IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received 2 September 2022, accepted 19 September 2022, date of publication 26 September 2022, date of current version 5 October 2022.

*Digital Object Identifier 10.1109/ACCESS.2022.3209812*

RESEARCH ARTICLE

# Globally Convergent Observer for the Rigid Body System on SE(3) in Presence of Intermittent Measurements

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This work was supported in part by the NRF Grant through the Korea Government Ministry of Science and Information Communication Technology (MSIT) under Grant 2021R1A2C2010585, in part by the Institute of Information and Communications Technology Planning and Evaluation (IITP) Grant by the Korea Government Ministry of Science and Information Communication Technology (MSIT) under Grant 2022000469, and in part by the Center for Applied Research in Artificial Intelligence (CARAI) Grant through the Defense Acquisition Program Administration (DAPA) and Agency for Defense Development(ADD) under Grant UD190031RD.

**ABSTRACT** We propose a globally asymptotically convergent hybrid observer for the rigid body rotation and translation system evolving on the special Euclidean group SE(3) in the presence of intermittent measurements of the pose and continuous measurements of the velocities. We embed the system into an ambient Euclidean space and design an observer on the Euclidean space for global convergence. We perform numerical simulation of the proposed observer to show convergence. We also perform simulation using data collected from an Intel Realsense T265 camera.

**INDEX TERMS** Observers for nonlinear systems, hybrid systems, sensor fusion.

## I. INTRODUCTION

This paper deals with designing observers for the system modelling translation and rotation of a rigid body in three dimensional space. The problem of nonlinear observer design for rotational kinematics has been extensively studied in the literature [1], [2], [3], [4]. The authors in those papers assume a continuous-time system with continuous measurements of the pose. However, due to the digital nature of sensors used in robotics, measurements are usually of a discrete nature while the system is modelled by an ordinary differential equation, hence being continuous. This requires designing a nonlinear observer for a continuous-time system which can handle measurements available at intermittent instants of time. Some recent results considering such a system with intermittent measurements can be seen in [5], [6], [7], [8], [9], and [10].

For the system under consideration, measurements are usually available from various sensors at varying frequency. Often, sensors such as IMUs and accelerometers have a considerably higher frequency of measurement as compared to sensors such as camera or lidars. As an example, the EuRoC dataset [11] which contains information about the flight of a quadcopter, contains IMUs which measure angular velocity at 200Hz while cameras, which are used to construct the pose measurement of the robot, operate at a frequency of 20Hz. Due to such a high ratio of frequencies, we assume that the measurements of the angular and linear velocity to be continuous in time while the measurements of the pose to be available intermittently for our system.

For the rigid body rotation and translation system with intermittent measurements, an almost global asymptotic observer has been designed in [8] on the special orthogonal group, SO(3), while a local exponential observer is designed on the special Euclidean group, SE(3) in [10]. As can be seen from these works, prior research in the field of design of estimators for continuous-time rigid body kinematics on SE(3) is restricted to designing observers on the special Euclidean group. This approach results in an inability to design globally asymptotic observers on SE(3) due to the topology of SO(3) [12]. To design globally convergent observers for systems evolving on SE(3), we design a globally convergent observer in $\mathbb{R}^{3\times 3} \times \mathbb{R}^3$ which is an ambient Euclidean space of SE(3). This observer system tracks the pose of the system evolving on SE(3) and does not suffer from the topological

The associate editor coordinating the review of this manuscript and approving it for publication was Haibin Sun.

obstruction. The technique of designing observers in ambient Euclidean space is well accepted as in [13], allowing the design of globally convergent observers while avoiding any topological obstruction of the original state space.

Since the authors in [8] and [10] consider a similar system and, to the best of our knowledge, their observers are the best performing observers in the literature, we compare our results with their observers. Since the presence of a repeller set in the observers proposed in [8] and [10] degrades their performance in a neighborhood of the repeller set, we compare the performance in this neighborhood to our observer. Since our observer exhibits global asymptotic convergence to the system when it starts from SE(3), the performance improvement over the previously proposed observers is significant.

This paper is structured as follows. Section II contains preliminary information about the problem. The notation used and the system and measurements considered are presented in this section. Section III presents the theorems pertaining to local asymptotic stability of the system. This section also contains results regarding global convergence on SE(3). Sections IV and V contain simulations(numerical and experimental) of the system and the comparison with observers present in the literature.

## II. PRELIMINARIES

Let $\{\mathcal{I}\}$ denote the inertial frame and $\{\mathcal{B}\}$ the body fixed frame. We denote the estimate of the state $A$ by $\bar{A}$. The state of the system after a discrete jump is denoted by $A^+$. The matrix representation of the cross product with a vector $v$ is denoted by $v_\times : \mathbb{R}^3 \to \text{skew}(3)$ such that for all $w \in \mathbb{R}^3$, $v \times w = v_\times w$, where skew(3) is the set of $3 \times 3$ skew symmetric matrices. Denote by SO(3) the special orthogonal group on $\mathbb{R}^3$. Denote by $\text{SE}(3) = \text{SO}(3) \rtimes \mathbb{R}^3$ the special Euclidean group on $\mathbb{R}^3$ where the product is regarded as a Cartesian product of the sets and a Semidirect product of the groups. The Euclidean inner product of two matrices in $\mathbb{R}^{m\times n}$ is denoted by $\langle\cdot,\cdot\rangle : \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n} \to \mathbb{R}$ such that for all $A, B \in \mathbb{R}^{m\times n}$, $\langle A, B\rangle = \text{trace}(A^T B)$. The Euclidean norm of a matrix $A \in \mathbb{R}^{m\times n}$ is denoted as $\|A\| = \sqrt{\langle A, A\rangle}$.

Consider the rigid body rotation and translation system

$$\dot{R} = R\Omega_\times, \tag{1a}$$
$$\dot{p} = Rv, \tag{1b}$$

where $R \in \text{SO}(3) \subset \mathbb{R}^{3\times 3}$ denotes the rotation of frame $\{\mathcal{B}\}$ with respect to frame $\{\mathcal{I}\}$ and $p \in \mathbb{R}^3$ is the position of the body in frame $\{\mathcal{I}\}$. The vector $\Omega \in \mathbb{R}^3$ denotes the angular velocity of the body in frame $\{\mathcal{B}\}$ and $v \in \mathbb{R}^3$ denotes the linear velocity of the body in frame $\{\mathcal{B}\}$.

We assume that continuous measurements of $\Omega$ and $v$ are available, denoted by $\Omega_m$ and $v_m$, respectively. We also assume that discrete measurements of $R$ and $p$ are available with the following assumption:

*Assumption 1: The measurements of $R$ and $p$ are available at intermittent instants of time $t_j, j \in \mathbb{N}$ such that $t_j$ is an increasing sequence for $j \in \mathbb{N}$ and there exists $T_M > 0$ such that $t_{j+1} - t_j \leq T_M$ for all $j \in \mathbb{N}$ with the first measurement available at time $t_1 \leq T_M$.*

We have assumed availability of measurements of $R$ and $p$. Alternatively, the locations of predetermined markers in the environment may be available. In this case, let $p_i \in \mathbb{R}^3$, $i = 1, \ldots, n$ denote the location of $n$ such markers in $\{\mathcal{I}\}$. Let $y_i \in \mathbb{R}^3$ denote the measurement of marker at $p_i$ in $\{\mathcal{B}\}$. Define $r_i = (p_i, 1)$ and $b_i = (y_i, 1)$ the corresponding homogeneous coordinates. We assume $n > 2$. If $n = 2$, a third marker can be generated as $p_1 \times p_2$, with the corresponding measurement as $y_1 \times y_2$. Then the measurements are related by the equation

$$b_i = X^{-1} r_i, \quad \text{where } X = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}.$$

Define $r = \begin{bmatrix} r_1 & r_2 & \ldots & r_n \end{bmatrix}$ and $b = \begin{bmatrix} b_1 & b_2 & \ldots & b_n \end{bmatrix}$. Then, the measured value of $X$ can be given by $X_m = rb^+$, where $b^+$ is the Moore Penrose pseudoinverse of $b$.

The presence of both continuous and discrete measurements requires us to consider a hybrid system for the observer. The measurements of the state at the discrete instants of time are used to correct any errors between the estimated state arrived at using a continuous update law and the true value of the state.

The following lemma will be used later in the proof of theorems.

*Lemma 1: If $\|A - R\| < 1$, where $R \in SO(3), A \in \mathbb{R}^{3\times 3}$, then $A$ is invertible and the norm of its inverse satisfies*

$$\|A^{-1}\| \leq \frac{\sqrt{3}}{1 - \|A - R\|}.$$

*Proof:* Define $T := R^T A$ such that $\|I - T\| = \|A - R\| < 1$. From Corollary 5.6.16 in [14], we see that $T$ is invertible and the inverse can be written as $T^{-1} = \sum_{k=0}^{\infty}(I - T)^k$. Taking norm on both sides of the equality, repeatedly applying the triangle inequality and using the submultiplicativity of the Euclidean norm,

$$\|T^{-1}\| \leq \|I\| + \sum_{k=1}^{\infty} \|I - T\|^k \leq \frac{\sqrt{3}}{1 - \|I - T\|},$$

where we have substituted $\|I\| = \sqrt{3}$ for the Euclidean norm. Substituting $T = R^T A$ and using the property $\|RB\| = \|B\|$ for all $B \in \mathbb{R}^{3\times 3}$ and $R \in \text{SO}(3)$, we prove the lemma. □

## III. HYBRID OBSERVER

We propose the following hybrid observer system for the rigid body rotation and translation system (1):

$$\dot{\bar{R}} = \bar{R}\Omega_{m_\times}, \tag{2a}$$
$$\dot{\bar{p}} = \bar{R}^{-T} v_m, \tag{2b}$$
$$\bar{R}^+ = (1 - k_p)\bar{R} + k_p R_m, \tag{2c}$$
$$\bar{p}^+ = ((1 - k_p)\bar{R}^T + k_p R_m^T)^{-1}((1 - k_e)\bar{R}^T \bar{p} + k_e R_m^T p_m), \tag{2d}$$

where the observer states $\bar{R} \in \mathbb{R}^{3\times 3}$ and $\bar{p} \in \mathbb{R}^3$ are estimates to $R$ and $p$, respectively. The discrete jumps in equations (2c) and (2d) occur at times $t = t_j, j \in \mathbb{N}$ when the measurements of $R$ and $p$ are available.

The following theorem proposes local convergence of the observer system defined in (2) to the system (1):

*Theorem 1: Suppose that Assumption 1 holds, and for all* $\mu > 0$, $\|p(t)\| \exp(-\mu t) \to 0$ *as* $t \to \infty$, *and let*

$$E_R = R - \bar{R}, \quad E_p = p - \bar{p}, \tag{3}$$

*where the rigid body system is given in equation (1), and the observer system is given in equation (2). Then, with* $0 < k_e < 2$ *and* $0 < k_p < 2$, *the error term* $\|E_R(t)\|$ *converges exponentially to 0 and* $\|E_p(t)\|$ *converges asymptotically to 0 as* $t \to \infty$ *for all* $(\bar{R}(0), \bar{p}(0)) \in \{(A, b) \in \mathbb{R}^{3\times3} \times \mathbb{R}^3 \mid \|R(0) - A\| < 0.99\}$.

*Proof:* We define the error term

$$e_p = R^T p - \bar{R}^T \bar{p}. \tag{4}$$

Using the defined error terms $E_R$ and $e_p$, the observer jump equations (2c) and (2d) can be simplified as

$$\bar{R}^+ = \bar{R} + k_p E_R, \quad \bar{p}^+ = (\bar{R}^+)^{-T}(\bar{R}^T \bar{p} + k_e e_p), \tag{5}$$

where we have substituted the measurement models. Using the above equations and the error terms $E_R$ and $e_p$, the following error equations are arrived at by differentiating the error terms with respect to time and substituting equations (1) and (2):

$$\dot{E}_R = E_R \Omega_\times, \tag{6a}$$
$$\dot{e}_p = -\Omega_\times e_p, \tag{6b}$$
$$E_R^+ = (1 - k_p) E_R, \tag{6c}$$
$$e_p^+ = (1 - k_e) e_p, \tag{6d}$$

where $E_R^+$ and $e_p^+$ are the error between the state and the estimate after the jump at $t = t_j, j \in \mathbb{N}$.

Define the function $V_1$ as

$$V_1(E_R) = \langle E_R, E_R \rangle. \tag{7}$$

Differentiating equation (7) with respect to time along the trajectory of system (6), we see that

$$\dot{V}_1(E_R) = 2\langle E_R, \dot{E}_R \rangle = 2\langle E_R, E_R \Omega_\times \rangle = 0,$$

where we have used the fact that $\langle E_R^T E_R, \Omega_\times \rangle = 0$ since $E_R^T E_R$ is a symmetric matrix and $\Omega_\times$ is a skew symmetric matrix. Hence, $V_1(E_R(t))$ is constant over the interval $(t_j, t_{j+1})$ for every $j \in \mathbb{N}$. Moreover, since $V_1(E_R(t)) = \|E_R(t)\|^2$, this leads to $\|E_R(t)\|$ being constant over the same intervals. Substituting the error jump map (6c) into (7) for the value of $V_1$ after the jump, we see that

$$V_1(E_R^+) = \langle E_R^+, E_R^+ \rangle = (1 - k_p)^2 V_1(E_R), \tag{8}$$

which decreases at the discrete jumps since $0 < k_p < 2$.

Similarly, considering a function $V_2$ for the error term $e_p$ defined as

$$V_2(e_p) = \langle e_p, e_p \rangle, \tag{9}$$

and using equation (6b), we see that the derivative of $V_2$ with respect to time along the trajectory of the system (6) is 0 for all $t \in (t_j, t_{j+1}), j \in \mathbb{N}$. Hence, $V_2(e_p(t))$ is constant over the interval $t \in (t_j, t_{j+1})$ for every $j \in \mathbb{N}$. Since $V_2(e_p(t)) = \|e_p(t)\|^2$, this leads to $\|e_p(t)\|$ being constant over the same intervals. The value of the function after the jump, arrived at using (6d),

$$V_2(e_p^+) = \langle e_p^+, e_p^+ \rangle = (1 - k_e)^2 V_2(e_p), \tag{10}$$

decreases from that before the jump since $0 < k_e < 2$.

The hybrid observer system equations (2b) and (2d) contain the terms $\bar{R}^{-1}$ and $((1 - k_p)\bar{R}^T + k_p R^T)^{-1}$, respectively. Note that the second term here is $(\bar{R}^+)^{-1}$. For the observer system to be defined for all $t \geq 0$, we now prove that $\bar{R}(t)$ and $\bar{R}^+(t)$ are invertible given that $\|E_R(0)\| < 0.99$.

Consider first the case where $t \leq t_1$. Since $\|E_R(0)\| < 0.99$ according to the choice of the initial condition and the fact that $\|E_R(t)\|$ is constant for all $t \leq t_1$, $\bar{R}(t)$ is invertible for all $t \leq t_1$ from Lemma 1. For the discrete time kinematics, note that $\|R(t) - \bar{R}^+(t)\| = \|E_R^+(t)\| = |1 - k_p| \|E_R(0)\| < 0.99$, since $0 < k_p < 2$ as specified in the theorem. Hence $\bar{R}^+(t_1)$ is invertible from Lemma 1. Since $\bar{R}^+(t_1)$ is the initial condition of the ordinary differential equation governing the update of the state in time $t \in (t_1, t_2)$, and $\|E_R^+(t_1)\| < 0.99$, it follows that $\bar{R}(t)$ is invertible for all $t \in (t_1, t_2)$. Also, $\bar{R}^+(t_2)$ is invertible and $\|E_R^+(t_2)\| < 0.99$. By induction, $\bar{R}(t)$ is invertible for all $t \geq 0$ and $\bar{R}^+(t_j)$ is invertible for all $j \in \mathbb{N}$ assuming $\|E_R(0)\| < 0.99$.

We now show exponential stability of the error terms $E_R$ and $e_p$. From the jump map (8), we can see that $V_1(E_R^+(t_j)) = (1 - k_p)^{2j} V_1(E_R(0))$ for all $j \in \mathbb{N}$. Defining $j_t$ the number of jumps till time $t \geq 0$, and using the property that $V_1(E_R(t))$ is constant over the interval $t \in (t_j, t_{j+1})$ for every $j \in \mathbb{N}$, and noting that $j_t + 1 \geq t/T_M$ for all $t \geq 0$ from Assumption 1 and $|1 - k_p| < 1$,

$$\begin{aligned} V_1(E_R(t)) &= V_1(E_R^+(t_{j_t})) = |1 - k_p|^{2j_t} V_1(E_R(0)) \\ &\leq |1 - k_p|^{2\left(\frac{t}{T_M} - 1\right)} V_1(E_R(0)) \\ &= V_1(E_R(0)) \exp(2\mu_1(T_M - t)), \end{aligned}$$

where

$$\mu_1 = -1/T_M \log_e |1 - k_p|. \tag{11}$$

Similarly, using the jump map (10) and the fact that $V_2(e_p(t))$ is constant over the interval $t \in (t_j, t_{j+1})$ for every $j \in \mathbb{N}$,

$$V_2(e_p(t)) \leq V_2(e_p(0)) \exp(2\mu_2(T_M - t)),$$

where

$$\mu_2 = -1/T_M \log_e |1 - k_e|. \tag{12}$$

Substituting $V_1(E_R)$ from equation (7) and $V_2(e_p)$ from equation (9) in the above equations, we get that

$$\|E_R(t)\| \leq \exp(\mu_1 T_M) \|E_R(0)\| \exp(-\mu_1 t), \tag{13}$$
$$\|e_p(t)\| \leq \exp(\mu_2 T_M) \|e_p(0)\| \exp(-\mu_2 t), \tag{14}$$

for all $t \geq 0$. Hence, $\|E_R(t)\| \to 0$ and $\|e_p(t)\| \to 0$ exponentially fast as $t \to \infty$.

We write $E_p$ defined in equation (3) in terms of $E_R$ and $e_p$ defined in equations (3) and (4) as

$$E_p = \bar{R}^{-T}(e_p - E_R^T p). \tag{15}$$

Taking norm on both sides of the equality, and from equations (13) and (14), we have for all $t \geq 0$,

$$\begin{aligned}\|E_p(t)\| &= \|\bar{R}(t)^{-T}(e_p(t) - E_R^T(t)p(t))\| \\ &\leq 100\sqrt{3}(\|e_p(t)\| + \|E_R^T(t)\|\|p(t)\|) \\ &\leq 100\sqrt{3}\exp(\max\{\mu_1, \mu_2\}T_M)(\exp(-\mu_2 t)\|e_p(0)\| \\ &\quad + \exp((\mu - \mu_1)t)\|E_R^T(0)\|\|p(t)\|\exp(-\mu t)),\end{aligned} \tag{16}$$

where $\|\bar{R}(t)^{-T}\| < 100\sqrt{3}$ from Lemma 1 and the fact that $\|R(t) - \bar{R}(t)\| \leq \|R(0) - \bar{R}(0)\| < 0.99$, and $\mu > 0$ is chosen such that $\mu - \mu_1 < 0$. Taking limit on both sides of the inequality as $t \to \infty$, and using the assumption $\lim_{t\to\infty} \|p(t)\|\exp(-\mu t) = 0$ for all $\mu > 0$, we get that $\|E_p\| \to 0$ as $t \to \infty$. □

The assumption in the statement of the theorem, i.e. $\|p(t)\|\exp(-\mu t) \to 0$ as $t \to \infty$ for all $\mu > 0$ includes the case where each component of $p(t)$ is a polynomial in $t$.

Note that here we need existence of finite $T_M$ for existence of $\mu_1$ and $\mu_2$ such that the exponential bounds on $V_1$ and $V_2$ can be defined. However, we do not need the information about the value of $T_M$ to design the observer. If the value of $T_M$ is known, the assumption on the position can be further relaxed as follows:

*Corollary 1: Suppose that Assumption 1 holds, and for some $\mu, c > 0$, $\|p(t)\|\exp(-\mu t) \to c$ as $t \to \infty$, and let*

$$E_R = R - \bar{R}, \quad E_p = p - \bar{p},$$

*where the rigid body system is given in equation (1), and the observer system is given in equation (2). Then, with $1 - \exp(-\mu T_M) < k_p < 1 + \exp(-\mu T_M)$ and $0 < k_e < 2$, the error term $\|E_R(t)\|$ converges exponentially to 0 and $\|E_p(t)\|$ converges asymptotically to 0 as $t \to \infty$ for all $(\bar{R}(0), \bar{p}(0)) \in \{(A, b) \in \mathbb{R}^{3\times3} \times \mathbb{R}^3 \mid \|R(0) - A\| < 0.99\}$.*

*Proof:* The part of the proof till local exponential convergence of $\|E_R(t)\|$ and $\|e_p(t)\|$ defined in equations (3) and (4) to 0 as $t \to \infty$ is not affected by the change in the assumption on the bounds on $\|p(t)\|$. Hence, it follows from the corresponding part of Theorem 1.

Writing $E_p$ in terms of $E_R$ and $e_p$ as defined in equation (15) and taking norm on both sides of the equality, we have from equation (16) for all $t \geq 0$,

$$\begin{aligned}\|E_p(t)\| &\leq 100\sqrt{3}\exp(\max\{\mu_1, \mu_2\}T_M)(\exp(-\mu_2 t)\|e_p(0)\| \\ &\quad + \exp((\mu - \mu_1)t)\|E_R^T(0)\|\|p(t)\|\exp(-\mu t)).\end{aligned}$$

The choice of $k_p$ ensures that $\mu - \mu_1 < 0$. Taking limit on both sides as $t \to \infty$, and using the assumption $\lim_{t\to\infty} \|p(t)\|\exp(-\mu t) = c$ for the given $\mu$, we get that $\|E_p\| \to 0$ as $t \to \infty$. □

The following modification to Theorem 1 enlarges the region of convergence.

*Theorem 2: Suppose that Assumption 1 holds, and for all $\mu > 0$, $\|p(t)\|\exp(-\mu t) \to 0$ as $t \to \infty$, and let*

$$E_R = R - \bar{R}, \quad E_p = p - \bar{p},$$

*where the rigid body system is given in equation (1), and the observer system is given in equation (2). Given any initial state $(\bar{R}(0), \bar{p}(0)) \in \{(A, b) \in \mathbb{R}^{3\times3} \times \mathbb{R}^3 \mid det(A) \neq 0\}$, with $0 < k_e < 2$ and*

$$|1 - k_p| < \frac{0.99}{\|R(0) - \bar{R}(0)\| + 1}, \tag{17}$$

*the error term $\|E_R(t)\|$ converges exponentially to 0 and $\|E_p(t)\|$ converges asymptotically to 0 as $t \to \infty$.*

*Proof:* The bounds on $k_p$ in (17) are a subset of the bounds on $k_p$ of Theorem 1. The initial condition and the bounds on the position also follow from Theorem 1. Hence, convergence for the initial condition $\|R(0) - \bar{R}(0)\| < 0.99$ has been shown in Theorem 1.

Consider the case where the given $\bar{R}(0)$ is such that $\alpha := \|R(0) - \bar{R}(0)\| \geq 0.99$. Since

$$\frac{d}{dt}\bar{R}\bar{R}^T = (\bar{R}\Omega_\times)\bar{R}^T + \bar{R}(\bar{R}\Omega_\times)^T = \bar{R}(\Omega_\times - \Omega_\times)\bar{R}^T = 0,$$

it follows that $\bar{R}(t)\bar{R}(t)^T = \bar{R}(0)\bar{R}(0)^T$ for all $0 \leq t < t_1$. The invertibility of $\bar{R}(0)$ implies that of $\bar{R}(0)\bar{R}(0)^T$ which implies the invertibility of $\bar{R}(t)\bar{R}(t)^T$ for all $0 \leq t < t_1$ which implies the invertibility of $\bar{R}(t)$ for all $0 \leq t < t_1$. By (17), we have that

$$\|R(t_1) - \bar{R}^+(t_1)\| = |1 - k_p|\|E_R(t_1)\| < \frac{0.99\alpha}{\alpha + 1} < 0.99.$$

Hence, $\bar{R}^+(t_1)$ is invertible and $\bar{p}^+$ is well defined. Note that there is no change in invertibility of $\bar{R}(t)$ for $t \leq t_1$. After the first jump, since $\|E_R^+(t_1)\| < 0.99$, Theorem 1 and the time-invariant nature of the system (1) and system (2) ensures that the error term $\|E_R(t)\|$ converges exponentially to 0 and $\|E_p(t)\|$ converges asymptotically to 0 as $t \to \infty$. □

Under the assumptions on the position and bounds on $k_e$ as specified in Theorem 2, as $k_p$ gets closer to 1, the region of convergence of the observer (2) to the system (1) *converges* to the set $\{(A, b) \in \mathbb{R}^{3\times3} \times \mathbb{R}^3 \mid \det(A) \neq 0\}$, which can be easily seen by re-writing (17) as

$$\|R(0) - \bar{R}(0)\| < \frac{0.99}{|1 - k_p|} - 1$$

for all $k_p$ near 1 but not equal to 1. Since the right-hand side of the above inequality diverges to infinity as $k_p \to 1$, the claims follows.

Based on Theorem 2, we present a globally convergent observer if the observer system starts from $\mathrm{SO}(3) \rtimes \mathbb{R}^3$:

*Theorem 3: Suppose that Assumption 1 holds, and for all $\mu > 0$, $\|p(t)\|\exp(-\mu t) \to 0$ as $t \to \infty$, and let*

$$E_R = R - \bar{R}, \quad E_p = p - \bar{p},$$

*where the rigid body system is given in equation (1), and the observer system is given in equation (2). Then, with* $0 < k_e < 2$ *and* $0.75 < k_p < 1.25$*, the error term* $\|E_R(t)\|$ *converges exponentially to 0 and* $\|E_p(t)\|$ *converges asymptotically to 0 as* $t \to \infty$ *for all* $(\bar{R}(0), \bar{p}(0)) \in SO(3) \rtimes \mathbb{R}^3$.

*Proof:* Note that $\det(\bar{R}(0)) \neq 0$. Hence it satisfies conditions of Theorem 2. Since $\|R(0) - \bar{R}(0)\| \leq 2\sqrt{2}$ and $0.75 < k_p < 1.25$, $k_p$ and $k_e$ satisfy the bounds as defined in Theorem 2. From Theorem 2, the errors $\|E_R(t)\|$ and $\|E_p(t)\|$ converge to 0 as $t \to \infty$. Moreover, the rate of convergence of $\|E_R(t)\|$ to 0 is exponential. Hence, the observer exhibits global convergence. □

The initial state of the observer is chosen by the designer. According to Theorem 3, irrespective of the initial state of the rigid body system, any choice of the initial state of the observer in $SO(3) \rtimes \mathbb{R}^3$ leads to convergence of the observer to the state of the rigid body. Hence, the result in Theorem 3 is global.

We can modify Theorem 2 and Theorem 3 by relaxing the condition on the bound of the position as in Corollary 1. The proof of the corollary and the theorem presented below follows from that of Theorem 2 and Theorem 3 with a similar modification as in Corollary 1. We leave the proof to the reader.

*Corollary 2: Suppose that Assumption 1 holds, and for some* $\mu, c > 0$, $\|p(t)\| \exp(-\mu t) \to c$ *as* $t \to \infty$*, and let*

$$E_R = R - \bar{R}, \quad E_p = p - \bar{p},$$

*where the rigid body system is given in equation (1), and the observer system is given in equation (2). Given any initial state* $(\bar{R}(0), \bar{p}(0)) \in \{(A, b) \in \mathbb{R}^{3\times 3} \times \mathbb{R}^3 \mid det(A) \neq 0\}$*, with* $0 < k_e < 2$ *and* $k_p^L < k_p < k_p^U$ *where the bounds* $k_p^L$ *and* $k_p^U$ *are defined as*

$$k_p^L = \max\left\{\frac{\|R(0) - \bar{R}(0)\| + 0.01}{\|R(0) - \bar{R}(0)\| + 1}, 1 - \exp(-\mu T_M)\right\},$$
$$k_p^U = \min\left\{\frac{\|R(0) - \bar{R}(0)\| + 1.99}{\|R(0) - \bar{R}(0)\| + 1}, 1 + \exp(-\mu T_M)\right\},$$

*the error term* $\|E_R(t)\|$ *converges exponentially to 0 and* $\|E_p(t)\|$ *converges asymptotically to 0 as* $t \to \infty$.

*Proof:* Omitted. □

The following theorem is a global result extending Theorem 3.

*Theorem 4: Suppose that Assumption 1 holds, and for some* $\mu, c > 0$, $\|p(t)\| \exp(-\mu t) \to c$ *as* $t \to \infty$*, and let*

$$E_R = R - \bar{R}, \quad E_p = p - \bar{p},$$

*where the rigid body system is given in equation (1), and the observer system is given in equation (2). Then, with* $0 < k_e < 2$ *and*

$$\max\{0.75, 1 - e^{-\mu T_M}\} < k_p < \min\{1.25, 1 + e^{-\mu T_M}\},$$

*the error term* $\|E_R(t)\|$ *converges exponentially to 0 and* $\|E_p(t)\|$ *converges asymptotically to 0 as* $t \to \infty$ *for all* $(\bar{R}(0), \bar{p}(0)) \in SO(3) \rtimes \mathbb{R}^3$.

*Proof:* Omitted. □

*Remark 1: To study the effect of varying the gains* $k_p$ *and* $k_e$ *on the observer dynamics, we note that* $\mu_1$ *and* $\mu_2$ *as defined in equations (11) and (12) are a function of* $\log_e |1 - k_p|$ *and* $\log_e |1 - k_e|$*, respectively. Consequently, as* $k_p$ *and* $k_e$ *are chosen closer to 1, the convergence of the observer to the measurement is faster. From equations (5), we see that the farther* $k_p$ *and* $k_e$ *are from 1, the higher their noise rejection effect is on the observer state.*

*Remark 2: We note that the system gains can be updated at every* $t_i, i \in \mathbb{N}$ *due to the structure of the observer. Hence, the designer may choose a varying gains strategy. An example of this is gains being chosen close to 1 initially, such that the bounds of Theorem 3 are satisfied, and then reducing the gains such that the system starts rejecting noise in the measurements. For an application of this scheme, see Sections IV and V.*

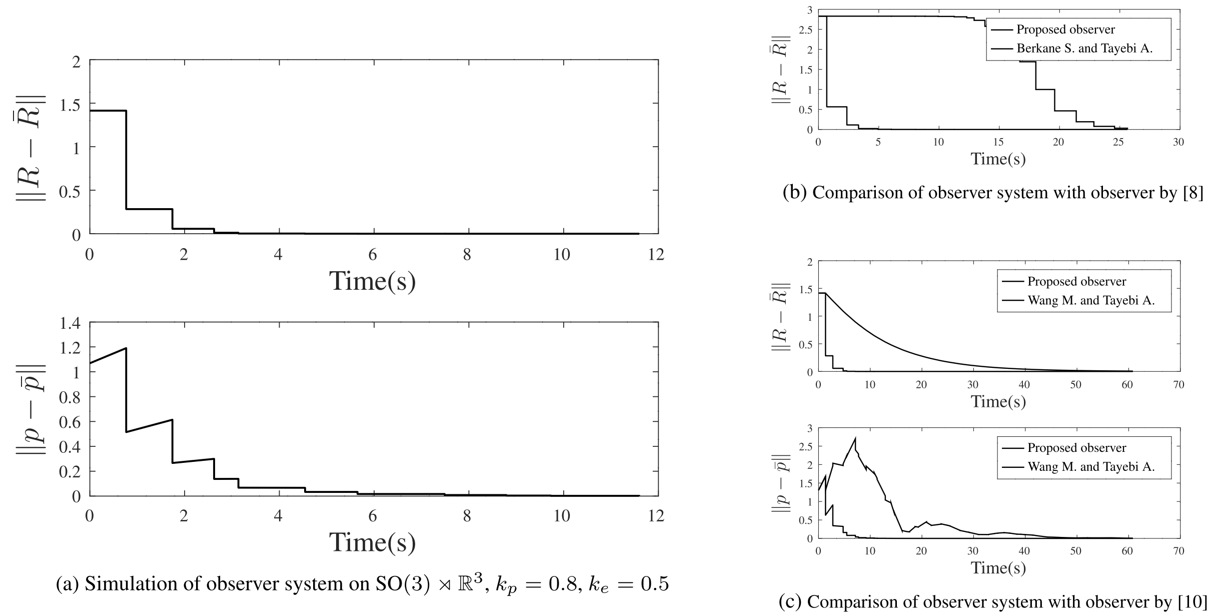
## IV. NUMERICAL SIMULATION

The simulation for the observer error system (6) is shown in Figure 1a. We assume that the initial state is $(\bar{R}, \bar{p}) = (\exp(\pi e_{3_\times}/3), 0_{3\times 1}) \in SO(3) \rtimes \mathbb{R}^3$. The angular velocity is assumed to be of the form $(\cos(t), \sin(t), \cos(2t))$ and the linear velocity $(\sin(t)\cos(t), \sin(2t), \cos(t/5))$. The measurements are assumed to arrive randomly between 0.5s to 2s. It can be seen that the observer system (2) converges exponentially fast for the choice of $k_p = 0.8$ and $k_e = 0.5$.

We compare the proposed observer (2) with the system designed in [8]. The major improvement of our paper over existing ones in the literature is the global convergence result, hence we choose the initial position for the estimate near the boundary of the repeller set in [8]. The simulation results can be seen in Figure 1b. We see that the proposed observer (2) does not suffer as the observer proposed by [8].

We also compare the same with the fixed-gain observer system designed in [10]. Note that the authors in [10] consider a system with the measurement of acceleration and angular velocity as compared to our proposed system with measurements of linear and angular velocity. To account for this, we calculate the acceleration corresponding to the velocity trajectory $[\sin(t)\cos(t), \sin(2t), \cos(t/5)]^T$ and use this acceleration in the simulation of the observer proposed in [10] while using the velocity in our proposed observer. The measurements are assumed to arrive randomly between 0.5s to 2s. The simulation results can be seen in Figure 1c. Since the choice of gains decides the size of the attractor set in [10], the proposed observer (2) improves upon the results presented by [10] due to its global nature.

We also simulate the system in the presence of noisy measurements. The noise is assumed to be Gaussian with a mean of 0 and a standard deviation of 0.1. The measurements are assumed to arrive randomly between 0.5s to 2s. The initial condition is assumed to be $(\bar{R}, \bar{p}) = (\exp(\pi e_{3_\times}/3), 0_{3\times 1}) \in SO(3) \rtimes \mathbb{R}^3$. To show the effect of $k_p$ and $k_e$ values on the results, we choose two sets of values, $k_p = 0.65$ and $k_e = 0.3$, and $k_p = 1$ and $k_e = 1$. As the bounds specified in Theorem 3

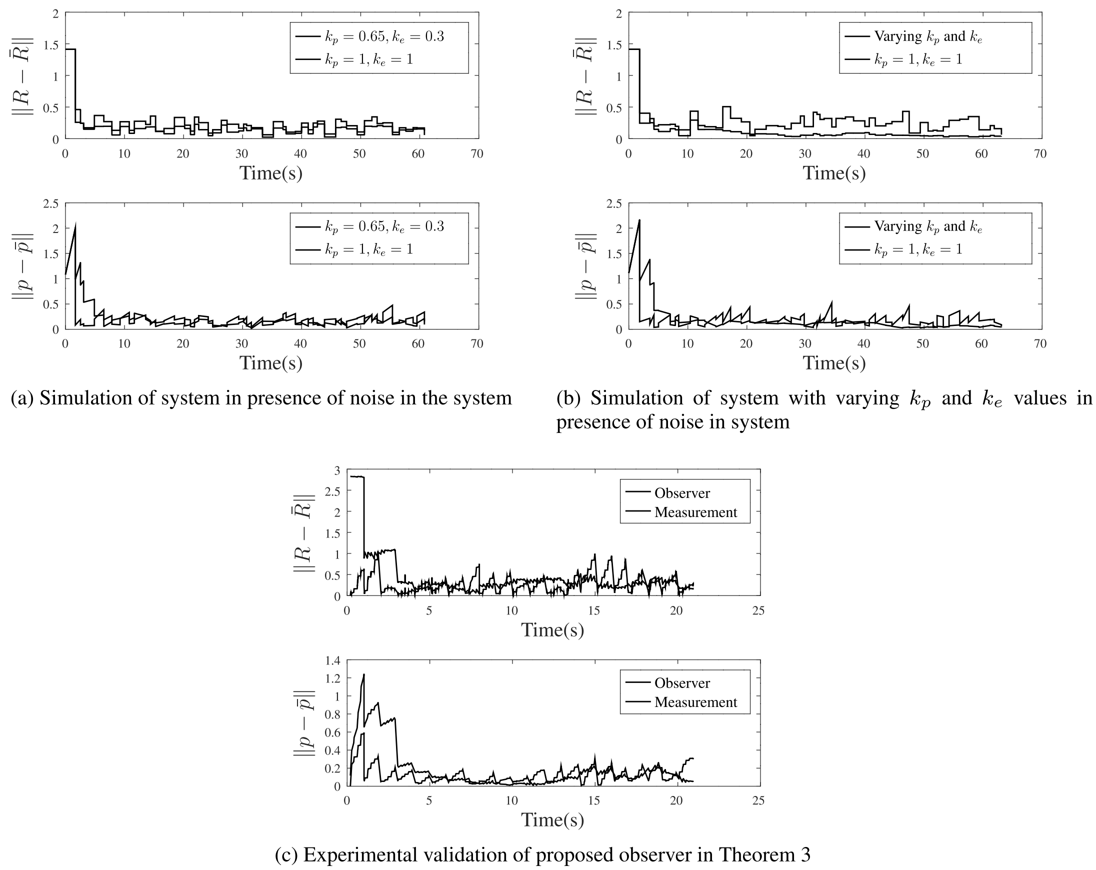



(a) Simulation of observer system on $\mathrm{SO}(3) \rtimes \mathbb{R}^3$, $k_p = 0.8$, $k_e = 0.5$

(b) Comparison of observer system with observer by [8]

(c) Comparison of observer system with observer by [10]

**FIGURE 1.** **Simulation of the observer system and comparison with observers in the literature.**




(a) Simulation of system in presence of noise in the system

(b) Simulation of system with varying $k_p$ and $k_e$ values in presence of noise in system

(c) Experimental validation of proposed observer in Theorem 3

**FIGURE 2.** **Simulation of the observer in presence of noise and experimental validation.**

correspond to the worst case scenario, we are permitted to use $k_p = 0.65$ here since $\|E_R(0)\| = 1.41$. The simulation of the system can be seen in Figure 2a.

We implement the varying $k_p$ and $k_e$ scheme as explained in Remark 2. We choose the same values of $k_p$ and $k_e$ as in the previous simulation, and change the value of $k_p$ and $k_e$

to 0.1 after 10 measurements. The effects of this can be seen in Figure 2b. This provides the designer of the control law greater flexibility in designing a robust observer system.

## V. EXPERIMENTAL SIMULATION

We perform a similar simulation as in Section V with data collected from a real system. We measure $R$ and $p$ using an Intel Realsense T265 camera and the ground truth is measured using a high accuracy motion tracking system OptiTrack.

The time difference between two consecutive measurements of angular velocity and linear velocity vary from $10^{-5}$s to 0.043s and that of the pose are around 1s. Since the initial estimate error is unknown, we choose $k_p = 0.8$ and $k_e = 0.2$ initially. We update the value of $k_p$ to 0.2 while keeping $k_e$ constant after one epoch according to Remark 2.

The simulation results for the observer are shown in Figure 2c. It can be seen that the proposed observer gives a better estimate of the true state as compared to the measurement values due to its hybrid nature.

## VI. CONCLUSION

In this paper, a hybrid measurement triggered observer to the rigid body rotation and translation kinematics problem is designed. It can be seen that the convergence of the rotation estimate of the state to the rotation of the body is exponential globally and the convergence of the position estimate to the position is asymptotic globally when the initial estimate is chosen from $SO(3) \rtimes \mathbb{R}^3$. Also, the improvements over the results of [8] and [10] are substantial since our proposed observer tracks the system and the error converges to 0 globally.

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