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RESEARCH ARTICLE

Gaussian-Beta Filters With Unknown Probability of Measurement Loss

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ABSTRACT Data loss is ubiquitous in practical engineering applications due to communication delay or congestion. Data loss rate is a key metric to evaluate the reliability of state estimation. To jointly estimate system state and data loss rate, we propose a class of Gaussian-Beta filters for linear and moderate nonlinear Gaussian state-space models with unknown probability of measurement loss. In the filters, the arrival of the measurement at each time is formulated as a binary random variable, which is determined by the classical threshold technology. In addition, the hidden state and the unknown probability of measurement loss are modeled as a product of Gaussian and Beta distributions, and the form remains unchanged through recursive operations. Simulation results verify the effectiveness of the proposed Gaussian-Beta filters compared with the existing filtering algorithms.

INDEX TERMS State-space model, measurement loss, threshold technology, Gaussian-Beta filter.

I. INTRODUCTION

The filtering problem for linear/nonlinear state-space models consists in recursively estimating the hidden state variables from the observable contaminated data online [1], [2], [3], [4]. Design, development and application of linear/nonlinear filters have received considerable attentions in the past few decades, since they have numerous significant applications in science and engineering, such as sonar ranging, radar tracking, missile orbit determination, and navigational and guidance systems, etc [5], [6], [7], [8], [9], [10], [11]. As is well known, the performance of a designed filter depends not only on the structure of itself, but also on the reliability of the sensing systems. A common feature of modern sensor network systems is the presence of severe communication delays and data loss through the network [12], [13], [14]. From the viewpoint of filtering theory, severe communication delay is equivalent to data loss, because only when data arrive in time can it be used for online estimation. For example, consider the navigation problem for a vehicle by utilizing the estimations of its current position and velocity from a sensor network. Measurements may be delayed or lost due to the unreliability of communication links. In order to reliably perform the navigation task, it is necessary to estimate the probability of measurement loss for the sensor network system to judge the reliability of the current state estimation according to the maximum data loss rate that the estimator can tolerate, since the covariance matrix of the state will tend to infinity if the measurement loss rate is greater than this threshold. When the unreliability occurs, we need to adjust the communication protocol to meet constraints. In this paper, we mainly focus on the filtering problem for Gaussian state-space models with unknown probability of measurement loss.

For a state-space model with known measurement loss, an intermittent Kalman filter (IKF) has been developed, in which measurement loss is modeled as a random process [14]. IKF is a minimum covariance estimator and has a recursive structure similar to the standard Kalman

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filter (KF) [5]. The unique difference between IKF and KF is that the former needs to choose the measurement noise covariance involved in the filtering gain via the known measurement loss variable. Then, the stability of the IKF is analyzed from the perspective of estimation error covariance. In [14], the authors prove that there exists a critical value for the probability of measurement loss, beyond which the expectation of estimation error covariance is unbounded and the performance of the IKF may severely deteriorate. In addition, explicit upper and lower bounds on the critical value are provided. In [15], an explicit expression of the critical value is obtained under Markovian measurement losses. In [16], the bound of the expected error covariance of the IKF has been derived. Therefore, the probability of measurement loss is a significant variable to evaluate the stability of a filter and it needs to be explicitly modeled.

The filtering problem for state-space models with unknown measurement loss becomes more complicated. In [17], two different Bayesian Kalman filters (BKFs) have been developed for a linear Gaussian state-space model with unknown sensor measurement loss. BKF1 is based on the estimation of the sensor measurement loss process, and BKF2 is based on the estimation of the posterior density of the sensor measurement loss. However, the two BKFs cannot adaptively estimate the probability of measurement loss, resulting in limited estimation accuracy. Recently, a variational Bayesian-based adaptive Kalman filter (VBAKF) has been proposed to solve the filtering problem for a linear Gaussian state-space model with unknown probability of sensor measurement loss [18], in which the joint posterior density of the state, the measurement arrival variable, and the measurement loss rate are derived by the variational Bayesian technology [19], [20], [21]. Although the VBAKF shows better estimation accuracy than the BKFs and can simultaneously estimate the sensor measurement loss rate to determine the reliability of the state estimation, it requires more time cost due to a large number of iterative operations, which degrades its real-time performance.

Motivated by the above discussions, a class of Gaussian-Beta filters are proposed in this paper for linear and moderate nonlinear Gaussian state-space models with unknown probability of measurement loss. They recursively estimate the joint posterior density of the hidden state and the unknown measurement loss rate, which are respectively formulated as a Gaussian distribution and a Beta distribution, conditionally on both the sensor measurement sequence and the estimated random process of the measurement arrival. There are three chief differences between the Gaussian-Beta filters and the VBAKF. Firstly, instead of computing the expected value of the measurement arrival variable in VBAKF, we utilize the threshold technology [22], [23], [24] to directly determine the value of the measurement arrival variable in Gaussian-Beta filters, since threshold technology has a complete theoretical basis and is widely used in practical applications. Secondly, when deriving the joint posterior density of the hidden state and the unknown measurement loss rate, the posterior density of measurement arrival needs to be estimated simultaneously in the VBAKF, while, in Gaussian-Beta filters, the estimated sequence of measurement arrivals is viewed as extra measurement information, leading to better estimation accuracy. Finally, the computational complexity of the VBAKF increases with the number of iterations, while Gaussian-Beta filters only require a single recursive operation and their structures are similar to that of the standard Gaussian filters [23]. Numerical results further validate the superiority of our Gaussian-Beta filters in estimation precision, convergence speed and computational complexity compared with the existing filtering algorithms.

The main contributions of the paper are as follows. Firstly, a novel Kalman-Beta filter (KBF) is proposed for linear Gaussian state-space models with unknown probability of measurement loss, and it can jointly estimate the hidden state and unknown probability of measurement loss. The KBF has a simple form, which facilitates its practical application. Secondly, the proposed KBF is further extended to the nonlinear case, and the extended Kalman-Beta filter (EKBF) and the unscented Kalman-Beta filter (UKBF) are developed. Finally, two target tracking scenarios are used to verify the effectiveness of the proposed filtering algorithms.

The rest of the paper is outlined as follows. In Section II, we formulate the filtering problem for a linear Gaussian state-space model with unknown probability of measurement loss. In Section III, the Kalman-Beta filter is designed to jointly estimate the hidden state and the unknown probability of measurement loss. In Section IV, we extend the KBF to moderate nonlinear Gaussian stochastic systems, and give the specific implementations of Gaussian-Beta filters in both extended-Kalman and unscented-Kalman filtering frameworks, named as extended-Kalman-Beta filter and unscented-Kalman-Beta filter, respectively. In Section V, numerical simulation is performed to validate the effectiveness of the proposed algorithms. Finally, we conclude the paper in Section VI.

II. PROBLEM FORMULATION

Consider the following linear Gaussian state-space system

$$\boldsymbol{x}_k = F_k \boldsymbol{x}_{k-1} + \boldsymbol{w}_k \tag{1}$$

$$\boldsymbol{y}_k = H_k \boldsymbol{x}_k + \boldsymbol{v}_k \tag{2}$$

where k is the time index, $\boldsymbol{x}_k \in \mathbb{R}^n$ is the hidden state vector, $F_k \in \mathbb{R}^{n\times n}$ is the state transition matrix, $\boldsymbol{w}_k \in \mathbb{R}^n$ is the process noise, subjected to a Gaussian distribution with zero means and covariance matrix Q_k, i.e., $\boldsymbol{w}_k \sim \mathcal{N}(\boldsymbol{w}_k; \mathbf{0}, Q_k)$, $\boldsymbol{y}_k \in \mathbb{R}^m$ is the measurement vector, $H_k \in \mathbb{R}^{m\times n}$ is the observation matrix, and $\boldsymbol{v}_k \in \mathbb{R}^m$ is the measurement noise.

To address the filtering problem for the linear Gaussian state-space model with intermittent measurements, the arrival of the measurement at time k is defined as a binary random variable $\gamma_k \in \{0, 1\}$ with probability $\Pr(\gamma_k = 0) = \tau_k$, where $\gamma_k = 0$ denotes the estimator does not receive the sensor measurement originated from state at time k, while

$\gamma_k = 1$ means that the estimator receives the sensor measurement originated from state at time k, and τ_k denotes the probability of measurement loss at time k. Then, the measurement noise $\boldsymbol{v}_k$ can be formulated as follows [14]:

$$p\left(\boldsymbol{v}_k|\gamma_k\right) = \begin{cases} \mathcal{N}\left(\mathbf{0}, R_k\right), & \gamma_k = 1 \\ \mathcal{N}\left(\mathbf{0}, \sigma_k^2 I\right), & \gamma_k = 0 \end{cases} \tag{3}$$

where σ a real scalar, and $I \in \mathbb{R}^{m\times m}$ is an m-dimensional identity matrix. From (3), the covariance matrix of measurement noise $\tilde{R}_k$ is related to the measurement arrival variable γ_k, i.e., $\tilde{R}_k = R_k$ if $\gamma_k = 1$, and $\tilde{R}_k = \sigma_k^2 I$ otherwise. The measurement loss in reality corresponds to the limiting case of $\sigma_k \to \infty$ [14]. This equivalent to the sensor receiving a very large outlier. In this paper, it is presumed that the initial state $\boldsymbol{x}_0$, $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$ are mutually independent.

The goal of this paper is to derive a Bayesian filter to adaptively estimate the joint posterior density of the hidden state and the unknown probability of measurement loss conditionally on both the sensor measurement sequence and the estimated measurement arrival sequence, i.e., $p\left(\boldsymbol{x}_k, \tau_k|\boldsymbol{y}_{1:k}, \gamma_{1:k}\right)$, where $\boldsymbol{y}_{1:k}$ and $\gamma_{1:k}$ are respectively the collections of the variables $\boldsymbol{y}$ and γ until time k. It should be emphasized that when computing the joint posterior density of the hidden state and the unknown measurement loss rate, the estimated measurement arrival sequence $\gamma_{1:k}$, determined by the threshold technology [22], [23], [24] (see in Section III-C), is also viewed as extra measurement information to obtain better estimation performance.

III. KALMAN-BETA FILTER

To solve the filtering problem for a linear Gaussian state-space model with unknown probability of measurement loss, a Kalman-Beta filter (KBF) is developed in this section. First, the time update is introduced, in which a heuristic dynamic model is employed for the probability of measurement loss. Then, a variational Bayesian approach is used to infer both the state and probability of measurement loss. Finally, we briefly introduce the threshold technology, which is used to estimate the binary variable of the measurement arrival.

A. TIME UPDATE

Assume that the joint posterior density of the hidden state $\boldsymbol{x}_{k-1}$ and the unknown probability of measurement loss τ_{k-1} at time $k-1$ has a product form of Gaussian and Beta distributions as follows:

$$\begin{aligned} p\left(\boldsymbol{x}_{k-1}, \tau_{k-1}|\boldsymbol{y}_{1:k-1}, \gamma_{1:k-1}\right) &= \mathcal{N}\left(\boldsymbol{x}_{k-1}; \boldsymbol{x}_{k-1|k-1}, P_{k-1|k-1}\right) \\ &\quad \times \mathrm{Beta}\left(\tau_{k-1}; \alpha_{k-1|k-1}, \beta_{k-1|k-1}\right) \end{aligned} \tag{4}$$

where $\mathrm{Beta}(\cdot; \alpha, \beta)$ denotes the Beta distribution with two positive shape parameters α and β, and its probability density function [25] is

$$\begin{aligned} p\left(\tau; \alpha, \beta\right) &= \frac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\tau^{\alpha-1}(1-\tau)^{\beta-1} \\ &= \frac{1}{\mathrm{B}\left(\alpha, \beta\right)}\tau^{\alpha-1}(1-\tau)^{\beta-1} \end{aligned} \tag{5}$$

where $\Gamma(\cdot)$ is the Gamma distribution, $\mathrm{B}(\cdot)$ is a normalization constant. If $\tau \sim \mathrm{Beta}(\tau; \alpha, \beta)$, then $\mathrm{E}[\tau] = \frac{\alpha}{\alpha+\beta}$, where $\mathrm{E}[\cdot]$ denotes the expectation operator.

The joint predicted density of the hidden state $\boldsymbol{x}_k$ and the unknown probability of measurement loss τ_k at time k can be obtained by the following Chapman-Kolmogorov equation [1],

$$\begin{aligned} p\left(\boldsymbol{x}_k, \tau_k|\boldsymbol{y}_{1:k-1}, \gamma_{1:k-1}\right) = \int & p\left(\boldsymbol{x}_k, \tau_k|\boldsymbol{x}_{k-1}, \tau_{k-1}\right) \\ & \times p\left(\boldsymbol{x}_{k-1}, \tau_{k-1}|\boldsymbol{y}_{1:k-1}, \gamma_{1:k-1}\right) \\ & \times d\boldsymbol{x}_{k-1} d\tau_{k-1} \end{aligned} \tag{6}$$

where $p\left(\boldsymbol{x}_k, \tau_k|\boldsymbol{x}_{k-1}, \tau_{k-1}\right)$ is the dynamical model of the state and the probability of measurement loss. In the framework of Byesian theory, it is necessary to construct an appropriate dynamic model for variables $\boldsymbol{x}_k$ and τ_k. In order to retain the product form of Gaussian and Beta distributions for the joint predicted density (6), the following modelling assumptions, which are also consistent with practical applications, need to be made.

Assumption 1: The dynamics of the hidden state and the probability of measurement loss are mutually independent, i.e.,

$$p\left(\boldsymbol{x}_k, \tau_k|\boldsymbol{x}_{k-1}, \tau_{k-1}\right) = p\left(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}\right) p\left(\tau_k|\tau_{k-1}\right) \tag{7}$$

Thus, the factored form will remain in the time update step, and the predicted density of the hidden state can be obtained according to the standard KF's time update [5], i.e., $p\left(\boldsymbol{x}_k|\boldsymbol{y}_{1:k}, \gamma_{1:k}\right) = \mathcal{N}\left(\boldsymbol{x}_k; \boldsymbol{x}_{k|k-1}, P_{k|k-1}\right)$, where the mean $\boldsymbol{x}_{k|k-1}$ and the covariance $P_{k|k-1}$ are computed by

$$\boldsymbol{x}_{k|k-1} = F_k \boldsymbol{x}_{k-1|k-1} \tag{8}$$

$$P_{k|k-1} = F_k P_{k-1|k-1} F_k^{\mathrm{T}} + Q_k \tag{9}$$

where $(\cdot)^{\mathrm{T}}$ denotes the transpose operator.

Assumption 2: The dynamic of the probability of measurement loss $p\left(\tau_k|\tau_{k-1}\right)$ is unknown and it is not straightforward to choose a dynamic model for the variable τ_k to yield a Beta distribution as its predicted distribution. Here, we adopt the heuristic dynamic model for τ_k suggested in [18] and [26], which simply propagates its previous approximate posterior distribution (4), i.e.,

$$\alpha_{k|k-1} = \rho\alpha_{k-1|k-1} \tag{10}$$

$$\beta_{k|k-1} = \rho\beta_{k-1|k-1} \tag{11}$$

where $\rho \in (0, 1]$ is a forgetting factor. The value $\rho = 1$ means a stationary probability of measurement loss and a smaller value means its fluctuation in time. Related studies have shown that choosing a constant close to 1 usually satisfies engineering applications.

The rationality of Assumption 2 can be explained from the following three aspects. On the one hand, the heuristic model ensures that the predicted density of the unknown probability of measurement loss also is also a Beta distribution, i.e.,

$p\left(\tau_k|\boldsymbol{y}_{1:k},\gamma_{1:k}\right) = \text{Beta}\left(\tau_k;\alpha_{k|k-1},\beta_{k|k-1}\right)$. On the other hand, the predicted expectation of the measurement loss rate τ_k at time k is equal to the posterior expectation of measurement loss rate τ_{k-1} at time $k-1$, i.e.,

$$\text{E}\left[\tau_{k|k-1}\right] = \text{E}\left[\tau_{k-1|k-1}\right] \tag{12}$$

It means that the measurement loss rate does not change drastically in a short period of time, which is consistent with practice. Finally, the heuristic model has a simple form, which facilitates its practical application.

Therefore, the joint predicted density of the hidden state and the unknown probability of measurement loss at time k remains a product form of Gaussian and Beta distributions based on the above Assumptions 1 and 2, and is given by

$$p\left(\boldsymbol{x}_k,\tau_k|\boldsymbol{y}_{1:k-1},\gamma_{1:k-1}\right) = \mathcal{N}\left(\boldsymbol{x}_k;\boldsymbol{x}_{k|k-1},P_{k|k-1}\right) \times \text{Beta}\left(\tau_k;\alpha_{k|k-1},\beta_{k|k-1}\right) \tag{13}$$

The involved parameters in (13) are obtained by (8), (9), (10) and (11).

B. MEASUREMENT UPDATE

Assume that the joint predicted density of the hidden state $\boldsymbol{x}_k$ and the unknown probability of measurement loss τ_k at time k has a product form of Gaussian and Beta distributions as shown in (13), then the joint posterior density is updated by the Bayesian's rule [1] as follows:

$$\begin{aligned} &p\left(\boldsymbol{x}_k,\tau_k|\boldsymbol{y}_{1:k},\gamma_{1:k}\right)\\ &\quad\propto p\left(\boldsymbol{y}_k,\gamma_k,\boldsymbol{x}_k,\tau_k|\boldsymbol{y}_{1:k-1},\gamma_{1:k-1}\right)\\ &\quad= p\left(\boldsymbol{y}_k,\gamma_k|\boldsymbol{x}_k,\tau_k\right)p\left(\boldsymbol{x}_k,\tau_k|\boldsymbol{y}_{1:k-1},\gamma_{1:k-1}\right) \end{aligned} \tag{14}$$

where the likelihood function $p\left(\boldsymbol{y}_k,\gamma_k|\boldsymbol{x}_k,\tau_k\right)$ can be factored by

$$p\left(\boldsymbol{y}_k,\gamma_k|\boldsymbol{x}_k,\tau_k\right) = p\left(\boldsymbol{y}_k|\boldsymbol{x}_k,\gamma_k\right)p\left(\gamma_k|\tau_k\right) \tag{15}$$

For the first term on the right-hand side of (15), it can be formulated by a product form of power functions (for the convenience of mathematical derivations in Proof 1) as follows [18]:

$$p\left(\boldsymbol{y}_k|\boldsymbol{x}_k,\gamma_k\right) = \left[\mathcal{N}\left(\boldsymbol{y}_k;H_k\boldsymbol{x}_k,R_k\right)\right]^{\gamma_k} \times\left[\mathcal{N}\left(\boldsymbol{y}_k;H_k\boldsymbol{x}_k,\sigma_k^2 I\right)\right]^{1-\gamma_k} \tag{16}$$

i.e.,

$$p\left(\boldsymbol{y}_k|\boldsymbol{x}_k,\gamma_k=1\right) = \mathcal{N}\left(\boldsymbol{y}_k;H_k\boldsymbol{x}_k,R_k\right) \tag{17}$$

$$p\left(\boldsymbol{y}_k|\boldsymbol{x}_k,\gamma_k=0\right) = \mathcal{N}\left(\boldsymbol{y}_k;H_k\boldsymbol{x}_k,\sigma_k^2 I\right) \tag{18}$$

which are consistent with the observation models of (2) and (3). For the second term of the right-hand side of (15), in which the binary stochastic variable γ_k is subjected to a Bernoulli distribution with probability $\Pr\left(\gamma_k=1\right)=1-\tau_k$, thus its probability density function is given by

$$p\left(\gamma_k|\tau_k\right) = \begin{cases} \left(1-\tau_k\right)^{\gamma_k}\left(\tau_k\right)^{1-\gamma_k}, & \gamma_k\in\{0,1\}\\ 0, & \text{otherwise}\end{cases} \tag{19}$$

Substituting (15), (16) and (19) in (14), the joint posterior density $p\left(\boldsymbol{x}_k,\tau_k|\boldsymbol{y}_{1:k},\gamma_{1:k}\right)$ is given by

$$\begin{aligned} &p\left(\boldsymbol{x}_k,\tau_k|\boldsymbol{y}_{1:k},\gamma_{1:k}\right)\\ &\quad\propto p\left(\boldsymbol{y}_k,\gamma_k,\boldsymbol{x}_k,\tau_k|\boldsymbol{y}_{1:k-1},\gamma_{1:k-1}\right)\\ &\quad=\left[\mathcal{N}\left(\boldsymbol{y}_k;H_k\boldsymbol{x}_k,R_k\right)\right]^{\gamma_k}\left[\mathcal{N}\left(\boldsymbol{y}_k;H_k\boldsymbol{x}_k,\sigma_k^2 I\right)\right]^{1-\gamma_k}\\ &\qquad\times\mathcal{N}\left(\boldsymbol{x}_k;\boldsymbol{x}_{k|k-1},P_{k|k-1}\right)\left(1-\tau_k\right)^{\gamma_k}\left(\tau_k\right)^{1-\gamma_k}\\ &\qquad\times\text{Beta}\left(\tau_k;\alpha_{k|k-1},\beta_{k|k-1}\right) \end{aligned} \tag{20}$$

With rigorous mathematical deduction, the joint posterior density $p\left(\boldsymbol{x}_k,\tau_k|\boldsymbol{y}_{1:k},\gamma_{1:k}\right)$ has a closed-form solution, and is still a product form of Gaussian and Beta distributions as follows:

$$p\left(\boldsymbol{x}_k,\tau_k|\boldsymbol{y}_{1:k},\gamma_{1:k}\right)=\mathcal{N}\left(\boldsymbol{x}_k;\boldsymbol{x}_{k|k},P_{k|k}\right)\text{Beta}\left(\tau_k;\alpha_{k|k},\beta_{k|k}\right) \tag{21}$$

where the mean $\boldsymbol{x}_{k|k}$ and the covariance $P_{k|k}$ of the state $\boldsymbol{x}_k$ are computed by

$$\boldsymbol{x}_{k|k} = \boldsymbol{x}_{k|k-1} + K_k\left(\boldsymbol{y}_k - H_k\boldsymbol{x}_{k|k-1}\right) \tag{22}$$

$$P_{k|k} = P_{k|k-1} - K_k H_k P_{k|k-1} \tag{23}$$

$$K_k = P_{k|k-1}H_k^{\text{T}}\left(H_k P_{k|k-1}H_k^{\text{T}} + \gamma_k R_k + \left(1-\gamma_k\right)\sigma_k^2 I\right)^{-1} \tag{24}$$

and the shape parameters $\alpha_{k|k}$ and $\beta_{k|k}$ of the probability of measurement loss τ_k are computed by

$$\alpha_{k|k} = \alpha_{k|k-1} - \gamma_k + 1 \tag{25}$$

$$\beta_{k|k} = \beta_{k|k-1} + \gamma_k \tag{26}$$

It can be seen that the measurement update of the hidden state x_k in KBF is consistent with that of the standard KF [5], except that the choice of the measurement noise covariance $\tilde{R}_k$ for the filtering gain is related to the estimation of γ_k, i.e., $\tilde{R}_k = R_k$ if $\gamma_k = 1$, and $\tilde{R}_k = \sigma_k^2 I$ otherwise. In addition, the shape parameters $\alpha_{k|k}$ and $\beta_{k|k}$ of the probability of measurement loss τ_k are adjusted through the estimation of the binary variable γ_k. The detailed mathematical derivations of the joint posterior density $p\left(\boldsymbol{x}_k,\tau_k|\boldsymbol{y}_{1:k},\gamma_{1:k}\right)$ is presented in the following Proof 1.

Proof 1: Since (20) has a product form of multiple functions, we utilize the idea of the variational Bayesian method [19], [20], [21] to seek its solution, i.e.,

$$p\left(\boldsymbol{x}_k,\tau_k|\boldsymbol{y}_{1:k},\gamma_{1:k}\right) \approx q_x\left(\boldsymbol{x}_k\right)q_\tau\left(\tau_k\right) \tag{27}$$

Variational Bayesian methods are widely used to compute approximate posterior inference. They derive a simple and analytical tractable form for the posterior distribution, which is either a factored form distribution or a fixed-form distribution. Using variational Bayesian technique, the Kullback-Leibler (KL) divergence [27] between the factorized densities and the true posterior density is minimized

$$\hat{q}_x\left(\boldsymbol{x}_k\right)\hat{q}_\tau\left(\tau_k\right) = \arg\min_{q_x,q_\tau}\text{KL}\left(q_x\left(\boldsymbol{x}_k\right)q_\tau\left(\tau_k\right)||p\left(\boldsymbol{x}_k,\tau_k|\boldsymbol{y}_{1:k},\gamma_{1:k}\right)\right) \tag{28}$$

where $\mathrm{KL}(q(\cdot)\|p(\cdot)) \triangleq \int q(x)\log\frac{q(x)}{p(x)}dx$ is the KL divergence. Then analytical solutions for $\hat{q}_x$ and $\hat{q}_\tau$ can be obtained by fixed point iterations, i.e.,

$$\log q_x(\boldsymbol{x}_k) \leftarrow \underset{q_\tau}{\mathrm{E}}\left[\log p\left(\boldsymbol{y}_k, \gamma_k, \boldsymbol{x}_k, \tau_k | \boldsymbol{y}_{1:k-1}, \gamma_{1:k-1}\right)\right] + c_x \tag{29}$$

$$\log q_\tau(\tau_k) \leftarrow \underset{q_x}{\mathrm{E}}\left[\log p\left(\boldsymbol{y}_k, \gamma_k, \boldsymbol{x}_k, \tau_k | \boldsymbol{y}_{1:k-1}, \gamma_{1:k-1}\right)\right] + c_\tau \tag{30}$$

where $\leftarrow$ denotes assign or reassign operators, the expected values on the right-hand sides of (29) and (30) are computed with regard to the current q_τ and q_x, c_x and c_τ are constants with regard to the variables $\boldsymbol{x}_k$ and τ_k, respectively.

According to (20), the logarithmic form of the joint posterior density $p(\boldsymbol{x}_k, \tau_k | \boldsymbol{y}_{1:k}, \gamma_{1:k})$, which is needed for the mathematical derivations, is given by

$$\begin{aligned}
&\log p\left(\boldsymbol{y}_k, \gamma_k, \boldsymbol{x}_k, \tau_k | \boldsymbol{y}_{1:k-1}, \gamma_{1:k-1}\right) \\
&= -\frac{\gamma_k}{2}\left(\boldsymbol{y}_k - H_k\boldsymbol{x}_k\right)^{\mathrm{T}} R_k^{-1}\left(\boldsymbol{y}_k - H_k\boldsymbol{x}_k\right) \\
&\quad - \frac{1-\gamma_k}{2}\left(\boldsymbol{y}_k - H_k\boldsymbol{x}_k\right)^{\mathrm{T}}\left(\sigma_k^2 I\right)^{-1}\left(\boldsymbol{y}_k - H_k\boldsymbol{x}_k\right) \\
&\quad + \gamma_k\log(1-\tau_k) + (1-\gamma_k)\log\tau_k \\
&\quad - \frac{1}{2}\left(\boldsymbol{x}_k - \boldsymbol{x}_{k|k-1}\right)^{\mathrm{T}} P_{k|k-1}^{-1}\left(\boldsymbol{x}_k - \boldsymbol{x}_{k|k-1}\right) \\
&\quad + \left(\alpha_{k|k-1} - 1\right)\log\tau_k + \left(\beta_{k|k-1} - 1\right)\log(1-\tau_k) + c
\end{aligned} \tag{31}$$

where c is a constant with respect to the variables $\boldsymbol{x}_k$ and τ_k.

Substituting (31) in (29), we obtain

$$\begin{aligned}
\log q_x(\boldsymbol{x}_k) = &-\frac{\gamma_k}{2}\left(\boldsymbol{y}_k - H_k\boldsymbol{x}_k\right)^{\mathrm{T}} R_k^{-1}\left(\boldsymbol{y}_k - H_k\boldsymbol{x}_k\right) \\
&- \frac{1-\gamma_k}{2}\left(\boldsymbol{y}_k - H_k\boldsymbol{x}_k\right)^{\mathrm{T}}\left(\sigma_k^2 I\right)^{-1}\left(\boldsymbol{y}_k - H_k\boldsymbol{x}_k\right) \\
&- \frac{1}{2}\left(\boldsymbol{x}_k - \boldsymbol{x}_{k|k-1}\right)^{\mathrm{T}} P_{k|k-1}^{-1}\left(\boldsymbol{x}_k - \boldsymbol{x}_{k|k-1}\right) + c_x
\end{aligned} \tag{32}$$

Hence,

$$q_x(\boldsymbol{x}_k) \propto \begin{cases} \mathcal{N}\left(\boldsymbol{y}_k; H_k\boldsymbol{x}_k, R_k\right)\mathcal{N}\left(\boldsymbol{x}_k; \boldsymbol{x}_{k|k-1}, P_{k|k-1}\right), & \gamma_k = 1 \\ \mathcal{N}\left(\boldsymbol{y}_k; H_k\boldsymbol{x}_k, \sigma_k^2 I\right)\mathcal{N}\left(\boldsymbol{x}_k; \boldsymbol{x}_{k|k-1}, P_{k|k-1}\right), & \gamma_k = 0 \end{cases} \tag{33}$$

Therefore, according to the standard KF's measurement update [5], $q_x(\boldsymbol{x}_k)$ of (33) still obeys Gaussian distribution, i.e., $q_x(\boldsymbol{x}_k) = \mathcal{N}(\boldsymbol{x}_k; \boldsymbol{x}_{k|k}, P_{k|k})$, where the mean $\boldsymbol{x}_{k|k}$ and the covariance matrix $P_{k|k}$ are obtained by (22), (23) and (24).

Substituting (31) in (30), we obtain

$$\begin{aligned}
\log q_\tau(\tau_k) &= \gamma_k\log(1-\tau_k) + (1-\gamma_k)\log\tau_k \\
&\quad + \left(\alpha_{k|k-1} - 1\right)\log\tau_k + \left(\beta_{k|k-1} - 1\right) \\
&\quad \times \log(1-\tau_k) + c_\tau \\
&= \left(\left(\alpha_{k|k-1} - \gamma_k + 1\right) - 1\right)\log\tau_k \\
&\quad + \left(\beta_{k|k-1} + \gamma_k - 1\right)\log(1-\tau_k)
\end{aligned} \tag{34}$$

Hence, $q_\tau(\tau_k)$ of (34) is still a Beta distribution, i.e., $q_\tau(\tau_k) = \mathrm{Beta}(\tau_k; \alpha_{k|k}, \beta_{k|k})$, where the shape parameters $\alpha_{k|k}$ and $\beta_{k|k}$ are obtained by (25) and (26).

It should be emphasized that the solution of (27) is strictly closed, i.e., $q_x(\boldsymbol{x}_k)\, q_\tau(\tau_k) = p(\boldsymbol{x}_k, \tau_k | \boldsymbol{y}_{1:k}, \gamma_{1:k})$. On one hand, since $q_x(\boldsymbol{x}_k)$ of (33) is a product form of two Gaussian densities, which are conjugate priors, and is unrelated to variable τ_k, we have

$$p\left(\boldsymbol{x}_k | \boldsymbol{y}_{1:k}, \gamma_{1:k}\right) = q_x(\boldsymbol{x}_k) = \mathcal{N}\left(\boldsymbol{x}_k; \boldsymbol{x}_{k|k}, P_{k|k}\right) \tag{35}$$

On the other hand, since $q_\tau(\tau_k)$ of (34) is a product form of Beta and Bernoulli distributions, which are conjugate priors [18], [28], and is unrelated to variable $\boldsymbol{x}_k$, we have

$$p\left(\tau_k | \boldsymbol{y}_{1:k}, \gamma_{1:k}\right) = q_\tau(\tau_k) = \mathrm{Beta}\left(\tau_k; \alpha_{k|k}, \beta_{k|k}\right) \tag{36}$$

Therefore, according to (35) and (36), the joint posterior density of (21) is strictly closed.

C. COMPUTING MEASUREMENT ARRIVAL VARIABLE

Instead of computing the expectation of the binary variable γ_k in [17] and [18], we utilize the threshold technology [22], [23], [24] to directly determine the value of γ_k, i.e., the process of the measurement arrival. Considering the state-space models based on Gaussian distribution, the Chi-square distribution, denoted by χ^2, is a useful tool to determine the threshold gate. Assume that the probability of the sensor measurement falling into the threshold gate is p_G, where G is the size of the gate, then p_G and G satisfied the following relationship:

$$p\left(\chi_\eta^2 > G\right) = 1 - p_G \tag{37}$$

where η is the degree of freedom (dof). Employing (37), for a certain value of p_G, the corresponding threshold gate value can be directly found in the Chi-square Distribution Table.

In the measurement update of the KBF, the innovation $\tilde{\boldsymbol{y}}_{k|k-1} = \boldsymbol{y}_k - \boldsymbol{y}_{k|k-1}$ and its covariance S_k can be easily obtained, and the measurement noise covariance involved in S_k is R_k. The norm of the innovation is defined as

$$D_k^2 = \tilde{\boldsymbol{y}}_{k|k-1}^{\mathrm{T}} S_k^{-1} \tilde{\boldsymbol{y}}_{k|k-1} \tag{38}$$

It has been proved that the variable D_k^2 obeys a χ^2 distribution with dof parameter $\eta = m$, where m is the dimension of the measurement. Then, γ_k can be determined by

$$\gamma_k = \begin{cases} 1, & D_k^2 \le G \\ 0, & D_k^2 > G \end{cases} \tag{39}$$

The detailed implementations of the KBF are presented in Algorithm 1.

IV. EXTENSION AND UNCENTED KALMAN-BETA FILTERS

In Section III, we have derived the KBF to solve the filtering problem for linear Gaussian state-space models with unknown probability of measurement loss. Obviously, KBF can be extended to the moderate nonlinear Gaussian

Algorithm 1 Kalman-Beta Filtering Approach

Require: F_k, H_k, Q_k, R_k, σ_k, $\boldsymbol{y}_k$, $\boldsymbol{x}_{k-1|k-1}$, $P_{k-1|k-1}$, $\alpha_{k-1|k-1}$, $\beta_{k-1|k-1}$, ρ, G

Time update: compute $\boldsymbol{x}_{k|k-1}$, $P_{k|k-1}$, $\alpha_{k|k-1}$, $\beta_{k|k-1}$

$\boldsymbol{x}_{k|k-1} \leftarrow F_k \boldsymbol{x}_{k-1|k-1}$

$P_{k|k-1} \leftarrow F_k P_{k-1|k-1} F_k^{\mathrm{T}} + Q_k$

$\alpha_{k|k-1} \leftarrow \rho \alpha_{k-1|k-1}$

$\beta_{k|k-1} \leftarrow \rho \beta_{k-1|k-1}$

Measurement update: compute $\boldsymbol{x}_{k|k}$, $P_{k|k}$, $\alpha_{k|k}$, $\beta_{k|k}$

$\tilde{\boldsymbol{y}}_{k|k-1} \leftarrow \boldsymbol{y}_k - H_k \boldsymbol{x}_{k|k-1}$

$S_k \leftarrow H_k P_{k|k-1} H_k^{\mathrm{T}} + R_k$

$D_k^2 \leftarrow \tilde{\boldsymbol{y}}_{k|k-1}^{\mathrm{T}} S_k^{-1} \tilde{\boldsymbol{y}}_{k|k-1}$

$\gamma_k \leftarrow \begin{cases} 1, & D_k^2 \leq G \\ 0, & D_k^2 > G \end{cases}$

$K_k \leftarrow P_{k|k-1} H_k^{\mathrm{T}} \left(H_k P_{k|k-1} H_k^{\mathrm{T}} + \gamma_k R_k + (1-\gamma_k)\, \sigma_k^2 I \right)^{-1}$

$\boldsymbol{x}_{k|k} \leftarrow \boldsymbol{x}_{k|k-1} + K_k \tilde{\boldsymbol{y}}_{k|k-1}$

$P_{k|k} \leftarrow P_{k|k-1} - K_k H_k P_{k|k-1}$

$\alpha_{k|k} \leftarrow \alpha_{k|k-1} - \gamma_k + 1$

$\beta_{k|k} \leftarrow \beta_{k|k-1} + \gamma_k$

$\tau_{k|k} \leftarrow \frac{\alpha_{k|k}}{\alpha_{k|k} + \beta_{k|k}}$

Ensure: $\boldsymbol{x}_{k|k}$, $P_{k|k}$, $\tau_{k|k}$

state-space models by introducing the general Gaussian filters [23]. Next, we first give the general recursive structure of the Gaussian-Beta filters, and then present the specific implementations under the standard extended-Kalman and unscented-Kalman filtering frameworks. The mathematical derivations of the Gaussian-Beta filter for moderate nonlinear Gaussian systems with unknown probability of measurement loss is similar to that of the KBF for the case of linear Gaussian systems, which will not be provided in the paper.

Consider the following moderate nonlinear Gaussian state-space model

$$\boldsymbol{x}_k = f(\boldsymbol{x}_{k-1}, \boldsymbol{u}_k) + \boldsymbol{w}_k \tag{40}$$

$$\boldsymbol{y}_k = h(\boldsymbol{x}_k) + \boldsymbol{v}_k \tag{41}$$

where k is the time index, $f(\cdot)$ and $h(\cdot)$ are nonlinear functions, $\boldsymbol{x}_k \in \mathbb{R}^n$ is the hidden state vector, $\boldsymbol{u}_k$ is the input, $\boldsymbol{w}_k \in \mathbb{R}^n$ is the process noise, subjected to a Gaussian distribution with zero means and covariance matrix Q_k, i.e., $\boldsymbol{w}_k \sim \mathcal{N}(\boldsymbol{w}_k; \boldsymbol{0}, Q_k)$, $\boldsymbol{y}_k \in \mathbb{R}^m$ is the measurement vector, and $\boldsymbol{v}_k \in \mathbb{R}^m$ is the measurement noise, subjected to the model of (3).

A. TIME UPDATE

Assume that the joint posterior density of the hidden state $\boldsymbol{x}_{k-1}$ and the unknown probability of measurement loss τ_{k-1} at time $k-1$ is approximated to a product form of Gaussian and Beta distributions as shown in (4). Then, the joint predicted density $p\left(\boldsymbol{x}_k, \tau_k | \boldsymbol{y}_{1:k-1}, \gamma_{1:k-1}\right)$ can be approximated to a product form of Gaussian and Beta distributions as shown in (13), where the mean $\boldsymbol{x}_{k|k-1}$ and the covariance $P_{k|k-1}$ of the hidden state $\boldsymbol{x}_k$ are computed by

$$\boldsymbol{x}_{k|k-1} = \int f(\boldsymbol{x}_{k-1}, \boldsymbol{u}_k) \times \mathcal{N}\left(\boldsymbol{x}_{k-1}; \boldsymbol{x}_{k-1|k-1}, P_{k-1|k-1}\right) d\boldsymbol{x}_{k-1} \tag{42}$$

$$P_{k|k-1} = \int \left(f(\boldsymbol{x}_{k-1}, \boldsymbol{u}_k) - \boldsymbol{x}_{k-1|k-1}\right) \times \left(f(\boldsymbol{x}_{k-1}, \boldsymbol{u}_k) - \boldsymbol{x}_{k-1|k-1}\right)^{\mathrm{T}} \times \mathcal{N}\left(\boldsymbol{x}_{k-1}; \boldsymbol{x}_{k-1|k-1}, P_{k-1|k-1}\right) d\boldsymbol{x}_{k-1} + Q_k \tag{43}$$

and the shape parameters $\alpha_{k|k-1}$ and $\beta_{k|k-1}$ of the unknown probability of measurement loss τ_k are computed by (25) and (26).

B. MEASUREMENT UPDATE

Assume that the joint predicted density of the hidden state $\boldsymbol{x}_k$ and the unknown probability of measurement loss τ_k is approximated to a product form of Gaussian and Beta distributions as shown in (13), then the joint posterior density can also be approximated to a product form of Gaussian and Beta distributions as shown in (21), where the mean $\boldsymbol{x}_{k|k}$ and the covariance $P_{k|k}$ of the unobservable state $\boldsymbol{x}_k$ are computed by

$$\boldsymbol{y}_{k|k-1} = \int h(\boldsymbol{x}_k)\, \mathcal{N}\left(\boldsymbol{x}_k; \boldsymbol{x}_{k|k-1}, P_{k|k-1}\right) d\boldsymbol{x}_k \tag{44}$$

$$P_{yy,k} = \int \left(h(\boldsymbol{x}_k) - \boldsymbol{y}_{k|k-1}\right)\left(h(\boldsymbol{x}_k) - \boldsymbol{y}_{k|k-1}\right)^{\mathrm{T}} \times \mathcal{N}\left(\boldsymbol{x}_k; \boldsymbol{x}_{k|k-1}, P_{k|k-1}\right) d\boldsymbol{x}_k + \gamma_k R_k + (1-\gamma_k)\, \sigma_k^2 I \tag{45}$$

$$P_{xy,k} = \int \left(\boldsymbol{x}_k - \boldsymbol{x}_{k|k-1}\right)\left(h(\boldsymbol{x}_k) - \boldsymbol{y}_{k|k-1}\right)^{\mathrm{T}} \times \mathcal{N}\left(\boldsymbol{x}_k; \boldsymbol{x}_{k|k-1}, P_{k|k-1}\right) d\boldsymbol{x}_k \tag{46}$$

$$K_k = P_{xy,k} P_{yy,k}^{-1} \tag{47}$$

$$\boldsymbol{x}_{k|k} = \boldsymbol{x}_{k|k-1} + K_k \left(\boldsymbol{y}_k - \boldsymbol{y}_{k|k-1}\right) \tag{48}$$

$$P_{k|k} = P_{k|k-1} - K_k P_{yy,k} K_k^{\mathrm{T}} \tag{49}$$

and the shape parameters $\alpha_{k|k}$ and $\beta_{k|k}$ of the unknown probability of measurement loss τ_k are computed by (25) and (26).

The Gaussian-Beta filters have recursive structures similar to the standard Gaussian filters, except for additional parameters γ_k, $\alpha_{k|k}$ and $\beta_{k|k}$, which are calculated in the same way as the KBF in previous section. The detailed implementations of the Gaussian-Beta filter in the extended-Kalman and unscented-Kalman filtering frameworks are presented in Algorithm 2 and 3.

V. SIMULATIONS

In this section, we demonstrate the proposed KBF, EKFB and UKBF in linear and moderate nonlinear Gaussian state-space models with unknown probability of measurement loss, respectively.

Algorithm 2 Extended-Kalman-Beta Filtering Approach

Require: $f(\cdot)$, $h(\cdot)$, Q_k, R_k, σ_k, $\boldsymbol{u}_k$, $\boldsymbol{y}_k$, $\boldsymbol{x}_{k-1|k-1}$, $P_{k-1|k-1}$, $\alpha_{k-1|k-1}$, $\beta_{k-1|k-1}$, ρ, G

Time update: compute $\boldsymbol{x}_{k|k-1}$, $P_{k|k-1}$, $\alpha_{k|k-1}$, $\beta_{k|k-1}$

1: $\boldsymbol{x}_{k|k-1} \leftarrow f\left(\boldsymbol{x}_{k-1|k-1}, \boldsymbol{u}_k\right)$

2: $F_k \leftarrow \left.\frac{\partial f(\boldsymbol{x}_{k-1})}{\partial \boldsymbol{x}_{k-1}}\right|_{\boldsymbol{x}_{k-1}=\boldsymbol{x}_{k-1|k-1}}$

3: $P_{k|k-1} \leftarrow F_k P_{k-1|k-1} F_k^{\mathrm{T}} + Q_k$

4: $\alpha_{k|k-1} \leftarrow \rho\alpha_{k-1|k-1}$

5: $\beta_{k|k-1} \leftarrow \rho\beta_{k-1|k-1}$

Measurement update: compute γ_k, $\boldsymbol{x}_{k|k}$, $P_{k|k}$, $\alpha_{k|k}$, $\beta_{k|k}$

6: $\tilde{\boldsymbol{y}}_{k|k-1} \leftarrow \boldsymbol{y}_k - h\left(\boldsymbol{x}_{k|k-1}\right)$

7: $H_k \leftarrow \left.\frac{\partial h(\boldsymbol{x}_k)}{\partial \boldsymbol{x}_k}\right|_{\boldsymbol{x}_k=\boldsymbol{x}_{k|k-1}}$

8: $S_k \leftarrow H_k P_{k|k-1} H_k^{\mathrm{T}} + R_k$

9: $D_k^2 \leftarrow \tilde{\boldsymbol{y}}_{k|k-1}^{\mathrm{T}} S_k^{-1} \tilde{\boldsymbol{y}}_{k|k-1}$

10: $\gamma_k \leftarrow \begin{cases} 1, & D_k^2 \le G \\ 0, & D_k^2 > G \end{cases}$

11: $K_k \leftarrow P_{k|k-1} H_k^{\mathrm{T}} \left(H_k P_{k|k-1} H_k^{\mathrm{T}} + \gamma_k R_k + (1-\gamma_k)\,\sigma_k^2 I\right)^{-1}$

12: $\boldsymbol{x}_{k|k} \leftarrow \boldsymbol{x}_{k|k-1} + K_k \tilde{\boldsymbol{y}}_{k|k-1}$

13: $P_{k|k} \leftarrow P_{k|k-1} - K_k H_k P_{k|k-1}$

14: $\alpha_{k|k} \leftarrow \alpha_{k|k-1} - \gamma_k + 1$

15: $\beta_{k|k} \leftarrow \beta_{k|k-1} + \gamma_k$

16: $\tau_{k|k} \leftarrow \frac{\alpha_{k|k}}{\alpha_{k|k}+\beta_{k|k}}$

Ensure: $\boldsymbol{x}_{k|k}$, $P_{k|k}$, $\tau_{k|k}$

Algorithm 3 Unscented-Kalman-Beta Filtering Approach

Require: $f(\cdot)$, $h(\cdot)$, Q_k, R_k, σ_k, $\boldsymbol{u}_k$, $\boldsymbol{y}_k$, $\boldsymbol{x}_{k-1|k-1}$, $P_{k-1|k-1}$, $\alpha_{k-1|k-1}$, $\beta_{k-1|k-1}$, ρ, G

Time update: compute $\boldsymbol{x}_{k|k-1}$, $P_{k|k-1}$, $\alpha_{k|k-1}$, $\beta_{k|k-1}$

Choose $2n+1$ sigma points, n is the dimension of $\boldsymbol{x}_k$

1: $\boldsymbol{\varsigma}_{k-1|k-1}^{0} \leftarrow \boldsymbol{x}_{k-1|k-1}$

2: $\boldsymbol{\varsigma}_{k-1|k-1}^{i} \leftarrow \boldsymbol{x}_{k-1|k-1} + \left(\sqrt{(n+\kappa)\,P_{k-1|k-1}}\right)_i$, $i = 1, \ldots, n$

3: $\boldsymbol{\varsigma}_{k-1|k-1}^{i} \leftarrow \boldsymbol{x}_{k-1|k-1} - \left(\sqrt{(n+\kappa)\,P_{k-1|k-1}}\right)_i$, $i = n+1, \ldots, 2n$

Propagate the sigma points $\boldsymbol{\varsigma}_{k-1|k-1}^{i}$ using $f(\cdot)$

4: $\boldsymbol{\varsigma}_{k|k-1}^{i} \leftarrow f\left(\boldsymbol{\varsigma}_{k-1|k-1}^{i}, \boldsymbol{u}_k\right)$, $\quad i = 0, 1, \ldots 2n$

5: $\omega^i \leftarrow \begin{cases} \kappa/(n+\kappa)\,, & \text{if } i = 0 \\ 0.5\kappa/(n+\kappa)\,, & \text{if } i = 1, \ldots, 2n \end{cases}$

6: $\boldsymbol{x}_{k|k-1} \leftarrow \sum_{i=0}^{2n} \omega^i \boldsymbol{\varsigma}_{k|k-1}^{i}$

7: $P_{k|k-1} \leftarrow \sum_{i=0}^{2n} \omega^i \left(\boldsymbol{\varsigma}_{k|k-1}^{i} - \boldsymbol{x}_{k|k-1}\right)\left(\boldsymbol{\varsigma}_{k|k-1}^{i} - \boldsymbol{x}_{k|k-1}\right)^{\mathrm{T}} + Q_k$

8: $\alpha_{k|k-1} \leftarrow \rho\alpha_{k-1|k-1}$

9: $\beta_{k|k-1} \leftarrow \rho\beta_{k-1|k-1}$

Measurement update: compute $\boldsymbol{x}_{k|k}$, $P_{k|k}$, $\alpha_{k|k}$, $\beta_{k|k}$

Propagate the sigma points $\boldsymbol{\varsigma}_{k|k-1}^{i}$ using $h(\cdot)$

10: $\boldsymbol{\xi}_k^i \leftarrow h\left(\boldsymbol{\varsigma}_{k|k-1}^{i}\right)$, $\quad i = 0, 1, \ldots 2n$

11: $\boldsymbol{y}_{k|k-1} \leftarrow \sum_{i=1}^{2n} \omega^i \boldsymbol{\xi}_k^i$

12: $\tilde{\boldsymbol{y}}_{k|k-1} \leftarrow \boldsymbol{y}_k - \boldsymbol{y}_{k|k-1}$

13: $S_k \leftarrow \sum_{i=1}^{2n} \omega^i \left(\boldsymbol{\xi}_k^i - \boldsymbol{y}_{k|k-1}\right)\left(\boldsymbol{\xi}_k^i - \boldsymbol{y}_{k|k-1}\right)^{\mathrm{T}} + R_k$

14: $D_k^2 \leftarrow \tilde{\boldsymbol{y}}_{k|k-1}^{\mathrm{T}} S_k^{-1} \tilde{\boldsymbol{y}}_{k|k-1}$

15: $\gamma_k \leftarrow \begin{cases} 1, & D_k^2 \le G \\ 0, & D_k^2 > G \end{cases}$

16: $P_{yy,k} \leftarrow \sum_{i=1}^{2n} \omega^i \left(\boldsymbol{\xi}_k^i - \boldsymbol{y}_{k|k-1}\right)\left(\boldsymbol{\xi}_k^i - \boldsymbol{y}_{k|k-1}\right)^{\mathrm{T}} + \gamma_k R_k + (1-\gamma_k)\,\sigma_k^2 I$

17: $P_{xy,k} \leftarrow \sum_{i=1}^{2n} \omega^i \left(\boldsymbol{\varsigma}_{k|k-1}^{i} - \boldsymbol{x}_{k|k-1}\right)\left(\boldsymbol{\xi}_k^i - \boldsymbol{y}_{k|k-1}\right)^{\mathrm{T}}$

18: $K_k \leftarrow P_{xy,k} P_{yy,k}^{-1}$

19: $\boldsymbol{x}_{k|k} \leftarrow \boldsymbol{x}_{k|k-1} + K_k \tilde{\boldsymbol{y}}_{k|k-1}$

20: $P_{k|k} \leftarrow P_{k|k-1} - K_k P_{yy,k} K_k^{\mathrm{T}}$

21: $\alpha_{k|k} \leftarrow \alpha_{k|k-1} - \gamma_k + 1$

22: $\beta_{k|k} \leftarrow \beta_{k|k-1} + \gamma_k$

23: $\tau_{k|k} \leftarrow \frac{\alpha_{k|k}}{\alpha_{k|k}+\beta_{k|k}}$

Ensure: $\boldsymbol{x}_{k|k}$, $P_{k|k}$, $\tau_{k|k}$

A. LINEAR EXAMPLE

Considering the linear Gaussian stochastic system shown in (1) and (2), the involved model parameters are set as

$$F_k = \begin{bmatrix} 0.6 & 0.4 \\ 0.1 & 0.9 \end{bmatrix}, \quad Q_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$H_k = \begin{bmatrix} 1 & -2 \end{bmatrix}$, $R_k = 150\ \mathrm{m}^2$, $\sigma_k = 500000$. The true probability of measurement loss is

$$p(\tau_k) = \begin{cases} 0.1, & 0 \le k \le K/3 \\ 0.3, & K/3 < k \le 2K/3 \\ 0.1, & 2K/3 < k \le K \end{cases} \tag{50}$$

where total simulation time is $K = 100$ s, the sampling period is $T = 0.01$ s.

In the simulation, the proposed KBF are compared with the existing IKF [14] with known measurement loss and the VBAKF in [18] over $N = 500$ Monte Carlo trials. The simulating parameters of the filtering algorithms with unknown probability of measurement loss are shown in Table 1. The averaged absolute error (AAE) at time k is defined by

$$\mathrm{AAE}_k \triangleq \frac{1}{N}\sum_{j=1}^{N} \mathrm{abs}\left(\boldsymbol{x}_{k,i}^{j} - \boldsymbol{x}_{k|k,i}^{j}\right), \quad i \in \{1, 2\} \tag{51}$$

where N is the total number of Monte Carlo trials, j is the j^{th} Monte Carlo trial, i is the dimension index, $\boldsymbol{x}_{k,i}^{j}$ and $\boldsymbol{x}_{k|k,i}^{j}$ are the true state and its estimation at time k, and $\mathrm{abs}(\cdot)$ denotes the absolute value operator.

TABLE 1. Simulating parameters.

Parameters	VBAKF	KBF
Nominal probability of measurement loss	0.5	0.5
Initial shape parameter α_0	5	5
Initial shape parameter β_0	5	5
The forgetting factor ρ	0.99	0.99
Iterative threshold δ	10^{-16}	–
Maximum number of iterations	30	–
The probability of Chi-square p_G	–	0.99

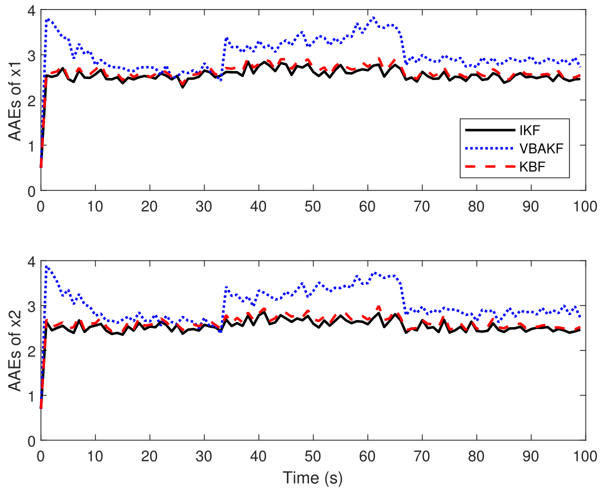

FIGURE 1. AAEs of $x1$ and $x2$ from different filters.

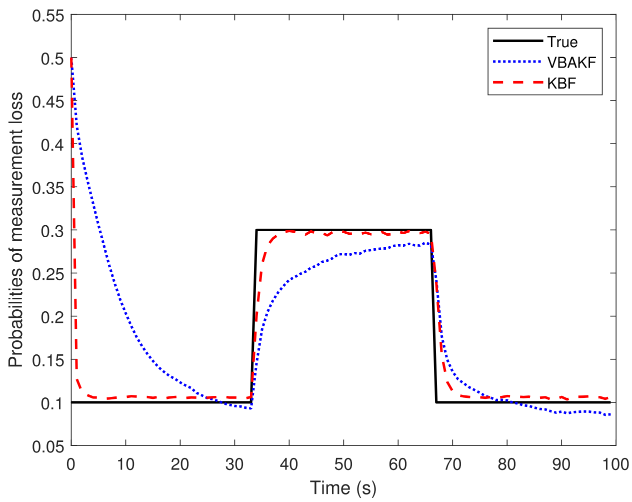

FIGURE 2. Estimations of the unknown probability of measurement loss τ_k.

Figure 1 shows the AAEs of $x1$ and $x2$ for different filters. It can be seen from Figure 1 that the estimation accuracy of the KBF proposed in this paper is very close to that of the IKF with known measurement loss, and better than that of the VBAKF.

Figure 2 shows the true and estimated probabilities of measurement loss. It can be seen from Figure 2 that, compared with the VBAKF, the proposed KBF has faster convergence speed and higher estimation accuracy for the measurement loss rate, which leading to higher estimation accuracy for the state variable as shown in Figure 1.

The averaged time costs of single Monte Carlo trial for different filtering algorithms are given in Table 2. Compared with the IKF, the proposed KBF needs to spend extra computational cost to estimate the measurement arrival variable and the measurement loss rate, while the VBAKF takes the most time cost because it needs many iterations to achieve stable state estimation.

TABLE 2. The averaged time costs of single Monte Carlo trial for different filters (10000 time samples).

Iterm	IKF	VBAKF	KBF
Time (s)	0.1478	1.7547	0.1945

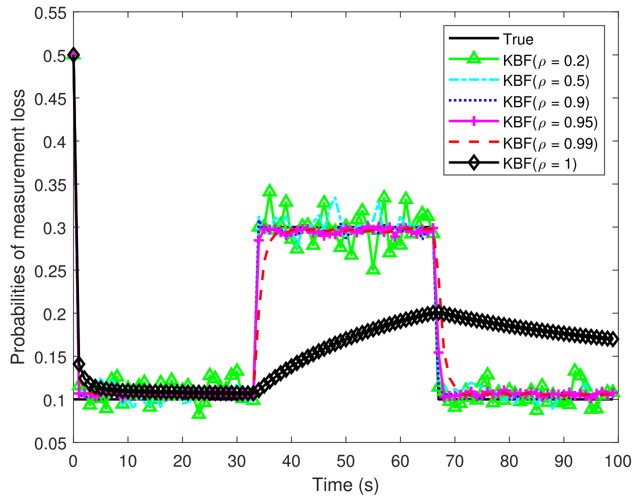

FIGURE 3. Estimations of the measurement loss rate τ_k with different ρ.

In addition, the estimated probabilities of measurement loss by the proposed KBF with different ρ are shown in Figure 3, and their associated errors are shown in Figure 4. It can be seen that when ρ is small, the estimated probabilities of measurement loss fluctuate significantly, resulting in large errors. With the increase of ρ, the KBF performs better. However, when the value ρ is infinitely close to 1, especially when $\rho = 1$, the performance of the KBF seriously deteriorates when the probability of measurement loss jumps. The main reason is that the probability of measurement loss is regarded as a stationary process when $\rho = 1$, which results in that the KBF cannot respond quickly when the variable τ_k changes significantly. In general, choosing a constant close to 1 for ρ usually satisfies engineering applications, and can respond quickly even when the probability of measurement loss jumps. The optimal forgetting factor is beyond the scope of this paper.

B. NONLINEAR EXAMPLE

Considering a target tracking application of the nonlinear stochastic dynamic and measurement models as shown in (40) and (41), the state vector is $\boldsymbol{x}_k = \left[\tilde{\boldsymbol{x}}_k^{\mathrm{T}}, \omega_k\right]^{\mathrm{T}}$, where $\tilde{\boldsymbol{x}}_k = \left[p_{x,k}, \dot{p}_{x,k}, p_{y,k}, \dot{p}_{y,k}\right]^{\mathrm{T}}$ consists of the planar position $\left(p_{x,k}, p_{y,k}\right)$ and velocity $\left(\dot{p}_{x,k}, \dot{p}_{y,k}\right)$, ω_k is the turn rate. The state transition model is

$$\tilde{\boldsymbol{x}}_{k+1} = F\left(\omega_k\right)\tilde{\boldsymbol{x}}_k + B\boldsymbol{w}_k \tag{52}$$

$$\omega_{k+1} = \omega_k + T\mu_k \tag{53}$$

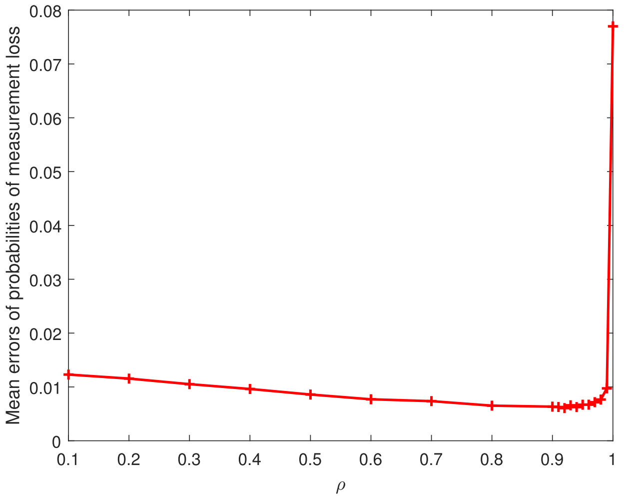

FIGURE 4. Errors of the measurement loss rate τ_k with different ρ.

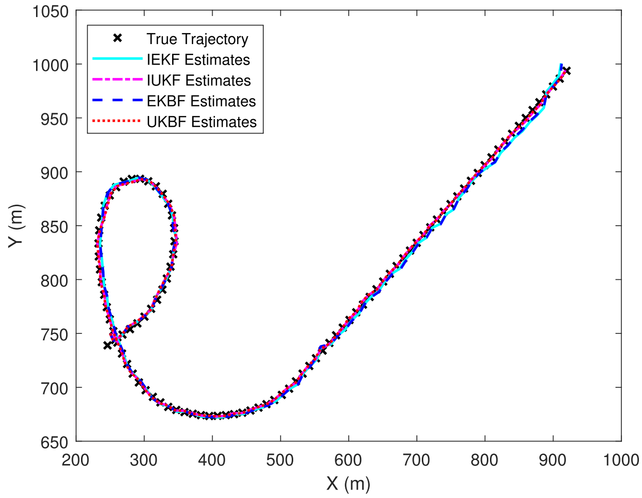

FIGURE 5. The true target trajectory and the estimations of different filters.

where

$$F(\omega)=\begin{bmatrix}1 & \frac{\sin(\omega T)}{\omega} & 0 & -\frac{\cos(\omega T)}{\omega}\\ 0 & \cos(\omega T) & 0 & -\sin(\omega T)\\ 0 & \frac{1-\cos(\omega T)}{\omega} & 1 & \frac{\sin(\omega T)}{\omega}\\ 0 & \sin(\omega T) & 0 & \cos(\omega T)\end{bmatrix},$$

$$B=\begin{bmatrix}\frac{T^2}{2} & 0\\ T & 0\\ 0 & \frac{T^2}{2}\\ 0 & T\end{bmatrix},$$

$\boldsymbol{w}_k \sim \mathcal{N}(\boldsymbol{w}_k; \boldsymbol{0}, \sigma_w^2 I)$, $\mu_k \sim \mathcal{N}(\mu_k; 0, \sigma_\mu^2 I)$, $T = 0.01$ s, $\sigma_w = 2$ m/s^2, $\sigma_\mu = (\pi/180)$ rad/s. The measurement model is

$$\boldsymbol{y}_k=\begin{bmatrix}\arctan(p_{x,k}/p_{y,k})\\ \sqrt{p_{x,k}^2+P_{y,k}^2}\end{bmatrix}+\boldsymbol{v}_k \tag{54}$$

where the statistic properties of v_k is shown in (3), $R_k = \mathrm{diag}([\sigma_{\theta 1}^2, \sigma_{r1}^2])$, $\sigma_k = [\sigma_{\theta 2}, \sigma_{r2}]$, $\sigma_{\theta 1} = (\pi/180)$ rad, $\sigma_{r1} = 5$ m, $\sigma_{\theta 2} = \pi$ rad, $\sigma_{r2} = 10000$ m. The true probability of measurement loss is set as (50), where the total simulation time is $K = 100$ s.

In the simulation, the proposed EKBF and UKBF are compared with the existing IEKF and IUKF with known measurement loss over $N = 500$ Monte Carlo trials. The simulating parameters with unknown probability of measurement loss are set as: the initial shape parameters $\alpha_0 = \beta_0 = 5$, the forgetting factor $\rho = 0.99$, and the probability of threshold gate $p_G = 0.99$. The averaged root mean square error (ARMSE) at time k is defined by

$$\mathrm{ARMSE} \triangleq \sqrt{\frac{1}{N}\sum_{j=1}^{N}\left(\left(p_{x,k}^j-\hat{p}_{x,k}^j\right)^2+\left(p_{y,k}^j-\hat{p}_{y,k}^j\right)^2\right)} \tag{55}$$

where $\left(p_{x,k}^j, p_{y,k}^j\right)$ and $\left(\hat{p}_{x,k}^j, \hat{p}_{y,k}^j\right)$ are the true and estimated positions of target at time k in the j^{th} Monte Carlo trial.

Figure 5 shows the true target trajectory and estimations of different filters for a single Monte Carlo trial. Figure 6 shows the ARMSEs of different filters over $N = 500$ Monte Carlo trails. It can be seen from Figure 6 that, on the one hand, the estimation accuracies of the IUKF and UKBF are obviously better than that of the IEKF and EKBF, since the latters have larger linearization errors for the nonlinear system. However, the formers take more time cost as shown in Table 3, since they need to propagate multiple sigma points in each recursive process. On the other hand, the estimation accuracy of the propose UKBF is very close to that of the IUKF with known measurement loss, while, as time goes on, the estimation accuracy of the proposed EKBF is worse than that of the IEKF with known measurement loss. The possible reason may be that the accumulated linearization approximation error leads to the worse estimation of the measurement arrival variable γ_k.

Figure 7 shows the true and estimated probabilities of measurement loss. It can be seen from Figure 7 that both the EKBF and UKBF can obtain accurate estimations for the measurement loss rate with fast convergence speed, and the estimation accuracy of the latter is slightly better than that of the former. In addition, the averaged time costs of single Monte Carlo trial for different filtering algorithms are given in Table 3. Although the estimation accuracies of the UKBF in both state and measurement loss rate are superior to that of the EKBF, the former takes more time cost. Therefore, it is necessary to make tradeoff between the estimation accuracy and time cost in practical applications.

It should be pointed out that, compared with the result of the KBF shown in Figure 2, both the EKBF and UKBF estimators have a biased estimation of the probability of measurement loss. By analyzing the three algorithms, the main reason is as follow. The key parameter that affects the

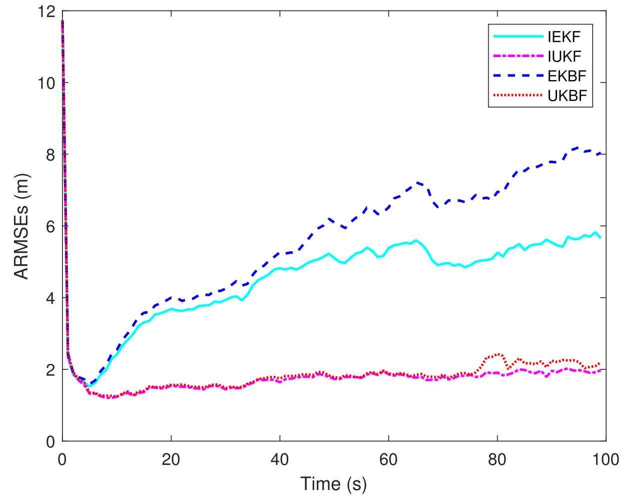

FIGURE 6. ARMSEs for different filters.

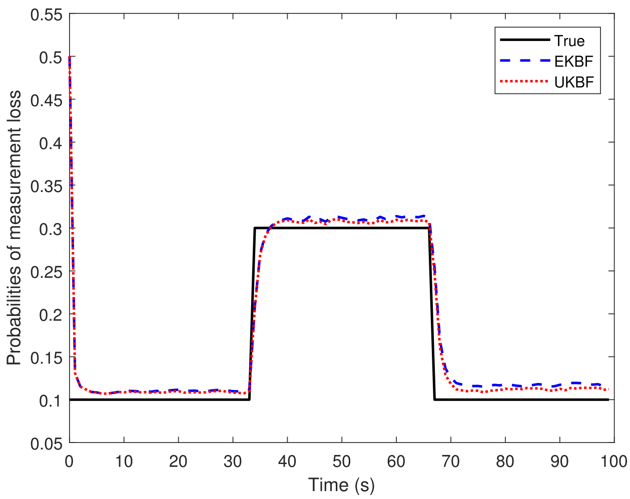

FIGURE 7. Estimations of the unknown probability of measurement loss τ_k.

TABLE 3. The averaged time costs of single Monte Carlo trial for different filters (10000 time samples).

Iterm	IEKF	EBKF	IUKF	UKBF
Time (s)	0.9612	0.9956	4.1346	4.2414

probability of measurement loss is the binary variable γ_k, which is determined by the residual (or called innovation) $\tilde{y}_{k|k-1}$. Compared to the KBF, which can obtain an accurate predicted measurement, both the EKBF and UKBF can only obtain an approximation of the predicted measurement due to the nonlinear properties of the system, which leads to a larger residual. A large residual may cause the measurement generated by the state to fall outside the designed threshold and be mistaken as an outlier, resulting in the overestimation of the probability of measurement loss by the EKBF and UKBF. Therefore, for a nonlinear system, how to improve the predicted values of the state and measurement based on the nonlinear model, so as to improve the estimation accuracies of the state and the probability of measurement loss, is still an important research orientation to solve the nonlinear filtering problem.

VI. CONCLUSION

In this paper, a class of Gaussian-Beta filters are proposed to solve the filtering problem for linear and moderate nonlinear Gaussian state-space models with unknown probability of measurement loss. The proposed Gaussian-Beta filters, where the hidden state and the unknown probability of measurement loss are respectively modeled by Gaussian and Beta distributions, have recursive structures similar to the standard Gaussian filters. Simulation results shows the superiority of the proposed Gaussian-Beta filters compared with the existing filtering algorithms for the systems with unknown probability of measurement loss. In addition, the proposed Gaussian-Beta filters can be directly extended to other variants of KF, such as cubature Kalman filter (CKF) [29], [30], quadratic Kalman filter (QKF) [31], etc. Next, we will investigate the filtering problem for nonlinear and non-Gaussian state-space models with unknown probability of measurement loss.

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