THEORY

Ellipsoidal Method for UAVs Target Tracking and Recognition

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ABSTRACT Based on our previous contributions about UAV target tracking and recognition, this paper continues to do the synthesis analysis for the ellipsoidal method for above practical problem. Given a nonlinear system or linear system, corresponding to the motion equation for the considered target, the UAV target tracking and recognition is to identify the unknown but bounded noise, then ellipsoidal method is proposed to achieve this goal. The main step for ellipsoidal method is to compute one intersection operation between one ellipsoid and a strip. Through our own mathematical derivations, this intersection part is also an ellipsoid. The center of the final ellipsoid is deemed as the accurate state estimation from the generated ellipsoidal sequences. To alleviate the deficiency of our previous contributions, synthesis analysis for ellipsoidal method is also derived to complete the efficiency. Finally, one simulation example confirms our ellipsoidal method for UAV target tracking and recognition.

INDEX TERMS UAV target tracking, recognition, ellipsoidal method, synthesis analysis.

I. INTRODUCTION

Unmanned aerial vehicles (UAVs), also known as aerial robot, is an aircraft with some nice properties, such as remote control, automatic, semi-autonomous and fully autonomous flight capabilities. Due to UAV's simple structure, low cost, good operation, strongly adaptive environment and load lots of equipment, research on UAV is becoming vast. Specifically, with the development of intelligent technology, sensor technology, microelectronics and digital communication, etc, some problems, existing for many years have been gradually overcome. For the aspect of military, UAV is often applied to perform tasks such as search and reconnaissance, battlefield situation monitoring and target tracking or detection. Some kinds of products of UAV have been in world war, and the more famous are listed as follows, American global hawk, predator, fire scout, etc. So far, 55 countries have been equipped with UAVm it means UAV has nice development prospects in both the military and civilian fields.

About major information, that human obtain from the outside world are received through the vision system. Therefore, the vision system can bring us the ability to obtain

more and enough external information. But with the development of society, human vision is unable to meet our urgent requirements on accuracy and efficiency, then human begin to develop the more advanced vision system of machine learning, helping the machine obtain real time information about the surrounding environment. The above idea is named as machine vision or computer vision. Through replying on the computer vision, the target recognition and tracking technology based on computer vision refer to the detection, recognition and tracking of the moving targets in the image sequence. Furthermore, other missions can be completed by using the computer vision for target tracking, for example, obtaining motion parameters,i.e.the position and speed of the target, and then further processing or analyzing them to calculate a higher level of task. Due to the low cost of image target recognition and tracking, high accuracy, strong anti-interference ability and good operation, computer vision for UAV is widely used in many fields, which include map matching, aircraft navigation, radar image tracking,and the end of weapon projecting systems. Some characteristics of hovering and low speed cruising for the micro-rotor aircraft are benefit as an ideal carrier for the camera to efficiently expand the camera's view field. More specifically, micro-UAVs equipped with visual sensors, i.e.cameras can conduct

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aerial surveillance and tracking flight of ground targets, then it effectively completes some tasks, for example, traffic monitoring, anti-terrorism, anti-rot, diaster relief and maritime search or rescue,etc. UAV target tracking system based on the computer vision means one UAV with a low flying height and a camera mounted pan-tilt control system can acquire the target motion information according to the sequence od the captured images, then automatically adjust the rotation and motion of the onboard pan-fit. As a consequence, the pitch angle and the attitude of the considered UAV make the tracked target always appear in the center of the camera's image, while using the tracking method to estimate the motion state of the target and regarding the estimated state as a feedback signal to control the considered UAV. The above process description is to let the UAV track the target, whatever ground target or flying target. Generally, computer vision is one of the hot research fields of artificial intelligence ata present, and it is an emerging discipline that studies how to use the computer system to realize the human visual system and explain the existence of the world. UAV target tracking and recognition based on computer vision has developed rapidly in recent decades, but there are still many problems that need to solved urgently. For example, among them, target tracking and recognition of sequence image is a typical and extremely challenging subject. Consider the civilian aspect, the most important applications of target tracking and recognition are the security monitoring of packing lots, banks and traffic, which satisfy the high security requirements. Specifically, the video images from UAV or other multiple ground platforms provide lots of useful information, such as abnormal behavior detection, intelligent vehicle identification and pedestrian behavior judgement, etc. Furthermore, within the field of military, the requirement for video surveillance system is more strict, as the battle field situation is changing rapidly and more complex than the normal civilian environment. Now target tracking and recognition is more widely applied in the military development, for example, in recent years the high resolution cameras and micro TVs are installed in USA or France.

Based on above detailed description about vision tracking system for UAVs target tracking and recognition, each UAV collects lots of observed data to form the image sequence through the computer vision system, then our mission, corresponding to the goal of target tracking and recognition, is to extract the most important information from these observed data or the formed image sequence, as the information for the considered target are embodied in the observed data. Then some statistical methods are applied directly from the probability theory to estimate or identify those useful information, i.e. system identification, statistical analysis, signal processing,etc. For convenience, one example is given to understand well. Assume an UAV is flying in sky, while one car is moving on the ground, the flying UAV wants to attack the moving car. Firstly, the flying UAV needs to get some information for the moving car, so the camera, installed on the head of the flying UAV, will take lots of photos about the

moving car, while guaranteeing the moving car in the center of each photo. Secondly, the computer control system will deal with these photos to extract some useful information for the moving car, such as its velocity, position, mass etc. This process corresponds to UAV target tracking and recognition, which depends on some nice estimation algorithms. To the best of our knowledge that before to the next practical engineering process, the approximated model or system is needed to express the considered plant,i.e. the moving car on the ground. Then based on this approximated model, the intrinsic computer controller system starts to use the statistical method to achieve the mission. If the approximated model is one nonlinear model, [1] applies it local approximation to describe the considered system, then all existed results about linear system can be used directly. A new correlation method was proposed in [2] to identify the unknown parameters, corresponding to the velocity for the considered target. The efficiency of this new correlation method was verified by the scenario strategy, coming from the data driven decision theory [3], when considering the external effects, the correlation method is not good at identifying the unknown parameters, as it is an ideal case without any external effects, so reference [4] proposed the sign perturbed sums to achieve the recognition within the case of undermodelling factors. On other extended model EIV system was designed to form the special target model, whose input and output are all corrupted by the statistical noise [5].The above process about extracting some useful information from the observed data of image sequence is similar to the novel research control community-data driven control [6], where the difference is that the observed data is dealt with to get state information or the controller respectively. From the theoretical respective, the number of observed data is derived to guarantee the efficiency of the target tracking result in [7] and [8]. Finally, our two newly published papers pave the way to deeply study this UAV target tracking and recognition. References [9] and [10] give the detailed analysis process for our continuously considered problem of UAV target tracking and recognition. Everybody, being interested in it, can refer to those two papers.

Based on above mentioned references and our two new contributions about UAV target tracking and recognition, this paper gives the recent result for the same subject, whose all variables ar unknown but bounded, i.e. observed noise and observed output are all included in their own sets. Whatever linear or nonlinear form are used to be the system for the target, all useful information are the elements, existing in the system state, so the problem of UAV target tracking and recognition is mainly concerned on estimating or identifying the unknown state, which is also the mission for system identification or adaptive control. For the case of linear form with statistical description on observed or external noise, the classical Kalman filters is nice to estimate the unknown system state. Then some improved Kalma filters are proposed to achieve the state estimation for the case of nonlinear form with statistical description on noise. Here statistical description on noise means its probabilistic density

function is known to each other. But you know this statistical description on noise is an ideal case, due to in practice white noise and color noise does not exist, and they are used in only theoretical research, so in order to extend the theoretical research and be more practical in engineering, the case of unknown but bounded noise is considered in recent years, i.e. the considered noise is known only in one set, but nothing about its probabilistic density function. Although some sets are proposed included the noise or other physical variable, here in this paper, ellipsoid is introduced. Other existing approaches hold on the condition that external noise is one white noise, but while noise does not exist in practice. So this paper considers more extended case for noise, being unknown but bounded noise.

Generally, the main contributions of this new paper are formulated as follows.
(1) Ellipsoidal method is proposed to estimate the unknown state, existing in the UAV target tracking and recognition. It means for the case of bounded noise, a sequence of ellipsoids are constructed to include the unknown state, then the center of the final ellipsoid is deemed as the state estimation.
(2) To alleviate the deficiency of our previous result in [11], synthesis analysis for this ellipsoidal method is yielded to complete the systematic analysis on the ellipsoidal method.

This paper is organized as follows. In section 2, the UAV vision tracking system is introduced to be the basis for the next target tracking and recognition. Then ellipsoidal method is proposed to identify the unknown state, which includes the useful information about the considered target in section 3, where our own derivation and analysis are given too. In section 4, synthesis analysis for ellipsoidal method is yielded to complete our previous results. Section 5 uses one numerical example to illustrate the effectiveness of our considered ellipsoidal method in UAV target tracking and recognition. Finally, section 6 ends the paper with final conclusion and points out the next subject.

II. UAV TARGET TRACKING SYSTEM

As the UAV vision tracking system ia a basis for the latter control and identification process, so here only some introductions about UAV vision tracking system is described to be the foundation. The whole UAV vision tracking system, plotted in Figure 1, includes two parts, i.e. the ground main control center and the airborne navigation system, where the information is transmitted through the wireless communication module. Specifically, the ground main control center is mainly used to monitor the flight status of UAV, and detect whether the flight data is normal of not. It can send commands to control UAV at any time to prevent UAV from being damaged due to equipment abnormalities. Similarly, the airborne navigation system is mainly applied to collect the image information and navigation information, and process the collect information to control the actuator, then finally the flight control and tracking task for UAV is realized. Among the whole UAV vision tracking system, the vision module is composed of a CCD camera, an image acquisition card,
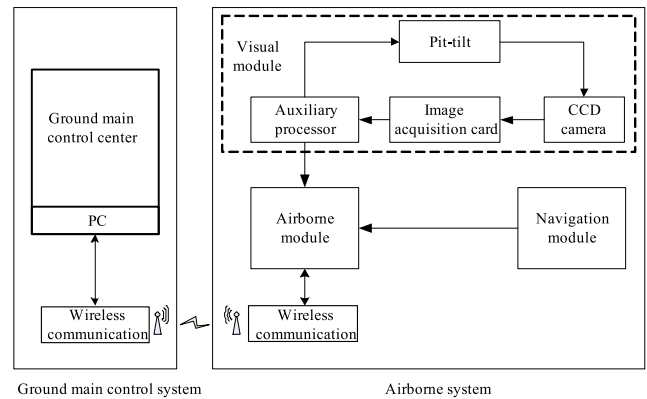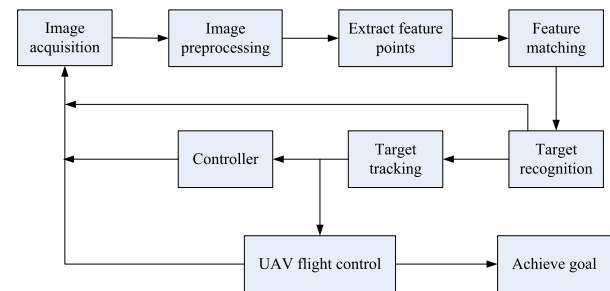

FIGURE 1. UAV vision tracking system.

FIGURE 2. UAV vision module.

a pan-tilt and an auxiliary processor. Roughly speaking, the navigation module consists of some sensors such as GPS, IMU inertial devices and ultrasonic ranging modules. Within the indoor environment, GPS always fails, so the three axis accelerometer of the IMU inertial device is applied to replace that GPS.

The visual module in the visual tracking system for UAV uses the collected images to process them. The choice and installation scheme for the camera, the image processing algorithm, and the computation speed all determine the performance of UAV visual tracking system. In the vision module, designed in the latter simulation example., a CCD camera is applied for image acquisition, and a single axis tilting pan tilt head is to expand the reconnaissance range of the camera. Then an image acquisition card is used to convert the analog signal into a digital signal that can be processed by the computer. In addition, an auxiliary processor is specially set up for image signal processing and pan-tilt rotation control, which will greatly improves the processing capability of the entire UAV vision tracking system.

Navigation is to guide the carrier to navigate. The automatic navigation of UAV based on computer vision technology means that the UAV uses the charge-coupled component CCD camera as the sensor to obtain the image during the flight, and then analyzes and understands the image, and then obtains the navigation information such as the position and motion status of the moving target. It enables the drone to have the ability to interact with the external environment, and further improves the automation level of flight control.

At present, there are two main ways to form a visual navigation system: one is to install an image processor on the aircraft, and the camera collects the image signal and transmits it to the onboard processor for processing. The hardware requirements are high and the implementation is more difficult; the other is to send the image signal to the ground computer through a high-speed wireless video transmitter. After the computer is processed, the required parameter signal is sent to the controller on the carrier. The structure of the visual tracking system is shown in Figure 2

The specific process mainly includes the following stages:

(1) Video image acquisition: After the aircraft receives the command to identify and track tasks during flight, it starts the onboard CCD camera for image acquisition, and then converts the analog signal into a digital signal that can be processed by the computer through the image acquisition card.

(2) Image preprocessing: Preprocessing the collected image, including image grayscale, histogram equalization, filtering and denoising, and other steps. (3) Target recognition: extract feature points from the preprocessed image, and match it with the pre-stored template image. If it can be successfully matched, it means that the target has been recognized, and proceed to the next step. If the matching fails, then Re-acquire images for processing.

(4) Target tracking: Using an improved tracking algorithm, the target can still be tracked in real time even when the background is disturbed or the target is occluded, and the motion state of the target is estimated to predict the position of the target in the next frame of image.

(5) PTZ and flight control: According to the predicted position of the target in the next frame of image, adjust the pitch angle of the PTZ and the yaw angle of the aircraft in time, so that the target is always kept near the center of the image. The aircraft keeps the flight altitude unchanged and keeps flying directly above the target. When the angle between the optical axis of the camera and the ground reaches about 90, it is considered that the aircraft has reached directly above the target, and the tracking task is completed.

III. ELLIPSOIDAL METHOD FOR UAV TARGET TRACKING AND RECOGNITION

UAV vision tracking scans the target to form some image sequences, then it is urgent to extract the most useful information from them. Then combining one statistical method and estimation theory to get one state space form, corresponding to the target, it means one state space equation is proposed to represent the considered target, i.e. Consider the following discrete time nonlinear stochastic system as that.

$$
\begin{cases}
x_{k+1} = f_k(x_k) \\
y_k = h_k(x_k), \quad k = 0, 1, 2, \cdots
\end{cases}
\tag{1}
$$

where in equation (1) $x_k \in R^{n_x}$ and $y_k \in R^{n_y}$ denote the state vector and observed output at instant instant k, respectively. Two maps $f_k : R^{n_x} \to R^{n_z}$; $k_k : R^{n_x} \to R^{n_y}$ are two unknown nonlinear functions.

As state vector x_k has many elements, and these elements correspond to their own physical variables, so our mission is to estimate or identify this unknown state vector x_k, then all physical variables are known to us, i.e. the useful information about the target are listed as the elements in that unknown state vector. For convenience, one linearized form of equation (1) is always studied in the theoretical research field, as the classical Kalman filter is applied to the following state space form.

$$
\begin{cases}
x_{k+1} = Ax_k + Bw_k \\
y_k = Cx_k + v_k, \quad k = 0, 1, 2, \cdots
\end{cases}
\tag{2}
$$

where in equation (2), w_k and v_k are two unknown noises, i.e. their probabilistic density functions are all unknown, so the classical Kalman filter is not well applied. Matrices $\{A, B, C, D\}$ are four systematic matrices with approximated dimension. Due to the unknown noises w_k and v_k, we use the ellipsoidal method to estimate the unknown state vector x_k, which is limited to be in one constructed ellipsoidal sequences. Then the final ellipsoid can be regarded as the state estimation.

Observing equation (1) and (2) again, we formulate them together as follows.

$$
\begin{cases}
x_{k+1} = f_k(x_k) \\
y_k = Cx_k + v_k, \quad k = 0, 1, 2, \cdots
\end{cases}
\tag{3}
$$

As noise v_k is unknown, but we assume it be unknown but bounded, i.e. there exists a scalar value σ such that.

$$
V_k \in \{v_k : |v_k| \leq \sigma\}
\tag{4}
$$

where $\sigma \in R$ is also named as the upper bound, i.e. noise v_k is unknown, but it is belonged to the set V_k.

Comment: All previous works on system identification and controller design are concerned on a zero mean random signal that is statistically independent from the input. This corresponds to the classical probabilistic identification algorithm. To relax this probabilistic description on noise, we investigate the unknown but bounded noise. This bounded noise is considered in set membership identification field widely, and it is a new deterministic identification algorithm.

For clarity of presentation about applying the ellipsoidal method for state estimation, corresponding to UAV target tracking and recognition, the definition of one feasible state set for the unknown state is deemed as the set of state that are consistent with the observed output and the given bound of the noise at time instant k. As a consequence, two definitions are given as follows.

Definition 1 [Feasible state set]: Suppose at time instant k, the state estimation x_k is known in one ellipsoidal set or ellipsoid, i.e.

$$
\begin{aligned}
X_k &= \{x_k : x_k = A_1 u | u^T u \leq 1\} \\
&= \{x_k : x_k^T A_1^{-T} A_1^{-1} x_k \leq 1\}
\end{aligned}
\tag{5}
$$

where two forms correspond to the same ellipsoid, A_1 is one matrix, and u is an virtual variable.

Definition 2[Information state set]: Given the observed output y_k, $k = 1, 2 \cdots N$ at time instant k, information state set I_k is a set of all feasible states, being consistent with the observed equation in equation (3) and that upper bound at time instant k, i.e.

$$
\begin{aligned}
I_k &= \{x_k : -\sigma \le y_k - Cx_k \le \sigma\} \\
&= \{x_k : |y_k - Cx_k| \le \sigma\} \\
&= \{x_k : |Cx_k - y_k| \le \sigma\}
\end{aligned}
\tag{6}
$$

observing that feasible state set X_k and information state set I_k simultaneously, we find that the state estimation must be satisfied that.

$$
x_k \in I_k \cap X_k
\tag{7}
$$

i.e. the state estimation x_k is in the intersection of feasible state set X_k and information state set I_k, so the next feasible state set X_{k+1} at time instant $k + 1$ can be defined as the following intersection set, i.e.

$$
X_{k+1} = I_k \cap X_k
\tag{8}
$$

Observing the feasible state set X_k and information state set I_k, we see that feasible state set X_k is an ellipsoid, and the information state set I_k is one strip at time instant k. Then the intersection operation $I_k \cap X_k$ is needed to compute, i.e. the intersection between an ellipsoid and a strip. Through our later mathematical derivations, the result about that intersection between an ellipsoid and a strip is also one ellipsoid, is formulated as the following Theorem 1.

Theorem 1: Given an ellipsoid $X_k = \{x_k : x_k = A_1 u | u^T u \le 1\}$ and a strip $I_k = \{x_k : |Cx_k - y_k| \le \sigma\}$ at time instant k, its intersection set is also represented as an other ellipsoid, being regarded as the next feasible state set X_{k+1} at time instant $k + 1$, i.e.

$$
\begin{aligned}
X_{k+1} &= I_k \cap X_k \\
&= \{x_{k+1} : (x_{k+1} + \frac{y_k}{\sigma})^T A_2 \sigma^2 (x_{k+1} + \frac{y_k}{\sigma}) \le 1; \\
&\quad -1 \le x_{k+1} \le 1\}
\end{aligned}
\tag{9}
$$

where

$$
A_2 = (CA_1)^{-T}(CA_1)^{-1}
$$

Proof: If $x \in I_k \cap X_k$, then $x \in X_k$, i.e.

$$
x = A_1 u, \text{ and } u^T u \le 1
\tag{10}
$$

As $x \in I_k$ too, there exists $w \in [-1, 1]$, such that

$$
Cx_k - y_k = \sigma w
\tag{11}
$$

substituting equation (11) into (10) to get

$$
\begin{aligned}
Cx_k - y_k &= CA_1 u - y_k \\
&= \sigma w, \text{ and } u^T u \le 1, w \in [-1, 1]
\end{aligned}
\tag{12}
$$

i.e.

$$
\begin{aligned}
CA_1 u &= y_k + \sigma w; \\
u &= (CA_1 u)^{-1}(y_k + \sigma w)
\end{aligned}
\tag{13}
$$

Then

$$
\begin{aligned}
u^T u &= (y_k + \sigma w)^T (CA_1 u)^{-T}(CA_1 u)^{-1}(y_k + \sigma w) \\
&= (y_k + \sigma w)^T A_2 (y_k + \sigma w) \le 1
\end{aligned}
\tag{14}
$$

Expanding equation (14), we have

$$
y_k^T A_2 y_k + (\sigma w)^T A_2 y_k + y_k^T A_2 \sigma w + (\sigma w)^T A_2 \sigma w \le 1
\tag{15}
$$

i.e.

$$
\begin{aligned}
(w + \frac{y_k}{\sigma})^T A_2 (w + \frac{y_k}{\sigma}) &\le \frac{1}{\sigma^2}; \\
(w + \frac{y_k}{\sigma})^T A_2 \sigma^2 (w + \frac{y_k}{\sigma}) &\le 1; \\
-1 \le x_{k+1} &\le 1
\end{aligned}
\tag{16}
$$

so it holds that

$$
\begin{aligned}
I_k \cap X_k = \{x : (x + \frac{y_k}{\sigma})^T A_2 \sigma^2 (x + \frac{y_k}{\sigma}) \le 1; \\
-1 \le x \le 1\}
\end{aligned}
\tag{17}
$$

Equation (17) is also an ellipsoid too, which is deemed as the next feasible state set X_{k+1} at time instant $k + 1$.

This completes the proof.

Finally, the detailed ellipsoidal method for the unknown state estimation, corresponding to UAV target tracking and recognition, is listed as follows, and its geometric form is seen in Figure 2.

Step 1: Collect the observed output data point $\{y_k\}_{k=1}^N$, where N is the number of output data points.

Step 2: Construct a strip that bounds the consistent state, i.e. information state set.

$$
\begin{aligned}
I_k &= \{x_k : -\sigma \le y_k - Cx_k \le \sigma\} \\
&= \{x_k : |y_k - Cx_k| \le \sigma\} \\
&= \{x_k : |Cx_k - y_k| \le \sigma\}
\end{aligned}
$$

Step 3: Set the initial ellipsoidal state set X_0 be one priori ellipsoid

$$
X_0 = \{x_0 : x_0 = A_1 u | u^T u \le 1\}
$$

Step 4: Calculate the first intersection $I_1 \cap X_9$ to be the next feasible state set X_1, i.e.

$$
\begin{aligned}
X_1 &= X_0 \cap I_1 \\
&= \{x_1 : (x_1 + \frac{y_k}{\sigma})^T A_2 \sigma^2 (x_1 + \frac{y_k}{\sigma}) \le 1; \\
&\quad -1 \le x \le 1\}
\end{aligned}
$$

Step 5: Iteratively calculate the following intersection operations to get a sequence of ellipsoids, i.e.

$$
X_0 \cap I_1, X_1 \cap I_2, \cdots, X_k \cap I_{k+1}, \cdots, X_{N-1} \cap I_N
$$

Then the final ellipsoid is that

$$
X_N X_{N-1} \cap I_N
$$

So the center of the final ellipsoid X_N is deemed as the state estimation x_N at time instant N.
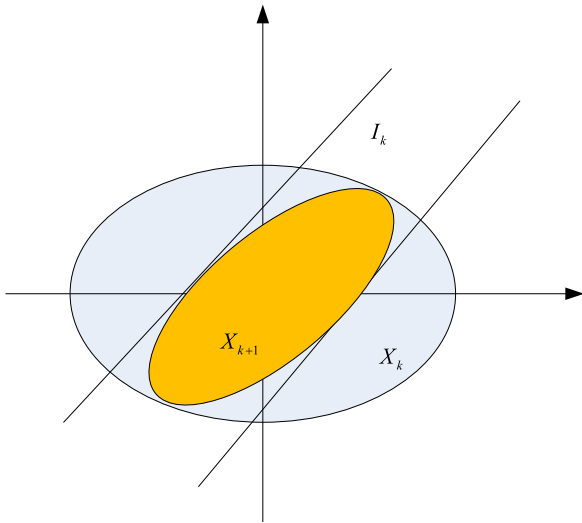
FIGURE 3. Ellipsoidal method.

IV. SYNTHESIS ANALYSIS FOR ELLIPSOIDAL METHOD

To give a synthesis analysis for our considered ellipsoidal method, that state space form (2) is considered in this section again, i.e.

$$\begin{cases} x_{k+1} = Ax_k + Bw_k \\ x_0 = 0, k = 0, 1, \cdots, N-1 \end{cases} \quad (18)$$

Noise w_k satisfying the following norm bound, i.e.

$$\|w_k\| \le 1, k = 0, 1, \cdots, N-1$$

Ellipsoidal method is to build the ellipsoidal approximation of the state estimation recursively. Let X_k be the set of all states, where the system can be driven in instant $k \le N$, and assume that we build inner and outer ellipsoidal approximations E_{in}^k and E_{out}^k of the set X_k.

$$E_{in}^k \subset X_k \subset E_{out}^k \quad (19)$$

According to the problem of how to construct the inner and outer ellipsoidal approximations E_{in}^k and E_{out}^k, the reader can refer to our previous paper (Wang Jianhong, Meng He, Ricardo.Ramirez-Mendoza, 2020). This section gives some synthesis analysis about constructing the inner and outer ellipsoidal approximations.

Firstly for convenience, the following Theorem 2 is given as follows.

Theorem 2: Let P_i be nonsingular, and Λ_i be positive definite $n \times n$ matrices, $i = 1, 2, \cdots m$, then for every collection $x^1, \cdots x^m$ of vectors from R^n, one has

$$[x^1 + \cdots + x^m]^T [\sum_{i=1}^m [P_i^T]^{-1} \Lambda_i P_i^{-1}]^{-1} [x^1 + \cdots + x^m]$$

$$\le \sum_{i=1}^m [x^i]^T P_i \Lambda_i P_i^T x^i \quad (20)$$

Proof: Construct the following $(nm+1) \times (nm+1)$ symmetric matrix as

$$C = \begin{bmatrix} P_1 \Lambda_1 P_1^T & \cdots & & \cdots & I_n \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & P_m \Lambda_m P_m^T & & I_n \\ I_n & \cdots & I_n & & \sum_{i=1}^m [P_i^T]^{-1} \Lambda_i P_i^{-1} \end{bmatrix} \quad (21)$$

Applying Schur Complement Lemma on matrix C, then we have.

$$\sum_{i=1}^m [P_i^T]^{-1} \Lambda_i P_i^{-1}$$

$$- \begin{bmatrix} I_n & \cdots & I_n \end{bmatrix} \begin{bmatrix} P_1 \Lambda_1 P_1^T & \cdots & & \cdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & P_m \Lambda_m P_m^T \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} I_n \\ \vdots \\ I_n \end{bmatrix}$$

$$= \sum_{i=1}^m [P_i^T]^{-1} \Lambda_i P_i^{-1} - \sum_{i=1}^m [P_i \Lambda_i P_i^T]^{-1} \quad (22)$$

Using the property about $\sum_{i=1}^m [P_i^T]^{-1} \Lambda_i P_i^{-1} - \sum_{i=1}^m [P_i \Lambda_i P_i^T]^{-1} \ge 0$, and the necessary and sufficient condition or the positive semidefinite, it is obvious that.

$$C \ge 0 \Leftrightarrow \sum_{i=1}^m [P_i^T]^{-1} \Lambda_i P_i^{-1} \ge 0$$

$$\times and \sum_{i=1}^m [P_i^T]^{-1} \Lambda_i P_i^{-1} - \begin{bmatrix} I_n & \cdots & I_n \end{bmatrix}$$

$$\times \begin{bmatrix} P_1 \Lambda_1 P_1^T & \cdots & & \cdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & P_m \Lambda_m P_m^T \end{bmatrix}^{-1} \begin{bmatrix} I_n \\ \vdots \\ I_n \end{bmatrix} \ge 0 \quad (23)$$

where the assumptions P_i be nonsingular and Λ_i be positive definite matrices are used to guarantee that.

$$\sum_{i=1}^m [P_i^T]^{-1} \Lambda_i P_i^{-1} \ge 0 \quad (24)$$

Then applying Schur Complement Lemma on positive semideginite matrix C again, it holds that.

$$\begin{bmatrix} P_1 \Lambda_1 P_1^T & \cdots & & \cdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & P_m \Lambda_m P_m^T \end{bmatrix}$$

$$- \begin{bmatrix} I_n \\ \vdots \\ I_n \end{bmatrix} [\sum_{i=1}^m [P_i^T]^{-1} \Lambda_i P_i^{-1}]^{-1} \begin{bmatrix} I_n & \cdots & I_n \end{bmatrix} \ge 0 \quad (25)$$

by the multiplication of the above inequality on the right

by $\begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^m \end{bmatrix}$ and on the left by $[\,x^1\ x^2\ \cdots\ x^m\,]$, the following

inequality is obtained.

$$
[\,x^1\ x^2\ \cdots\ x^m\,]\begin{bmatrix} P_1\Lambda_1 P_1^T & \cdots & & \cdots \\ \vdots & \vdots & & \vdots \\ \vdots & & \vdots & P_m\Lambda_m P_m^T \end{bmatrix}
$$

$$
\times \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^m \end{bmatrix} = \sum_{i=1}^{m} [x^i]^T P_i \Lambda_i P_i^T x^i
$$

$$
\times [\,x^1\ x^2\ \cdots\ x^m\,]\begin{bmatrix} I_n \\ \vdots \\ I_n \end{bmatrix} = x^1 + \cdots + x^m
$$

$$
\times [\,I_n\ \cdots\ I_n\,]\begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^m \end{bmatrix} = x^1 + \cdots + x^m \tag{26}
$$

substituting equation (26) into inequality (25), which completes the proof the Theorem 2.

Based on above Theorem 2, a parametric family of ellipsoids containing the arithmetic sum of m given ellipsoids are generated as other new alternative way. This new alternative form is formulated as the following Theorem 3.

Theorem 3: Given m full dimensional ellipsoids centered at the origin.

$$
W_i = \{x \in R^n | x^T B_i x \le 1\}, i = 1 \cdots m \tag{27}
$$

For every collection Λ of positive definite $n \times n$ matrices Λ_i such that

$$
\sum_i \lambda_{max}(\Lambda_i) \le 1 \tag{28}
$$

the ellipsoid

$$
E_\Lambda = \{x | x^T [\sum_{i=1}^{m} B_i^{-\frac{1}{2}} \Lambda_i^{-1} B_i^{-\frac{1}{2}}]^{-1} x \le \} \tag{29}
$$

contains the sum $W_1 + \cdots + W_m$ of the ellipsoids W_i.

Proof: Set $x = [x^1 + \cdots + x^m]$, then

$$
[x^1 + \cdots + x^m]^T [\sum_{i=1}^{m} B_i^{-\frac{1}{2}} \Lambda_i^{-1} B_i^{-\frac{1}{2}}]^{-1} [x^1 + \cdots + x^m]
$$

$$
\le \sum_{i=1}^{m} [x^i]^T B_i^{\frac{1}{2}} \Lambda_i B_i^{\frac{1}{2}} [x^i]
$$

$$
\le \sum_{i=1}^{m} \lambda_{max}(\Lambda_i)[x^i]^T B_i^{\frac{1}{2}} \Lambda_i B_i^{\frac{1}{2}} [x^i]
$$

$$
\le \sum_{i=1}^{m} [x^i]^T B_i[x^i] = x^T B_i x \le 1 \tag{30}
$$

where in above derivation process, the inequality from Theorem 2 is used.

Let us start with the observation that an ellipsoid $W[Z] = \{x | x^T Z x \le 1\}$ contains $W_1 + \cdots + W_m$ if and only if the following implication holds.

$$
\{\{x^i \in R^n\}_{i=1}^m, [x^i]^T B_i[x^i] \le 1, i = 1, 2 \cdots m\}
$$
$$
\Downarrow
$$
$$
(x^1 + \cdots + x^m)^T Z(x^1 + \cdots + x^m) \le 1 \tag{31}
$$

Then we choose $Z = [\sum_{i=1}^{m} B_i^{-\frac{1}{2}} \Lambda_i^{-1} B_i^{-\frac{1}{2}}]^{-1}$ to guarantee $W[\Lambda_i] = \{x | x^T [\sum_{i=1}^{m} B_i^{-\frac{1}{2}} \Lambda_i^{-1} B_i^{-\frac{1}{2}}]^{-1} x \le \}$ contains the sum $W_1 + \cdots + W_m$ of the ellipsoids W_i, and every collection Λ of positive definite $n \times n$ matrices Λ_i such that $\sum_i \lambda_{max}(\Lambda_i) \le 1$.

Which completes the proof of the Theorem 3.

After this parametric family of ellipsoids (29) are obtained to contain the sum $W_1 + \cdots + W_m$ of the ellipsoids W_i, then the smallest volume ellipsoid within this parametric family can be also reduced to one semidefinite program, which is seed as Theorem 4.

Theorem 4: In order to find the smallest volume ellipsoid in the family $\{E_\Lambda\}_\Lambda$ defined in equation (29), it suffices to solve the following semidefinite program as (32), shown at

$$
\max\ t
$$
$$
Subject\ to\ t \le Det^{\frac{1}{n}}(Z) \begin{bmatrix} \Lambda_1 & \cdots & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & \cdots & \Lambda_m \end{bmatrix}
$$
$$
\ge \begin{bmatrix} B_1^{-1}ZB_1^{-1} & B_1^{-\frac{1}{2}}ZB_2^{-\frac{1}{2}} & \cdots & B_1^{-\frac{1}{2}}ZB_m^{-\frac{1}{2}} \\ B_2^{-\frac{1}{2}}ZB_1^{-\frac{1}{2}} & B_2^{-\frac{1}{2}}ZB_2^{-\frac{1}{2}} & \cdots & B_2^{-\frac{1}{2}}ZB_m^{-\frac{1}{2}} \\ \vdots & \vdots & \vdots & \vdots \\ B_m^{-\frac{1}{2}}ZB_1^{-\frac{1}{2}} & B_m^{-\frac{1}{2}}ZB_2^{-\frac{1}{2}} & \cdots & B_m^{-\frac{1}{2}}ZB_m^{-\frac{1}{2}} \end{bmatrix}\ Z \ge 0\ \ \Lambda_i \le \lambda_i I_n, i = 1, 2 \cdots m\ \ \sum_i \lambda_i \le 1 \tag{32}
$$

the bottom of the previous page, whose decision variables are $Z, \Lambda_i, t, \lambda_i \in R$.

Proof: For every collection Λ of positive definite $n \times n$ matrices Λ_i such that $\sum_i \lambda_{max}(\Lambda_i) \leq 1$, and $\Lambda_i = \lambda_i I_n$. Given m positive reals λ_i with unit sum, one defines the ellipsoid

$$E(x) = \{x | x^T [\sum_i \lambda_i^(- 1)] B_i^(- 1) x \leq 1\}$$

This ellipsoid contains the arithmetic sum W of the ellipsoids $\{x | x^T B_i X \leq 1\}$, and in order to approximate the smallest volume ellipsoid containing W. We only minimize $DetE(x)$ over λ varying in the standard simplex $\{\lambda \geq 0, \sum_i \lambda_i = 1\}$.

Comment: As here this paper considered the case of bounded noise, the final result is to derive one ellipsoidal sequence, which include the accurate state estimation. Figure 2 shows the ellipsoidal method, and the obtained ellipsoids are three-dimension shape. The always used method to measure the size of this three-dimension shape-ellipsoid is dependent of the volume. By the way, other specifications are used, for example, P-radius or robust radius. Due to the equivalent property among the volume, P-radius or robust radius, so we apply the volume to measure the size of the obtained ellipsoids.

The above Theorem 2,3,and 4 are three related results on outer ellipsoidal approximation, similarly the further result about inner ellipsoidal approximation is easily obtained as the following Theorem 5.

Theorem 5: Let A_i be nonsingular $n \times n$ matrices, $i = 1, 2 \cdots m$, and let $W_i = \{x = A_i u | u^T u \leq 1\}$ be the associated ellipsoids in R_n. Let $\Delta_m = \{\lambda \in R_+^m | \sum_i \lambda_i = 1\}$, then

(1) Whenever $\lambda \in \Delta_m$ and A is such that

$$AA^T \geq F(\lambda) = \sum_{i=1}^m \lambda_i^{-1} A_i A_i^T \tag{33}$$

the ellipsoid $E[A] = \{x = Au | u^T u \leq 1\}$ contains $W = W_1 + \cdots + W_m$.

(2) Whenever A is such that

$$AA^T \leq F(\lambda) = \sum_{i=1}^m \lambda_i^{-1} A_i A_i^T, \forall \lambda \in \Delta_m \tag{34}$$

the ellipsoid $E[A] = \{x = Au | u^T u \leq 1\}$ contained in $W = W_1 + \cdots + W_m$, and vice versa.

Proof: (1) As the associated ellipsoids

$$W_i = \{x = A_i u | u^T u \leq 1\}$$

i.e.

$$u = A_i^{-1} x \rightarrow u^T u = x^T A_i^{-T} A_i^{-1} x \leq 1$$

Similarly the ellipsoid satisfies that.

$$E[A] = \{x = A_i u | u^T u \leq 1\};$$
$$u = A^{-1} x; u^T u = x^T A^{-T} A^{-1} x \leq 1;$$
$$u^T u = x^T [AA^T]^{-1} x \leq 1$$

As

$$AA^T \geq F(\lambda) = \sum_{i=1}^m \lambda_i^{-1} A_i A_i^T;$$

$$(AA^T)^{-1} \leq [\sum_{i=1}^m \lambda_i^{-1} A_i A_i^T]^{-1}$$

Similarly

$$AA^T \geq F(\lambda) = \sum_{i=1}^m \lambda_i^{-1} A_i A_i^T;$$

$$\sum_{i=1}^m \lambda_i^{-1} A_i A_i^T AA^T \leq 1$$

and

$$\sum_{i=1}^m \lambda_i A_i^T A_i^{-T} A_i^{-1} AA^T A_i^T = \sum_{i=1}^m \lambda_i A^T A = A^T A$$

where $\sum_{i=1}^m \lambda_i = 1$ is used in above mathematical derivation. Then we have

$$W_i = \{x = A_i u | u^T u \leq 1\} = \{x | x^T A_i^{-T} A_i^{-1} x \leq 1\};$$
$$W = \{x = Au | u^T u \leq 1\} = \{x | x^T A^{-T} A^{-1} x \leq 1\}$$

Which completes the proof.

The above new results give some improvements for our previous results, which will be applied in our latter research on multi-UAVs formation flight control.

(2) Represent the given centered at the original ellipsoids W_i as $W_i = \{x = A_i u | u^T u \leq 1\}$, $i = 1, \cdots m$. Due to the fact that an ellipsoid $E[Z] = \{x = Zu | u^T u \leq 1\}$ is contained in the sum $W_1 + \cdots + W_m$ of the ellipsoids W_i if and only if one has $\forall x : \|Z^T x\|_2 \leq \sum_{i=1}^m \|A_i^T x\|_2$. Then we obtain that $W_i = \{x | x^T [A_i A_i^T]^{-1} x \leq 1\}$, $i = 1, 2 \cdots m$. For the ellipsoid contained in the sum, we still have.

$$E[A] = \{x = Au | u^T u \leq 1\} = \{x | x^T [AA^T]^{-1} x \leq 1\}$$

$$\Downarrow$$

$$\|[AA^T]^{-1} x\|_2 \leq \sum_{i=1}^m \|[A_i A_i^T]^{-1} x\|_2$$

$$\Downarrow$$

$$x^T [AA^T]^{-1} x \leq \sum_{i=1}^m x^T [A_i A_i^T]^{-1} x \tag{35}$$

Furthermore, we have

$$\|[AA^T]^{-1} x\|_2 \leq \sum_{i=1}^m \|[A_i A_i^T]^{-1} x\|_2$$

$$= \min_{\lambda \in \Delta_m} \sum_{i=1}^m x^T \frac{\|A_i A_i^T\|_2}{\lambda_i} x \tag{36}$$

It means that

$$\|A^T x\|_2 \leq \sum_{i=1}^m \|A_i^T x\|_2$$

FIGURE 4. UAV model.

$$\Downarrow$$

$$x^T AA^T x \le \sum_{i=1}^{m} x^T A_i A_i^T x = x^T [\sum_{i=1}^{m} A_i A_i^T] x \qquad (37)$$

Rewriting equation (37) as that

$$AA^T \le \sum_{i=1}^{m} A_i A_i^T = \sum_{i=1}^{m} \|A_i\|^2 \le (\sum_{i=1}^{m} |A_i|)^2$$

$$= \min_{\lambda \in \Delta_m} \sum_{i=1}^{m} \frac{|A_i|^2}{\lambda_i} = \min_{\lambda \in \Delta_m} \sum_{i=1}^{m} \frac{A_i A_i^T}{\lambda_i} \qquad (38)$$

It equals to

$$AA^T \le \sum_{i=1}^{m} \lambda_i^{-1} A_i A_i^T \qquad (39)$$

Which completes the proof of the Theorem 5.

The above analyzed results can not only be further alternative forms for outer ellipsoidal approximation and inner ellipsoidal approximation, but also be used as the auxiliary steps for section 3, where the number of ellipsoids is $m = 2$, so this further analysis process extends the results in section 3.

V. SIMULATION

In this simulation example, a quadratic UAV is used, as it can take off and land vertically. It has strong air control capability, good static flight and low speed flight characteristics, which is shown in Figure 4 to achieve various of flight attitudes. This quadratic UAV adopts the basic axis-symmetric layout, the four crisscrossed robots are evenly distributed, and the structure is simple. Furthermore, the quadratic UAV is mainly composed of rotor, body, flight control equipment, motor, power supply and landing gear,etc.

The quadrotor UAV has four controllable basic motion states during flight, namely vertical flight, longitudinal flight, lateral flight and horizontal rotation. Its four flight module is plotted in Figure 5, whose vertical flight is the easiest to control among the four states. At the same time, the output power of the four motors is increased, so that the rotor speed increases and the total pulling force increases. When the total
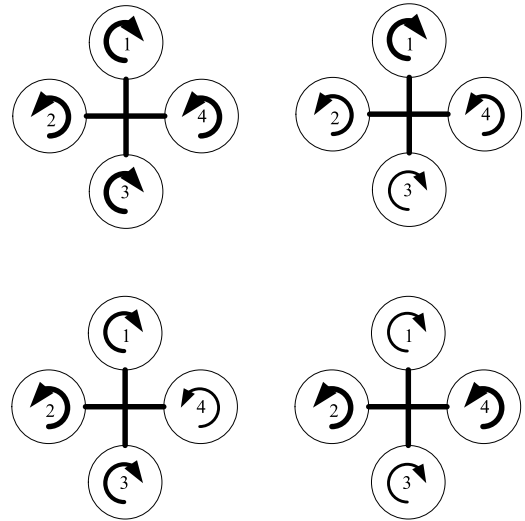

FIGURE 5. Four flight modules.

（a）Kalman filter tracking results

（b）Improved Kalman filter tracking results
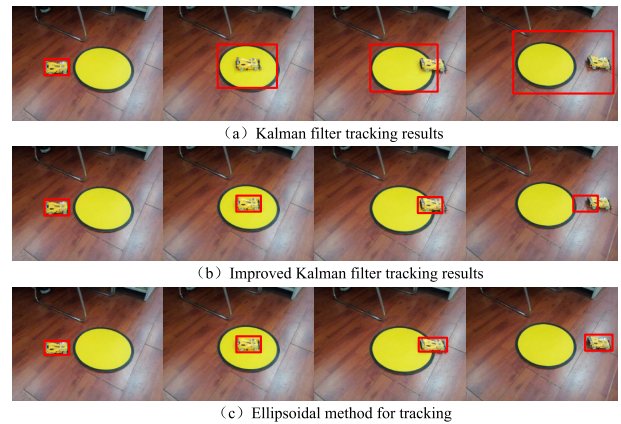
（c）Ellipsoidal method for tracking

FIGURE 6. Comparisons with kalman filter method.

pulling force is greater than the gravity, the quadrotor flies vertically upward. The output power of the small four motors slows down the rotor speed, reduces the total pulling force, and the quadrotor flies vertically downward.

To compare the tracking results of applying our proposed ellipsoidal method and classical Kalman filter, we let the quadratic UAV track one moving car, whose average velocity os set to be 0.2-5m/s. Figure 6 shows the image sequences for three methods, i.e. classical Kalman filter, improved Kalman filter and ellipsoidal method, then only frames (1,30,46,53) are extracted from the first image sequence. In this image sequence, there is a large yellow circle with a color very similar to the four-wheel drive car as a background noise, and the car starts to move slowly and almost uniformly, and suddenly accelerates at frame 52.

After observing Figure 6 for the above Kalman filter, when at the 17th frame, the car enters near the center of the yellow circle. Due to the interference of similar colors, the target cannot be tracked correctly. The tracking window is enlarged to include the entire yellow circle. In the 46th frame, part of the car leaves the yellow circle, and the tracking window

continues to enlarge. But for the tracking results from our ellipsoidal method, through image matching, the tracking target is firstly identified and the window position and size are automatically determined. Ellipsoidal method combines the advantages of the two methods to enhance the real-time and robustness of tracking. No matter the target object is disturbed by similar colors or the speed is suddenly accelerated, that quadratic UAV can track the target very accurately and effectively.

VI. CONCLUSION

In this paper, ellipsoidal method is proposed in UAV target tracking and recognition, and its synthesis analysis is also given too. The problem about why the ellipsoidal method is applied is that the noise, existing in the nonlinear system or linear system, is unknown but bounded, i.e. its probabilistic density function of the considered noise is unknown, so the classical Kalman filter can not work well. Within the synthesis analysis for the ellipsoidal method, the intersection operation between an ellipsoid and a strip is represented as another ellipsoid. Through the generated ellipsoid sequences, the center of the final ellipsoid is chosen as the nice state estimation. Then other applications of our considered ellipsoidal method is the interesting direction of future work.

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