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RESEARCH ARTICLE

A Reduced-Order Extrapolation (ROE) Method for Solution Coefficient Vectors in the Mixed Finite Element (MFE) Method for the Two-Dimensional (2D) Fourth-Order Hyperbolic Equation

CHANGBIAO YU, WENWEN XU, (Member, IEEE), AND XINDONG LI, (Member, IEEE)

School of Mathematics and Statistics, Qilu University of Technology (Shandong Academy of Sciences), Jinan 250353, China

Corresponding author: Xindong Li (lxd851268@126.com)

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ABSTRACT This study focuses on a reduced-order extrapolation method for the coefficient vectors of the mixed finite element solution for two-dimensional fourth-order hyperbolic equation. We first establish the mixed finite element scheme for the equation and give the matrix model of the mixed finite element scheme and the existence, stability and error estimates of its solutions. Then, we derive a reduced-order extrapolation mixed finite element matrix model with a small number of unknowns, where the proper orthogonal decomposition method is used to save central processing unit time, and prove the existence, stability and error estimates of the reduced-order extrapolation mixed finite element solutions with the help of matrix knowledge. More importantly, the reduced-order extrapolation mixed finite element matrix model have the same basis functions and error accuracy as the mixed finite element matrix model. Finally, some numerical experiments confirm the effectiveness of the reduced-order extrapolation mixed finite element matrix model, where the central processing unit time is greatly reduced and the accuracy is maintained.

INDEX TERMS Fourth-order hyperbolic equation, mixed finite element method, reduced-order extrapolation, proper orthogonal decomposition, existence and stability as well as error analysis.

I. INTRODUCTION

Consider the following two-dimensional (2D) fourth-order hyperbolic equation.

$$\begin{cases} u_{tt} + \Delta^2 u = f, \quad \text{in } \Omega \times J, \\ u = g_1, \ \Delta u = g_2, \quad \text{on } \partial\Omega \times J, \\ u(x, y, 0) = u_0(x, y), \quad \text{at } t = 0 \text{ and in } \overline{\Omega}, \\ u_t(x, y, 0) = u_1(x, y), \quad \text{at } t = 0 \text{ and in } \overline{\Omega}, \end{cases} \tag{1}$$

where Ω is an interconnected domian with bounded boundary $\partial\Omega$, $\overline{\Omega} = \Omega \cup \partial\Omega$, $J = [0, T]$, T is the final moment, $f(x, y, t)$ is the given sufficiently smooth source function, $g_1(x, y, t)$, $g_2(x, y, t)$, $u_0(x, y)$ and $u_1(x, y)$ are given sufficiently smooth boundary functions and initial functions respectively.

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The 2D fourth-order hyperbolic equation is an important partial differential equation describing vibration or wave phenomena, which has important application value in aerospace, petroleum exploration, urban construction, crustal sounding and so on [1], [2], [3]. For practical problems, due to the complexity of the physical problem itself and the solution region, it is often difficult to obtain the exact solution of the problem (1). In the last few years, many scholars have deeply studied the numerical solution of the problem (1) and put forward a variety of numerical calculation methods. Li [4] constructed a two-layer implicit Crank-Nicolson (CN) compact difference scheme for the problem (1). Zhang [5] proposed a lower order conforming mixed finite element (MFE) approximation scheme with the bilinear element Q_{11} for a type of nonlinear fourth-order hyperbolic equation. However, these numerical methods contain too many unknowns, which lead to very high computation and complexity, as well as the accumulation of rounded-off errors will affect the accuracy of numerical solutions.

The proper orthogonal decomposition (POD) method has played an important role in reducing the number of unknowns in the numerical methods [6], [7], [8], [9]. Kunisch and volkwein first proposed to apply the POD method to the reduced-order of the Galerkin method for the parabolic problems [10] and Luo extended the POD-based reduced-order method to other finite element (FE) methods and also to finite difference (FD) mthod, finite volume element (FVE) method [11], [12], [13], but this resulted in the repeated calculation. The POD-based reduced-order extrapolation methods [14], [15], [16], [17] don't have to repeat large-scale calculations because they only need to select a few classical numerical solutions as snapshots to formulate the continuous POD basis, but it requires a lot of abstract mathematical knowledge and the original space, such as FE space, is replaced by subspaces spanned with few continuous POD basic functions, resulting in large errors in the process of reduced-order.

In order to overcome the above two problems caused by the continuous POD basic functions, a reduced-order extrapolation method for coefficient vectors of the classical numerical solutions is proposed in [18], [19], [20], and [21], which not only has the same basis function and accuracy as the classical numerical methods, but also the theoretical analysis is easy. In this paper, we will establish the reduced-order extrapolation MFE (ROEMFE) matrix model for the problem (1) by reducing the order of coefficient vectors of the MFE solutions by means of POD basis vectors, in which the POD basis vectors are formed by the initial coefficient vectors of the MFE solutions. The ROEMFE matrix model has the same basis functions and accuracy as the MFE method owing to the basis functions in the MFE subspace are absorbed into the stiffness matrix and mass matrix of the MFE matrix model and the unknown solution coefficient vectors in the MFE matrix model are reduced with the linear combinations of the few POD basic vectors. Besides, the stability and error estimates of the ROEMFE matrix model are analyzed with the help of the matrix idea, which makes the theoretical analysis simple.

The rest of the paper is organized as follows. In Section II, the existence, uniqueness and error estimates of the MFE solutions are given. We write the MFE scheme as matrix form and prove the stability of the MFE matrix model and the MFE scheme. In Section III, we establish the ROEMFE matrix model by the POD basis vectors produced by the initial coefficient vectors of the MFE solutions and prove the stability and error estimates of the ROEMFE solutions by the matrix idea. Some numerical experiment which confirms the theoretical results is presented in Section IV. Section V summarizes the main conclusions.

In this article, we adopt the classical Sobolev spaces $W^{m,p}$ and their norms $\|\cdot\|_{m,p}$. When $p = 2$, we will briefly note $W^{m,2}$ as H^m and $\|\cdot\|_{m,2}$ as $\|\cdot\|_m$. C is a general positive constant independent of h and Δt, which may be different in different places.

II. THE MFE METHOD FOR THE 2D FOURTH-ORDER HYPERBOLIC EQUATION

Let $w = -\Delta u$, problem (1) is equivalent to

$$\begin{cases} u_{tt} - \Delta w = f, & \text{in } \Omega \times J, \\ w = -\Delta u, & \text{in } \partial\Omega \times J, \\ u(x, y, 0) = u_0(x, y), & \text{at } t = 0 \text{ and in } \overline{\Omega}, \\ u_t(x, y, 0) = u_1(x, y), & \text{at } t = 0 \text{ and in } \overline{\Omega}, \end{cases} \tag{2}$$

The weak formulation for the problem (2) is: Find $\{u, w\}$: $[0, T] \to H_0^1 \times H_0^1$ such that

$$\begin{cases} (u_{tt}, \phi) + (\nabla w, \nabla\phi) = (f, \phi), \forall\phi \in H_0^1, \\ (w, \psi) = (\nabla u, \nabla\psi), \forall\psi \in H_0^1, \\ u(x, y, 0) = u_0(x, y), (x, y) \in \overline{\Omega} \\ u_t(x, y, 0) = u_1(x, y), (x, y) \in \overline{\Omega}. \end{cases} \tag{3}$$

Let $\Im_h$ be a uniformly regular rectangular partition of rectangle Ω with mesh size h, The bilinear finite element subspace V_h, spanned by the basis $\{N_i(x, y)\}_{i=1}^M$, be defined as follows:

$$V_h = \{v_h \in H_0^1(\Omega) \cap C(\Omega) : v_h|_K \in Q_{11}(K), K \in \Im_h\},$$

where $Q_{11} = span\{1, x, y, xy\}$.

Let $R_h : H_0^1(\Omega) \to V_h$ be the Ritz projection [5], [22], [23], that is, for $\forall u \in H_0^1(\Omega)$ such that

$$(\nabla(R_h u - u), \nabla v) = 0, \forall v \in V_h.$$

If $u \in H^2(\Omega) \cap H_0^1(\Omega)$, then the Ritz projection has the following boundedness and error estimates [5], [22], [23]

$$\|\nabla R_h u\|_0 \le \|\nabla u\|_0. \tag{4}$$

Furthermore, let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition with step size $\Delta t = \frac{T}{N}$ on interval $[0, T]$ and $t_n = n\Delta t$, $n = 0, 1, 2, \ldots, N$. $u^n = u(x, y, t_n)$, $w^n = w(x, y, t_n)$, u_h^n and w_h^n be the approximation of $u(t_n)$ and $w(t_n)$ in V_h, respectively.

Thus, The MFE scheme of the problem (3) is to find $\{u_h^n, w_h^n\} \in V_h \times V_h$ such that

$$\begin{cases} (w_h^{n-1}, \psi) = (\nabla u_h^{n-1}, \nabla\psi), 1 \le n \le N+1, \forall\psi \in V_h, \\ \dfrac{1}{\Delta t^2}(u_h^{n+1} - 2u_h^n + u_h^{n-1}, \phi) + \dfrac{1}{4}(\nabla(w_h^{n+1} + 2w_h^n \\ + w_h^{n-1}), \nabla\phi) = (f^n, \phi), 1 \le n \le N, \forall\phi \in V_h, \end{cases} \tag{5}$$

where the initial values $u_h^0 = R_h u_0$, $u_h^1 = R_h(u_0 + \Delta t u_1 + \frac{1}{2}\Delta t^2 u_{tt}(0))$ and $w_h^0 = R_h(-\Delta u_0)$, $w_h^1 = R_h((-\Delta u_0) + \Delta t(-\Delta u_1) + \frac{1}{2}\Delta t^2(-\Delta u_{tt}(0)))$, $u_{tt}(0) = f(0) - \Delta^2 u_0$.

The following results for the existence, uniqueness and error estimates for the solutions to the problem (5) were proved in [5].

Lemma 1: If $u, w \in L^{\infty}(J; H^3(\Omega))$, the problem (5) has a unique set of solutions $\{u_h^n, w_h^n\} \in V_h \times V_h$ $(1 \le n \le N)$ satisfying the following error estimates:

$$\|u^n - u_h^n\|_1 + \|w^n - w_h^n\|_1 \le C(h + \Delta t^2), 1 \le n \le N.$$

Set $\boldsymbol{U}^n = (u_1^n, u_2^n, \ldots, u_M^n)^{\mathrm{T}}$, $\boldsymbol{W}^n = (w_1^n, w_2^n, \ldots, w_M^n)^{\mathrm{T}}$. Therefore, the problem (5) can be rewritten into the following matrix model by the basis $\{N_i(x, y)\}_{i=1}^M$.

The matrix model for the problem (5) is to find $\{\boldsymbol{U}^n, \boldsymbol{W}^n\} \in \mathbb{R}^M \times \mathbb{R}^M$ and $\{u_h^n, w_h^n\} \in V_h \times V_h$ such that

$$\begin{cases} \boldsymbol{A}\boldsymbol{W}^{n-1} = \boldsymbol{B}\boldsymbol{U}^{n-1}, 1 \le n \le N+1, \\ \boldsymbol{A}\boldsymbol{U}^{n+1} + \dfrac{\Delta t^2}{4}\boldsymbol{B}\boldsymbol{W}^{n+1} = 2\boldsymbol{A}\boldsymbol{U}^n - \boldsymbol{A}\boldsymbol{U}^{n-1} - \dfrac{\Delta t^2}{2}\boldsymbol{B}\boldsymbol{W}^n \\ \quad -\dfrac{\Delta t^2}{4}\boldsymbol{B}\boldsymbol{W}^{n-1} + \Delta t^2 \boldsymbol{F}^n, 1 \le n \le N-1, \\ u_h^n = \displaystyle\sum_{i=1}^M u_i^n N_i = \boldsymbol{U}^n \cdot \boldsymbol{N}, w_h^n = \sum_{i=1}^M w_i^n N_i = \boldsymbol{W}^n \cdot \boldsymbol{N}, \end{cases} \tag{6}$$

where $\boldsymbol{A} = ((N_i, N_j))_{M\times M}$ and $\boldsymbol{B} = ((\nabla N_i, \nabla N_j))_{M\times M}$ are both positive definite matrices [5], $\boldsymbol{F}^n = ((f^n, N_i))_{M\times 1}$, $\boldsymbol{N} = (N_1, N_2, \ldots, N_M)^{\mathrm{T}}$. $\boldsymbol{U}^1 = \boldsymbol{U}^0 + \Delta t \boldsymbol{U}_1 + \frac{1}{2}\Delta t^2 \boldsymbol{U}_{tt}$, $\boldsymbol{W}^1 = \boldsymbol{W}^0 + \Delta t \boldsymbol{W}_1 + \frac{1}{2}\Delta t^2 \boldsymbol{W}_{tt}$, $\boldsymbol{U}^0, \boldsymbol{U}_1, \boldsymbol{U}_{tt}, \boldsymbol{W}^0, \boldsymbol{W}_1$ and $\boldsymbol{W}_{tt}$ are the Ritz projection values of $u_0(x, y), u_1(x, y), u_{tt}(0), -\Delta u_0(x, y), -\Delta u_1(x, y)$ and $-\Delta u_{tt}(0)$ at grid points, respectively.

Lemma 2: The positive definite matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ in the problem (6) satisfies the following inequalities (see [24], Lemma 1.22 and [25], Lemma 1.4.1 and Lemma 1.4.2):

$$\|\boldsymbol{A}\|_\infty \le Ch, \quad \|\boldsymbol{A}^{-1}\|_\infty \le Ch,$$
$$|\boldsymbol{B}\|_\infty \le C, \quad \|\boldsymbol{B}^{-1}\|_\infty \le C.$$

Theorem 1: The coefficient vectors $\{\boldsymbol{U}^n, \boldsymbol{W}^n\} \in \mathbb{R}^M \times \mathbb{R}^M$ $(1 \le n \le N)$ of the MFE solutions in the problem (6) are unconditionally stable, so that the solutions $\{u_h^n, w_h^n\} \in V_h \times V_h$ $(1 \le n \le N)$ of the problem (5) are also unconditionally stable.

Proof: Because the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive definite matrices, set $\boldsymbol{D}_1 = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}$, $\boldsymbol{D}_2 = \boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B}^{-1}\boldsymbol{A}$, then the problem (6) can be rewritten as

$$\boldsymbol{U}^{n+1} - 2\boldsymbol{U}^n + \boldsymbol{U}^{n-1} = -\frac{\Delta t^2}{4}\boldsymbol{D}_1(\boldsymbol{U}^{n+1} + 2\boldsymbol{U}^n + \boldsymbol{U}^{n-1}) + \Delta t^2 \boldsymbol{A}^{-1}\boldsymbol{F}^n, 1 \le n \le N-1. \tag{7}$$

Noting that $\boldsymbol{U}^1 = \boldsymbol{U}^0 + \Delta t \boldsymbol{U}_1 + \frac{1}{2}\Delta t^2 \boldsymbol{U}_{tt}$ and summing from 1 to n $(n \ge 1)$ for (7), we have

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n + \Delta t \boldsymbol{U}_1 + \frac{\Delta t^2}{2}\boldsymbol{U}_{tt} - \frac{\Delta t^2}{4}\boldsymbol{D}_1 \sum_{i=1}^n (\boldsymbol{U}^{i+1} + 2\boldsymbol{U}^i + \boldsymbol{U}^{i-1}) + \Delta t^2 \boldsymbol{A}^{-1}\sum_{i=1}^n \boldsymbol{F}^i, 1 \le n \le N-1. \tag{8}$$

Summing from 1 to $n-1$ $(n \ge 2)$ for (8), we obtain

$$\boldsymbol{U}^n = \boldsymbol{U}^0 + n\Delta t \boldsymbol{U}_1 + \frac{n\Delta t^2}{2}\boldsymbol{U}_{tt} - \frac{\Delta t^2}{4}\boldsymbol{D}_1 \sum_{j=1}^{n-1}\sum_{i=1}^{j}(\boldsymbol{U}^{i+1} + 2\boldsymbol{U}^i + \boldsymbol{U}^{i-1}) + \Delta t^2 \boldsymbol{A}^{-1}\sum_{j=1}^{n-1}\sum_{i=1}^{j}\boldsymbol{F}^i, 2 \le n \le N.$$

By Lemma 2, we have

$$\begin{aligned} \|\boldsymbol{U}^n\|_1 &\le \|\boldsymbol{U}^0\|_1 + n\Delta t\|\boldsymbol{U}_1\|_1 + Ch\Delta t^2 \sum_{j=1}^{n-1}\sum_{i=1}^{j}\|\boldsymbol{F}^i\|_1 \\ &\quad + \frac{n\Delta t^2}{2}\|\boldsymbol{U}_{tt}\|_1 + \frac{Ch^2\Delta t^2}{4}\sum_{j=1}^{n-1}\sum_{i=1}^{j} \\ &\quad \times \|\boldsymbol{U}^{i+1} + 2\boldsymbol{U}^i + \boldsymbol{U}^{i-1}\|_1 \\ &\le \|\boldsymbol{U}^0\|_1 + n\Delta t\|\boldsymbol{U}_1\|_1 + nCh\Delta t^2 \sum_{i=1}^{n-1}\|\boldsymbol{F}^i\|_1 \\ &\quad + \frac{n\Delta t^2}{2}\|\boldsymbol{U}_{tt}\|_1 + \frac{nCh^2\Delta t^2}{4}\sum_{i=1}^{n-1} \\ &\quad \times \|\boldsymbol{U}^{i+1} + 2\boldsymbol{U}^i + \boldsymbol{U}^{i-1}\|_1 \\ &\le \|\boldsymbol{U}^0\|_1 + T\|\boldsymbol{U}_1\|_1 + CT\Delta t\|\boldsymbol{U}_{tt}\|_1 \\ &\quad + ChT\Delta t\sum_{i=1}^{n-1}\|\boldsymbol{F}^i\|_1 + Ch^2 T\Delta t\sum_{i=0}^{n-1} \\ &\quad \times \|\boldsymbol{U}^i\|_1 + Ch^2 T\Delta t\|\boldsymbol{U}^n\|_1, 2 \le n \le N. \end{aligned}$$

Further, the above inequality is equivalent to

$$(1 - Ch^2 T\Delta t)\|\boldsymbol{U}^n\|_1 \le \|\boldsymbol{U}^0\|_1 + T\|\boldsymbol{U}_1\|_1 + CT\Delta t\|\boldsymbol{U}_{tt}\|_1 + ChT\Delta t\sum_{i=1}^{n-1}\|\boldsymbol{F}^i\|_1 + Ch^2 T\Delta t\sum_{i=0}^{n-1}\|\boldsymbol{U}^i\|_1, 2 \le n \le N. \tag{9}$$

Apply the discrete Gronwall inequality (see [26], Lemma 1.4.1) to (9) and using the smoothness of $f(x, y, t)$, $u_0(x, y)$ and $u_1(x, y)$, we have

$$\|\boldsymbol{U}^n\|_1 \le \Big(\|\boldsymbol{U}^0\|_1 + T\|\boldsymbol{U}_1\|_1 + ChT\Delta t\|\sum_{i=1}^{n-1}\|\boldsymbol{F}^i\|_1 + CT\Delta t \boldsymbol{U}_{tt}\|_1\Big)\exp(Ch^2 Tn\Delta t) \le C, 2 \le n \le N. \tag{10}$$

We can know from (10) that the solution $\boldsymbol{U}^n$ of the problem (6) is unconditionally stable, and $\|\boldsymbol{N}\|_1 \le C$, then we get

$$\|u_h^n\|_1 = \|\boldsymbol{U}^n \cdot \boldsymbol{N}\|_1 \le C\|\boldsymbol{N}\|_1\|\boldsymbol{U}^n\|_1 \le C, 1 \le n \le N. \tag{11}$$

The derivation process similar to (11) yields

$$\|w_h^n\|_1 \le C, 1 \le n \le N. \tag{12}$$

Thus, the solutions $\{u_h^n, w_h^n\} \in V_h \times V_h$ $(1 \le n \le N)$ of the problem (5) are also unconditionally stable. □

Remark 1: As long as h, Δt, $f(x, y, t)$, $u_0(x, y)$ and $u_1(x, y)$ are given, we can get two sequences of the coefficient vectors $\{\boldsymbol{U}^n, \boldsymbol{W}^n\}$ and the MFE solutions $\{u_h^n, w_h^n\}$ $(n = 1, 2, \ldots, N)$

for the equation (1) by computing the problem (5) or the problem (6). But the MFE scheme contains many unknowns and so we need to decrease the unknowns for the problem (5) by the POD method.

III. THE ROEMFE MATRIX MODEL FOR THE 2D FOURTH-ORDER HYPERBOLIC EQUATION

A. GENERATION OF POD BASIS

We firstly choose two sets of the first L vectors $\boldsymbol{U}^n$ and $\boldsymbol{W}^n$ ($n = 1, 2, \ldots, L$) from the series of the coefficient vectors L vectors $\boldsymbol{U}^n$ and $\boldsymbol{W}^n$ ($n = 1, 2, \ldots, N$) for problem (6), forming two snapshot matrices $\boldsymbol{Q}_1 = (\boldsymbol{U}^1, \boldsymbol{U}^2, \ldots, \boldsymbol{U}^L, \tilde{\boldsymbol{U}})_{M\times(L+1)}$ and $\boldsymbol{Q}_2 = (\boldsymbol{W}^1, \boldsymbol{W}^2, \ldots, \boldsymbol{W}^L, \tilde{\boldsymbol{W}})_{M\times(L+1)}$, respectively, where $\tilde{\boldsymbol{U}} = (\boldsymbol{U}^L - \boldsymbol{U}^{L-1})/\Delta t$ and $\tilde{\boldsymbol{W}} = (\boldsymbol{W}^L - \boldsymbol{W}^{L-1})/\Delta t$. Then, we can obtain the positive eigenvalues $\lambda_{i,j}$ ($j = 1, 2, \ldots, r_i = rank(\boldsymbol{Q}_i), i = 1, 2$) with $\lambda_{i,1} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,r_i}$ and the corresponding orthonormal eigenvectors $\tilde{\boldsymbol{X}}_i = (\boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}, \ldots, \boldsymbol{x}_{ir_i})$ of $\boldsymbol{Q}_i\boldsymbol{Q}_i^{\mathrm{T}}$. Finally, we accquire two sets of POD basis $\boldsymbol{X}_i = (\boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}, \ldots, \boldsymbol{x}_{id})$ ($d \leq r_i, i = 1, 2$) from the foremost d orthonormal eigenvectors in $\tilde{\boldsymbol{X}}_i$ and satisfying the following properties [26], [27], [28]:

$$\|\boldsymbol{Q}_i - \boldsymbol{X}_i\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{Q}_i\|_{2,2} = \sqrt{\lambda_{i,d+1}}, \quad i = 1, 2, \tag{13}$$

where $\|\boldsymbol{Q}_i\|_{2,2} = sup_{\boldsymbol{U}\neq 0}\|\boldsymbol{Q}_i\boldsymbol{U}\|_2/\|\boldsymbol{U}\|_2$ and $\|\boldsymbol{U}\|_2$ is the L^2 norm for vector $\boldsymbol{U}$.

From (13), when $n = 1, 2, \ldots, L$, we have

$$\begin{aligned}\|\boldsymbol{U}^n - \boldsymbol{X}_1\boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{U}^n\| &= \|(\boldsymbol{Q}_1 - \boldsymbol{X}_1\boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{Q}_1)\boldsymbol{e}^n\|\\ &\leq \|\boldsymbol{Q}_1 - \boldsymbol{X}_1\boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{Q}_1\|_{2,2}\|\boldsymbol{e}^n\| \leq \sqrt{\lambda_{1,d+1}}, \quad (14)\\ \|\tilde{\boldsymbol{U}} - \boldsymbol{X}_1\boldsymbol{X}_1^{\mathrm{T}}\tilde{\boldsymbol{U}}\| &= \|(\boldsymbol{Q}_1 - \boldsymbol{X}_1\boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{Q}_1)\boldsymbol{e}^{L+1}\|\\ &\leq \|\boldsymbol{Q}_1 - \boldsymbol{X}_1\boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{Q}_1\|_{2,2}\|\boldsymbol{e}^{L+1}\| \leq \sqrt{\lambda_{1,d+1}}, \quad (15)\\ \|\boldsymbol{W}^n - \boldsymbol{X}_2\boldsymbol{X}_2^{\mathrm{T}}\boldsymbol{W}^n\| &= \|(\boldsymbol{Q}_2 - \boldsymbol{X}_2\boldsymbol{X}_2^{\mathrm{T}}\boldsymbol{Q}_2)\boldsymbol{e}^n\|\\ &\leq \|\boldsymbol{Q}_2 - \boldsymbol{X}_2\boldsymbol{X}_2^{\mathrm{T}}\boldsymbol{Q}_2\|_{2,2}\|\boldsymbol{e}^n\| \leq \sqrt{\lambda_{2,d+1}}, \quad (16)\\ \|\tilde{\boldsymbol{W}} - \boldsymbol{X}_2\boldsymbol{X}_2^{\mathrm{T}}\tilde{\boldsymbol{W}}\| &= \|(\boldsymbol{Q}_2 - \boldsymbol{X}_2\boldsymbol{X}_2^{\mathrm{T}}\boldsymbol{Q}_2)\boldsymbol{e}^{L+1}\|\\ &\leq \|\boldsymbol{Q}_2 - \boldsymbol{X}_2\boldsymbol{X}_2^{\mathrm{T}}\boldsymbol{Q}_2\|_{2,2}\|\boldsymbol{e}^{L+1}\| \leq \sqrt{\lambda_{2,d+1}}, \quad (17)\end{aligned}$$

where $\boldsymbol{e}^n$ ($n = 1, 2, \ldots, L, L+1$) are the unit vectors with the nth component is 1.

Remark 2: Because $M \gg (L+1)$ and the positive eigenvalues $\lambda_{i,j}$ ($j = 1, 2, \ldots, r_i, i = 1, 2$) of $\boldsymbol{Q}_i\boldsymbol{Q}_i^{\mathrm{T}}$ and $\boldsymbol{Q}_i^{\mathrm{T}}\boldsymbol{Q}_i$ are identical, so we may first obtain the foremost d eigenvalues $\lambda_{i,j}$ ($1 \leq j \leq d, i = 1, 2$) of $\boldsymbol{Q}_i^{\mathrm{T}}\boldsymbol{Q}_i$ and the corresponding eigenvectors $\boldsymbol{y}_{i,j}$ ($1 \leq j \leq d, i = 1, 2$). Then, we can easily acquire the eigenvectors $\boldsymbol{x}_{i,j} = \boldsymbol{Q}_i\boldsymbol{y}_{i,j}/\sqrt{\lambda_{i,j}}$ ($1 \leq j \leq d, i = 1, 2$) corresponding to the positive eigenvalues $\lambda_{i,j}$ for $\boldsymbol{Q}_i\boldsymbol{Q}_i^{\mathrm{T}}$ to make up the POD basis.

B. ROEMFE MODEL

Let $\boldsymbol{U}_d^n = (u_{d1}^n, u_{d2}^n, \ldots, u_{dM}^n)^{\mathrm{T}} = \boldsymbol{X}_1\boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{U}^n =: \boldsymbol{X}_1\boldsymbol{a}_d^n$ and $\boldsymbol{W}_d^n = (w_{d1}^n, w_{d2}^n, \ldots, w_{dM}^n)^{\mathrm{T}} = \boldsymbol{X}_2\boldsymbol{X}_2^{\mathrm{T}}\boldsymbol{W}^n =: \boldsymbol{X}_2\boldsymbol{b}_d^n$ be the first L ($L \leq N$) coefficient vectors of the ROEMFE solutions, where $\boldsymbol{a}_d^n = (a_1^n, a_2^n, \ldots, a_d^n)^{\mathrm{T}}$, $\boldsymbol{b}_d^n = (b_1^n, b_2^n, \ldots, b_d^n)^{\mathrm{T}}$. Then we acquire the first L ROEMFE solutions $u_d^n = \boldsymbol{U}_d^n \cdot \boldsymbol{N}$ and $w_d^n = \boldsymbol{W}_d^n \cdot \boldsymbol{N}$ ($1 \leq n \leq L$). Substituting the solutions $\boldsymbol{U}^n$ and $\boldsymbol{W}^n$ in the problem (6) for $\boldsymbol{U}_d^n = \boldsymbol{X}_1\boldsymbol{a}_d^n$ and $\boldsymbol{W}_d^n = \boldsymbol{X}_2\boldsymbol{b}_d^n$ ($L+1 \leq n \leq N$), respectively, we get the following the ROEMFE model.

Find $\{\boldsymbol{a}^n, \boldsymbol{b}^n\} \in \mathbb{R}^d \times \mathbb{R}^d$ and $\{u_d^n, w_d^n\} \in V_h \times V_h$ such that

$$\begin{cases}\boldsymbol{a}_d^n = \boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{U}^n, \quad \boldsymbol{b}_d^n = \boldsymbol{X}_2^{\mathrm{T}}\boldsymbol{W}^n, 1 \leq n \leq L,\\ \boldsymbol{A}\boldsymbol{X}_2\boldsymbol{b}_d^{n-1} = \boldsymbol{B}\boldsymbol{X}_1\boldsymbol{a}_d^{n-1}, L+1 \leq n \leq N+1,\\ \boldsymbol{A}\boldsymbol{X}_1\boldsymbol{a}_d^{n+1} + \dfrac{\Delta t^2}{4}\boldsymbol{B}\boldsymbol{X}_2\boldsymbol{b}_d^{n+1} = 2\boldsymbol{A}\boldsymbol{X}_1\boldsymbol{a}_d^n - \boldsymbol{A}\boldsymbol{X}_1\boldsymbol{a}_d^{n-1} -\\ \dfrac{\Delta t^2}{2}\boldsymbol{B}\boldsymbol{X}_2\boldsymbol{b}_d^n - \dfrac{\Delta t^2}{4}\boldsymbol{B}\boldsymbol{X}_2\boldsymbol{b}_d^{n-1} + \Delta t^2\boldsymbol{F}^n, L \leq n \leq N-1,\\ u_d^n = \displaystyle\sum_{i=1}^{M} u_{di}^n N_i = \boldsymbol{U}_d^n \cdot \boldsymbol{N}, w_d^n = \sum_{i=1}^{M} w_{di}^n N_i = \boldsymbol{W}_d^n \cdot \boldsymbol{N},\end{cases} \tag{18}$$

where $\boldsymbol{U}^n$ and $\boldsymbol{W}^n$ ($1 \leq n \leq L$) are two sequence of the initial L coefficient vectors in the problem (6) and the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{F}^n$ are given in the problem (6).

*Remark 3: The solutions of the problem (18) exists and is unique due to **A** and **B** are both positive definite matrices. It is not difficult to find that the problem (6) has M unknowns in each level, but the problem (18) has only d unknowns at the same time level ($d \ll M$), which means that the problem (18) can greatly reduce unknowns, so that enormously reduce the CPU time, reduce the accumulation of round-off errors, and more importantly, compared with the problem (6), it improves the accuracy of numerical solutions in the practical calculation (see Section IV). Therefore, the problem (18) is clearly superior to the problem (6). In addition, since the problem (18) and the problem (6) has the same basis, this ensures that they have the same error accuracy.*

C. STABILITY AND ERROR ESTIMATES OF THE ROEMFE SOLUTIONS

Theorem 2: Under the same hypotheses in Lamma 1, the ROMEFE solutions $\{u_d^n, w_d^n\} \in V_h \times V_h$ ($1 \leq n \leq N$) in the problem (18) are unconditionally stable and satisfy the following error estimates:

$$\begin{aligned}\|u^n - u_d^n\|_1 + \|w^n - w_d^n\|_1 \leq C(h + \Delta t^2\\ + \sqrt{\lambda_{1,d+1}} + \sqrt{\lambda_{2,d+1}}), 1 \leq n \leq N.\end{aligned} \tag{19}$$

Proof:

1) STABILITY OF THE SOLUTIONS OF THE PROBLEM (18)

When $1 \leq n \leq L$, using the orthonormality of vectors in $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, we have

$$\begin{aligned}\|u_d^n\|_1 + \|w_d^n\|_1 &= \|\boldsymbol{U}_d^n \cdot \boldsymbol{N}\|_1 + \|\boldsymbol{W}_d^n \cdot \boldsymbol{N}\|_1\\ &= \|\boldsymbol{X}_1\boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{U}^n \cdot \boldsymbol{N}\|_1 + \|\boldsymbol{X}_2\boldsymbol{X}_2^{\mathrm{T}}\boldsymbol{W}^n \cdot \boldsymbol{N}\|_1\\ &\leq C(\|u_h^n\|_1 + \|w_h^n\|_1), 1 \leq n \leq L.\end{aligned} \tag{20}$$

Therefore, according to the unconditional stability of $\{u_h^n\}_{n=1}^N$ and $\{w_h^n\}_{n=1}^N$ in Theorem 1, we can find that $\{u_d^n\}_{n=1}^N$ and $\{w_d^n\}_{n=1}^N$ are unconditionally stable.

When $L+1 \le n \le N$, due to $\boldsymbol{A}$ is a positive definite matrix, we can rewrite (18) as

$$\boldsymbol{U}_d^n - 2\boldsymbol{U}_d^{n-1} + \boldsymbol{U}_d^{n-2} = -\frac{\Delta t^2}{4}\boldsymbol{D}_1(\boldsymbol{U}_d^n + 2\boldsymbol{U}_d^{n-1} + \boldsymbol{U}_d^{n-2}) + \Delta t^2\boldsymbol{A}^{-1}\boldsymbol{F}^{n-1}, L+1 \le n \le N. \quad (21)$$

Summing from L to n ($n \ge L+1$) for (21), we have

$$\boldsymbol{U}_d^n = \boldsymbol{U}_d^{n-1} + \boldsymbol{U}_d^L - \boldsymbol{U}_d^{L-1} - \frac{\Delta t^2}{4}\boldsymbol{D}_1\sum_{i=L}^{n}(\boldsymbol{U}_d^i + 2\boldsymbol{U}_d^{i-1} + \boldsymbol{U}_d^{i-2}) + \Delta t^2\boldsymbol{A}^{-1}\sum_{i=L}^{n}\boldsymbol{F}^{i-1}, L+1 \le n \le N. \quad (22)$$

Summing from L to n ($n \ge L+1$) for (22), using $\boldsymbol{U}_d^n = \boldsymbol{X}_1\boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{U}^n$ and (8), we obtain

$$\begin{aligned}\boldsymbol{U}_d^n &= (n-L)(\boldsymbol{U}_d^L - \boldsymbol{U}_d^{L-1}) - \frac{\Delta t^2}{4}\boldsymbol{D}_1\sum_{j=L}^{n}\sum_{i=L}^{j}(\boldsymbol{U}_d^i + 2\boldsymbol{U}_d^{i-1} + \boldsymbol{U}_d^{i-2}) + \Delta t^2\boldsymbol{A}^{-1}\sum_{j=L}^{n}\sum_{i=L}^{j}\boldsymbol{F}^{i-1}\\
&= (n-L)\boldsymbol{X}_1\boldsymbol{X}_1^{\mathrm{T}}\Big\{\Delta t\boldsymbol{U}_1 + \frac{\Delta t^2}{2}\boldsymbol{U}_{tt} - \frac{\Delta t^2}{4}\boldsymbol{D}_1\sum_{i=1}^{L-1}(\boldsymbol{U}^{i+1} + 2\boldsymbol{U}^i + \boldsymbol{U}^{i-1}) + \Delta t^2\boldsymbol{A}^{-1}\sum_{i=1}^{L-1}\boldsymbol{F}^{i-1}\Big\}\\
&\quad - \frac{\Delta t^2}{4}\boldsymbol{D}_1\sum_{j=L}^{n}\sum_{i=L}^{j}(\boldsymbol{U}_d^i + 2\boldsymbol{U}_d^{i-1} + \boldsymbol{U}_d^{i-2}) + \Delta t^2\boldsymbol{A}^{-1}\sum_{j=L}^{n}\sum_{i=L}^{j}\boldsymbol{F}^{i-1}, L+1 \le n \le N.\end{aligned}$$

Using Lemma 2 and (10), we have

$$\begin{aligned}|\boldsymbol{U}_d^n\|_1 &\le (n-L)\Delta t\|\boldsymbol{U}_1\|_1 + \frac{(n-L)\Delta t^2}{2}\|\boldsymbol{U}_{tt}\|_1 + Ch^2(n-L)\Delta t^2 + Ch(n-L)\Delta t^2\sum_{i=1}^{n}\|\boldsymbol{F}^{i-1}\|_1\\
&\quad + Ch^2(n-L)\Delta t^2\sum_{i=L}^{n}\|\boldsymbol{U}_d^i + 2\boldsymbol{U}_d^{i-1} + \boldsymbol{U}_d^{i-2}\|_1\\
&\le T\|\boldsymbol{U}_1\|_1 + CT\Delta t\|\boldsymbol{U}_{tt}\|_1 + Ch^2T\Delta t + ChT\Delta t\sum_{i=1}^{n}\|\boldsymbol{F}^{i-1}\|_1 + Ch^2T\Delta t\sum_{i=L-2}^{n-1}\|\boldsymbol{U}_d^i\|_1\\
&\quad + Ch^2T\Delta t\|\boldsymbol{U}_d^n\|_1, L+1 \le n \le N.\end{aligned}$$

From the above inequality, we can get

$$(1 - Ch^2T\Delta t)\|\boldsymbol{U}_d^n\|_1 \le T\|\boldsymbol{U}_1\|_1 + CT\Delta t\|\boldsymbol{U}_{tt}\|_1 + Ch^2T\Delta t + ChT\Delta t\sum_{i=1}^{n}\|\boldsymbol{F}^{i-1}\|_1 + Ch^2T\Delta t\sum_{i=L-2}^{n-1}\|\boldsymbol{U}_d^i\|_1, L+1 \le n \le N. \quad (23)$$

Apply the discrete Gronwall inequality (see [26], Lemma 1.4.1) to (23) and using the smoothness of $f(x,y,t)$ and $u_1(x,y)$, we have

$$\begin{aligned}\|\boldsymbol{U}_d^n\|_1 &\le \Big(T\|\boldsymbol{U}_1\|_1 + Ch^2T\Delta t + ChT\Delta t\sum_{i=1}^{n}\|\boldsymbol{F}^{i-1}\|_1 + CT\Delta t\|\boldsymbol{U}_{tt}\|_1\Big)\exp(Ch^2Tn\Delta t)\\
&\le C, L+1 \le n \le N. \end{aligned} \quad (24)$$

Noting that $\|\boldsymbol{N}\|_1 \le C$, from (24) we obtain

$$\|u_d^n\|_1 = \|\boldsymbol{U}^d \cdot \boldsymbol{N}\|_1 \le C\|\boldsymbol{N}\|_1\|\boldsymbol{U}^d\|_1 \le C, L+1 \le n \le N. \quad (25)$$

Similarly, we can prove that the solution w_d^n of the problem (18) have the following result

$$\|w_d^n\|_1 \le C, L+1 \le n \le N. \quad (26)$$

It can be seen from (20), (25) and (26) that the solutions $\{u_d^n, w_d^n\} \in V_h \times V_h$ ($1 \le n \le N$) of the problem (18) is unconditionally stable.

2) ERROR ESTIMATION OF THE SOLUTIONS OF THE PROBLEM (18)

When $1 \le n \le L$, noting that $u_h^n = \boldsymbol{N}\cdot\boldsymbol{U}^n$, $w_h^n = \boldsymbol{N}\cdot\boldsymbol{W}^n$, $\|\boldsymbol{N}\|_1 \le C$, by (14) and (16), we have

$$\begin{aligned}\|u_h^n - u_d^n\|_1 &\le \|\boldsymbol{U}^n - \boldsymbol{U}_d^n\|_\infty\|\boldsymbol{N}\|_1\\
&\le C\|\boldsymbol{U}^n - \boldsymbol{X}_1\boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{U}^n\| \le C\sqrt{\lambda_{1,d+1}}, \quad (27)\\
\|w_h^n - w_d^n\|_1 &\le \|\boldsymbol{W}^n - \boldsymbol{W}_d^n\|_\infty\|\boldsymbol{N}\|_1\\
&\le C\|\boldsymbol{W}^n - \boldsymbol{X}_2\boldsymbol{X}_2^{\mathrm{T}}\boldsymbol{W}^n\| \le C\sqrt{\lambda_{2,d+1}}. \quad (28)\end{aligned}$$

Set $\boldsymbol{E}_1^n = \boldsymbol{U}^n - \boldsymbol{U}_d^n$ and $\boldsymbol{E}_2^n = \boldsymbol{W}^n - \boldsymbol{W}_d^n$. When $L+1 \le n \le N$, subtract (21) from (7), we obtain the following error equation about $\boldsymbol{E}_1^n$

$$\boldsymbol{E}_1^n - 2\boldsymbol{E}_1^{n-1} + \boldsymbol{E}_1^{n-2} = -\frac{\Delta t^2}{4}\boldsymbol{D}_1(\boldsymbol{E}_1^n + 2\boldsymbol{E}_1^{n-1} + \boldsymbol{E}_1^{n-2}), L+1 \le n \le N. \quad (29)$$

Summing from L to n ($n \ge L+1$) for (29), we have

$$textsl\boldsymbol{E}_1^n - \boldsymbol{E}_1^{n-1} - (\boldsymbol{E}_1^L - \boldsymbol{E}_1^{L-1}) = -\frac{\Delta t^2}{4}\boldsymbol{D}_1\sum_{i=L}^{n}(\boldsymbol{E}_1^i + 2\boldsymbol{E}_1^{i-1} + \boldsymbol{E}_1^{i-2}), L+1 \le n \le N. \quad (30)$$

Summing from L to n ($n \ge L+1$) for (30), we have

$$\begin{aligned}\boldsymbol{E}_1^n &= (n-L)(\boldsymbol{E}_1^L - \boldsymbol{E}_1^{L-1}) - \frac{\Delta t^2}{4}\boldsymbol{D}_1\sum_{j=L}^{n}\sum_{i=L}^{j}(\boldsymbol{E}_1^i + 2\boldsymbol{E}_1^{i-1} + \boldsymbol{E}_1^{i-2})\\
&= (n-L)[(\boldsymbol{U}^L - \boldsymbol{U}^{L-1}) - \boldsymbol{X}_1\boldsymbol{X}_1^{\mathrm{T}}(\boldsymbol{U}^L - \boldsymbol{U}^{L-1})]\end{aligned}$$

TABLE 1. **The error and CPU time of u_h^n and u_d^n for example 1 at $t = 1$.**

	MFE model			ROEMFE model		
mesh	$\|u^n - u_h^n\|_1$	order	CPU time (s)	$\|u^n - u_d^n\|_1$	order	CPU time (s)
8×8	9.2911e-02	–	0.2616	9.2910e-02	–	0.1829
16×16	4.6360e-02	1.0021	1.0845	4.6359e-02	1.0021	0.7074
32×32	2.3165e-02	1.0006	15.2990	2.3165e-02	1.0006	4.6487
64×64	1.1581e-02	1.0001	639.5934	1.1580e-02	1.0002	115.4294

TABLE 2. **The error and CPU time of u_h^n and u_d^n for example 1 at $t = 2$.**

	MFE model			ROEMFE model		
mesh	$\|u^n - u_h^n\|_1$	order	CPU time (s)	$\|u^n - u_d^n\|_1$	order	CPU time (s)
8×8	3.4047e-02	–	0.4212	3.4047e-02	–	0.2409
16×16	1.7035e-02	0.9993	1.6604	1.7035e-02	0.9993	0.7619
32×32	8.5197e-03	0.9997	36.1407	8.5197e-03	0.9997	5.4753
64×64	4.2601e-03	0.9999	1350.9701	4.2601e-03	1.0000	124.6507

TABLE 3. **The error and CPU time of w_h^n and w_d^n for example 1 at $t = 1$.**

	MFE model			ROEMFE model		
mesh	$\|w^n - w_h^n\|_1$	order	CPU time (s)	$\|w^n - w_d^n\|_1$	order	CPU time (s)
8×8	1.9204e+00	–	0.2574	1.9204e+00	–	0.1739
16×16	9.2600e-01	1.0369	1.1000	9.2600e-01	1.0369	0.6863
32×32	4.5866e-01	1.0094	18.3554	4.5866e-01	1.0094	4.7448
64×64	2.2878e-02	1.0024	659.3240	2.2878e-02	1.0024	130.5486

TABLE 4. **The error and CPU time of w_h^n and w_d^n for example 1 at $t = 2$.**

	MFE model			ROEMFE model		
mesh	$\|w^n - w_h^n\|_1$	order	CPU time (s)	$\|w^n - w_d^n\|_1$	order	CPU time (s)
8×8	9.1018e-01	–	0.4106	9.1018e-01	–	0.2193
16×16	3.7048e-01	1.2283	1.8429	3.7048e-01	1.2283	0.7545
32×32	1.7259e-01	1.0732	34.3898	1.7259e-01	1.0732	5.0977
64×64	8.4636e-02	1.0196	1128.6204	8.4636e-02	1.0196	139.0451

$$-\frac{\Delta t^2}{4}\boldsymbol{D}_1\sum_{j=L}^{n}\sum_{i=L}^{j}(\boldsymbol{E}_1^i+2\boldsymbol{E}_1^{i-1}+\boldsymbol{E}_1^{i-2})$$
$$=(n-L)\Delta t(\tilde{\boldsymbol{U}}-\boldsymbol{X}_1\boldsymbol{X}_1^{\mathrm{T}}\tilde{\boldsymbol{U}})-\frac{\Delta t^2}{4}\boldsymbol{D}_1\sum_{j=L}^{n}\sum_{i=L}^{j}(\boldsymbol{E}_1^i$$
$$+2\boldsymbol{E}_1^{i-1}+\boldsymbol{E}_1^{i-2}), L+1\le n\le N. \qquad (31)$$

From Lemma 2, (15) and (31), we have

$$\|\boldsymbol{E}_1^n\|_1 \le (n-L)\Delta t\sqrt{\lambda_{1,d+1}}$$
$$+C(n-L)\Delta t^2\sum_{i=L}^{n}\|\boldsymbol{E}_1^i+2\boldsymbol{E}_1^{i-1}+\boldsymbol{E}_1^{i-2}\|_1$$
$$\le CT\sqrt{\lambda_{1,d+1}}+CT\Delta t\|\boldsymbol{E}_1^n\|_1$$
$$+CT\Delta t\sum_{i=L-2}^{n-1}\|\boldsymbol{E}_1^i\|_1, L+1\le n\le N. \qquad (32)$$

Further, (32) can be rewritten as the following form

$$(1-CT\Delta t)\|\boldsymbol{E}_1^n\|_1 \le CT\sqrt{\lambda_{1,d+1}}$$
$$+CT\Delta t\sum_{i=L-2}^{n-1}\|\boldsymbol{E}_1^i\|_1, L+1\le n\le N. \qquad (33)$$

According to discrete Gronwall inequality (see [26], Lemma 1.4.1) for (33), we obtain

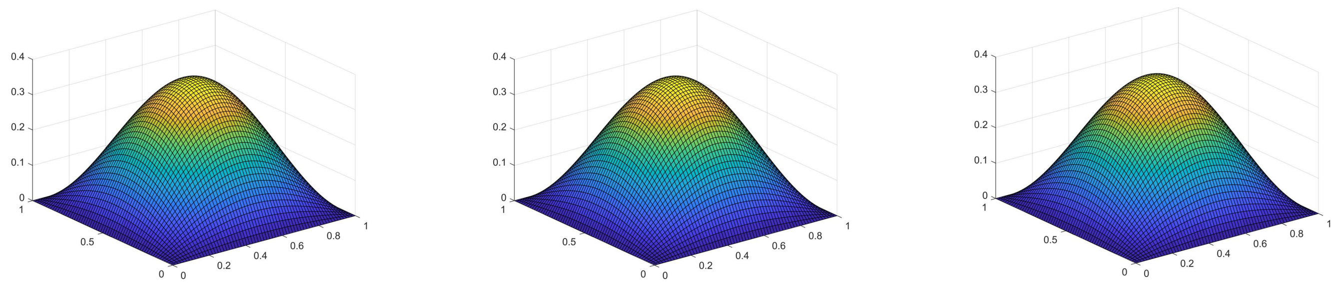
$$\|\boldsymbol{E}_1^n\|_1 \le C\sqrt{\lambda_{1,d+1}}, \quad L+1\le n\le N. \qquad (34)$$

Similar to the process of $\|\boldsymbol{E}_1^n\|$, we have

$$\|\boldsymbol{E}_2^n\|_1 \le C\sqrt{\lambda_{2,d+1}}, \quad L+1\le n\le N. \qquad (35)$$

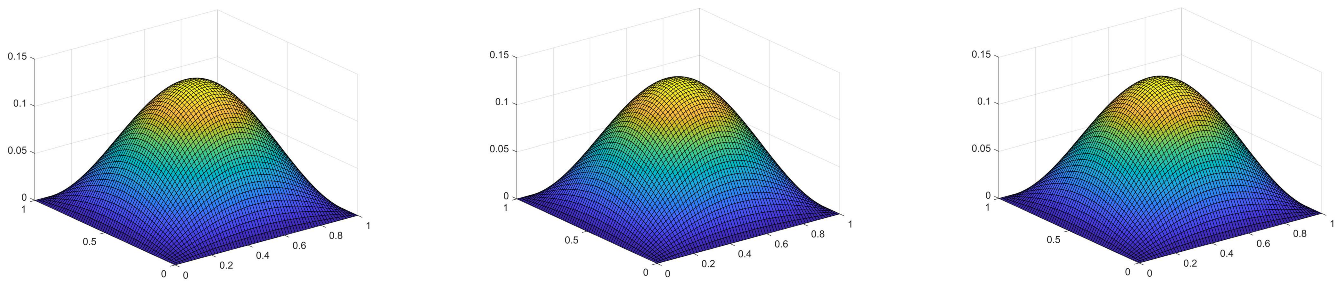
From (27), (28), (34) and (35) with Lemma (1), we acquire (19). □

Remark 4: The error term $\sqrt{\lambda_{1,d+1}}+\sqrt{\lambda_{2,d+1}}$ *in Theorem (2) is generated by the reduced-order for the MFE matrix scheme, which can be used to determine how many POD*



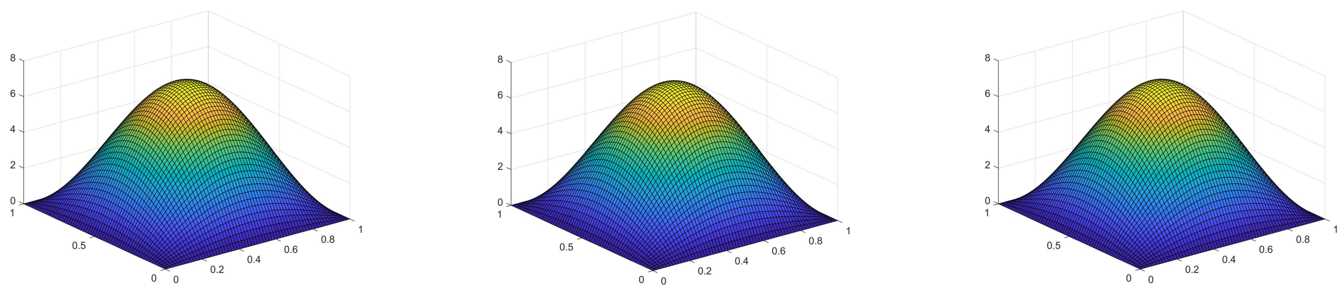
(a) Exact solution u^n (b) MFE solution u_h^n (c) ROEMFE solution u_d^n

FIGURE 1. The graphics of u^n, u_h^n and u_d^n for example 1 at $t = 1$.



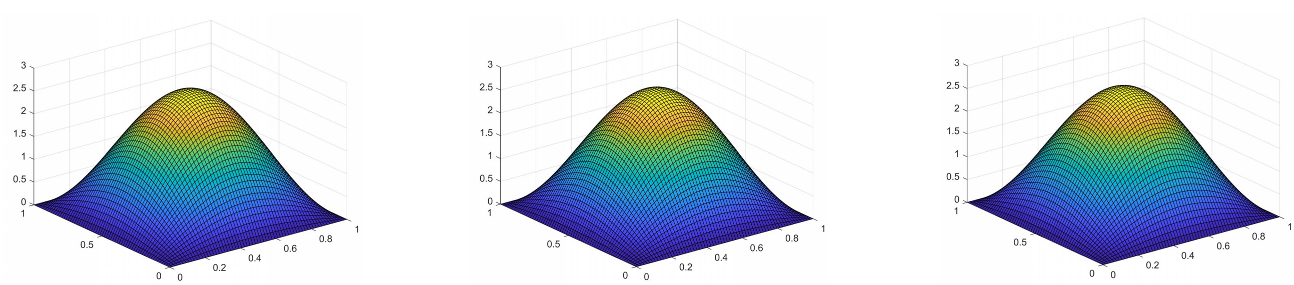
(a) Exact solution u^n (b) MFE solution u_h^n (c) ROEMFE solution u_d^n

FIGURE 2. The graphics of u^n, u_h^n and u_d^n for example 1 at $t = 2$.



(a) Exact solution w^n (b) MFE solution w_h^n (c) ROEMFE solution w_d^n

FIGURE 3. The graphics of w^n, w_h^n and w_d^n for example 1 at $t = 1$.



(a) Exact solution w^n (b) MFE solution w_h^n (c) ROEMFE solution w_d^n

FIGURE 4. The graphics of w^n, w_h^n and w_d^n for example 1 at $t = 2$.

basis to select, namely, the number d of POD basis must satisfy $\sqrt{\lambda_{1,d+1}} + \sqrt{\lambda_{2,d+1}} \leq h + \Delta t^2$. A large number of numerical experiments have indicated that the eigenvalues $\sqrt{\lambda_{1,d+1}}$ and $\sqrt{\lambda_{2,d+1}}$ will quickly tend to 0. Usually, when $d = 5$ or 6, $\sqrt{\lambda_{1,d+1}}$ and $\sqrt{\lambda_{2,d+1}}$ are very small and satisfies $\sqrt{\lambda_{1,d+1}} + \sqrt{\lambda_{2,d+1}} \leq h + \Delta t^2$.

TABLE 5. The error and CPU time of u_h^n and u_d^n for example 2 at $t = 1$.

mesh	MFE model $\|u^n - u_h^n\|_1$	order	CPU time (s)	ROEMFE model $\|u^n - u_d^n\|_1$	order	CPU time (s)
8×8	1.9925e-02	–	0.2643	1.9902e-02	–	0.1854
16×16	1.0025e-02	0.9937	1.1999	9.9888e-03	0.9962	0.7890
32×32	5.0482e-03	0.9930	13.4973	5.0118e-03	0.9965	4.9117
64×64	2.5070e-03	1.0068	541.1359	2.5022e-02	1.0014	119.8134

TABLE 6. The error and CPU time of u_h^n and u_d^n for example 2 at $t = 2$.

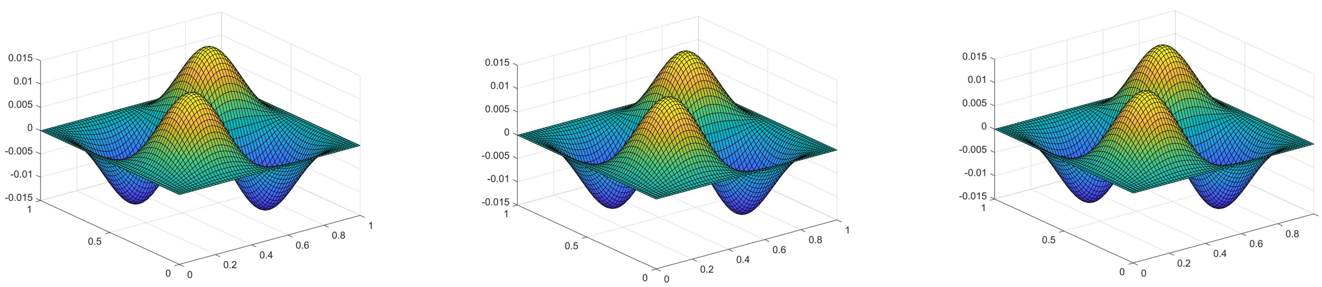
mesh	MFE model $\|u^n - u_h^n\|_1$	order	CPU time (s)	ROEMFE model $\|u^n - u_d^n\|_1$	order	CPU time (s)
8×8	9.8803e-03	–	0.4450	8.0973e-03	–	0.2128
16×16	4.0919e-03	1.2072	1.6163	3.9214e-03	1.0324	0.7928
32×32	1.9157e-03	1.0679	23.5288	1.8770e-03	1.0445	5.1370
64×64	9.2594e-04	1.0344	1029.9082	9.2205e-04	1.0178	134.5199

TABLE 7. The error and CPU time of w_h^n and w_d^n for example 2 at $t = 1$.

mesh	MFE model $\|w^n - w_h^n\|_1$	order	CPU time (s)	ROEMFE model $\|w^n - w_d^n\|_1$	order	CPU time (s)
8×8	4.0217e+00	–	0.2794	3.3134e+00	–	0.2092
16×16	1.5935e+00	1.2619	1.0679	1.7567e+00	0.9430	0.7648
32×32	6.8178e-01	1.1686	15.4672	7.2380e-01	1.2135	4.9219
64×64	3.3896e-01	1.0056	535.3630	3.3697e-01	1.0739	135.2695

TABLE 8. The error and CPU time of w_h^n and w_d^n for example 2 at $t = 2$.

mesh	MFE model $\|w^n - w_h^n\|_1$	order	CPU time (s)	ROEMFE model $\|w^n - w_d^n\|_1$	order	CPU time (s)
8×8	1.8442e+00	–	0.4729	2.1281e+00	–	0.1929
16×16	9.9001e-01	0.9314	1.6245	9.8238e-01	1.0831	0.7357
32×32	3.5226e-01	1.4180	26.4556	3.8138e-01	1.2879	5.0984
64×64	1.5107e-01	1.1658	1111.5471	1.2530e-01	1.5218	140.9505

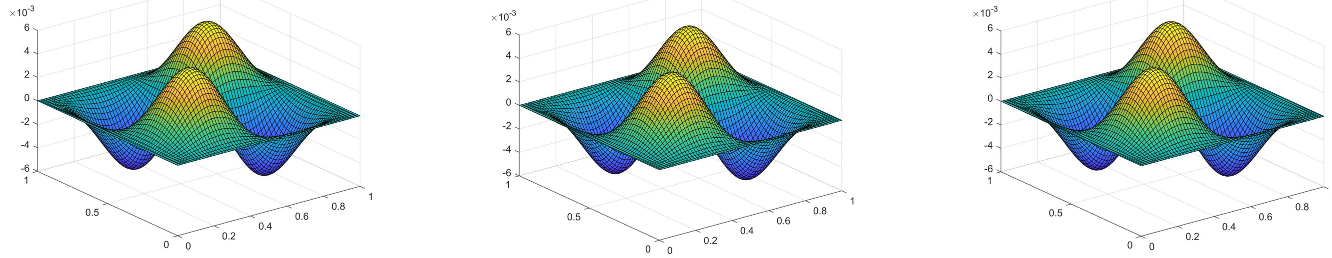


(a) Exact solution u^n (b) MFE solution u_h^n (c) ROEMFE solution u_d^n

FIGURE 5. The graphics of u^n, u_h^n and u_d^n for example 2 at $t = 1$.

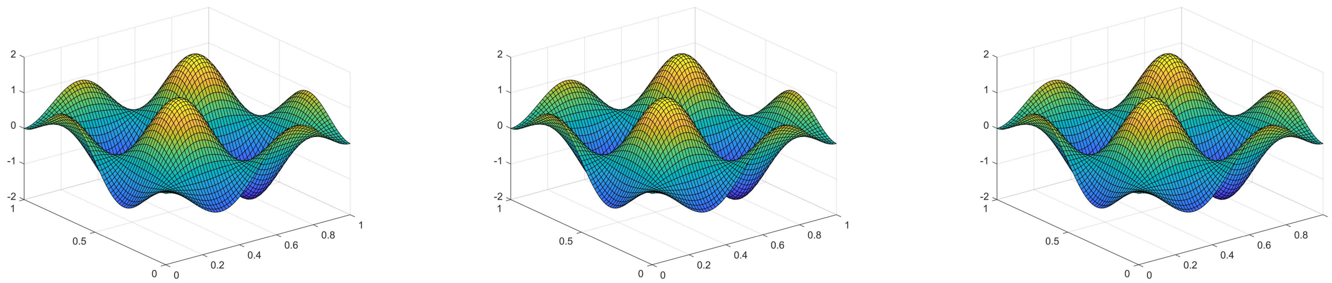
IV. NUMERICAL EXPERIMENT

In this section, we present some numerical examples to illustrate the superiority of the ROEMFE matrix model. Example 1 and Example 2 are used to reflect the calculation time and verify the error accuracy of the original variable u_d^n and the intermediate variable w_d^n of the ROEMFE matrix model, in which the smooth exact solution is constructed and the corresponding right term, initial and boundary conditions and the intermediate variable w are computed from the exact solution respectively.



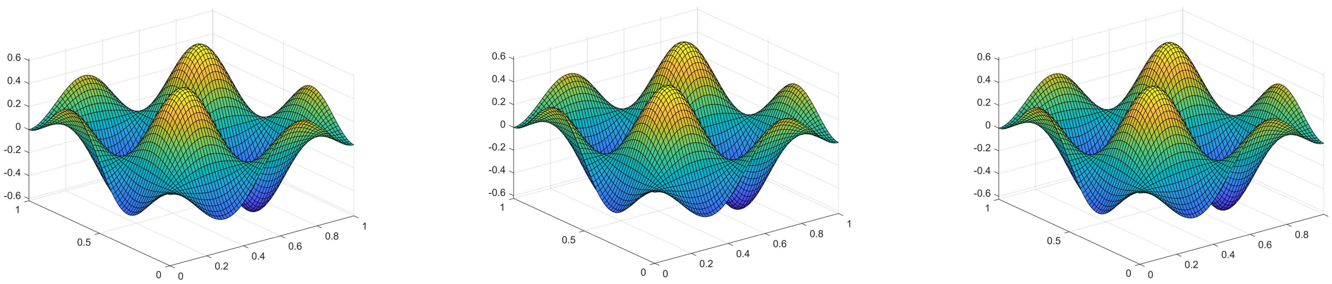
(a) Exact solution u^n (b) MFE solution u_h^n (c) ROEMFE solution u_d^n

FIGURE 6. The graphics of u^n, u_h^n and u_d^n for example 2 at $t = 2$.



(a) Exact solution w^n (b) MFE solution w_h^n (c) ROEMFE solution w_d^n

FIGURE 7. The graphics of w^n, w_h^n and w_d^n for example 2 at $t = 1$.



(a) Exact solution w^n (b) MFE solution w_h^n (c) ROEMFE solution w_d^n

FIGURE 8. The graphics of w^n, w_h^n and w_d^n for example 2 at $t = 2$.

Take $\overline{\Omega} = [0, 1] \times [0, 1]$ and the domain is divided into uniform rectangles in each direction. When the spatial step $h = 1/64$ and the time step $\Delta t = 1/100$, then the theoretical error is $O(10^{-2})$ according to Lemma 1 and Theorems 2.

Example 1: $u = \exp^{-t} \sin(\pi x) \sin(\pi y)$.

In order to use the ROEMFE matrix model, taking $L = 20$ and $d = 1$, we can find $\sqrt{\lambda_{1,2}} + \sqrt{\lambda_{2,2}} \leq 1.5 \times 10^{-2}$ by calculating the positive eigenvalues of $\boldsymbol{Q}_i^{\mathrm{T}} \boldsymbol{Q}_i$ $(i = 1, 2)$. Tables 1-2 show the error estimates of u_h^n and u_d^n in H^1-norm and the CPU time, we can see that $\|u^n - u_h^n\|_1$ and $\|u^n - u_d^n\|_1$ have the same accuracy and basically satisfies the first-order convergence rate, which conforms to our theoretical analysis. More importantly, when the domain is finely divided, $\|u^n - u_d^n\|_1$ is smaller than $\|u^n - u_h^n\|_1$, which implies that the solution u_d^n of the ROEMFE matrix model is more accurate than the solution u_h^n of the MFE matrix model.

We also can see from Tables 1-2 that as the spatial step decreases and the time increases, the CPU time of the ROEMFE matrix model is shorter and increases slower than that of the MFE matrix model. When the grid is 64×64 and $t = 2$, the CPU time for the ROEMFE matrix model is about $1/10$ times that of the MFE matrix model. When the grid is 64×64, from $t = 1$ to $t = 2$, the ROEMFE matrix model takes only 9.2213s, while the MFE matrix model takes 711.3767s, which means that the ROEMFE matrix model can greatly reduce the CPU time. Tables 3-4 show the error estimates of w_h^n and w_d^n in H^1-norm and the CPU time, we can see that $\|w^n - w_h^n\|_1$ and $\|w^n - w_d^n\|_1$ have the same accuracy and basically satisfies the first-order convergence rate, and w_d^n gets faster than w_h^n.

For clarify, we present the graphics of exact solution u^n, the MFE solution u_h^n and the ROEMFE solution u_d^n in Figures 1-2 and graphics of exact solution w^n, the MFE solution w_h^n and the ROEMFE solution w_d^n in Figures 3-4 at $t = 1, 2$ for Example 1, respectively. We can see from Figures 1-4 that the graphics in Figure (a), Figure (b) and Figure (c) look very much alike, but the ROEMFE solutions are better than the MFE solutions due to the accumulation of small round-off errors in the calculation process of the ROEMFE algorithm.

Example 2: $u = \exp^{-t} x(1-x)y(1-y)\sin(2\pi x)\sin(2\pi y)$.

In order to use the ROEMFE matrix model, taking $L = 20$ and $d = 2$, we can find $\sqrt{\lambda_{1,3}} + \sqrt{\lambda_{2,3}} \leq 1.5 \times 10^{-2}$ by calculating the positive eigenvalues of $\boldsymbol{Q}_i^{\mathrm{T}}\boldsymbol{Q}_i$ $(i = 1, 2)$. Tables 5-8 shows the error estimates of u_h^n, u_d^n, w_h^n and w_d^n in H^1-norm and the CPU time, we can see that $\|u^n - u_h^n\|_1$ and $\|u^n - u_d^n\|_1$ has the same accuracy and basically satisfies the first-order convergence rate and so do those in $\|w^n - w_h^n\|_1$ and $\|w^n - w_d^n\|_1$, which conforms to our theoretical analysis. In addition, when the domain is finely divided, $\|u^n - u_d^n\|_1$ is smaller than $\|u^n - u_h^n\|_1$, which implies that the solution u_d^n of the ROEMFE matrix model is more accurate than the solution u_h^n of the MFE matrix model. For clarify, we present the graphics of exact solution u^n, the MFE solution u_h^n and the ROEMFE solution u_d^n in Figures 5-6 and graphics of exact solution w^n, the MFE solution w_h^n and the ROEMFE solution w_d^n in Figures 7-8 at $t = 1, 2$ for Example 2, respectively. Similarly, Tables 5-8 and Figures 5-8 also shows that we can quickly obtain more accurate numerical solutions using the ROEMFE matrix model.

V. CONCLUSION

In this paper, we have studied the reduced-order of solution coefficient vectors for the MFE method for the 2D fourth-order hyperbolic equation by means of the POD method. The ROEMFE matrix model for the equation has been proposed with the POD basic vectors consisting of the first few known MFE solution coefficient vectors, the existence, stability and error estimates of the ROEMFE solutions has been readily proved with the help of matrix analysis tools, and some numerical experiments have confirmed the correctness of the theoretical analysis and the superiority of the ROEMFE matrix model. Since the unknowns of the ROEMFE matrix model are far less than those of the MFE matrix model, which not only greatly reduces the accumulation of round-off errors, but also greatly shortens CPU time and maintains accuracy in the calculation. In addition, the intermediate variable w also have been successfully reduced-order, which means that the method can be extended to more variable problems. Particularly, the ROEMFE matrix model for higher order problem is come up for the first time, thus it is totally different from the existing POD-based reduced-order MFE methods. Therefore, this study on the reduced-order of solution coefficient vectors for the MFE method for the equation is meaningful.

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CHANGBIAO YU received the B.S. degree in mathematics and applied mathematics from Huaibei Normal University, Anhui, China, in 2020. He is currently pursuing the master's degree with the Qilu University of Technology (Shandong Academy of Sciences), China. His research interest includes the numerical solution of partial differential equations, and he has published many articles in related fields.

WENWEN XU (Member, IEEE) received the Ph.D. degree from Shandong University, China, in 2016.

She is currently an Associate Professor in mathematics and artificial intelligence at the Qilu University of Technology (Shandong Academy of Sciences), China. She is mainly engaged in the research of numerical methods of partial differential equations, including mixed finite element and multi-point flux mixed finite element decoupling algorithm. She has presided over one youth fund of the National Natural Science Foundation of China and one training fund of the Natural Science Foundation of Shandong Province, and published more than ten papers.

XINDONG LI (Member, IEEE) was born in China, in 1985. He received the Ph.D. degree in computational mathematics from Shandong University, China.

He is currently an Associate Professor in mathematics and artificial intelligence at the Qilu University of Technology (Shandong Academy of Sciences), China. His research interests include numerical solution of partial differential equations and reservoir numerical simulation. He has presided over and completed one National Natural Science Foundation and one Natural Science Foundation of Shandong Province, participated in many National Natural Science Foundation and national science and technology major projects, and published more than ten SCI papers.