IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 21 September 2022, accepted 12 October 2022, date of publication 17 October 2022, date of current version 21 October 2022.
Digital Object Identifier 10.1109/ACCESS.2022.3215232

RESEARCH ARTICLE

A Novel Similarity-Based Multi-Attribute Group Decision-Making Method in a Probabilistic Hesitant Fuzzy Environment

YUBO HU AND ZHIQIANG PANG
School of Statistics, Lanzhou University of Finance and Economics, Lanzhou 730020, China
Corresponding author: Yubo Hu (huyubo202209@163.com)

ABSTRACT Probabilistic hesitant fuzzy sets (PHFSs), a significant expansion of hesitant fuzzy sets (HFSs), were suggested and intensively explored in order to address the problem of missing preference information. Based on previous research on PHFS, we discovered that there are still unresolved issues in the probabilistic hesitant fuzzy environment, such as (i) the similarity between probabilistic hesitant fuzzy elements (PHFEs) has not been studied, (ii) in the study of entropy, the uncertainty resulting from the inner hesitancy of decision makers (DMs) has been neglected; in addition, DMs may obtain different decision results by using different entropy formulas; (iii) the relationship between the similarity and entropy of PHFE has not been researched, and (iv) there is no multi-attribute group decision-making (MAGDM) method that uses similarity in a probabilistic hesitant fuzzy environment. In order to address the aforementioned issues, in this study, we attempt to incorporate similarity into a probabilistic hesitant fuzzy environment and offer a novel similarity-based MAGDM method. First, we define the similarity in probabilistic hesitant fuzzy environments and present some similarity formulas. Furthermore, considering the limitations of entropy presented by other researchers, we redefine the entropy of PHFEs and discuss the relationship between similarity and entropy in probabilistic hesitant fuzzy settings for the first time. Based on the similarity measure and entropy, we offer a new method for MAGDM with unknown attribute weights, which can be effectively applied to the assessment of small and medium-sized enterprises' (SMEs) credit risk. Finally, we demonstrated the effectiveness and robustness of the proposed decision-making process.

INDEX TERMS Probabilistic hesitant fuzzy set, similarity, entropy, multi-criteria decision-making.

I. INTRODUCTION

In daily life, owing to the limitations of decision makers (DMs) in knowledge, ability, and experience, it is difficult for them to use accurate numbers to make decisions on plans, but they can only give fuzzy judgments. When making evaluations, DMs must choose an appropriate form of expressing the decision-making evaluation information. To address the aforementioned problem, it is critical to investigate the expression form of the decision evaluation information that is more in line with the human thinking process. Several scholars have conducted extensive research in this area. Zadeh proposed a fuzzy set [1] and believed that the fuzzy set, rather than precise numbers, can express the uncertainty and subjective ambiguity about objective things. However, fuzzy sets are not without flaws. As more scholars study fuzzy sets, their corresponding extended forms such as L-type fuzzy sets [2], 2-type fuzzy sets [3], and fuzzy interval sets [4] have been proposed. However, these new fuzzy set forms cannot fully express fuzziness from additional dimensions, but only broaden the value range of the membership functions. Therefore, Atanassov proposed an intuitionistic fuzzy set in [5]. Subsequently, extended forms of intuitionistic fuzzy sets, such as interval intuitionistic fuzzy sets [6], were proposed. However, further research has revealed that DMs cannot directly and accurately assign

The associate editor coordinating the review of this manuscript and approving it for publication was Yilun Shang.

membership or non-membership degrees when evaluating a scheme. DMs frequently waver between multiple membership values. Different experts offer different perspectives during group decision-making. Therefore, it is critical to find an appropriate way to integrate diverse perspectives. Torra [7] proposed a hesitant fuzzy set (HFS) to solve the aforementioned hesitation and group decision problem, and since then, many scholars have conducted extensive research on HFS [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27]. Despite significant theoretical and practical advances, HFSs are not ideal. When assigning different membership degrees, the preferences of DMs for each membership degree may differ. Consequently, the expression form of the hesitant fuzzy element (HFE) indicates that each membership degree is equally important, which does not conform to the most realistic decision-making scenarios, resulting in the loss of DMs' preference information. Therefore, Xu and Zhou [28] first proposed the concept of a probabilistic hesitant fuzzy set (PHFS) by applying probability information to hesitant fuzzy sets, which can better overcome the defects of HFS. A PHFS is a set composed of multiple probabilistic hesitant fuzzy elements (PHFEs), where the PHFEs are composed of real numbers in [0, 1] and corresponding probabilities. The probability information corresponding to the degree of membership is added; thus, the preference information of DMs is better expressed, helping avoid the loss of decision information. However, in some cases, DMs cannot fully offer probability information in the decision-making process. Therefore, Zhang et al. [29] improved the PHFS proposed in [28] to make it suitable for more general situations. In recent years, numerous researchers have focused their attention on the PHFS due to its usefulness as a tool for expressing evaluative information; despite this, there are still certain research voids to be filled. (1) the similarity between PHFEs has not been studied, (2) in the study of entropy, the uncertainty resulting from the inner hesitancy of DMs has been neglected; in addition, DMs may receive different decision results by using different entropy formulas; (3) the relationship between the similarity and entropy of PHFE has not been researched, and (4) there is no MAGDM method that uses similarity in a probabilistic hesitant fuzzy environment.

As a method to quantify the relationship between elements, similarities have been researched extensively in intuitionistic fuzzy sets [30], [31], interval-valued fuzzy sets [32], type-2 fuzzy sets [33], [34], spherical fuzzy sets [35], [36], [37], q-rung orthopair fuzzy sets [38], [39], pythagorean fuzzy sets [40], [41], picture fuzzy sets [42],[43], [44], fuzzy soft sets [45], [46], bipolar complex fuzzy sets [47], and hesitant fuzzy sets [48], [49]. With the increasing amount of research on extended forms of hesitant fuzzy sets in recent years, the similarity of other extended hesitant fuzzy sets has also become a research hotspot, such as dual hesitant fuzzy sets [50], [51], [52], [53], hesitant fuzzy linguistic term sets [54], hesitant interval-valued fuzzy sets [55], [56], typical hesitant fuzzy sets [57], cubic hesitant fuzzy sets [58], higher-order hesitant fuzzy sets [59], and complex hesitant fuzzy sets [60], [61]. However, PHFS, as a special extended form of hesitant fuzzy sets, based on a survey of the current literature, is still uncommon for studies on its similarity. This study defined the similarity of PHFEs and provided a distance-based similarity formula. Furthermore, Shannon [62] initially introduced entropy, a concept in physics, into information theory, and De Luca and Termini [63] first proposed the definition and formula of classical fuzzy entropy based on this concept. With the introduction of PHFS, the entropy of PHFS has also been investigated in depth. Su et al. [64] presented two types of PHFE entropies. On the one hand, influenced by De Luca and Termini [63], Su et al. [64] offered membership degree-based entropies for PHFEs, and on the other hand, influenced by Farhadinia [56], they presented distance-based entropy for PHFEs. Membership degree-based entropies for PHFEs can only be used to evaluate uncertainty in the form of $\{0.5|1\}$, but $\{1|0.5, 0|0.5\}$, the uncertainty resulting from the inner hesitancy of DMs is disregarded at the same time. For the distance-based entropy for PHFEs, the final value depends on the function we choose such that the entropy of the PHFE will change with our subjectivity. Therefore, further research on the entropy of PHFEs is required. Moreover, numerous researchers have explored the link between similarity and entropy in other contexts, including interval-valued fuzzy sets [65], [66], [67], interval-valued intuitionistic fuzzy sets [68], [69], interval-valued neutrosophic sets [70], and hesitant fuzzy sets [49]. However, the relationship between similarity and entropy in a probabilistic hesitant fuzzy environment has not been examined by other researchers owing to the paucity of studies on similarity in a probabilistic hesitant fuzzy environment. In this study, a new definition and measure of the entropy of the PHFE is proposed based on the new similarity, and the relationship between similarity and entropy is investigated. Under certain conditions, we can prove that the new distance-based similarity is the entropy of the PHFE.

Many methods for the multi-attribute decision-making (MADM) in hesitant fuzzy environments have been studied by many scholars [71], [72], [73], [74], [75], [76], [77], [78], [79], but there has been little research on probabilistic hesitant fuzzy decision-making method. Ding et al. [80] proposed the first distance measure for PHFSs and created an interactive method for solving a probabilistic hesitant fuzzy multi-attribute group decision-making (MAGDM) problem with incomplete weight information. Zhou et al. [81] incorporated the financial concept of value at risk (VaR) into decision-making and proposed a tail group decision-making process using the expected hesitant VaR(EHVaR) and programming model in a probabilistic hesitant fuzzy environment. Tian et al. [82] considered the bounded rationality of DMs, introduced prospect theory, established a consensus process based on the probability hesitant fuzzy preference relation and prospect theory, and proposed a sequential investment problem method based on the bounded rationality of DMs. He et al. [83] applied the reference ideal method to a probabilistic hesitant fuzzy environment, proposed three different decision-making methods to solve the reference ideal MADM problem, and used them in intelligent system research. Wu et al. [84] proposed a dynamic emergency response method in a probabilistic hesitant fuzzy environment based on the GM(1, 1) model and TOPSIS method. Gao et al. [85] considered the uncertain probability in the external environment and proposed a dynamic decision-making method based on the PHFS according to the characteristics of emergency decision-making in crisis management. Gupta et al. [86] introduced hesitant probability fuzzy sets in time-series forecasting and proposed a PHFS-based time-series forecasting method. Li et al. [87] proposed a Hausdorff distance measure for a PHFS and a maximum deviation method. Accordingly, the probabilistic hesitant fuzzy environment was subjected to the ELECTRE method. Li et al. [88] proposed a PHFS-based ORESTE method. Liu et al. [89] proposed and applied a probabilistic hesitant fuzzy MADM method based on the regret theory for the evaluation of venture capital projects. Tian et al. [90] proposed and used a probabilistic hesitant fuzzy TODIM method for selecting green suppliers. Krishankumar et al. [91] introduced the VIKOR method into the probabilistic hesitant fuzzy environment. Krishankumar et al. [92] proposes a novel ranking model under PHFS by extending the idea of evidence theory (ET) and applies this method to renewable energy technology selection. Despite the above-mentioned findings, there is no MAGDM method that uses similarity in a probabilistic hesitant fuzzy environment. Using similarity measures and entropy, this study proposes a new method for MAGDM with unknown attribute weights.

The major contributions of this study are as follows. 1. Inspired by HFS, we define the similarity of PHFEs and develop similarity formulas. 2. Considering the shortcomings of entropy of PHFE proposed by Su et al. [64], we offer a new definition of entropy of PHFE, further study the relationship between the newly proposed similarity and entropy, and finally prove that the similarity based on distance is a type of entropy of PHFE under certain conditions. 3. We propose a MAGDM method based on our newly proposed similarity and entropy with unknown attribute weights and apply this method to banks' assessment of the credit risk of small and medium-sized enterprises (SMEs), which provides banks with a new perspective under soft information [96] and helps alleviate the financing difficulties of SMEs.

The remainder of this paper is organized as follows:

In Section 2, we review the relevant concepts of HFS and PHFS. In Section 3, we define the similarity of PHFEs and present similarity formulas based on this description. To address the drawbacks of entropy suggested by other scholars, we present a new definition of entropy, analyze the relationship between similarity and entropy, and establish a new entropy based on similarity. In Section 4, a new MAGDM approach based on the similarity of PHFEs is presented. In Section 5, we use a credit decision example to demonstrate the use of our proposed method. In Section 6, we discuss the efficiency, robustness, and advancement of the proposed decision-making method. In Section 7, we conclude this study and offer suggestions for further research.

II. PRELIMINARY

In this section, we will review the relevant concepts of HFS and PHFS.

A. CONCEPT OF HFS

To solve the problem of group decision-making and the situation in which DMs are hesitant to face multiple membership degrees, Torra [7] introduced the HFS concept.

Definition 1 [7]: Assuming that X is a reference set, HFS A on X is defined in terms of a function $h_A(x)$that returns a finite subset of [0, 1] when applied to X.

Following that, Xia et al. [93] investigated HFS further and expressed it mathematically:

$$A = \{\langle x, h_A(x)\rangle \,|x \in X\} \tag{1}$$

Here, the function $h_A(x)$ is a set of different values in [0, 1], representing the possible membership degrees of the element x in X to A. For convenience of application and description, $h_A(x)$ is called an HFE.

Xu and Xia [48] first developed distance measures for HFSs to describe the relationships between HFEs. Subsequently, Xu and Xia [94] proposed a distance definition and distance measures for HFEs.

Definition 2 [94]: The HFEs h_1 and h_2, and the distance between h_1 and h_2, denoted as $d(h_1, h_2)$, should satisfy the following properties:

(1) $0 \leq d(h_1, h_2) \leq 1$;

(2) $d(h_1, h_2)=0$ if and only if$h_1 = h_2$;

(3) $d(h_1, h_2) = d(h_2, h_1)$.

Based on Definition 2, Xu and Xia [94] proposed the following distance measures for HFEs.

$$d(h_1, h_2) = \frac{1}{l}\sum_{i=1}^{l}\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right| \tag{2}$$

$$d(h_1, h_2) = \sqrt{\frac{1}{l}\sum_{i=1}^{l}\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right|^2} \tag{3}$$

$$d(h_1, h_2) = \max_i\left\{\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right|\right\} \tag{4}$$

$$d(h_1, h_2) = \max_i\left\{\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right|^2\right\} \tag{5}$$

$$d(h_1, h_2) = \frac{1}{2}\left(\begin{array}{l}\frac{1}{l}\sum_{i=1}^{l}\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right| \\ + \max_i\left\{\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right|\right\}\end{array}\right) \tag{6}$$

$$d(h_1, h_2) = \frac{1}{2}\left(\begin{array}{l}\sqrt{\frac{1}{l}\sum_{i=1}^{l}\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right|^2} \\ + \max_i\left\{\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right|^2\right\}\end{array}\right) \tag{7}$$

Furthermore, Xu and Xia [49] proposed a definition of the similarity and distance-based similarity formulas of HFEs based on Definition 2 as follows:

Definition 3 [49]: For h_1 and h_2, the similarity between h_1 and h_2, denoted as $s(h_1, h_2)$, should satisfy the following properties:

(1) $s(h_1, h_2) = 0$ *iff* $h_1 = \{0\}, h_2 = \{1\}$or $h_1 = \{1\}, h_2 = \{0\}$;

(2) $s(h_1, h_2) = 1$ *iff* $h_1^{\sigma(i)} = h_2^{\sigma(i)}, i = 1, 2, \cdots, l$;

(3) $s(h_1, h_3) \le s(h_1, h_2), s(h_1, h_3) \le s(h_2, h_3)$; *If* $h_1^{\sigma(i)} \le h_2^{\sigma(i)} \le h_3^{\sigma(i)}$ *or* $h_1^{\sigma(i)} \ge h_2^{\sigma(i)} \ge h_3^{\sigma(i)}, i = 1, 2, \cdots, l$;

(4) $s(h_1, h_2) = s(h_2, h_1)$.

Based on Definition 3, Xu and Xia [49] developed distance-based hesitant fuzzy similarity formulas, as follows:

$$s(h_1, h_2) = 1 - \frac{1}{l}\sum_{i=1}^{l}\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right| \tag{8}$$

$$s(h_1, h_2) = 1 - \sqrt{\frac{1}{l}\sum_{i=1}^{l}\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right|^2} \tag{9}$$

$$s(h_1, h_2) = \max_i\left\{\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right|\right\} \tag{10}$$

$$s(h_1, h_2) = \max_i\left\{\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right|^2\right\} \tag{11}$$

$$s(h_1, h_2) = 1 - \frac{1}{2}\left(\frac{1}{l}\sum_{i=1}^{l}\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right| + \max_i\left\{\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right|\right\}\right) \tag{12}$$

$$s(h_1, h_2) = 1 - \frac{1}{2}\left(\sqrt{\frac{1}{l}\sum_{i=1}^{l}\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right|^2} + \max_i\left\{\left|h_1^{\sigma(i)} - h_2^{\sigma(i)}\right|^2\right\}\right) \tag{13}$$

Subsequently, Xu and Xia [49] studied the entropy of an HFE and provided an axiomatic definition:

Definition 4 [49]: Entropy on HFE h is a real-valued function $E : H \to [0, 1]$ that satisfies the following axiomatic requirements:

(1) $E(h) = 0$ *iff* $h = \{0\}$ *or* $h = \{1\}$;

(2) $E(h) = 1$ *iff* $h^{\sigma(i)} + h^{\sigma(l-i+1)} = 1, i = 1, 2, \cdots, l$;

(3) $E(h) = E(h^c)$;

(4) $E(h_1) \le E(h_2)$; *If* $h_1^{\sigma(i)} \le h_2^{\sigma(i)}$ for $h_2^{\sigma(i)} + h_2^{\sigma(l-i+1)} \le 1$ *or* $h_1^{\sigma(i)} \ge h_2^{\sigma(i)}$ for $h_2^{\sigma(i)} + h_2^{\sigma(l-i+1)} \ge 1$, $i = 1, 2, \cdots, l$.

Based on Definition 4, Xu and Xia [49] studied the relationship between the similarity and entropy of HFE and concluded that $s(h, h^c)$ is an entropy measure of h, which can be expressed mathematically as:

$$E(h) = s\left(h, h^c\right) \tag{14}$$

B. CONCEPT OF PHFS

Although HFS plays an important and effective role in MADM, it has limitations. In the decision-making process of a single decision maker, for example, DM examines alternatives based on criteria. Evaluation values are 0.3, 0.5, and 0.8, with the DM preferring 0.8. If the evaluation information is represented by a HFE {0.3, 0.5, 0.8}, it does not reflect the DM's preference for 0.8. Second, it is assumed in group decision making that there are five experts to evaluate a certain plan; three experts give 0.7, one expert gives 0.6, and the final expert gives 0.4. At the moment, we can only achieve {0.7, 0.6, 0.4}. Obviously, this data cannot fully express the initial information given by DMs, and the information provided by two DMs is completely ignored. As a result, using HFEs to describe the evaluation information is insufficient in this circumstance.To overcome the loss of information in HFSs, Xu and Zhou[28] first proposed the concept of a PHFS.

Definition 5 [28]: Let X be a fixed set; then, a PHFS on X is expressed by a mathematical symbol:

$$A = \{\langle x, h_A(\gamma_i(x)|p_i(x))\rangle\, | x \in X\} \tag{15}$$

where $h_A(\gamma_i(x)|p_i(x))$ is a set of elements, $\gamma_i(x)|p_i(x)$ denotes the hesitant fuzzy information with probabilities to the set A, $0 \le \gamma_i(x) \le 1, i = 1, 2, \cdots, l_h$, where $l_h$is the number of possible elements in $h_A(\gamma_i(x)|p_i(x))$, $p_i(x) \in [0, 1]$ is the hesitant probability of $\gamma_i(x)$, and $\sum_{i=1}^{l_h} p_i = 1$.

For convenience of application and description, Xu and Zhou [28] call$h_A(\gamma_i(x)|p_i(x))$a PHFE and A the set of all PHFEs. However, in this study, $h_A(\gamma_i(x)|p_i(x))$ will be shortened to $h(\gamma_i|p_i)$ or $h(p)$, and $\gamma_i(x)|p_i(x), \gamma_i(x)$ and $p_i(x)$ will be abbreviated as $\gamma_i|p_i$, γ_i, and p_i, respectively.

Subsequently, Zhang et al[29] considered the situation of incomplete probability information, which is the case of $\sum_{i=1}^{l_h} p_i < 1$, improved the PHFS, and proposed the concept of weak PHFE. In these cases, the weak PHFE can normalize the associated probabilities using $p = p_i \Big/ \sum_{i=1}^{l_h} p_i$ and $i = 1, 2, \cdots, l_h$. Therefore, in this study, we only considered the normalized case, which is the case of $\sum_{i=1}^{l_h} p_i = 1$.

In the decision-making process, Xu and Zhou [28] proposed score and deviation functions to rank the PHFEs as follows:

Definition 6 [28]: For PHFE $h(\gamma_i|p_i)$, where $i = 1, 2, \cdots, l_h$, the score function of$h(\gamma_i|p_i)$ can be expressed as:

$$s(h) = \sum_{i=1}^{l_h} \gamma_i p_i \tag{16}$$

where l_h is the number of possible elements in $h(\gamma_i|p_i)$, and the deviation function of $h(\gamma_i|p_i)$ can be expressed as:

$$v(h) = \sum_{i=1}^{l_h}(\gamma_i - s(h))^2 p_i \tag{17}$$

Thus, the following rules are provided to compare h_1 and h_2:

If $s(h_1) > s(h_2)$, *then* $h_1 > h_2$;
If $s(h_1) = s(h_2)$,
(1) *if* $v(h_1) > v(h_2)$, *then* $h_1 < h_2$;
(2) *if* $v(h_1) < v(h_2)$, *then* $h_1 > h_2$; *and*
(3) *if* $v(h_1) = v(h_2)$, *then* $h_1 = h_2$.

Inspired by the operations of HFEs, Zhang et al[29] defined some basic operations of PHFEs, which are listed as follows:

Definition 7 [29]: Let h, h_1, and h_2 be three normalized PHFEs, $\lambda > 0$; then,

(1) $\lambda h = \bigcup_{\gamma_l \in h}\left\{\left[1-(1-\gamma_l)^{\lambda}\right]|p_l\right\}$;
(2) $h^{\lambda} = \bigcup_{\gamma_l \in h}\left\{\gamma_l^{\lambda}|p_l\right\}$;
(3) $h_1 \oplus h_2 = \bigcup_{\gamma_{1l}\in h_1,\gamma_{2k}\in h_2}\{[\gamma_{1l}+\gamma_{2k}-\gamma_{1l}\gamma_{2k}]|p_{1l}\cdot p_{2k}\}$;
(4) $h_1 \otimes h_2 = \bigcup_{\gamma_{1l}\in h_1,\gamma_{2k}\in h_2}\{[\gamma_{1l}\gamma_{2k}]|p_{1l}\cdot p_{2k}\}$.

Based on the operations of PHFEs, Zhang et al. [29] further proposed a probabilistic hesitant fuzzy weighted averaging (PHFWA) operator for PHFEs, which is critical for dealing with information fusion in probabilistic hesitant fuzzy group decision-making.

Definition 8 [29]: Let $h_i\,(i = 1, 2\cdots n)$ be n normalized PHFEs, and $\omega = (\omega_1, \omega_2\cdots\omega_n)$ be the weight vector of $h_i\,(i = 1, 2\cdots n)$ with $\omega_i \in [0, 1]$, $i = 1, 2\cdots n$, and $\sum_{i=1}^{n}\omega_i = 1$. Then, the probabilistic hesitant fuzzy weighted averaging (PHFWA) operator has the following form:

$$\begin{aligned} &PHFWA\,(h_1, h_2, \cdots, h_n) \\ &= \bigoplus_{i=1}^{n}\omega_i h_i \\ &= \bigcup_{\gamma_{1l}\in h_1,\gamma_{2l}\in h_2,\cdots,\gamma_{nl}\in h_n}\left\{\left[1-\prod_{i=1}^{n}(1-\gamma_{il})^{\omega_i}\right]\left|\prod_{i=1}^{n}p_{il}\right.\right\} \end{aligned} \tag{18}$$

In particular, if $\omega = \left(\frac{1}{n}, \frac{1}{n}\cdots\frac{1}{n}\right)$, the PHFWA operator reduces to a probabilistic hesitant fuzzy averaging (PHFA) operator.

C. DISTANCE MEARSURE FOR PHFEs

Distance has attracted the attention of many scholars as an excellent instrument for measuring and describing the relationship between PHFEs. Su et al. [64] provided an axiomatic definition of the distance measure for PHFEs based on the distance measure for PHFSs proposed by Ding et al. [80].

Definition 9 [64]: Let h_1 and h_2 be two PHFEs on A. Then, the distance measure $d(h_1, h_2)$ between h_1 and h_2 should satisfy the following properties:

(1) $d(h_1, h_2) \geq 0$;
(2) $d(h_1, h_2) = 0$ *iff* $h_1 = h_2$;
(3) $d(h_1, h_2) = d(h_2, h_1)$.

Based on the above axiomatic definitions, Su et al. [64] further developed the distance formula for PHFEs, as follows:

$$d_1(h_1, h_2) = \sum_{i=1}^{l}\left|p_1^{\sigma(i)}\gamma_1^{\sigma(i)} - p_2^{\sigma(i)}\gamma_2^{\sigma(i)}\right| \tag{19}$$

$$d_2(h_1, h_2) = \left[\sum_{i=1}^{l}\left|p_1^{\sigma(i)}\gamma_1^{\sigma(i)} - p_2^{\sigma(i)}\gamma_2^{\sigma(i)}\right|^2\right]^{\frac{1}{2}} \tag{20}$$

$$d_3(h_1, h_2) = \max_i\left\{\left|p_1^{\sigma(i)}\gamma_1^{\sigma(i)} - p_2^{\sigma(i)}\gamma_2^{\sigma(i)}\right|\right\} \tag{21}$$

where (19), (20), and (21) are the Hamming distance, Euclidean distance, and Hamming Hausdorff distance between two PHFEs, respectively.

Where $l = \max\{l_1, l_2\}$, $p_1^{\sigma(i)}\gamma_1^{\sigma(i)}$ and $p_2^{\sigma(i)}\gamma_2^{\sigma(i)}$ are the ith maximum values in h_1 and h_2 Meanwhile, $\gamma_1^{\sigma(i)}$, $\gamma_2^{\sigma(i)}$ and $p_1^{\sigma(i)}, p_2^{\sigma(i)}$ are the corresponding membership degrees and the associated probabilities, respectively. When $l_1 \neq l_2$ (assuming $l_1 \leq l_2$), several terms $\gamma_i|p_i$ with probability 0 are added by using the conservative or optimistic criteria proposed in [29].

The aforementioned distance formula is extensively used to measure the relationship between PHFEs and to solve MADM problems. Despite its wonderful effect, it also has shortcomings.

Inspired by Farhadinia [56], Su et al. [64] proposed a distance-based entropy for PHFEs; however, the key to obtaining this entropy is to make the PHFE and its complement symmetric with respect to $\{0.5|1\}$, namely $d(h, \{0.5|1\}) = d(h^c, \{0.5|1\})$. Su et al. [64] discussed the symmetry of the distance measure of PHFEs and found that the distance formula (19) proposed in Definition 9 could not obtain $d(h, \{0.5|1\}) = d(h^c, \{0.5|1\})$, we use an example to illustrate this problem as follows:

Example 1: For $h = \{0.6|0.8, 0.3|0.2\}$, we use the distance formula (19) and calculate $d_1(h, \{0.5|1\})$ and $d_1(h^c, \{0.5|1\})$ as follows:

$$\begin{aligned} d_1(h, \{0.5|1\}) &= |0.6\times 0.8 - 0.5\times 1| \\ &\quad + |0.3\times 0.2 - 0.5\times 0| = 0.08 \\ d_1\left(h^c, \{0.5|1\}\right) &= |0.4\times 0.8 - 0.5\times 1| \\ &\quad + |0.7\times 0.2 - 0.5\times 0| = 0.32 \end{aligned}$$

Obviously, $d_1(h, \{0.5|1\}) \neq d_1(h^c, \{0.5|1\})$, that is, h and h^c cannot be symmetric about $\{0.5|1\}$.

Therefore, Su et al. [64] introduced the concept of probability and expectation in mathematical statistics theory to describe the relationship between PHFEs to obtain a distance measure that can satisfy symmetry, and then proposed a new distance measure formula called the like-distance.

Definition 10 [64]: Let h_1 and h_2 be two PHFEs on A; then, the like-distance $d(h_1, h_2)$ has the following properties:

(1) $d(h_1, h_2) \geq 0$;
(2) *If* $h_1 = h_2$, $d(h_1, h_2) = 0$;
(3) $d(h_1, h_2) = d(h_2, h_1)$.

Based on Definition 10, Su et al[64] proposed the following:

$$d_4(h_1,h_2)=\left|\sum_{i=1}^{l_1}p_1^{\sigma(i)}\gamma_1^{\sigma(i)}-\sum_{i=1}^{l_2}p_2^{\sigma(i)}\gamma_2^{\sigma(i)}\right| \tag{22}$$

Su et al. [64] demonstrated its symmetry and successfully utilized it to measure the entropy. We use the case in *Example 1* to illustrate its symmetry as follows:

Example 2: For $h = \{0.6|0.8, 0.3|0.2\}$, we use the distance formula (22) and calculate $d_4(h,\{0.5|1\})$ and $d_4(h^c,\{0.5|1\})$ as follows:

$$d_4(h,\{0.5|1\}) = |0.6\times 0.8+0.3\times 0.2-0.5\times 1| = 0.04$$
$$d_4\left(h^c,\{0.5|1\}\right) = |0.4\times 0.8+0.7\times 0.2-0.5\times 1| = 0.04$$

Obviously, $d_4(h,\{0.5|1\}) = d_4(h^c,\{0.5|1\})$, that is, the distance formula (22) can make h and h^c symmetrical about $\{0.5|1\}$.

In addition, the distance formula is helpful when dealing with two PHFEs of different lengths. Unlike the formula mentioned in Definition 9, if the two PHFEs have different lengths, the length of the PHFE with the lower length will increase according to the DMs' risk attitude, which strengthens their subjectivity. By contrast, the like-distance can ignore the length of the PHFEs, thus avoiding DMs' subjectivity of the DMs. In addition, although Su et al[64] proposed $d(h_1,h_2)\geq 0$ in Definition 10, it can be proven that formula (22) satisfies $0\leq d(h_1,h_2)\leq 1$. For PHFEs, $\{0|1\}$ implies that all experts disagree, and $\{1|1\}$ implies that all experts agree. In this case, the distance between the two PHFEs is the maximum. If we place $\{0|1\}$ and $\{1|1\}$ into distance formula (22), the maximum distance is 1, and hence, the like-distance formula (22) can satisfy $0\leq d(h_1,h_2)\leq 1$.

D. ENTROPY FOR PHFE

Shannon [62] initially applied the concept of entropy to information theory. Based on this concept, De Luca and Termini [66] initially proposed a definition and formula for classical fuzzy entropy. With the introduction of PHFS, its entropy has also been thoroughly researched. Su et al. [64] introduced two types of entropies, the first of which was influenced by De Luca and Termini [63]. Su et al. [64] proposed membership degree-based entropy for PHFEs as follows:

Definition 11 [64]: The entropy in PHFE h is a real-valued function E: H → [0,1], satisfying the following conditions:

(1) $E(h)=0$ *iff* $h=\{0|1\}$ *or* $h=\{1|1\}$;
(2) $E(h)=1$ *iff* $h=\{0.5|1\}$;
(3) $E(h)=E(h^c)$, *where* h^c *complements* h;
(4) $E(h_1)\leq E(h_2)$,
if $l_{h_1}=l_{h_2}$, *for*
$\gamma_2^{\sigma(i)}\leq 0.5, \gamma_1^{\sigma(i)}\leq\gamma_2^{\sigma(i)}, p_1^{\sigma(i)}=p_2^{\sigma(i)}$, *or* $\gamma_2^{\sigma(i)}\geq 0.5, \gamma_1^{\sigma(i)}\geq\gamma_2^{\sigma(i)}, p_1^{\sigma(i)}=p_2^{\sigma(i)}, i=1,2,\cdots,l_{h_1}$.
Or if $l_{h_1}\neq l_{h_2}$, *for*
$\gamma_1^{\sigma(i)}\leq\gamma_2^{\sigma(l_{h_2})}\leq 0.5$ *or* $\gamma_1^{\sigma(i)}\geq\gamma_2^{\sigma(1)}\geq 0.5$, $i=1,2,\cdots,l_{h_1}$.

The parameters $\gamma_1^{\sigma(i)}$ and $\gamma_2^{\sigma(i)}$ are the i th maximum membership degrees in h_1 and h_2, and $p_1^{\sigma(i)}$ and $p_2^{\sigma(i)}$ are the corresponding probabilities. Furthermore, h^c is a complementary operation of the PHFE h introduced in [28] and $h^c=\underset{i=1,2,\cdots,l_h}{\cup}\{(1-\gamma_i)\,|p_i\}$.

According to the above definition, the corresponding useful membership degree-based entropy for the normalized probabilistic hesitant fuzzy information can be obtained as:

$$E_1(h) = -\frac{1}{\ln 2}\sum_{i=1}^{l_h}[\gamma_i\ln\gamma_i+(1-\gamma_i)\ln(1-\gamma_i)]p_i \tag{23}$$

$$E_2(h) = \frac{1}{(\sqrt{e}-1)}\sum_{i=1}^{l_h}\left[\gamma_i e^{1-\gamma_i}+(1-\gamma_i)e^{\gamma_i}-1\right]p_i \tag{24}$$

Obviously, the above definition and formula of entropy are valid only in such uncertain situations as $h=\{0.5|1\}$, but the uncertainty, such as $h=\{1|0.5, 0|0.5\}$, is ignored, which is caused by the inner hesitation of the DMs.

Motivated by Farhadinia [56], Su et al. [64] proposed another type of entropy, namely, distance-based entropy for PHFEs. The definitions and formulas are as follows:

Definition 12 [64]: The entropy on the PHFE h is a real-valued function E: H → [0,1] that satisfies the following requirements:

(1) $E(h)=0$ *iff* $h=\{0|1\}$ or $h=\{1|1\}$;
(2) $E(h)=1$ *iff* $h=\{0.5|1\}$;
(3) $E(h)=E(h^c)$;
(4) *E monotonically decreases with respect to* $d(h,\{0.5|1\})$.

Then the entropy based on like-distance can be expressed as:

$$E_3(h)=f(d_4(h,\{0.5|1\})) \tag{25}$$

The key to this distance-based entropy is that the selected distance measure must satisfy $d_4(h,\{0.5|1\}) = d_4(h^c,\{0.5|1\})$. Su et al. [64] proposed a like-distance to satisfy this condition, which we discussed in Definition 10. Another key point is to choose a suitable function $f(x)$, to represent the relationship between the entropy and like-distance measures. According to this requirement, the functions are: (1) $f_1(x)=1-2x$, (2) $f_2(x)=1-4x^2$, (3) $f_3(x)=\cos\pi x$, and (4) $f_4(x)=\frac{1-2x}{1+2x}$.

Obviously, the distance-based entropy for the PHFEs depends on the chosen function. For example, for $h=\{0.6|0.7, 0.3|0.3\}$, to obtain $E_3(h)$, we use functions $f_1(x)=1-2x$ and $f_4(x)=\frac{1-2x}{1+2x}$, respectively, and obtain $E_3^1(h)=0.98$ and $E_3^4(h)=0.96$ condescendingly. Obviously, $E_3^1(h)\neq E_3^4(h)$. Therefore, The distance-based entropy for PHFEs, proposed by Su et al. [64], depends on our subjectivity.

In short, to avoid subjectivity in obtaining entropy for PHFEs and to consider the type of uncertainty

$h = \{1|0.5, 0|0.5\}$, it is necessary to further explore the entropy for PHFEs.

III. SIMILARITY AND SIMILARITY-BASED ENTROPY FOR PHFEs

In this section, inspired by the similarity proposed by Xu and Xia. [48] in HFEs, we offer a definition of the similarity for PHFEs and propose several similarity formulas based on this definition. Next, to overcome the shortcomings of entropy in [64], we propose a new definition of entropy, discuss the relationship between similarity and entropy, and obtain a new similarity-based entropy.

A. SIMILARITY FOR PHFEs

Definition 13: Let h_1 and h_2 be two PHFEs on A. Then, the similarity measure between h_1 and h_2 has the following properties:

(1) $s(h_1, h_2) = 0$ *iff* $h_1 = \{0|1\}, h_2 = \{1|1\}$;

(2) $s(h_1, h_2) = 1$ *iff* $p_1^{\sigma(i)} = p_2^{\sigma(i)}$ *and* $\gamma_1^{\sigma(i)} = \gamma_2^{\sigma(i)}$;

(3) $s(h_1, h_2) = s(h_2, h_1)$;

(4) $s(h_1, h_3) \le s(h_1, h_2), s(h_1, h_3) \le s(h_2, h_3)$; *if* $l_1 = l_2 = l_3$, *for* $p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \le p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \le p_3^{\sigma(i)}\gamma_3^{\sigma(i)}$ *or* $p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \ge p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \ge p_3^{\sigma(i)}\gamma_3^{\sigma(i)}$;
if $l_1 \ne l_2 \ne l_3$, *for* $\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \le \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \le \sum_{i=1}^{l_3} p_3^{\sigma(i)}\gamma_3^{\sigma(i)}$ *or* $\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \ge \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \ge \sum_{i=1}^{l_3} p_3^{\sigma(i)}\gamma_3^{\sigma(i)}$.

where γ_i is the degree of membership in h_i, p_i is the corresponding probability, and l_i is the number of elements.

According to our proposed Definition 13, we propose a similarity measure for PHFEs as follows:

Definition 14: Let h_1 and h_2 be two PHFEs on A, then

$$s_1(h_1, h_2) = 1 - \left|\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} - \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)}\right| \tag{26}$$

can be called the similarity measure between the PHFEs h_1 and h_2.

Example 3: Let $h_1 = \{0.5|0.7, 0.3|0.3\}$, $h_2 = \{0.2|0.6, 0.1|0.4\}$, $h_3 = \{0.6|0.5, 0.4|0.5\}$ Then, the similarity between h_1 and h_2, h_1 andh_3, and h_2 and h_3 can be calculated as follows:

$$\begin{aligned} s_1(h_1, h_2) &= 1 - |(0.5 \times 0.7 + 0.3 \times 0.3) \\ &\quad - (0.2 \times 0.6 + 0.1 \times 0.4)| = 0.72 \\ s_1(h_1, h_3) &= 1 - |(0.5 \times 0.7 + 0.3 \times 0.3) \\ &\quad - (0.6 \times 0.5 + 0.4 \times 0.5)| = 0.94 \\ s_1(h_2, h_3) &= 1 - |(0.2 \times 0.6 + 0.1 \times 0.4) \\ &\quad - (0.6 \times 0.5 + 0.4 \times 0.5)| = 0.66 \end{aligned}$$

The above example shows that the similarity between h_1 and h_3 is the largest, and the similarity between h_2 and h_3 is the smallest, which means that in decision-making, the evaluation represented by h_1 and h_3 are relatively close for the same alternative or attribute.

It is obvious that the distance measure formula (22) proposed by Su et al. in [64] is used in the above formula (26), and we have also proved $0 \le \left|\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} - \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)}\right| \le 1$ in this paper. In addition, it is evident that our proposed formula (26) satisfies (1), (2), and (3) in Definition 13. Now, we prove whether formula (26) satisfies (4) in Definition 13.

Proof: if $l_1 = l_2 = l_3$,

$$\text{for } p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \le p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \le p_3^{\sigma(i)}\gamma_3^{\sigma(i)}$$

or

$$p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \ge p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \ge p_3^{\sigma(i)}\gamma_3^{\sigma(i)}$$

Obviously,

$$\left|\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} - \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)}\right| \le \left|\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} - \sum_{i=1}^{l_3} p_3^{\sigma(i)}\gamma_3^{\sigma(i)}\right|$$

$$\left|\sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} - \sum_{i=1}^{l_3} p_3^{\sigma(i)}\gamma_3^{\sigma(i)}\right| \le \left|\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} - \sum_{i=1}^{l_3} p_3^{\sigma(i)}\gamma_3^{\sigma(i)}\right|$$

Thus,

$$1 - \left|\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} - \sum_{i=1}^{l_3} p_3^{\sigma(i)}\gamma_3^{\sigma(i)}\right| \le 1 - \left|\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} - \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)}\right|$$

$$1 - \left|\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} - \sum_{i=1}^{l_3} p_3^{\sigma(i)}\gamma_3^{\sigma(i)}\right| \le 1 - \left|\sum_{i=1}^{l_1} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} - \sum_{i=1}^{l_3} p_3^{\sigma(i)}\gamma_3^{\sigma(i)}\right|$$

Then we can get,

$$s(h_1, h_3) \le s(h_1, h_2), s(h_1, h_3) \le s(h_2, h_3)$$

If $l_1 \ne l_2 \ne l_3$,

$$\text{for } \sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \le \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \le \sum_{i=1}^{l_3} p_3^{\sigma(i)}\gamma_3^{\sigma(i)}$$

or

$$\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \ge \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \ge \sum_{i=1}^{l_3} p_3^{\sigma(i)}\gamma_3^{\sigma(i)}$$

Obviously, we can get,

$$s(h_1,h_3) \le s(h_1,h_2), \quad s(h_1,h_3) \le s(h_2,h_3).$$

Based on (26), other similarity measure formulas for PHFEs can be obtained as follows:

$$s_2(h_1,h_2) = 1 - \left(\left|\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} - \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)}\right|^2\right)^{\frac{1}{2}} \tag{27}$$

$$s_3(h_1,h_2) = 1 - \max_i \left\{\left|\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} - \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)}\right|\right\} \tag{28}$$

Note 1: The corresponding similarity measure formula between PHFSs can be obtained as follows:

$$s_5(A,B) = 1 - \frac{1}{n}\sum_{j=1}^{n}\left|\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)}(x_j) - \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)}(x_j)\right| \tag{29}$$

Here, A and B are two PHFSs.

B. SIMILARITY -BASED ENTROPY FOR PHFEs

Many scholars have studied the relationship between similarity and entropy in different environments such as interval-valued fuzzy sets [65], [66], [67], interval-valued intuitionistic fuzzy sets [68], [69], interval-valued neutrosophic sets [70], and hesitant fuzzy sets [49]. The relationship between similarity and entropy in the PHFS environment has not yet been examined because of the paucity of studies on similarity in the PHFS environment. Inspired by these studies, we discuss and explore the relationship between the similarity and entropy of PHFEs.

First, we redefine the entropy of PHFEs as follows:

Definition 15: Entropy on PHFE h is a real-valued function E: H → [0, 1] satisfying the following conditions:

(1) $E(h) = 0$ *iff* $h = \{0|1\}$ or $h^c = \{1|1\}$;

(2) $E(h) = 1$ *iff* $h = \{0.5|1\}$ or $h = \{0|0.5, 1|0.5\}$;

(3) $E(h) = E(h^c)$;

(4) $E(h_1) \le E(h_2)$,

if $l_1 = l_2$, *for* $\gamma_1^{\sigma(i)} \le \gamma_2^{\sigma(i)} \le 0.5, p_1^{\sigma(i)} = p_2^{\sigma(i)}$ *or* $\gamma_1^{\sigma(i)} \ge \gamma_2^{\sigma(i)} \ge 0.5, p_1^{\sigma(i)} = p_2^{\sigma(i)}, i = 1,2,\cdots,l_1$;

or

if $l_1 \ne l_2$, *for* $\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \le \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \le 0.5$ *or* $\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \ge \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \ge 0.5$

Clearly, in Definition 15, we overcome the shortcomings of Definitions 11 and 12 and add $h = \{1|0.5, 0|0.5\}$, the uncertainty caused by the hesitation of DMs.

Theorem 1: Let h be an PHFE, then $s(h,h^c)$ is an entropy of h.

Then, according to Theorem 1, we obtain a new entropy measure formula of the PHFEs as follows:

$$E(h) = s(h,h^c) \tag{30}$$

Example 4: For $h = \{0.4|0.8, 0.3|0.2\}$, we use the entropy formula (30) and calculate $E(h)$ as follows:

$$E(h) = 1 - |(0.4\times 0.8 + 0.3\times 0.2) - (0.6\times 0.8 + 0.7\times 0.2)| = 0.76$$

Next, we prove whether formula (30) satisfies the properties in Definition 15. First, we prove that it satisfies (1), (2), and (3) in Definition 15. The details are as follows.

Proof:

(1) $E(h) = s(h,h^c) = 0 \Leftrightarrow h = \{0|1\}$ *and* $h^c = \{1|1\}$ *or* $h = \{1|1\}$ *and* $h^c = \{0|1\}$

(2) $E(h) = s(h,h^c) = 1 \Leftrightarrow h = \{0.5|1\}$ *or* $h = \{0|0.5, 1|0.5\}$

(3) $E(h) = E(h^c) \Leftrightarrow s(h,h^c) = s(h^c,h)$

Next, we prove whether formula (30) satisfies (4) in Definition 15, that is, to prove $E(h_1) \le E(h_2) \Leftrightarrow s(h_1,h_1^c) \le s(h_2,h_2^c)$ as follows:

Proof: if $l_1 = l_2$, *for* $\gamma_1^{\sigma(i)} \le \gamma_2^{\sigma(i)} \le 0.5, p_1^{\sigma(i)} = p_2^{\sigma(i)}$

Or $\gamma_1^{\sigma(i)} \ge \gamma_2^{\sigma(i)} \ge 0.5, p_1^{\sigma(i)} = p_2^{\sigma(i)}, i = 1,2,\cdots,l_1$;

Then,

$$\gamma_1^{\sigma(i)} \le \gamma_2^{\sigma(i)} \le 1-\gamma_2^{\sigma(i)} \le 1-\gamma_1^{\sigma(i)}$$

Or

$$\gamma_1^{\sigma(i)} \ge \gamma_2^{\sigma(i)} \ge 1-\gamma_2^{\sigma(i)} \ge 1-\gamma_1^{\sigma(i)}$$

It can be further obtained:

$$p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \le p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \le p_2^{\sigma(i)}\left(1-\gamma_2^{\sigma(i)}\right) \le p_1^{\sigma(i)}\left(1-\gamma_1^{\sigma(i)}\right)$$

Or

$$p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \ge p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \ge p_2^{\sigma(i)}\left(1-\gamma_2^{\sigma(i)}\right) \ge p_1^{\sigma(i)}\left(1-\gamma_1^{\sigma(i)}\right)$$

According to Definition 13 with respect to similarity, it can be obtained:

$$s(h_1,h_1^c) \le s(h_2,h_2^c)$$

Namely,

$$E(h_1) \le E(h_2)$$

If $l_1 \ne l_2$, *for* $\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \le \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \le 0.5$

or $\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \ge \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \ge 0.5$;

Then,

$$1-\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \geq 1-\sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \geq 0.5$$

$$\Rightarrow \sum_{i=1}^{l_1} p_1^{\sigma(i)} - \sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \geq \sum_{i=1}^{l_2} p_2^{\sigma(i)} - \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \geq 0.5$$

$$\Rightarrow \sum_{i=1}^{l_1} p_1^{\sigma(i)}\left(1-\gamma_1^{\sigma(i)}\right) \geq \sum_{i=1}^{l_2} p_2^{\sigma(i)}\left(1-\gamma_2^{\sigma(i)}\right) \geq 0.5$$

Or

$$1-\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \leq 1-\sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \leq 0.5$$

$$\Rightarrow \sum_{i=1}^{l_1} p_1^{\sigma(i)} - \sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \leq \sum_{i=1}^{l_2} p_2^{\sigma(i)} - \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \leq 0.5$$

$$\Rightarrow \sum_{i=1}^{l_1} p_1^{\sigma(i)}\left(1-\gamma_1^{\sigma(i)}\right) \leq \sum_{i=1}^{l_2} p_2^{\sigma(i)}\left(1-\gamma_2^{\sigma(i)}\right) \leq 0.5$$

It can be further obtained:

$$\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \leq \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \leq \sum_{i=1}^{l_2} p_2^{\sigma(i)}\left(1-\gamma_2^{\sigma(i)}\right) \leq \sum_{i=1}^{l_1} p_1^{\sigma(i)}\left(1-\gamma_1^{\sigma(i)}\right)$$

Or

$$\sum_{i=1}^{l_1} p_1^{\sigma(i)}\gamma_1^{\sigma(i)} \geq \sum_{i=1}^{l_2} p_2^{\sigma(i)}\gamma_2^{\sigma(i)} \geq \sum_{i=1}^{l_2} p_2^{\sigma(i)}\left(1-\gamma_2^{\sigma(i)}\right) \geq \sum_{i=1}^{l_1} p_1^{\sigma(i)}\left(1-\gamma_1^{\sigma(i)}\right)$$

According to the Definition 13, it can be obtained:

$$s(h_1, h_1^c) \leq s(h_2, h_2^c)$$

Namely,

$$E(h_1) \leq E(h_2)$$

In summary,

$$E(h_1) \leq E(h_2) \Leftrightarrow s(h_1, h_1^c) \leq s(h_2, h_2^c).$$

Based on the above discussion, it is clear that the greatest benefit of our proposed similarity formula (26) is that a new entropy measure formula may be constructed from it. This new entropy measure not only compensates for the shortcoming of ignoring $h = \{1|0.5, 0|0.5\}$, the uncertainty caused by the inner reluctance of DMs, but it also eliminates the subjectivity we exhibit in order to calculate the entropy of a PHFE. However, we also note that the similarity presented by us is based on the distance formula proposed by Su et al. [64], and there may be other distance measures with better properties for researching similarity, which we will investigate further in the future.

IV. A NEW MAGDM METHOD BASED ON SIMILARITY OF PHFE

This section proposes a new MAGDM method based on the similarity of the PHFEs. The algorithmic stages are as follows:

For a MAGDM problem, let $A = \{A_1, A_2 \cdots A_n\}$ be a set of alternatives, $C = \{C_1, C_2 \cdots C_m\}$ be a set of attributes, where the attribute weights$w_j (j = 1, 2, \cdots, m)$are unknown and the attributes are independent of each other, $D = \{D_1, D_2 \cdots D_k\}$ is the set of experts and the weights ω_q $(q = 1, 2, \cdots, k)$ of each expert are the same. The evaluation value for the alternative A_i with respect to the attribute C_i is expressed by PHFE. In PHFE, A^* represents the ideal alternative because $\{0|1\}$ means that the experts completely disagree and $\{1|1\}$ means that the experts completely agree. Therefore, A^*implies that k experts agree, and A^* can be expressed mathematically as $A^* = \left\{(1|1)^1, (1|1)^2 \cdots (1|1)^k\right\}$.

Step 1: Each expert $D_q (q = 1, 2, \cdots, k)$ provides an individual decision matrix $IDM_q = \left(h_{ij}^q(p)\right)_{n\times m}$ $(q = 1, 2, \cdots, k)$ by PHFE for alternatives $A_i (i = 1, 2, \cdots, n)$ under attribute $C_j (j = 1, 2, \cdots, m)$, where $h_{ij}^q(p)$ is expressed by PHFE.

Step 2: The standardized individual decision matrix $IDM_q^* = \left(h_{ij}^{q*}(p)\right)_{n\times m}$ $(q = 1, 2, \cdots, k)$ is obtained, in which for the benefit attribute J_1, the corresponding decision information remains unchanged. For the cost attribute J_2, the membership degree in the decision information takes its complement with the corresponding probability remaining unchanged, namely:

$$h_{ij}^*(p) = \begin{cases} h_{ij}(p) = \left\{\gamma_{ij}^t|p_{ij}^t|t = 1, 2, \cdots, l\right\}, & C_j \in J_1 \\ h_{ij}^c(p) = \left\{1-\gamma_{ij}^t|p_{ij}^t|t = 1, 2, \cdots, l\right\}, & C_j \in J_2 \end{cases}$$

Step 3: Using our proposed entropy formula (30) and the normalized individual decision matrix $IDM_q^* = \left(h_{ij}^{q*}(p)\right)_{n\times m}$ of each expert, the probability hesitant fuzzy entropy matrix $E_q = \left[E\left(h_{ij}^{*q}(p)\right)\right]_{n\times m}$ $(q = 1, 2, \cdots, k)$ of each expert was obtained.

Step 4: Based on the entropy matrix of each expert $D_q (q = 1, 2, \cdots, k)$, we calculate the weight w_j^q of each expert under the attribute C_j using the entropy weight formula (31) [95].

$$w_j^q = \frac{1-E_j}{\sum_{j=1}^{m}\left(1-E_j\right)}, \quad j = 1, 2, \cdots, m \tag{31}$$

Here, $E_j = \frac{1}{n}\sum_{i=1}^{n} E\left(h_{ij}^{q*}(p)\right)$.

Step 5: Obtain the group decision matrix $GDM = \left(r_{ij}\right)_{n\times k}$, where r_{ij} is obtained using the PHFWA operator in Definition 8.

TABLE 1. The detailed description of attributes.

Attribute	Description
Personal morality	The level of morality shown by business owners to others and society.
Business ability	The business owner's ability to manage the business and create profits.
Customer relationships	The ability of business owners to maintain good relationships with customers in order to stabilize and expand the sales of business products.
Community image	Impressions or comments of others in the community about the business owner in various aspects.

Step 6: Based on the aforementioned group decision matrix and our proposed similarity formula (26), the similarity $S(A_i, A^*)$ between any alternative and the ideal alternative is calculated.

Step 7: Obtain the final ranking of alternatives based on similarity $S(A_i, A^*)$ calculated in *Step 6.*

V. CASE STUDY

In this section, we describe the MAGDM problem, which was solved using the method proposed in the previous section.

A. DESCRIPTION OF THE PROBLEM

Currently, SMEs in China face more credit constraints than large enterprises. Banks can more easily collect and quantify information about the latter such as production data, enterprise size, and financial data. On the contrary, it is more difficult for banks to collect business data about SMEs because of their small scale and short history, and these data can be easily forged because of the lack of standardized management systems for SMEs, which leads to an aggravation of information asymmetry between banks and enterprises. To control SMEs' credit risk, banks collect soft information [96] that reflects the business owners themselves, such as personal morality, business ability, customer relationships, and community image. Therefore, it is necessary to design an algorithm to help banks evaluate SME loans using soft information. Assuming that five SMEs apply for a bank loan, four experts in the bank are designated to evaluate the credit risk of the three SMEs according to four major attributes: personal morality (C_1), business ability (C_2), customer relationships (C_3), and community image (C_4). A detailed description of these attributes is provided in Table 1. The attribute weights $w_j\,(j = 1, 2, 3, 4)$ are unknown and the attributes are independent of each other. The weights $\omega_q\,(q = 1, 2, 3, 4)$ of the four experts $D_q(q = 1, 2, 3, 4)$ are equal, so the vector is $\omega = (0.25, 0.25, 0.25, 0.25)^T$.

B. ILUSTRATION OF THE PROPOSED MODEL

Based on the proposed model, the above problems can be solved using the following steps.

Step 1: Construct individual probabilistic hesitant fuzzy decision matrix.

TABLE 2. Evaluation information of expert 1.

<table>
<tr><th></th><th>C1</th><th>C2</th><th>C3</th><th>C4</th></tr>
<tr><td>A1</td><td>0.5|0.6
0.3|0.4</td><td>0.6|0.7
0.3|0.3</td><td>0.8|1</td><td>0.6|1</td></tr>
<tr><td>A2</td><td>0.8|0.5
0.7|0.5</td><td>0.4|1</td><td>0.6|1</td><td>0.4|0.7
0.2|0.3</td></tr>
<tr><td>A3</td><td>0.4|0.5
0.2|0.5</td><td>0.6|1</td><td>0.6|0.8
0.8|0.2</td><td>0.4|1</td></tr>
<tr><td>A4</td><td>0.9|0.7
0.8|0.3</td><td>0.2|0.6
0.1|0.4</td><td>0.8|1</td><td>0.9|1</td></tr>
<tr><td>A5</td><td>0.5|0.6
0.6|0.4</td><td>0.9|1</td><td>0.3|0.8
0.4|0.2</td><td>0.8|1</td></tr>
</table>

TABLE 3. Evaluation information of expert 2.

<table>
<tr><th></th><th>C1</th><th>C2</th><th>C3</th><th>C4</th></tr>
<tr><td>A1</td><td>0.8|0.5
0.7|0.5</td><td>0.4|0.7
0.6|0.3</td><td>0.5|0.6
0.7|0.4</td><td>0.5|0.4
0.3|0.6</td></tr>
<tr><td>A2</td><td>0.6|0.6
0.4|0.4</td><td>0.7|1</td><td>0.6|0.7
0.4|0.3</td><td>0.6|0.8
0.5|0.2</td></tr>
<tr><td>A3</td><td>0.8|0.7
0.7|0.3</td><td>0.8|1</td><td>0.9|1</td><td>0.3|0.6
0.2|0.4</td></tr>
<tr><td>A4</td><td>0.5|0.6
0.3|0.4</td><td>0.6|0.7
0.3|0.3</td><td>0.8|1</td><td>0.7|1</td></tr>
<tr><td>A5</td><td>0.9|0.7
0.8|0.3</td><td>0.2|0.6
0.1|0.4</td><td>0.8|1</td><td>0.9|1</td></tr>
</table>

TABLE 4. Evaluation information of expert 3.

<table>
<tr><th></th><th>C1</th><th>C2</th><th>C3</th><th>C4</th></tr>
<tr><td>A1</td><td>0.7|0.7
0.6|0.3</td><td>0.5|0.6
0.6|0.4</td><td>0.7|0.6
0.8|0.4</td><td>0.8|1</td></tr>
<tr><td>A2</td><td>0.5|0.5
0.4|0.5</td><td>0.3|0.7
0.2|0.3</td><td>0.6|1</td><td>0.5|0.7
0.6|0.3</td></tr>
<tr><td>A3</td><td>0.9|0.7
0.8|0.3</td><td>0.2|0.6
0.1|0.4</td><td>0.8|0.5
0.7|0.5</td><td>0.9|1</td></tr>
<tr><td>A4</td><td>0.5|0.6
0.6|0.4</td><td>0.9|1</td><td>0.3|0.8
0.4|0.2</td><td>0.8|1</td></tr>
<tr><td>A5</td><td>0.5|0.7
0.4|0.3</td><td>0.7|0.7
0.6|0.3</td><td>0.7|0.6
0.8|0.4</td><td>0.7|1</td></tr>
</table>

Four experts used PHFE to obtain individual decision matrices for five SMEs with four attributes. The decision information is presented in Tables 2-5

Step 2: Normalize the individual decision matrix.

In this case, the four attributes are no longer normalized because they are all benefit attributes.

Step 3: Obtain the individual probabilistic hesitant fuzzy entropy matrix.

TABLE 5. Evaluation information of expert 4.

	C_1	C_2	C_3	C_4
A_1	0.6|0.7 0.7|0.3	0.4|0.6 0.3|0.4	0.8|0.6 0.9|0.4	0.7|1
A_2	0.5|0.6 0.6|0.4	0.8|1	0.2|0.8 0.3|0.2	0.9|1
A_3	0.4|0.5 0.3|0.5	0.7|0.7 0.6|0.3	0.6|1	0.8|0.6 0.7|0.4
A_4	0.5|0.7 0.4|0.3	0.7|0.7 0.6|0.3	0.7|0.6 0.8|0.4	0.7|1
A_5	0.5|0.6 0.3|0.4	0.6|0.7 0.3|0.3	0.8|1	0.7|1

TABLE 6. The probabilistic hesitant fuzzy entropy matrix of expert 1.

	C_1	C_2	C_3	C_4
A_1	0.84	0.98	0.4	0.8
A_2	0.5	0.8	0.8	0.68
A_3	0.6	0.8	0.72	0.8
A_4	0.26	0.32	0.4	0.2
A_5	0.92	0.2	0.64	0.4

TABLE 7. The probabilistic hesitant fuzzy entropy matrix of expert 2.

	C_1	C_2	C_3	C_4
A_1	0.5	0.92	0.84	0.76
A_2	0.96	0.6	0.92	0.84
A_3	0.46	0.4	0.2	0.52
A_4	0.84	0.98	0.4	0.6
A_5	0.26	0.32	0.4	0.2

TABLE 8. The probabilistic hesitant fuzzy entropy matrix of expert 3.

	C_1	C_2	C_3	C_4
A_1	0.66	0.92	0.52	0.4
A_2	0.9	0.54	0.8	0.94
A_3	0.26	0.32	0.5	0.2
A_4	0.92	0.2	0.64	0.4
A_5	0.99	0.66	0.52	0.6

We used our proposed entropy formula (30) and obtained the probabilistic hesitant fuzzy entropy matrix $E_q = \left[E\left(h_{ij}^q(p)\right)\right]_{5\times 4}$ $(q = 1, 2, 3, 4)$ of each expert. The results are presented in Tables 6-9.

Step 4: Derive the attribute weights matrix.

TABLE 9. The probabilistic hesitant fuzzy entropy matrix of expert 4.

	C_1	C_2	C_3	C_4
A_1	0.74	0.72	0.32	0.6
A_2	0.92	0.4	0.44	0.2
A_3	0.7	0.66	0.8	0.48
A_4	0.94	0.66	0.52	0.6
A_5	0.84	0.98	0.4	0.6

TABLE 10. The attribute weights matrix.

	C_1	C_2	C_3	C_4
D_1	0.323	0.128	0.329	0.220
D_2	0.265	0.265	0.255	0.216
D_3	0.234	0.242	0.234	0.290
D_4	0.128	0.243	0.287	0.343

Based on the four probabilistic hesitant fuzzy entropy matrices, we obtain the weight matrix $\left(w_{ij}^q\right)_{4\times 4}$ under attribute $C_j(j = 1, 2, 3, 4)$ according to formula (31). The results are presented in Table 10.

Step 5: The individual probabilistic hesitant fuzzy decision matrix is integrated to obtain the group decision matrix.

Using the PHFWA operator in Definition 8, we obtain a group decision matrix, $GDM = \left(r_{ij}\right)_{5\times 4}$. The results are presented in Table 11.

Step 6: Calculate the similarity between any alternative and the ideal.

Based on the group decision matrix $GDM = \left(r_{ij}\right)_{5\times 4}$ obtained in *Step 5*, we can use similarity formula (26) to calculate the similarity $S(A_i, A^*)$ between the alternative A_i and the ideal alternative A^*. Table 12 presents the results.

Step 7: Rank all alternatives.

$$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$$

Finally, because the similarity between A_3 and the ideal alternative A^* is maximum, we can conclude that A_4 is the firm with the least credit risk.

VI. DISCUSSION

To prove the effectiveness, robustness, and advancement of our proposed decision-making method, we first compared it with the decision-making method proposed by Su et al. [64] by simulating the above cases. Second, we referred to the practice of Su et al. [64], assuming no probability information, to discuss whether our proposed method is more advanced in a probabilistic hesitant fuzzy environment than in a hesitant fuzzy environment that does not consider probability information.

First, we compare our method with the method proposed by Su et al. [64]. The procedure is as follows.

TABLE 11. Group decision matrix.

<table>
<tr><th></th><th>D₁</th><th>D₂</th></tr>
<tr><td>A₁</td><td>0.6471|0.42
0.5966|0.18
0.6178|0.28
0.5631|0.12</td><td>0.5839|0.084,0.6194|0.036,0.6388|0.056
0.6062|0.084
0.5850|0.054,0.6696|0.024,0.6398|0.036
0.5463|0.126
0.5404|0.084,0.6011|0.056,0.4989|0.126
0.5650|0.084
0.5797|0.036,0.5417|0.054,0.6351|0.024
0.6022|0.036</td></tr>
<tr><td>A₂</td><td>0.5833|0.35
0.5500|0.15
0.5413|0.35
0.5047|0.15</td><td>0.6242|0.336,0.5849|0.224,0.5795|0.144
0.5356|0.096
0.6020|0.084,0.5604|0.056,0.5547|0.036
0.5082|0.024</td></tr>
<tr><td>A₃</td><td>0.5093|0.4
0.5894|0.1
0.4747|0.4
0.5604|0.1</td><td>0.7719|0.420
0.7639|0.280
0.7481|0.180
0.7393|0.120</td></tr>
<tr><td>A4</td><td>0.8036|0.42
0.7980|0.28
0.7685|0.18
0.7619|0.12</td><td>0.6759|0.42
0.6334|0.18
0.6480|0.28
0.6019|0.12</td></tr>
<tr><td>A5</td><td>0.7094|0.48
0.7207|0.12
0.7244|0.32
0.7351|0.08</td><td>0.8081|0.42
0.8031|0.28
0.7726|0.18
0.7666|0.12</td></tr>
</table>

<table>
<tr><th></th><th>D₃</th><th>D₄</th></tr>
<tr><td>A₁</td><td>0.6922|0.252,0.7217|0.168
0.7115,0.168,0.7392|0.112
0.6779|0.108,0.7089|0.072
0.6982|0.072,0.7272|0.048</td><td>0.6869|0.252,0.7521|0.168
0.6766|0.168,0.7439|0.112
0.6971|0.108,0.7602|0.072
0.6871|0.072,0.7523|0.048</td></tr>
<tr><td>A₂</td><td>0.4784|0.245,0.5125|0.105
0.4577|0.105,0.4932|0.045
0.4632|0.245,0.4983|0.105
0.4420|0.105,0.4784|0.045</td><td>0.7193|0.480
0.7317|0.120
0.7264|0.320
0.7385|0.080</td></tr>
<tr><td>A₃</td><td>0.7824|0.210,0.7592|0.210
0.7748|0.140,0.7508|0.140
0.7573|0.090,0.7315|0.090
0.7489|0.060,0.7222|0.060</td><td>0.6878|0.210,0.5018|0.140
0.6682|0.090,0.4706|0.060
0.6822|0.210,0.4929|0.140
0.6623|0.090,0.4612|0.060</td></tr>
<tr><td>A4</td><td>0.7421|0.48
0.7519|0.12
0.7510|0.32
0.7604|0.08</td><td>0.6818|0.294,0.7225|0.196
0.6619|0.126,0.7051|0.084
0.6751|0.126,0.7166|0.084
0.6548|0.054,0.6989|0.036</td></tr>
<tr><td>A5</td><td>0.6749|0.294,0.7062|0.196
0.6466|0.126,0.6805|0.084
0.6655|0.126,0.6976|0.084
0.6363|0.054,0.6712|0.036</td><td>0.7051|0.42
0.6682|0.18
0.6935|0.28
0.6551|0.12</td></tr>
</table>

TABLE 12. Probabilistic hesitant fuzzy similarity matrix.

A_1	A_2	A_3	A4	A5
0.653298	0.585287	0.658047	0.720084	0.719655

Step 1: Calculate the entropy of each expert for the five SMEs and obtain the group decision entropy matrix.

We used the entropy formula (25) proposed by Su et al. [64] to transform the group decision matrix (Table 11) into four different group decision entropy matrices under $f_1(x) = 1 - 2x, f_2(x) = 1 - 4x^2, f_3(x) = \cos \pi x$, and $f_4(x) = \frac{1-2x}{1+2x}$, respectively.The results are presented in Table 13-16.

Step 2: Calculate the entropy of each SME.

TABLE 13. The group decision entropy matrix1.

	D_1	D_2	D_3	D_4
A_1	0.7606	0.8520	0.5851	0.5759
A_2	0.8964	0.8205	0.9500	0.5508
A_3	0.9829	0.4771	0.4808	0.7948
A_4	0.4186	0.6969	0.5048	0.6190
A_5	0.5647	0.4094	0.6471	0.6216

TABLE 14. The group decision entropy matrix2.

	D_1	D_2	D_3	D_4
A_1	0.9427	0.9781	0.8279	0.8201
A_2	0.9893	0.9678	0.9975	0.7982
A_3	0.9997	0.7266	0.7305	0.9579
A_4	0.6620	0.9081	0.7548	0.8549
A_5	0.8105	0.6511	0.8755	0.8568

TABLE 15. The group decision entropy matrix3.

	D_1	D_2	D_3	D_4
A_1	0.9301	0.9731	0.7951	0.7862
A_2	0.9868	0.9605	0.9969	0.7612
A_3	0.9996	0.6812	0.6855	0.9485
A_4	0.6112	0.8888	0.7124	0.8262
A_5	0.7752	0.5996	0.8502	0.8285

TABLE 16. The group decision entropy matrix4.

	D_1	D_2	D_3	D_4
A_1	0.4320	0.7422	0.4136	0.4044
A_2	0.4727	0.6957	0.9048	0.3801
A_3	0.4957	0.3133	0.3165	0.6595
A_4	0.2951	0.5348	0.3376	0.4483
A_5	0.3609	0.2574	0.4783	0.4510

TABLE 17. The final entropy for five SMEs under different functions.

	A_1	A_2	A_3	A4	A5
$f_1(x)$	0.6934	0.8044	0.6839	0.5598	0.5607
$f_2(x)$	0.8922	0.9382	0.8537	0.7949	0.7985
$f_3(x)$	0.8711	0.9264	0.8287	0.7596	0.7634
$f_4(x)$	0.4980	0.6133	0.4462	0.4039	0.3869

When the weights ω_q $(q = 1, 2, 3, 4)$ of the four experts are equal, we use the calculation results in Tables 13-16 to obtain the entropy of each SME under $f_1(x) = 1 - 2x$, $f_2(x) = 1 - 4x^2, f_3(x) = \cos \pi x$, and $f_4(x) = \frac{1-2x}{1+2x}$, respectively. The results are listed in Table 17.

Step 3: Rank all alternatives.

According to the principle proposed by Su et al. [64], the smaller the entropy, the smaller the uncertainty and the smaller the risk, the final ranking results are listed in Table 18.

TABLE 18. The final ranking results for five SMEs under different functions.

	Ranking
$f_1(x)$	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
$f_2(x)$	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
$f_3(x)$	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
$f_4(x)$	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$

TABLE 19. Evaluation information of expert 1.

	C_1	C_2	C_3	C_4
A_1	0.5,0.3	0.6,0.3	0.8	0.6
A_2	0.8,0.7	0.4	0.6	0.4,0.2
A_3	0.4,0.2	0.6	0.6,0.8	0.4
A_4	0.9,0.8	0.2,0.1	0.8	0.9
A_5	0.5,0.6	0.9	0.3,0.4	0.8

TABLE 20. Evaluation information of expert 2.

	C_1	C_2	C_3	C_4
A_1	0.8,0.7	0.6,0.4	0.7,0.5	0.5,0.3
A_2	0.6,0.4	0.7	0.6,0.4	0.6,0.5
A_3	0.8,0.7	0.8	0.9	0.3,0.2
A_4	0.5,0.3	0.6,0.3	0.8	0.7
A_5	0.9,0.8	0.2,0.1	0.8	0.9

We can see that when $f_1(x)$, $f_2(x)$ and $f_3(x)$ are adopted, the ranking result is completely consistent with those of our proposed method. At this time, A_2 has the maximum entropy, A_4 has the minimum entropy, which means thatA_2 has greater uncertainty or risk, whereas the uncertainty ofA_4 is relatively low.Whereas when $f_4(x)$ is adopted, A_5 is regarded as optimal solution and A_4 is regarded as second-best solution, and the remaining ranking results are consistent with those of our proposed method, which shows that the method proposed by Su et al. [64] may obtain different results due to the use of different functions. In comparison, the results of our proposed method are unique, which demonstrates that the effectiveness and robustness of our research.

Next, we consider using similar steps to our proposed method for decision-making when no probabilistic information is provided by the experts, and then compare it with the final result of our proposed method. The procedure is as follows.

Step 1: Convert the original individual decision matrix of each expert into an individual decision matrix expressed by the HFE.

Decision information is displayed in Tables 19-22.

Step 2: Normalize the individual decision matrix.

In this case, the four attributes are no longer normalized because they are all benefit attributes.

Step 3: Derive the attribute weights matrix.

We use formula (14) to obtain the entropy matrix of each expert, where the similarity formula uses formula (8).

TABLE 21. Evaluation information of expert 3.

	C_1	C_2	C_3	C_4
A_1	0.7,0.6	0.6,0.5	0.8,0.7	0.8
A_2	0.5,0.4	0.3,0.2	0.6	0.6,0.5
A_3	0.9,0.8	0.2,0.1	0.8,0.7	0.9
A_4	0.6,0.5	0.9	0.4,0.3	0.8
A_5	0.5,0.4	0.7,0.6	0.8,0.7	0.7

TABLE 22. Evaluation information of expert 4.

	C_1	C_2	C_3	C_4
A_1	0.7,0.6	0.4,0.3	0.9,0.8	0.7
A_2	0.6,0.5	0.8	0.3,0.2	0.9
A_3	0.4,0.3	0.7,0.6	0.6	0.8,0.7
A_4	0.5,0.4	0.7,0.6	0.8,0.7	0.7
A_5	0.5,0.3	0.6,0.3	0.8	0.7

TABLE 23. The attribute weights matrix.

	C_1	C_2	C_3	C_4
D_1	0.317	0.233	0.233	0.217
D_2	0.328	0.224	0.207	0.241
D_3	0.200	0.308	0.292	0.200
D_4	0.169	0.220	0.356	0.254

Based on the four hesitant fuzzy entropy matrices, we obtain the weight matrix $\left(w_{ij}^q\right)_{4\times 4}$ under attribute $C_j (j = 1, 2, 3, 4)$ according to formula (31). Table 23 presents the results.

Step 4: Integrate individual hesitant fuzzy decision matrix to obtain group decision matrix.

Using the hesitant fuzzy weighted averaging (HFWA) operator $HFWA\,(h_1, h_2, \cdots, h_n) = \overset{n}{\underset{i=1}{\oplus}} \omega_i h_i = \bigcup_{\gamma_{1l}\in h_1,\gamma_{2l}\in h_2,\cdots,\gamma_{nl}\in h_n} \left\{1 - \prod_{i=1}^{n} (1-\gamma_{il})^{\omega_i}\right\}$ proposed by Xu and Xia[93], a group decision matrix $GDM = \left(r_{ij}\right)_{3\times 4}$ is obtained. The results are presented in Table 24.

Step 5: Calculate the similarity between any alternative and the ideal.

Based on the group decision matrix $GDM = \left(r_{ij}\right)_{5\times 4}$ obtained in Step 4, we can use similarity formula (8) proposed by Xu and Xia[93] to calculate the similarity $S\,(A_i, A^{**})$ between the alternative A_i and the ideal alternative A^{**}, where $A^{**} = \left\{(1)^1, (1)^2, (1)^3, (1)^4\right\}$ means that four experts fully agree without considering the probability information. The results are presented in Table 25.

Step 6: Rank all alternatives.

$$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$$

By comparison, we find that the ranking order above is different from $A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$ obtained by our proposed method. When making decisions in a hesitant fuzzy environment, A_2 is regarded as an SME with the least

TABLE 24. Group decision matrix.

	D_1	D_2
A_1	0.6347,0.5838 0.5936,0.5370	0.6144,0.6478,0.6531,0.6238 0.6181,0.6832,0.6564,0.5818 0.5595,0.6037,0.5223,0.5702 0.5978,0.5638,0.6381,0.6075
A_2	0.6146,0.5898 0.5618,0.5335	0.6250,0.5921,0.6042,0.5696 0.5716,0.5341,0.5479,0.5084
A_3	0.5033,0.5774 0.4559,0.5370	0.7657,0.7580 0.7323,0.7236
A4	0.8092,0.8039 0.7623,0.7557	0.6521,0.6075 0.6115,0.5597
A5	0.6953,0.7061 0.7162,0.7262	0.8161,0.8112 0.7691,0.7630

	D_3	D_4
A_1	0.6762,0.7124,0.6977,0.7315 0.6571,0.6953,0.6798,0.7156	0.6821,0.7516,0.6711,0.7430 0.6972,0.7634,0.6867,0.7552
A_2	0.4804,0.5031,0.4586,0.4822 0.4611,0.4846,0.4385,0.4630	0.6787,0.6937 0.6906,0.7050
A_3	0.7677,0.7385,0.7591,0.7288 0.7332,0.6996,0.7233,0.6885	0.6625,0.5201,0.6405,0.4887 0.6536,0.5074,0.6310,0.4752
A4	0.7203,0.7326 0.7325,0.7442	0.6726,0.7166,0.6512,0.6981 0.6623,0.7077,0.6402,0.6886
A5	0.6677,0.7048,0.6369,0.6775 0.6554,0.6939,0.6235,0.6655	0.6981,0.6585 0.6804,0.6385

TABLE 25. Hesitant fuzzy similarity matrix.

A_1	A_2	A_3	A4	A5
0.45674	0.51913	0.39481	0.38931	0.37561

credit risk. This result is completely different from the decision-making results in a probabilistic hesitant fuzzy environment. The difference between the two results is mainly due to the probability information. Because the DM's preference information is not expressed through probability in the hesitant fuzzy environment, it is possible that the decision results are inaccurate,which indicates that the PHFS is more in line with people's thinking process after adding probabilistic information. Obviously, our proposed method takes into account more preference information, thus making the final decision result more credible.

VII. CONCLUSION

In this research, we attempt to incorporate similarity into a probabilistic hesitant fuzzy environment and offer a novel similarity-based multi-attribute group decision-making method. Following the definition of similarity, we present the distance-based similarity formulas. Second, considering the shortcomings of entropy proposed by other scholars, we redefine the entropy of PHFEs. Inspired by other studies on the relationship between similarity and entropy in different fuzzy environments, we discuss the relationship between similarity and entropy in probabilistic hesitant fuzzy environments for the first time. Finally, we prove that this newly proposed similarity is a type of entropy of probabilistic hesitant fuzzy elements under certain conditions, which enriches research on the relationship between probabilistic hesitant fuzzy elements. Based on the similarity measure and entropy, we propose a new method for MAGDM with unknown attribute weights, which can be effectively applied to the assessment of SMEs' credit risk. Finally, we demonstrated the effectiveness, robustness, and advancement of the proposed decision-making method.

However, the aforementioned research has several shortcomings. For example, we only proposed distance-based similarity. As in the hesitant fuzzy set, there may be other types of similarity worth discussing. In addition, the similarity presented by us is based on the distance formula proposed by Su et al. [64], and there may be other distance measures with better properties for researching similarity. Finally, in recent years, spherical fuzzy set (TSFS) and T-spherical fuzzy set (TSFS), as the extended forms of new fuzzy sets, have attracted the attention of many researchers, among which a T-spherical fuzzy set (TSFS) is an extended form of a spherical fuzzy set(SFS) because it can express the uncertainty of DMs from more dimensions, which may be more in line with the thinking process of DMs in real decision-making. Recently, TSFS has gradually become a new research hotspot, and future research on its similarity, entropy, decision-making method and application is a worthy research direction. We will conduct further research on this topic in the future.

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YUBO HU received the M.S. degree in finance from the Lanzhou University of Finance and Economics, China, in 2017, where he is currently pursuing the Ph.D. degree in statistics. He has published several original articles in some academic journals. His research interests include multiple attribute decision making and aggregation operators.

ZHIQIANG PANG received the B.S. degree in statistics from the Lanzhou University of Finance and Economics, China, in 1987. He is currently a Professor with the School of Statistics, Lanzhou University of Finance and Economics. His research interests include statistical survey techniques, economic statistics, and comprehensive evaluation research. He has authored five books and published more than 90 articles in professional journals, including the *Ecological Indicators*, *Natural Hazards*, *Journal of Statistical Research*, and *Journal of Economic Issues*.