathscr{G}$ such that $\underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e}) \neq \emptyset$, for all $\mathfrak{e} \in \mathscr{E}$. Then, we have

$$\underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e}) = \mathscr{X} = \overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e})$$

for each parameter $\mathfrak{e} \in \mathscr{E}$.

Proof: According to Lemma 1, it follows that

$$\underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e}) \subseteq \mathscr{X} \subseteq \overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}) \text{ for each } \mathfrak{e} \in \mathscr{E}. \quad (15)$$

Now to prove $\overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}) \subseteq \underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e})$, we claim that $1_{\mathscr{G}} \in \underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e})$. Since, $\underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e}) \neq \emptyset$, there exists $\mathfrak{t} \in \underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e})$. By Equation 15, $\mathfrak{t} \in \mathscr{X}$. Therefore,

$$1_{\mathscr{G}} \cdot \eta_1(\mathfrak{e}) = \mathfrak{t}^{-1} \cdot \mathfrak{t}\eta_1(\mathfrak{e}) \subseteq \mathscr{X} \cdot \subseteq \mathscr{X}$$

Hence, $1_{\mathscr{G}} \in \underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e})$. Now let $\mathfrak{t} \in \overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e})$, $\mathfrak{e} \in \mathscr{E}$. There exists $\mathfrak{x} \in \mathfrak{t}\eta_1(\mathfrak{e}) \cap \mathscr{X}$ and $\mathfrak{y} \in \mathfrak{t}\eta_2(\mathfrak{e}) \cap \mathscr{X}$. Since, $1_{\mathscr{G}} \in \underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e})$, so $\eta_1(\mathfrak{e}) \subseteq \mathscr{X}$ or $\eta_2(\mathfrak{e}) \subseteq \mathscr{X}$. If $\eta_1(\mathfrak{e}) \subseteq \mathscr{X}$, then $\mathfrak{t}\eta_1(\mathfrak{e}) \subseteq \mathfrak{t}.\mathscr{X} = \mathscr{X}$. Hence, $\mathfrak{t}\eta_1(\mathfrak{e}) \subseteq \mathscr{X}$. Similarly, if $\eta_2(\mathfrak{e}) \subseteq \mathscr{X}$, then $\mathfrak{t}\eta_2(\mathfrak{e}) \subseteq \mathscr{X}$. Thus, $\mathfrak{t} \in \underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e})$. Therefore, $\overline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e}) \subseteq \underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e})$.
This completes the proof. ∎

The converse of Theorem 6 may not be true, which can be justified by the following example.

Example 5: Let $\mathscr{G} = \mathcal{S}_3$ and $\mathscr{E} = \{\mathfrak{e}_1, \mathfrak{e}_2\}$. Define two NSGs $(\eta_i, \mathscr{E})$, $i = 1, 2$ as follows:

$$\eta_1(\mathfrak{e}) = \{1_{\mathscr{G}}\},$$
$$\eta_2(\mathfrak{e}) = \begin{cases} \{1_{\mathscr{G}}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \mathcal{S}_3, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{E}$. Let $\mathscr{X} = \{1_{\mathscr{G}}, (123), (132), (12), (13)\}$. Then,

$$\underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e}) = \overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}) = \mathscr{X} \text{ for all } \mathfrak{e} \in \mathscr{E},$$
$$\underline{\mathscr{X}}_{(\eta_1,\eta_2)}(\mathfrak{e}) \neq \emptyset \text{ for all } \mathfrak{e} \in \mathscr{E}.$$

But $\mathscr{X}$ is not a subgroup of $\mathscr{G}$.
In the following result, an essential characterization of the upper SRA-space is given.

Lemma 2: Let $(\eta_1, \mathscr{E})$ and $(\eta_2, \mathscr{E})$ be two NSGs over $\mathscr{G}$. Then, for any non-empty subset $\mathscr{X}$ of $\mathscr{G}$, the following property hold:

$$\overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}) = \mathscr{X} \cdot \eta_1(\mathfrak{e}) \cap \mathscr{X} \cdot \eta_2(\mathfrak{e})$$

for all $\mathfrak{e} \in \mathscr{E}$.

Proof: Let $\mathfrak{x} \in \overline{X}^{(\eta_1,\eta_2)}(\mathfrak{e})$, $\mathfrak{e} \in \mathscr{E}$. By Definition 13, $\mathfrak{a} \in \mathfrak{x}\eta_1(\mathfrak{e}) \cap \mathscr{X}$ and $\mathfrak{b} \in \mathfrak{x}\eta_2(\mathfrak{e}) \cap \mathscr{X}$. Since $(\eta_i, \mathscr{E})$ are NSGs, for all $i = 1, 2$, we have $\mathfrak{x} \in \mathfrak{a}\eta_1(\mathfrak{e}) \subseteq \mathscr{X}\eta_1(\mathfrak{e})$ and $\mathfrak{x} \in \mathfrak{b}\eta_2(\mathfrak{e}) \subseteq \mathscr{X}\eta_2(\mathfrak{e})$. Hence, $\mathfrak{x} \in \mathscr{X}\eta_1(\mathfrak{e}) \cap \mathscr{X}\eta_2(\mathfrak{e})$.

Conversely, assume that $\mathfrak{x} \in \mathscr{X}\eta_1(\mathfrak{e}) \cap \mathscr{X}\eta_2(\mathfrak{e})$. Then, $\mathfrak{x} = \mathfrak{a}\mathfrak{t}_1$ and $\mathfrak{x} = \mathfrak{b}\mathfrak{t}_2$, for some $\mathfrak{a}, \mathfrak{b} \in \mathscr{X}$, $\mathfrak{t}_1 \in \eta_1(\mathfrak{e})$ and $\mathfrak{t}_2 \in \eta_2(\mathfrak{e})$. It follows that $\mathfrak{a} = \mathfrak{x}\mathfrak{t}_1^{-1} \in \mathfrak{x}\eta_1(\mathfrak{e}) \cap \mathscr{X}$ and $\mathfrak{b} = \mathfrak{x}\mathfrak{t}_2^{-1} \in \mathfrak{x}\eta_2(\mathfrak{e}) \cap \mathscr{X}$. This proves that $\mathfrak{x} \in \overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e})$. ∎

Theorem 7: Let $(\eta_i, \mathscr{E})$ and $(\theta_i, \mathscr{E})$ be NSGs over $\mathscr{G}$, for all $i = 1, 2$. Suppose that $\mathscr{X}$ is any non-empty subset of $\mathscr{G}$ and $1_{\mathscr{G}} \in \mathscr{X}$. Then

$$(\overline{\mathscr{X}}^{(\eta_1,\eta_2)}, \mathscr{E})\hat{\circ}(\overline{\mathscr{X}}^{(\theta_1,\theta_2)}, \mathscr{E})\tilde{\subseteq}(\overline{\mathscr{X}}^{(\zeta_1,\zeta_2)}, \mathscr{E}),$$

where $(\zeta_i, \mathscr{E}) = (\eta_i, \mathscr{E})\hat{\circ}(\theta_i, \mathscr{E})$ such that $\zeta_i(\mathfrak{e}) = \eta_i(\mathfrak{e}) \cdot \theta_i(\mathfrak{e})$, for all $\mathfrak{e} \in \mathscr{E}$ and $i = 1, 2$ (see Definition 9).

Proof: Let $\mathfrak{e} \in \mathscr{E}$. Then,

$$\begin{aligned}
&\overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}) \cdot \overline{\mathscr{X}}^{(\theta_1,\theta_2)}(\mathfrak{e}) \\
&= \big(\mathscr{X}\eta_1(\mathfrak{e}) \cap \mathscr{X}\eta_2(\mathfrak{e})\big) \\
&\quad \cdot \big(\mathscr{X}\theta_1(\mathfrak{e}) \cap \mathscr{X}\theta_2(\mathfrak{e})\big), \text{ by Lemma 2} \\
&\subseteq \big(\mathscr{X}\eta_1(\mathfrak{e}) \cdot \mathscr{X}\theta_1(\mathfrak{e})\big) \\
&\quad \cap \big(\mathscr{X}\eta_1(\mathfrak{e}) \cdot \mathscr{X}\theta_2(\mathfrak{e})\big) \\
&\quad \cap \big(\mathscr{X}\eta_2(\mathfrak{e}) \cdot \mathscr{X}\theta_1(\mathfrak{e})\big) \\
&\quad \cap \big(\mathscr{X}\eta_2(\mathfrak{e}) \cdot \mathscr{X}\theta_2(\mathfrak{e})\big) \\
&= \big(\mathscr{X}(\eta_1(\mathfrak{e}) \cdot \mathscr{X})\theta_1(\mathfrak{e})\big) \\
&\quad \cap \big(\mathscr{X}(\eta_1(\mathfrak{e}) \cdot \mathscr{X})\theta_2(\mathfrak{e})\big) \\
&\quad \cap \big(\mathscr{X}(\eta_2(\mathfrak{e}) \cdot \mathscr{X})\theta_1(\mathfrak{e})\big) \\
&\quad \cap \big(\mathscr{X}(\eta_2(\mathfrak{e}) \cdot \mathscr{X})\theta_2(\mathfrak{e})\big) \\
&= \big(\mathscr{X}(\mathscr{X} \cdot \eta_1(\mathfrak{e}))\theta_1(\mathfrak{e})\big) \\
&\quad \cap \big(\mathscr{X}(\mathscr{X} \cdot \eta_1(\mathfrak{e}))\theta_2(\mathfrak{e})\big) \\
&\quad \cap \big(\mathscr{X}(\mathscr{X} \cdot \eta_2(\mathfrak{e}))\theta_1(\mathfrak{e})\big) \\
&\quad \cap \big(\mathscr{X}(\mathscr{X} \cdot \eta_2(\mathfrak{e}))\theta_2(\mathfrak{e})\big) \\
&= \big((\mathscr{X}\mathscr{X})\eta_1(\mathfrak{e}) \cdot \theta_1(\mathfrak{e})\big) \\
&\quad \cap \big((\mathscr{X}\mathscr{X})\eta_1(\mathfrak{e}) \cdot \theta_2(\mathfrak{e})\big) \\
&\quad \cap \big((\mathscr{X}\mathscr{X})\eta_2(\mathfrak{e}) \cdot \theta_1(\mathfrak{e})\big) \\
&\quad \cap \big((\mathscr{X}\mathscr{X})\eta_2(\mathfrak{e}) \cdot \theta_2(\mathfrak{e})\big) \\
&\subseteq \big((\mathscr{X}\mathscr{X})\eta_1(\mathfrak{e}) \cdot \theta_1(\mathfrak{e})\big) \\
&\quad \cap \big((\mathscr{X}\mathscr{X})\eta_2(\mathfrak{e}) \cdot \theta_2(\mathfrak{e})\big) \\
&\subseteq \big(\mathscr{X}\eta_1(\mathfrak{e}) \cdot \theta_1(\mathfrak{e})\big) \\
&\quad \cap \big(\mathscr{X}\eta_2(\mathfrak{e}) \cdot \theta_2(\mathfrak{e})\big), \text{ by Hypothesis} \\
&= \mathscr{X}\zeta_1(e) \cap \mathscr{X}\zeta_2(e), \text{ by Definition 9} \\
&= \overline{\mathscr{X}}^{(\zeta_1,\zeta_2)}(e), \text{ by Lemma 2 .}
\end{aligned}$$

Hence, the proof is complete. ∎

In the following example, it is shown that the inclusion in Theorem 7 might be strict.

Example 6: Consider a group of matrices $\mathscr{G} = \{\mathscr{I}, \mathscr{A}, \mathscr{B}, \mathscr{C}\}$, where

$$\mathscr{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\mathscr{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$\mathscr{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\mathscr{C} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

with the following multiplication table:

$\cdot$	$\mathscr{I}$	$\mathscr{A}$	$\mathscr{B}$	$\mathscr{C}$
$\mathscr{I}$	$\mathscr{I}$	$\mathscr{A}$	$\mathscr{B}$	$\mathscr{C}$
$\mathscr{A}$	$\mathscr{A}$	$\mathscr{I}$	$\mathscr{C}$	$\mathscr{B}$
$\mathscr{B}$	$\mathscr{B}$	$\mathscr{C}$	$\mathscr{I}$	$\mathscr{A}$
$\mathscr{C}$	$\mathscr{C}$	$\mathscr{B}$	$\mathscr{A}$	$\mathscr{I}$

Let $\mathscr{E} = \{\mathfrak{e}_1, \mathfrak{e}_2\}$. Define the NSGs $(\eta_i, \mathscr{E})$ and $(\theta_i, \mathscr{E})$, for $i = 1, 2$ characterized by the mappings $\eta_i : \mathscr{E} \to \mathcal{P}(\mathscr{G})$ and $\theta_i : \mathscr{E} \to \mathcal{P}(\mathscr{G})$, $i = 1, 2$ as follows:

$$\eta_1(\mathfrak{e}) = \begin{cases} \{\mathscr{I}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{\mathscr{I}, \mathscr{A}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2, \end{cases}$$

$$\eta_2(\mathfrak{e}) = \begin{cases} \{\mathscr{I}, \mathscr{A}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{\mathscr{I}, \mathscr{C}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2, \end{cases}$$

$$\theta_1(\mathfrak{e}) = \begin{cases} \{\mathscr{I}, \mathscr{A}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{\mathscr{I}, \mathscr{C}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2, \end{cases}$$

$$\theta_2(\mathfrak{e}) = \begin{cases} \{\mathscr{I}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{\mathscr{I}, \mathscr{A}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{E}$. Using the Definition 9, we obtain two NSGs $(\zeta_i, \mathscr{E})$, where $\zeta_i(\mathfrak{e}) = \eta_i(\mathfrak{e}) \cdot \theta_i(\mathfrak{e})$, for all $\mathfrak{e} \in \mathscr{E}$ and $i = 1, 2$ given as follows:

$$\zeta_1(\mathfrak{e}) = \zeta_2(\mathfrak{e}) = \begin{cases} \{\mathscr{I}, \mathscr{A}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \mathscr{G}, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{E}$. Let $\mathscr{X} = \{\mathscr{A}, \mathscr{B}\}$. Then,

$$\overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}_1) = \overline{\mathscr{X}}^{(\theta_1,\theta_2)}(\mathfrak{e}_1) = \{\mathscr{A}, \mathscr{B}\},$$
$$\overline{\mathscr{X}}^{(\zeta_1,\zeta_2)}(\mathfrak{e}_1) = \mathscr{G}.$$

Since $\{\mathscr{A}, \mathscr{B}\}.\{\mathscr{A}, \mathscr{B}\} = \{\mathscr{I}, \mathscr{C}\} \subsetneq \mathscr{G}$, therefore, $\big(\overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}_1)\big) \cdot \big(\overline{\mathscr{X}}^{(\theta_1,\theta_2)}(\mathfrak{e}_1)\big) \not\supseteq \overline{\mathscr{X}}^{(\zeta_1,\zeta_2)}(\mathfrak{e}_1)$.

Theorem 8: Let $(\eta_i, \mathscr{E})$ and $(\theta_i, \mathscr{E})$ be NSGs over $\mathscr{G}$, $1 \leq i \leq 2$. Then, for a non-empty subset $\mathscr{X}$ of $\mathscr{G}$, we have:

$$(\overline{\mathscr{X}}^{(\eta_1,\eta_2)}, \mathscr{E})\hat{\circ}(\theta_i, \mathscr{E}) \Cap (\eta_j, \mathscr{E})\hat{\circ}(\overline{\mathscr{X}}^{(\theta_1,\theta_2)}, \mathscr{E}) \,\widetilde{\subseteq}(\overline{\mathscr{X}}^{(\zeta_1,\zeta_2)}, \mathscr{E}),$$

for all $i, j = 1, 2$, where $(\zeta_i, \mathscr{E}) = (\eta_i, \mathscr{E})\hat{\circ}(\theta_i, \mathscr{E})$ such that $\zeta_i(\mathfrak{e}) = \eta_i(\mathfrak{e}) \cdot \theta_i(\mathfrak{e})$, for all $\mathfrak{e} \in \mathscr{E}$ (see Definition 9).

Proof: Let $\mathfrak{e} \in \mathscr{E}$. Then,

$$\begin{aligned} \overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}) \cdot \theta_1(\mathfrak{e}) &= \big(\mathscr{X}\eta_1(\mathfrak{e}) \cap \mathscr{X}\eta_2(\mathfrak{e})\big) \cdot \theta_1(\mathfrak{e}), \quad \text{by Lemma 2} \\ &\subseteq \mathscr{X}\big(\eta_1(\mathfrak{e}) \cdot \theta_1(\mathfrak{e})\big) \cap \mathscr{X}\big(\eta_2(\mathfrak{e}) \cdot \theta_1(\mathfrak{e})\big) \\ &\subseteq \mathscr{X}\big(\eta_1(\mathfrak{e}) \cdot \theta_1(\mathfrak{e})\big) \\ &= \mathscr{X}\zeta_1(\mathfrak{e}), \quad \text{by Definition 9.} \end{aligned}$$

Similarly, $\overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}) \cdot \theta_2(\mathfrak{e}) \subseteq \mathscr{X}\zeta_2(\mathfrak{e})$, $\eta_2(\mathfrak{e}) \cdot \overline{\mathscr{X}}^{(\theta_1,\theta_2)}(\mathfrak{e}) \subseteq \mathscr{X}\zeta_2(\mathfrak{e})$ and $\eta_1(\mathfrak{e}) \cdot \overline{\mathscr{X}}^{(\theta_1,\theta_2)}(\mathfrak{e}) \subseteq \mathscr{X}\zeta_1(\mathfrak{e})$. Thus,

$$\begin{aligned} \overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}) \cdot \theta_1(\mathfrak{e}) \cap \eta_2(\mathfrak{e}) \cdot \overline{\mathscr{X}}^{(\theta_1,\theta_2)}(\mathfrak{e}) &\subseteq \mathscr{X}\zeta_1(\mathfrak{e}) \cap \mathscr{X}\zeta_2(\mathfrak{e}) \\ &= \overline{\mathscr{X}}^{(\zeta_1,\zeta_2)}(\mathfrak{e}) \end{aligned}$$

for all $\mathfrak{e} \in \mathscr{E}$ (see Lemma 2).

Also,

$$\begin{aligned} \overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}) \cdot \theta_2(\mathfrak{e}) \cap \eta_1(\mathfrak{e}) \cdot \overline{\mathscr{X}}^{(\theta_1,\theta_2)}(\mathfrak{e}) &\subseteq \mathscr{X}\zeta_2(\mathfrak{e}) \cap \mathscr{X}\zeta_1(\mathfrak{e}) \\ &= \mathscr{X}\zeta_1(\mathfrak{e}) \cap \mathscr{X}\zeta_2(\mathfrak{e}) \\ &= \overline{\mathscr{X}}^{(\zeta_1,\zeta_2)}(\mathfrak{e}) \end{aligned}$$

for all $\mathfrak{e} \in \mathscr{E}$ (see Lemma 2). Thus, $\overline{\mathscr{X}}^{(\eta_1,\eta_2)}(\mathfrak{e}) \cdot \theta_i(\mathfrak{e}) \cap \eta_j(\mathfrak{e}) \cdot \overline{\mathscr{X}}^{(\theta_1,\theta_2)}(\mathfrak{e}) \subseteq \overline{\mathscr{X}}^{(\zeta_1,\zeta_2)}(\mathfrak{e})$, for all $\mathfrak{e} \in \mathscr{E}$, $i, j = 1, 2$. This completes the proof. ■

V. DECISION MAKING BASED ON SMGRSs

In the recent years, both the SS theory and the MGRS theory have been applied to address various DM problems. In this section, we give a novel DM framework using the theory of SMGRSs. In almost all kinds of data analysis, DM plays an imperative role in collecting an appropriate substitute among various choices. In order to find a wise decision, several commodious techniques have constructed by innumerable experts and researchers (see [5], [19], [39], [41], [45], [46], [48], [49], and [50]). This section presents an algorithm for DM by using our proposed concept of SMGRSs over groups defined in the previous section. With the help of this scheme, one can easily find a feasible parameter $\mathfrak{e}$ of a SS $(F, \mathscr{A})$. In other words, one can obtain the nearest accurate group $F(\mathfrak{e})$ on $(F, \mathscr{A})$ by using multi NSGs over a group $\mathscr{G}$. First, we mention the ratio based choice values for the approximate groups $F(\mathfrak{e})$ in the sequel.

Definition 14: Let $\mathscr{G}$ be a multiplicative group with an identity element $1_{\mathscr{G}}$ and $\mathscr{E}$ be the related set of parameters. Let $\mathscr{A} = \{\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_m\} \subseteq \mathscr{E}$ and $(F, \mathscr{A})$ be an original description SS over $\mathscr{G}$. Let $(\eta_1, \mathscr{B})$ and $(\eta_2, \mathscr{B})$ be two NSGs over $\mathscr{G}$, where $\mathscr{B} = \{\mathfrak{e}_1, \mathfrak{e}_2\} \subseteq \mathscr{A}$ and $(\mathscr{G}, \eta_1, \eta_2, \mathscr{B})$ be the SMGRA-space. Define the decision choice values $C_i(\mathfrak{e}_j)$ as follows:

$$C_i(\mathfrak{e}_j) = \frac{\left|\overline{F(\mathfrak{e}_i)}^{(\eta_1,\eta_2)}(\mathfrak{e}_j)\right| - \left|\underline{F(\mathfrak{e}_i)}_{(\eta_1,\eta_2)}(\mathfrak{e}_j)\right|}{\left|F(\mathfrak{e}_i)\right|} \tag{16}$$

for all $1 \leq i \leq |\mathscr{A}|$ and $1 \leq j \leq 2$.

With the similar notations as in Definition 14, an algorithm to determine an optimal parameter $\mathfrak{e}$ of the SS $(F, \mathscr{A})$ is designed as follows:

A. ALGORITHM

Here, we will put forward the step-by-step procedure to find the best parameter of a given SS $(F, \mathscr{A})$. The corresponding steps are listed as follows:

Step 1: Input the original description group $\mathscr{G}$, a SS $(F, \mathscr{A})$ over $\mathscr{G}$ and the SMGRA-space $(\mathscr{G}, \eta_1, \eta_2, \mathscr{B})$, where $(\eta_1, \mathscr{B})$ and $(\eta_2, \mathscr{B})$ are two NSGs over $\mathscr{G}$ such that $\eta_h(\mathfrak{e}_j) \neq 1_{\mathscr{G}}$, for any $1 \leq h, j \leq 2$.

Step 2: Evaluate $\underline{F(\mathfrak{e}_i)}_{(\eta_1,\eta_2)}(\mathfrak{e}_j)$ and $\overline{F(\mathfrak{e}_i)}^{(\eta_1,\eta_2)}(\mathfrak{e}_j)$ for the SS $(F, \mathscr{A})$ for all $1 \leq i \leq |\mathscr{A}|$ and $1 \leq j \leq 2$.

Step 3: Compute different decision choice values $C_i(\mathfrak{e}_j)$ for all $1 \leq i \leq |\mathscr{A}|$ and $1 \leq j \leq 2$ according to Definition 14.

Step 4: Find the minimum value $C_k(\mathfrak{e}_j)$ of $C_i(\mathfrak{e}_j)$, where $1 \leq i \leq |\mathscr{A}|$ and $1 \leq j \leq 2$, that is,

$$C_k(\mathfrak{e}_j) = min\Big\{C_i(\mathfrak{e}_j) : 1 \leq i \leq |\mathscr{A}|, 1 \leq j \leq 2\Big\}. \quad (17)$$

Step 5: The optimal parameter is $\mathfrak{e}_k$ and hence the corresponding $F(\mathfrak{e}_k)$ is accurate approximation of $(F, \mathscr{A})$. If $F(\mathfrak{e}_k)$ are more than one, then any one of $F(\mathfrak{e}_k)$ may be chosen.

Step 6: If $\eta_h(\mathfrak{e}_j) = 1_{\mathscr{G}}$, for any $h, j = 1, 2$, then $C_i(\mathfrak{e}_j) = 0$ for all $i \in \{1, 2, \ldots, |\mathscr{A}|\}$. In this case, redefine $\eta_h(\mathfrak{e}_j)$ and repeat from step 2.

Figure 1 depicts a flow chart representation of the aforementioned algorithm.

B. NUMERICAL EXAMPLES

In this segment, several examples are solved for elaboration of the proposed technique and comparison purpose with some existing techniques given by Maji et al. [5], Pan and Zhan [46], and Pan and Zan [47]. The results are intuitive and satisfactory. First, for demonstration of our interpreted technique and comparison with the scheme of Pan and Zhan [46], consider the following two examples.

Example 7:

Step-1: Suppose that we want to determine the nearest accurate normal group on a SS $(F, \mathscr{A})$. Consider $\mathscr{G} = \mathcal{S}_3 = \{1, (12), (13), (23), (123), (132)\}$, $\mathscr{E} = \{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_4, \mathfrak{e}_5\}$ and $\mathscr{A} = \{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_4\} \subseteq \mathscr{E}$ same as in Example 5.1 of [46]. Input a SS $(F, \mathscr{A})$ over $\mathscr{G}$ as follows:

$$F(\mathfrak{e}) = \begin{cases} \{1, (12), (13), (23)\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{1, (12), (13)\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \\ \{1\}, & \text{if } \mathfrak{e} = \mathfrak{e}_3 \\ \{1, (13), (123), (132)\}, & \text{if } \mathfrak{e} = \mathfrak{e}_4 \end{cases}$$

Let $\mathscr{B} = \{\mathfrak{e}_1, \mathfrak{e}_2\} \subseteq \mathscr{A}$. Define the NSGs $(\eta_i, \mathscr{B})$, where $i = 1, 2$ as follows:

$$\eta_1(\mathfrak{e}) = \begin{cases} \{1, (12)\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{1, (13)\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2, \end{cases}$$

$$\eta_2(\mathfrak{e}) = \begin{cases} \{1\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \mathscr{A}_3, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{A}$.

Step-2: Using Definition 9, we get:

$$\underline{F(\mathfrak{e}_i)}_{(\eta_1,\eta_2)}(\mathfrak{e}_1) = \overline{F(\mathfrak{e}_i)}^{(\eta_1,\eta_2)}(\mathfrak{e}_1) = F(\mathfrak{e}_i)$$

for all $1 \leq i \leq |\mathscr{A}|$. Hence, $C_i(\mathfrak{e}_1) = 0$ for all $1 \leq i \leq |\mathscr{A}|$. But

$$\underline{F(\mathfrak{e})}_{(\eta_1,\eta_2)}(\mathfrak{e}_2) = \begin{cases} F(\mathfrak{e}_1), & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{1, (13)\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \\ \emptyset, & \text{if } \mathfrak{e} = \mathfrak{e}_3 \\ F(\mathfrak{e}_4), & \text{if } \mathfrak{e} = \mathfrak{e}_4, \end{cases}$$

$$\overline{F(\mathfrak{e})}^{(\eta_1,\eta_2)}(\mathfrak{e}_2) = \begin{cases} \mathscr{G}, & \text{if } \mathfrak{e} = \mathfrak{e}_1, \mathfrak{e}_4 \\ \{1, (12), (13), (132)\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \\ \{1\}, & \text{if } \mathfrak{e} = \mathfrak{e}_3 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{A}$. So, $C_1(\mathfrak{e}_2) = 0.5$, $C_2(\mathfrak{e}_2) = \frac{2}{3} = 0.6$, $C_3(\mathfrak{e}_2) = 1$ and $C_4(\mathfrak{e}_2) = 0.5$. Since, $C_i(\mathfrak{e}_1) = 0$ for all $i \in \{1, \ldots, |\mathscr{A}|\}$, therefore we need to redefine the NSG $(\eta_2, \mathscr{B})$ as follows:

$$\eta_2(\mathfrak{e}) = \begin{cases} \{1, (23)\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \mathscr{A}_3, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \end{cases}$$

Using the Definition 9, we obtain the following two SSs:

$$\underline{F(\mathfrak{e})}_{(\eta_1,\eta_2)}(\mathfrak{e}_1) = \begin{cases} \{1, (12), (13)\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{1, (12)\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \\ \emptyset, & \text{if } \mathfrak{e} = \mathfrak{e}_3 \\ \{1, (13), (132)\}, & \text{if } \mathfrak{e} = \mathfrak{e}_4 \end{cases}$$

$$\overline{F(\mathfrak{e})}^{(\eta_1,\eta_2)}(\mathfrak{e}_1) = \begin{cases} \mathscr{G}, & \text{if } \mathfrak{e} = \mathfrak{e}_1, \mathfrak{e}_4 \\ \{1, (12), (13), (123)\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \\ \{1\}, & \text{if } \mathfrak{e} = \mathfrak{e}_3 \end{cases}$$

Step-3: Compute the decision choice values $C_i(\mathfrak{e}_j)$ according to Definition 14,
$C_1(\mathfrak{e}_1) = 0.75, C_2(\mathfrak{e}_1) = 0.6, C_3(\mathfrak{e}_1) = 1$ and $C_4(\mathfrak{e}_1) = 0.75$. Since $C_1(\mathfrak{e}_2) = 0.5$, $C_2(\mathfrak{e}_2) = \frac{2}{3} = 0.6$, $C_3(\mathfrak{e}_2) = 1$, $C_4(\mathfrak{e}_2) = 0.5$.

Step-4: Thus,

$$\begin{aligned} C_k(\mathfrak{e}_j) &= min\big\{C_i(\mathfrak{e}_j) : 1 \leq i \leq |\mathscr{A}|, 1 \leq j \leq 2\big\} \\ &= C_1(\mathfrak{e}_2) = C_4(\mathfrak{e}_2) = 0.5. \end{aligned}$$

Step-5: Hence, $\mathfrak{e}_1$ and $\mathfrak{e}_4$ are the best parameters, therefore the nearest accurate groups are $F(\mathfrak{e}_1)$ and $F(\mathfrak{e}_4)$, or the best approximations on the SS $(F, \mathscr{A})$ concerning the normal subgroups $\eta_i(\mathfrak{e}_2)$, $i = 1, 2$. So, any one of $F(\mathfrak{e}_1)$ and $F(\mathfrak{e}_4)$ can be chosen.

Comparatively, in [46] Example 5.1, we notice that the optimal parameter is $\mathfrak{e}_2$, and hence $F(\mathfrak{e}_2)$ is the nearest accurate group on the SS $(F, \mathscr{A})$ with respect to the normal subgroup $\{1, (12)\}$. But $|F(\mathfrak{e}_1)| = |F(\mathfrak{e}_4)| = 4$ and $|F(\mathfrak{e}_2)| = 3$. So, $F(\mathfrak{e}_1)$ and $F(\mathfrak{e}_4)$ are most nearest groups on $(F, \mathscr{A})$ with respect to the normal subgroups $\eta_h(\mathfrak{e}_2)$, $h = 1, 2$ than that of $F(\mathfrak{e}_2)$ with respect to only one normal subgroup.

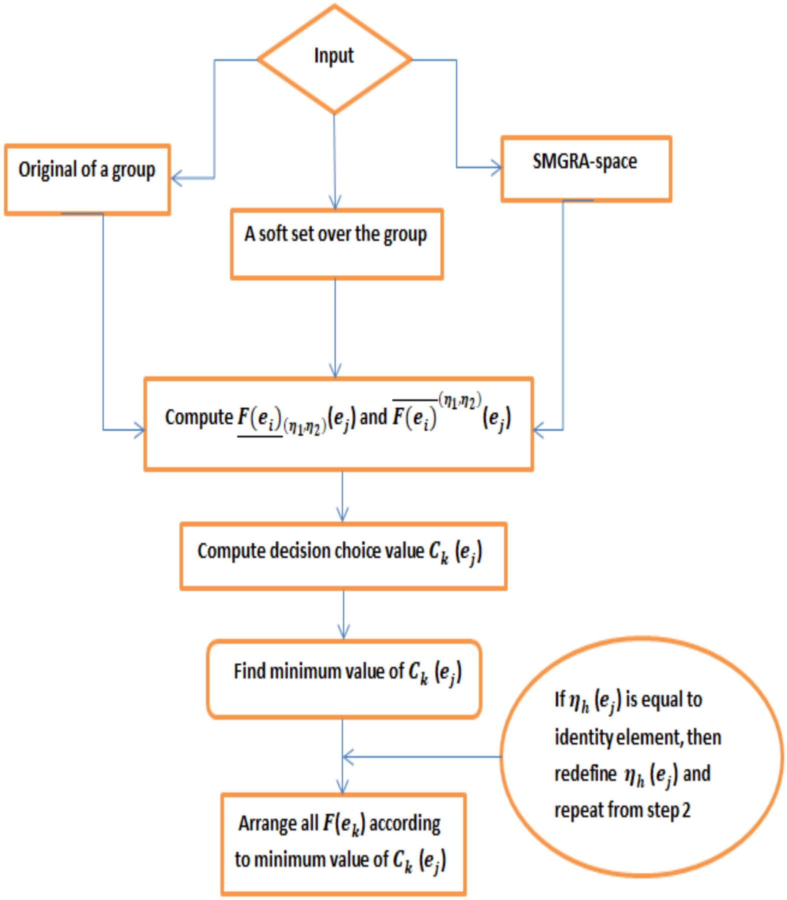


FIGURE 1. **The summary of the proposed DM algorithm.**

Example 8:

Step-1: Let $\mathscr{G} = \mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ and $\mathscr{A} = \{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_4\}$. Define a SS $(F, \mathscr{A})$ as follows:

$$F(\mathfrak{e}) = \begin{cases} \{\bar{0}, \bar{1}, \bar{3}, \bar{5}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{\bar{0}, \bar{1}, \bar{3}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \\ \{\bar{0}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_3 \\ \{\bar{0}, \bar{2}, \bar{3}, \bar{4}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_4 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{A}$. Define two NSGs $(\eta_h, \mathscr{B})$, where $\mathscr{B} = \{\mathfrak{e}_1, \mathfrak{e}_2\}$ and $1 \leq h \leq 2$, as follows:

$$\eta_1(\mathfrak{e}) = \begin{cases} \{\bar{0}, \bar{3}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{\bar{0}, \bar{2}, \bar{4}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2, \end{cases}$$

$$\eta_2(\mathfrak{e}) = \begin{cases} \{\bar{0}, \bar{2}, \bar{4}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \mathbb{Z}_6, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{B}$.

Step-2: By using the Definition 9, we accomplish the following SSs:

$$\underline{F(\mathfrak{e})}_{(\eta_1,\eta_2)}(\mathfrak{e}_1) = \begin{cases} \{\bar{0}, \bar{1}, \bar{3}, \bar{5}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{\bar{0}, \bar{3}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \\ \emptyset, & \text{if } \mathfrak{e} = \mathfrak{e}_3 \\ \{\bar{0}, \bar{2}, \bar{3}, \bar{4}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_4, \end{cases}$$

$$\overline{F(\mathfrak{e})}^{(\eta_1,\eta_2)}(\mathfrak{e}_1) = \begin{cases} \mathbb{Z}_6, & \text{if } \mathfrak{e} = \mathfrak{e}_1, \mathfrak{e}_4 \\ \{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \\ \{\bar{0}, \bar{2}, \bar{3}, \bar{4}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_3 \end{cases}$$

$$\underline{F(\mathfrak{e})}_{(\eta_1,\eta_2)}(\mathfrak{e}_2) = \begin{cases} \{\bar{1}, \bar{3}, \bar{5}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \emptyset, & \text{if } \mathfrak{e} = \mathfrak{e}_2, \mathfrak{e}_3 \\ \{\bar{0}, \bar{2}, \bar{4}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_4, \end{cases}$$

$$\overline{F(\mathfrak{e})}^{(\eta_1,\eta_2)}(\mathfrak{e}_2) = \begin{cases} \mathbb{Z}_6, & \text{if } \mathfrak{e} = \mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_4 \\ \{\bar{0}, \bar{2}, \bar{4}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_3 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{A}$.

Step-3: By using the Definition 14, the different choice values are attained as follows:

$$C_1(\mathfrak{e}_1) = 0.5,\ \ C_2(\mathfrak{e}_1) = 0.6,\ \ C_3(\mathfrak{e}_1) = 4,$$

$C_4(\mathfrak{e}_1) = 0.5$ and
$C_1(\mathfrak{e}_2) = 0.75,\ C_2(\mathfrak{e}_2) = 2,\ C_3(\mathfrak{e}_2) = 3,$
$C_4(\mathfrak{e}_2) = 0.75$

Step-4: Hence,

$$C_k(\mathfrak{e}_j) = min\{C_i(\mathfrak{e}_j) : 1 \le i \le |\mathscr{A}|, 1 \le j \le 2\} = C_1(\mathfrak{e}_1) = C_4(\mathfrak{e}_1).$$

Step-5: Therefore, optimal parameters are $\mathfrak{e}_1$, $\mathfrak{e}_4$, and hence $F(\mathfrak{e}_1)$ and $F(\mathfrak{e}_4)$ are the best or nearest approximations or groups on the SS $(F, \mathscr{A})$.
On the other hand, if we take the normal subgroup $\mathscr{N} = \{\bar{0}, \bar{3}\}$ of $\mathbb{Z}_6$. Then, we obtain the following two SSs according to Pan and Zhan's [46] Definition 4.1,

$$\underline{F(\mathfrak{e})}_{\mathscr{N}} = \begin{cases} \mathscr{N}, & \text{if } \mathfrak{e} = \mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_4 \\ \emptyset, & \text{if } \mathfrak{e} = \mathfrak{e}_3, \end{cases}$$

$$\overline{F(\mathfrak{e})}^{\mathscr{N}} = \begin{cases} \mathbb{Z}_6, & \text{if } \mathfrak{e} = \mathfrak{e}_1, \mathfrak{e}_4 \\ \{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \\ \mathscr{N}, & \text{if } \mathfrak{e} = \mathfrak{e}_3 \end{cases}$$

Then, by using formula $C_i = \frac{\left|\overline{F(\mathfrak{e}_i)}^{\mathscr{N}}\right| - \left|\underline{F(\mathfrak{e}_i)}_{\mathscr{N}}\right|}{\left|F(\mathfrak{e}_i)\right|}$ given in [46], where $1 \le i \le |\mathscr{A}|$, We get:

$$C_1 = C_4 = 1, \quad C_2 = 0.6, \quad C_3 = 2$$

Hence, $C_k = min\{C_i : 1 \le i \le |\mathscr{A}|\} = C_2 = 0.6$. Therefore, $\mathfrak{e}_2$ is the optimal parameter. Thus, $F(\mathfrak{e}_2)$ is the corresponding nearest group to $\mathbb{Z}_6$ with respect to the normal subgroup $\mathscr{N} = \{\bar{0}, \bar{3}\}$. We observe that $|F(\mathfrak{e}_2)| = 3$ but $|F(\mathfrak{e}_1)| = |F(\mathfrak{e}_4)| = 4$, so $F(\mathfrak{e}_1)$ and $F(\mathfrak{e}_4)$ are most nearest and hence accurate groups (or the approximations) on the SS $(F, \mathscr{A})$ over $\mathscr{G} = \mathbb{Z}_6$ with respect to the normal subgroups $\eta_h(\mathfrak{e}_1)$, where $h = 1, 2$.

In the sequel, we construct an example to compare our technique with Pan and Zan [47].

Example 9: **Step-1:** Let $\mathscr{G} = \mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ and $\mathscr{A} = \{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_4\}$. Consider the same SS $(F, \mathscr{A})$ as in above Example 8 given as follows:

$$F(\mathfrak{e}) = \begin{cases} \{\bar{0}, \bar{1}, \bar{3}, \bar{5}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{\bar{0}, \bar{1}, \bar{3}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \\ \{\bar{0}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_3 \\ \{\bar{0}, \bar{2}, \bar{3}, \bar{4}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_4 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{A}$. Define another SS $(\mathscr{X}, \mathscr{B})$, where $\mathscr{B} = \{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3\}$ as follows:

$$\mathscr{X}(\mathfrak{e}) = \begin{cases} \{\bar{0}, \bar{2}, \bar{3}, \bar{4}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{\bar{0}, \bar{1}, \bar{3}, \bar{4}, \bar{5}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \\ \{\bar{0}, \bar{1}, \bar{2}, \bar{5}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_3 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{B}$. According to the Definition 15 given in Shabir et al. [44], the mapping $\varphi : \mathscr{G} \to P(\mathscr{B})$ is given as:

$$\varphi(\mathfrak{u}) = \begin{cases} \mathscr{B}, & \text{if } \mathfrak{u} = \bar{0} \\ \{\mathfrak{e}_2, \mathfrak{e}_3\}, & \text{if } \mathfrak{u} = \bar{1}, \bar{5} \\ \{\mathfrak{e}_1, \mathfrak{e}_3\}, & \text{if } \mathfrak{u} = \bar{2} \\ \{\mathfrak{e}_1, \mathfrak{e}_2\}, & \text{if } \mathfrak{u} = \bar{3}, \bar{4} \end{cases}$$

for all $\mathfrak{u} \in \mathbb{Z}_6$.

Step-2: By using the Definition 4.1 of [47], we reckon the lower SR-approximation $\underline{(\mathscr{X}, \mathscr{B})}_{\varphi}$ and upper SR-approximation $\overline{(\mathscr{X}, \mathscr{B})}^{\varphi}$ as follows:

$$\underline{F(\mathfrak{e})}_{\varphi}(\mathfrak{e}) = \begin{cases} \{\bar{0}, \bar{1}, \bar{5}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{\bar{0}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2, \mathfrak{e}_3 \\ \{\bar{0}, \bar{2}, \bar{3}, \bar{4}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_4 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{A}$.

$$\overline{F(\mathfrak{e})}^{\varphi}(\mathfrak{e}) = \begin{cases} \{\bar{0}, \bar{1}, \bar{3}, \bar{4}, \bar{5}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1, \mathfrak{e}_2 \\ \{\bar{0}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_3 \\ \{\bar{0}, \bar{2}, \bar{3}, \bar{4}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_4 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{A}$.

Step-3: By using $C_i = \left|F(\mathfrak{e}_i)\right| = \frac{\left|\underline{F(\mathfrak{e})}_{\varphi}(\mathfrak{e}_i)\right|}{\left|\overline{F(\mathfrak{e})}^{\varphi}(\mathfrak{e}_i)\right|}$, we get:

$$C_1 = \left|F(\mathfrak{e}_1)\right| = 0.6,$$
$$C_2 = \left|F(\mathfrak{e}_2)\right| = 0.2,$$
$$C_3 = \left|F(\mathfrak{e}_3)\right| = C_4 = \left|F(\mathfrak{e}_4)\right| = 1.$$

Step-4: Since, $C_k = max\left\{\left|F(\mathfrak{e}_i)\right| : i = 1, 2, 3, 4\right\} = C_3 = C_4 = 1$.

Step-5: Thus, $F(\mathfrak{e}_3) = \{\bar{0}\}$ and $F(\mathfrak{e}_4) = \{\bar{0}, \bar{2}, \bar{3}, \bar{4}\}$ are closest accurate groups.
It can be easily seen that $|F(\mathfrak{e}_3)| = 1$, so $F(\mathfrak{e}_3) = \{\bar{0}\}$ is not the nearest accurate group to given SS $(F, \mathscr{A})$ which is defined in step 1 of the above example. Next, we compare our results with Maji et al.'s technique [5] in the following example.

Example 10: Let $\mathscr{G} = \mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ and $\mathscr{A} = \{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_4\}$. Consider the same SS $(F, \mathscr{A})$ as in Example 8 given as follows:

$$F(\mathfrak{e}) = \begin{cases} \{\bar{0}, \bar{1}, \bar{3}, \bar{5}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_1 \\ \{\bar{0}, \bar{1}, \bar{3}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_2 \\ \{\bar{0}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_3 \\ \{\bar{0}, \bar{2}, \bar{3}, \bar{4}\}, & \text{if } \mathfrak{e} = \mathfrak{e}_4 \end{cases}$$

for all $\mathfrak{e} \in \mathscr{A}$. The tabular representation of $(F, \mathscr{A})$ is as follows:

$\mathscr{G} \backslash \mathscr{A}$	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{e}_3$	$\mathfrak{e}_4$
$\bar{0}$	1	1	1	1
$\bar{1}$	1	1	0	0
$\bar{2}$	0	0	0	1
$\bar{3}$	1	1	0	1
$\bar{4}$	0	0	0	1
$\bar{5}$	1	0	0	0

We can easily see that there is no reduct of $\mathscr{A}$.

Thus we can easily find the choice values of row by using the Definition of [10], we obtain

$$\sigma(\mathfrak{u}) = \begin{cases} 4, & \text{if } \mathfrak{u} = \bar{0} \\ 2, & \text{if } \mathfrak{u} = \bar{1} \\ 1, & \text{if } \mathfrak{u} = \bar{2}, \bar{4}, \bar{5} \\ 3, & \text{if } \mathfrak{u} = \bar{3} \end{cases}$$

for all $\mathfrak{u} \in \mathscr{G}$. Thus, $max\sigma(\mathfrak{u}) = \sigma(\mathfrak{u}_1) = 4$, so $\mathfrak{u}_1 = \bar{0}$ is optimal object.

In Example 10, we see that the method of choice value proposed by Maji et al. [5] facilitates us to choose an optimal object, not a group of optimal objects. Thus, our method is more robust than Maji et al. [5].

C. COMPARATIVE ANALYSIS AND DISCUSSION

In SSs, the ranking of objects is natural and easy according to Maji et al. [5] technique. But the situation is not straightforward in the case of SRSs proposed by Feng et al. [8] and MSRSs of Shabir et al. [56]. However, there are certain techniques proposed by Ma et al. [45], Pan and Zan [46], [47], Wang et al. [48], and Zan et al. [49]. These techniques are independent of each other. These ranking techniques are based on various kinds of ratios defined by their SRA-spaces. Here, we have a new hybrid model SMRSs by using multi S-BRs and also have applied it to group theory to see its influence on groups with its influences on DM. The proposed scheme is based on two NSGs and it is also valid for $n-$NSGs. So, the ranking of approximations $F(\mathfrak{e}_k)$ on a SS $(F, \mathscr{A})$ in the case of SMGRSs is not an easy task. Naturally, the most nearest group of objects $F(\mathfrak{e}_k)$ on a SS $(F, \mathscr{A})$ which has more number of objects. In the view of the results accomplished in Examples 7, 8, 9 and 10, we conclude that the results by using our technique are more accurate due to the flexibility and novelty of our proposed technique by using SMGRSs concerning multi NSGs $(\eta_h, \mathscr{B})$ in DM. The use of multigranules, that is, multi NSGs makes SMGRS a more generalized and powerful hybrid structure than the existing techniques of [5], [46], and [47]. Thus, we conclude that our proposed model of SMGRS is more effective and robust to solve DM problems as they are the more general form of SRS.

VI. CONCLUSION

The RS and SS theories are incredible mathematical tools to deal with uncertainty. One of the desired directions in RS theory is MGRS, which approximates lower and upper approximations via granular structures obtained by multiple binary relations. Based on RS, it offers a novel approach for decision analysis. In this paper, the idea of SMGRS has been introduced based on the notion of after sets of two S-BRs. Some important structural properties of SRA-spaces have been studied with illustrative Examples. Further, the concept of SMGRS has been applied to group theory, where the SRA-spaces were defined by using multi NSGs. Some significant results related to SMGRSs in groups have been studied in detail. Finally, an algorithm of DM has been established by utilizing SMGRSs over groups and supported by some constructive examples.

The notion of SMGRS is more robust and has a more extensive hybrid structure than the existing concept of SRSs. The results of this study enrich decision analysis. Bearing in mind as mentioned earlier, future research studies will focus on:

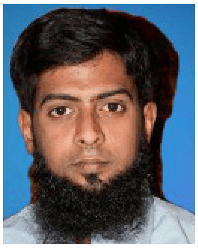
- The practical applications of the suggested technique in solving a wider variety of selection problems, like TOSIS, VIKOR, ELECTRE, AHP, COPRAS, PROMETHEE, etc.
- Researchers may study the algebraic structures of SMGMRS.
- The attribute reduction of SMGRS should be analyzed, and comprehensive experimental investigations, and comparisons with existing methodologies should also be justified and explored.
- Strategies for decision support in real-time and dynamic DM tasks are also our next goal.
- The idea of SMGRS can be extended in a fuzzy environment, and effective DM techniques might be developed.
- Further study can be done to establish fruitful algorithms for different kinds of DM problems.
- Another direction is to investigate the topological properties and similarity measures of SMGRS to create a solid foundation for future perspectives.
- We will also explore the possible hybridization of the suggested approach to improve precision in results and apply these strategies to real-world problems with large data sets. In this manner, we can acquire and demonstrate the use of our suggested framework.

REFERENCES

[1] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353, Jun. 1965.
[2] Z. Pawlak, "Rough sets," *Int. J. Comput. Inf. Sci.*, vol. 11, no. 5, pp. 341–356, 1982.
[3] D. Molodtsov, "Soft set theory—First results," *Comput. Math. Appl.*, vol. 37, nos. 4–5, pp. 19–31, Feb. 1999.
[4] P. K. Maji, R. Biswas, and A. Roy, "Soft set theory," *Comput. Math. Appl.*, vol. 45, nos. 4–5, pp. 555–562, 2003.
[5] P. K. Maji, A. R. Roy, and R. Biswas, "An application of soft sets in a decision making problem," *Comput. Math. Appl.*, vol. 44, nos. 8–9, pp. 1077–1083, Oct. 2002.
[6] M. I. Ali, F. Feng, X. Liu, W. K. Min, and M. Shabir, "On some new operations in soft set theory," *Comput. Math. Appl.*, vol. 57, no. 9, pp. 1547–1553, May 2009.
[7] H. Aktaş and N. Çağman, "Soft sets and soft groups," *Inf. Sci.*, vol. 177, no. 13, pp. 2726–2735, Jul. 2007.
[8] F. Feng, X. Liu, V. L.-Fotea, and Y. B. Jun, "Soft sets and rough sets," *Inf. Sci.*, vol. 181, no. 6, pp. 1125–1137, 2011.
[9] S. Alkhazaleh and E. A. Marei, "New soft rough set approximations," *Int. J. Fuzzy Log. Intell. Syst.*, vol. 21, no. 2, pp. 123–134, Jun. 2021.
[10] F. Feng, Y. Li, and N. Çağman, "Generalized uni–int decision making schemes based on choice value soft sets," *Eur. J. Oper. Res.*, vol. 220, no. 1, pp. 162–170, Jul. 2012.
[11] B.-J. Fan, E.-C.-C. Tsang, W.-T. Li, and X.-P. Xue, "Multigranulation soft rough sets," in *Proc. Int. Conf. Wavelet Anal. Pattern Recognit. (ICWAPR)*, Jul. 2017, pp. 1–6.
[12] A. Sezgin and A. O. Atagün, "Soft groups and normalistic soft groups," *Comput. Math. Appl.*, vol. 62, no. 2, pp. 685–698, Jul. 2011.
[13] M. Aslam and S. M. Qurashi, "Some contributions to soft groups," *Ann. Fuzzy Math. Inform.*, vol. 1, no. 4, pp. 177–195, 2012.

[14] N. Çağman, F. Çıtak, and H. Aktaş, "Soft int-group and its applications to group theory," *Neural Comput. Appl.*, vol. 21, no. S1, pp. 151–158, Aug. 2012.
[15] K. Kaygisiz, "On soft int-groups," *Ann. Fuzzy Math. Inform.*, vol. 4, no. 2, pp. 363–375, 2012.
[16] K. Kaygisiz, "Normal soft int-group," 2012, *arXiv:1209.3157*.
[17] M. Shabir and M. Naz, "On soft toplogical spacses," *Comput. Math. Appl.*, vol. 61, no. 7, pp. 1786–1799, 2011.
[18] F. Feng, I. Ali, and M. Shabir, "Soft relations applied to semigroups," *Filomat*, vol. 27, no. 7, pp. 1183–1196, 2013.
[19] M. I. Ali, "A note on soft sets, rough soft sets and fuzzy soft sets," *Appl. Soft Comput.*, vol. 11, no. 4, pp. 3329–3332, Jun. 2011.
[20] Y. Shang, "Subgraph robustness of complex networks under attacks," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 49, no. 4, pp. 821–832, Apr. 2017.
[21] Y. Qian, J. Liang, Y. Yao, and C. Dang, "MGRS: A multi-granulation rough set," *Inf. Sci.*, vol. 180, no. 6, pp. 949–970, Mar. 2010.
[22] R. Biswas and S. Nanda, "Rough groups and rough subgroups," *Bull. Polish Acadamey Sci., Math.*, vol. 42, no. 3, pp. 251–254, 1994.
[23] N. Kuroki and P. P. Wang, "The lower and upper approximations in a fuzzy group," *Inf. Sci.*, vol. 90, nos. 1–4, pp. 203–220, Apr. 1996.
[24] W. Mahmood, M. S. Ahmad, W. Nazeer, and S. M. Kang, "A comparision between lower and upper approximations in groups with respect to group homomorphisms," *J. Intell. Fuzzy Syst.*, vol. 35, nos. 2–3, pp. 1–11, 2018.
[25] W. Mahmood, W. Nazeer, and S. M. Kang, "The lower and upper approximations and homomorphisms between lower approximations in quotient groups," *J. Intell. Fuzzy Syst.*, vol. 33, no. 4, pp. 2585–2594, Sep. 2017.
[26] B. Davvaz, "Roughness in rings," *Inf. Sci.*, vol. 164, nos. 1–4, pp. 147–163, Aug. 2004.
[27] B. Davvaz and M. Mahdavipour, "Roughness in modules," *Inf. Sci.*, vol. 176, no. 24, pp. 3658–3674, Dec. 2006.
[28] Z. Chen, S. Ayub, W. Mahmood, A. Mahboob, and C. Y. Jung, "A study of roughness in modules of fractions," *IEEE Access*, vol. 7, pp. 93088–93099, 2019.
[29] S. Ayub, W. Mahmood, F. G. Nabi, and M. Shabir, "Applications of roughness in soft-intersection groups," *Comput. Appl. Math.*, vol. 38, no. 4, pp. 1–16, Dec. 2019.
[30] S. Ayub, M. Shabir, and W. Mahmood, "New types of soft rough sets in groups based on normal soft groups," *Comput. Appl. Math.*, vol. 39, no. 2, pp. 1–15, May 2020.
[31] S. Ayub, W. Mahmood, M. Shabir, and R. Gul, "A study of modules of fractions under fuzzy sets, soft sets and rough sets," *New Math. Natural Comput.*, submitted for publication.
[32] L. Wang and H. Garg, "Algorithm for multiple attribute decision-making with interactive Archimedean norm operations under Pythagorean fuzzy uncertainty," *Int. J. Comput. Intell. Syst.*, vol. 4, no. 1, pp. 503–527, 2021.
[33] M. Gulzar, G. Abbas, and F. Dilwar, "Algebraic propoerties of w-Q-fuzzy subgroups," *Int. J. Math. Comput. Sci.*, vol. 15, no. 1, pp. 265–274, 2020.
[34] M. Gulzar, F. Dilwar, D. Alghazzaawi, and M. H. Mateen, "A note on a complex fuzzy subfield," *Indonesian J. Elect. Eng. Comput. Sci.*, vol. 21, no. 2, pp. 1048–1056, 2021.
[35] S. Bhunia, G. Ghorai, M. A. Kutbi, M. Gulzar, and M. A. Alam, "On the algebraic characteristics of fuzzy sub-e-groups," *J. Function Spaces*, vol. 2021, pp. 1–7, Oct. 2021.
[36] M. Gulzar, M. H. Mateen, D. Alghazzaawi, and N. Kausar, "A novel application of complex intuintionistic fuzzy sets in group theory," *IEEE Access*, vol. 8, pp. 196075–196085, 2020.
[37] M. Gulzar, D. Alghazzawi, M. H. Mateen, and N. Kausar, "A certain class of t-Intuitionistic fuzzy subgroups," *IEEE Access*, vol. 8, pp. 163260–163268, 2020, doi: 10.1109/ACCESS.2020.3020366.
[38] Z. Li, N. Xie, and N. Gao, "Rough approximations based on soft binary relations and knowledge bases," *Soft Comput.*, vol. 21, pp. 839–852, Feb. 2017.
[39] F. Feng, Y. B. Jun, X. Liu, and L. Li, "An adjustable approach to fuzzy soft set based decision making," *J. Comput. Appl. Math.*, vol. 234, pp. 10–20, May 2010.
[40] P. K. Maji, R. Biswas, and A. R. Roy, "Fuzzy soft sets," *J. Fuzzy Math.*, vol. 9, no. 3, pp. 589–602, 2001.
[41] A. R. Roy and P. K. Maji, "A fuzzy soft set theoretic appraoch to decision making problems," *J. Comput. Appl. Math.*, vol. 203, pp. 412–418, Jun. 2007.
[42] R. S. Kanwal and M. Shabir, "Roughness in semigroups by soft binary relations," *J. Intell. Fuzzy Syst.*, vol. 35, nos. 5–6, pp. 1–14, 2018.
[43] F. Feng, C. Li, B. Davvaz, and M. I. Ali, "Soft sets combined with fuzzy sets and rough sets: A tentative approach," *Soft Comput.*, vol. 14, no. 9, pp. 899–911, Jul. 2010.
[44] M. Shabir, M. I. Ali, and T. Shaheen, "Another approach to soft rough sets," *Knowl. Based Syst.*, vol. 40, pp. 72–78, Mar. 2013.
[45] X. Ma, Q. Liu, and J. Zhan, "A survey of decision making methods based on certain hybrid soft set models," *Artif. Intell. Rev.*, vol. 47, no. 4, pp. 507–530, Apr. 2017.
[46] W. Pan and J. Zhan, "Rough fuzzy groups and rough soft groups," *Italian J. Pure Appl. Math.*, vol. 38, pp. 158–178, Jul. 2017.
[47] W. Pan and J. Zhan, "Soft rough groups and corresponding decision making," *Italian J. Pure Appl. Math.*, vol. 38, pp. 158–171, Jul. 2017.
[48] Q. Wang, J. Zhan, and R. A. Borzooei, "A study on soft rough semigroups and corresponding decision making applications," *Open Math.*, vol. 15, no. 1, pp. 1400–1413, Dec. 2017.
[49] J. Zhan and K. Zhu, "A novel soft rough fuzzy set: Z-soft rough fuzzy ideals of hemirings and corresponding decision making," *Soft Comput.*, vol. 21, no. 8, pp. 1923–1936, Apr. 2017.
[50] J. Zhan and J. C. R. Alcantud, "A novel type of soft rough covering and its application to multicriteria group decision making," *Artif. Intell. Rev.*, vol. 52, no. 4, pp. 2381–2410, Dec. 2019.
[51] Y. Yao, "Constructive and algebraic methods of the theory of rough sets," *Inf. Sci.*, vol. 109, nos. 1–4, pp. 21–47, Aug. 1998.
[52] W. Zhu, "Generalized rough sets based on relations," *Inf. Sci.*, vol. 177, no. 22, pp. 4997–5011, 2007.
[53] B. Davvaz, "A short note on algebraic T-rough sets," *Inf. Sci.*, vol. 178, no. 16, pp. 3247–3252, Aug. 2008.
[54] M. I. Ali, B. Davvaz, and M. Shabir, "Some properties of generalized rough sets," *Inf. Sci.*, vol. 224, pp. 170–179, Mar. 2013.
[55] A. Skowron and J. Stepaniuk, "Tolerance approximation spaces," *Fundamenta Informaticae*, vol. 27, nos. 2–3, pp. 245–253, 1996.
[56] R. Slowinski and D. Vanderpooten, "Similarity relation as a basis for rough approximations," *ICS Res. Rep.*, vol. 53, pp. 249–250, Jun. 1995.
[57] M. Shabir, J. Din, and I. A. Ganie, "Multigranulation roughness based on soft relations," *J. Intell. Fuzzy Syst.*, vol. 40, no. 6, pp. 10893–10908, Jun. 2021.
[58] Y. Qian, J. Liang, and C. Dang, "Knowledge structure, knowledge granulation and knowledge distance in a knowledge base," *Int. J. Approx. Reasoning*, vol. 50, no. 1, pp. 174–188, Jan. 2009.
[59] G. Lin, Y. Qian, and J. Li, "NMGRS: Neighborhood-based multigranulation rough sets," *Int. J. Approx. Reasoning*, vol. 53, no. 7, pp. 1080–1093, Oct. 2012.
[60] W. Xu, X. Zhang, and Q. Wang, "A generalized multi-granulation rough set approach," in *Proc. Int. Conf. Intell. Comput.* Berlin, Germany: Springer, Aug. 2011, pp. 681–689.
[61] W. Li, X. Zhang, and W. Sun, "Further study of multigranulation-fuzzy rough sets," *Sci. World J.*, 2014.
[62] Y. She and X. He, "On the structure of the multigranulation rough set model," *Knowl.-Based Syst.*, vol. 36, pp. 81–92, Dec. 2012.
[63] X.-B. Yang, Y.-H. Qian, and J.-Y. Yang, "Hierarchical structures on multigranulation spaces," *J. Comput. Sci. Technol.*, vol. 27, no. 6, pp. 1169–1183, Nov. 2012.
[64] A. Ali, M. I. Ali, and N. Rehman, "New types of dominance based multi-granulation rough sets and their applications in conflict analysis problems," *J. Intell. Fuzzy Syst.*, vol. 35, no. 3, pp. 3859–3871, Oct. 2018.

SABA AYUB received the M.Sc., M.Phil., and Ph.D. degrees in mathematics from Quaid-i-Azam University, Islamabad, in 2014, 2016, and 2019, respectively. She is a Research Scholar of pure mathematics with the Department of Mathematics, Quaid-i-Azam University. She has published eight articles. Her main research interests include commutative algebra, fuzzy algebra, rough set theory, soft set theory, hyperstructures, decision making, and conflict analysis.



WAQAS MAHMOOD received the M.Sc. degree in mathematics from the University of the Punjab, Lahore, and the Ph.D. degree from the Abdus Salaam School of Mathematical Sciences (ASSMS), Government College University Lahore. He is an Associate Professor of pure mathematics with the Department of Mathematics, Quaid-i-Azam University, Islamabad. He has published 30 articles. His main research interests include homological algebra, commutative algebra, and fuzzy algebra.

MUHAMMAD SHABIR received the M.Sc., M.Phil., and Ph.D. degrees in mathematics from Quaid-i-Azam University, Islamabad, Pakistan, in 1981, 1984, and 1996, respectively. Currently, he is a Visiting Professor with the Department of Mathematics, Quaid-i-Azam University. He has published more than 200 research articles in international journals. He has coauthored of the book *Fuzzy Semirings with Applications to Automata Theory* (Springer, New York, NY, USA). He has supervised 17 Ph.D. and 120 M.Phil. Thesis. His current research interests include fuzzy algebraic structures, soft algebraic structures, and their applications.

ALI N. A. KOAM received the B.Sc. degree in mathematics from King Abdul Aziz University, Jeddah, KSA, in 2007, and the Ph.D. degree in mathematics from Leicester, U.K., in 2016. He is an Assistant Professor of pure mathematics with the College of Science, Mathematics Department, Jazan University, Saudi Arabia. He is the Vice-Dean of academic development of quality with Jazan University. He has authored several journal articles in his research areas. His research interests include logical algebra, associative algebras, homological algebra, modern algebra, decision making, fuzzy set theory, and graph theory.

RIZWAN GUL received the B.S. degree in mathematics from the Kohat University of Science and Technology, Kohat, Pakistan, in 2017, and the M.Phil. degree in pure mathematics from Quaid-i-Azam University, Islamabad, Pakistan, in 2020. He is currently a Ph.D. Scholar with the Department of Mathematics, Quaid-i-Azam University. His current research interests include fuzzy algebraic structures, decision analysis, soft sets, rough sets, and their hybrid algebraic structures with their applications.