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RESEARCH ARTICLE

A Study of Complex Dombi Fuzzy Graph With Application in Decision Making Problems

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ABSTRACT A complex fuzzy set (CFS) is a generalization of a fuzzy set (FS) in which a limit of degrees occurs on the complex plane with unit disc. The averaging operators are a key part of turning all the data into one value. Dombi operators have exceptional flexibility with operating factors, and they are particularly efficient in decision-making problems. In this paper, we establish a complex dombi fuzzy graph (CDFG). We implement dombi operators on CFSs to extend graph nomenclature. We define the complement of CDFG with an example. The idea of self-complementary in CDFG is discussed. The concepts of homomorphism, isomorphism, weak isomorphism, and co-weak isomorphism of two CDFGs are discussed. We define regular and entirely regular graphs with sufficient elaboration and examine their key properties. Furthermore, significant characteristics are used to explain the edge regularity of CDFG. Lastly, we establish an application of CDFG in decision making problems.

INDEX TERMS CDFG, complement, self complementary of CDFG, homomorphism, isomorphism, weak isomorphism, co-weak isomorphism, regular and totally regular CDFG, application.

I. INTRODUCTION

Due to the existence of unclear data, Zadeh [20] created the concept of a FS, which is an extension of the crisp set theory. A FS consists of a true membership function that belongs to a closed interval [0, 1]. The FS has many applications in the area of science.

Menger [12] introduced triangle norms and conorms in the context of probabilistic metric spaces, later defined and analysed by Schweizer and Sklar [13]. Numerous additional researchers have proposed alternative T-operators [5], [10]. Zadeh's conventional T-operators, min and max, are widely employed in fuzzy logic, especially in decision-making and fuzzy graph theory. It is commonly recognised that alternative T-operators function better in specific contexts, especially in decision-making procedures. Examples of preferable operators include products [6]. When choosing T-operators for a certain application, one must evaluate their features, model applicability, simplicity, software and hardware implementation, etc. As the study of these operators has grown, more alternatives for selecting T-operators have emerged.

Graphs have many applications in the field of operational research and computer science. A graph is a visual representation of links between several items that is useful for elaborating on information. However, haziness turns a graph into a fuzzy graph. Fuzzy graphs are intended to portray as a matter of degree structures of connections (in the form of edges) between tangible objects (nodes). Fuzzy graphs have a wide variety of applications, including decision-making, database theory, cluster analysis, and network optimization. Kaufman [11] firstly proposed the concept of fuzzy graph. After that, Rosenfeld [14] studied fuzzy relations on fuzzy sets and used max and min operations to construct the structure of fuzzy graphs, resulting in analogues of numerous graph-theoretical ideas. Bhattacharya [4] made some comments on fuzzy graphs. Reference [16] described new

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operations on picture fuzzy graph. Recently, a few researchers are contributing their efforts in filed of fuzzy theory [7], [8], [9], [17], [18], [19].

Fuzzy graph theory makes it simple to structure and model uncertain decision-making issues. In the discipline of graph theory, only a small amount of work is put towards using the Dombi operator. As a result, Ashraf *et al.* [3] proposed the Dombi fuzzy graph (DFG). Ramot *et al.* [15] presented the concept of CFS in which a range of degrees occurs in the complex plane with unit disk. Akram and Khan [1] studied complex pythagorean fuzzy graph in decision making problem.

The following is a summary of the motivation for this paper:

- When faced with one-dimensional phenomena of imprecise and intuitive knowledge, a CFS is capable of coping with the situation effectively. There is no information loss due to the phase term of the CFS.
- As a result of incorporating the qualities of numerous frequently used operators, Dombi operators have a broader range of applications and are extremely efficient in decision-making.

The following are the key points of this paper:

- The notion of CDFG is initiated.
- The concept of the degree and total degree of a node in both phase terms and amplitude terms are discussed with examples.
- We define complement, self-complementary, homomorphism, isomorphism, weak isomorphism and co-weak isomorphism with their properties.
- We define strong CDFG and complete CDFG.
- We introduce regular and totally regular graphs with appropriate elaboration, and their pivotal properties are discussed.

The following is the structure of this paper:

We presented some basic definitions which will help to understand the paper in Section II. In section III, we study the notion of CDFG, the degree and total degree of a node, complement, self-complementary, homomorphism, isomorphism, weak isomorphism, co-weak isomorphism, strong CDFG, complete CDFG, regular and totally regular graphs with appropriate elaboration, and their pivotal properties are discussed. In Section IV, application of CDFG is discussed. At the end, we write the conclusion and some future plans in Section V.

II. PRELIMINARIES

Definition 1 [1]: A FS on a universe ψ is an object of the following form $\mathcal{C} = \{< g, \varsigma_{\mathcal{C}}(g) > | g \in \psi\}$, where $\varsigma_A : \psi \to [0,1]$ denotes the membership value of $\mathcal{C}$.

Definition 2 [1]: A FS on $\psi \times \psi$ is called fuzzy relation on ψ, denoted by $\mathcal{F} = \{< xy, \varsigma_{\mathcal{F}}(xy) > | xy \in \psi \times \psi\}$, where $\varsigma_B : \psi \times \psi \to [0,1]$ denotes the membership value of $\mathcal{F}$.

Definition 3 [1]: A fuzzy graph on $\psi \neq \phi$ is a pair $\tau = (\mathcal{C}, \mathcal{F})$ with $\mathcal{C}$ a FS on ψ and $\mathcal{F}$ a fuzzy relation on ψ such that $\varsigma_{\mathcal{F}}(xy) \leq \varsigma_{\mathcal{C}}(x) \wedge \varsigma_{\mathcal{C}}(y)$ for all $x, y \in \psi$.

Definition 4 [1]: $\mathcal{A} : [0,1] \times [0,1] \to [0,1]$ binary function known as triangular norm (t-norm) if for all x, y, r ∈ [0,1], it follows the following axioms:

1. $\mathcal{A}(x, 1) = x$.
2. $\mathcal{A}(x, y) = \mathcal{A}(y, x)$.
3. $\mathcal{A}(x, \mathcal{A}(y, r)) = \mathcal{A}(\mathcal{A}(x, y), r)$.
4. $\mathcal{A}(x, y) \leq \mathcal{A}(r, s)$ if $x \leq r$ and $y \leq s$.

Interchanging 1 by 0 in axiom (1), we get the idea of triangular conorm (t-conorm).

Following are some popular t-norms:

- $\mathcal{M}(x, y) = min(x, y)$. (minimum operator $\mathcal{M}$)
- $\mathcal{P}(x, y) = xy$. (product operator $\mathcal{P}$)
- $\mathcal{W}(x, y) = max(x+y-1, 0)$. (Lukasiewicz's t-norm $\mathcal{W}$)
- $\mathcal{D}(x, y) = \frac{1}{1+[(\frac{1-x}{x})^{\lambda}+(\frac{1-y}{y})^{\lambda}]^{1/\lambda}}$: $\lambda > 0$. (Dombi's t-norm $\mathcal{D}$)

By putting $\lambda = 1$ in dombi's t-norm, we obtain one more T-operator that is $T(x, y) = \frac{xy}{x+y-xy}$.

The corresponding t-conorms are as follows:

- $\mathcal{M}^*(x, y) = max(x, y)$. (maximum operator $\mathcal{M}^*$)
- $\mathcal{P}^*(x, y) = x + y - xy$. (probabilistic sum $\mathcal{P}^*$)
- $\mathcal{W}^*(x, y) = min(x+y, 1)$. (Lukasiewicz's t-conorm $\mathcal{W}^*$)
- $\mathcal{D}^*(x, y) = \frac{1}{1+[(\frac{1-x}{x})^{-\lambda}+(\frac{1-y}{y})^{-\lambda}]^{1/-\lambda}}$: $\lambda > 0$. (Dombi's t-conorm $\mathcal{D}^*$)

By putting $\lambda = 1$ in dombi's t-conorm, we obtain one more T-operator that is $S(g, h) = \frac{g+h-2gh}{1-gh}$.

Definition 5 [1]: A DFG on $\mathcal{V}$ (underlying set) is an ordered pair $\tau = (\mathcal{C}, \mathcal{F})$, where $\mathcal{F} : \mathcal{V} \times \mathcal{V} \to [0,1]$ is a symmetric fuzzy relation on $\mathcal{C}$ and $\mathcal{C} : \mathcal{V} \to [0,1]$ is a fuzzy subset in $\mathcal{V}$ such that

$$\varsigma_{\mathcal{F}}(gh) \leq \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}$$

for all g,h∈ $\mathcal{V}$, where $\varsigma_{\mathcal{C}}$ and $\varsigma_{\mathcal{F}}$ denotes the membership values of $\mathcal{C}$ and $\mathcal{F}$ respectively.

Definition 6 [1]: A complex fuzzy set (CFS) on a universe ψ is an object of the form $\mathcal{C} = \{< g, \varsigma_{\mathcal{C}}(g)e^{i\vartheta_{\mathcal{C}}(g)} > | g \in \psi\}$, $i = \sqrt{-1}$, where $\varsigma_{\mathcal{C}} : \psi \to [0,1]$ is a real valued function represents the membership value and $\vartheta_{\mathcal{C}}(g) \in [0, 2\pi]$, for all $g \in \psi$. Note that $\varsigma_{\mathcal{C}}(g)$ is called amplitude term and $\vartheta_{\mathcal{C}}(g)$ is called phase term.

Definition 7 [1]: A CFS on $\psi \times \psi$ is said to be complex fuzzy relation (CFR) denoted by $\mathcal{F} = \{< gh, \varsigma_{\mathcal{F}}(gh)e^{i\vartheta_{\mathcal{F}}(gh)} > | gh \in \psi \times \psi\}$, $i = \sqrt{-1}$, where $\varsigma_{\mathcal{F}} : \psi \times \psi \to [0,1]$ represents the membership value and $\vartheta_{\mathcal{F}}(gh) \in [0, 2\pi]$, for all $gh \in \psi$. Note that $\varsigma_{\mathcal{F}}(gh)$ is called amplitude term and $\vartheta_{\mathcal{F}}(gh)$ is called phase term.

III. CDFG

Definition 8: A CDFG on a universe ψ is an ordered pair $\tau = (\mathcal{C}, \mathcal{F})$, where $\mathcal{C} = (g, \varsigma_{\mathcal{C}}e^{i\vartheta_{\mathcal{C}}}) : \psi \to \{z : z \in C, |Z| \leq 1\}$ is a CFS subset in ψ and $\mathcal{F} = (gh, \varsigma_{\mathcal{F}}e^{i\vartheta_{\mathcal{F}}}) : \psi \times \psi \to \{z : z \in C, |Z| \leq 1\}$ is a complex fuzzy relation (CFR) on $\mathcal{C}$ such that for amplitude term

$$\varsigma_{\mathcal{F}}(gh) \leq \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)},$$

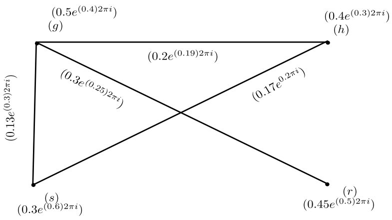

FIGURE 1. $G_1 \circ G_2$.

and for phase term

$$\vartheta_{\mathcal{F}}(gh) \leq \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g) + \vartheta_{\mathcal{C}}(h) - \vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)},$$

for all g,h$\in \psi$, where $i = \sqrt{-1}$, and $\vartheta_{\mathcal{F}}(gh) \in [0, 2\pi]$. We call $\mathcal{C}$ and $\mathcal{F}$ the complex fuzzy node set and complex fuzzy edge set, respectively.

Example 1: Let $\tau = (\mathcal{C}, \mathcal{F})$ be a CDFG on $\tau^* = (\psi, \mathcal{A})$ as shown in Figure 1, where $\psi = \{g, h, r, s\}$ and $\mathcal{A} = \{gh, pr, ps, qs\}$. The set of nodes $\mathcal{C}$ and set of arcs $\mathcal{F}$ of τ are defined on ψ and $\mathcal{A}$, respectively.

$$\mathcal{C} = < \frac{g}{0.5e^{i2\pi(0.4)}}, \frac{h}{0.4e^{i2\pi(0.3)}}, \frac{r}{0.45e^{i2\pi(0.5)}}, \frac{s}{0.3e^{i2\pi(0.6)}} >$$

and

$$\mathcal{F} = < \frac{gh}{0.2e^{i2\pi(0.19)}}, \frac{pr}{0.3e^{i2\pi(0.25)}}, \frac{ps}{0.13e^{i2\pi(0.3)}}, \frac{qs}{0.17e^{i2\pi(0.2)}} >$$

By calculations, one can see that $\tau = (\mathcal{C}, \mathcal{F})$ is a CDFG.

Definition 9: Let $\mathcal{F} = \{(gh, \varsigma_{\mathcal{F}}(gh)e^{i\vartheta_{\mathcal{F}}(gh)}) | gh \in \mathcal{A}\}$ be a set of arcs in CDFG τ, then

- *The degree of node $g \in \psi$ for amplitude term is denoted by $\mathcal{D}_\tau(g) = \mathcal{D}_\varsigma(g)$, where*

$$\mathcal{D}_\varsigma(g) = \sum_{g,h\neq g\in\psi} \varsigma_{\mathcal{F}(gh)} = \sum_{g,h\neq g\in\psi} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}.$$

The degree of a node $g \in \psi$ for a phase term is expressed by $\mathcal{D}_\tau(g) = \mathcal{D}_{e^{i\vartheta}}(g)$, where

$$\mathcal{D}_{e^{i\vartheta}}(g) = \sum_{g,h\neq g\in\psi} \vartheta_{\mathcal{F}(gh)} = \sum_{g,h\neq g\in\psi} \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g) + \vartheta_{\mathcal{C}}(h) - \vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}.$$

- *The total degree of node $g \in \psi$ for amplitude term is expressed by $\mathcal{TD}_\tau(g) = \mathcal{TD}_\varsigma(g)$, where*

$$\mathcal{TD}_\varsigma(g) = \sum_{g,h\neq g\in\psi} \varsigma_{\mathcal{F}(gh)} + \varsigma_A(g) = \sum_{g,h\neq g\in\psi} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} + \varsigma_A(g).$$

The total degree of a node $g \in \psi$ for a phase term is expressed by $\mathcal{TD}_\tau(g) = \mathcal{TD}_{e^{i\vartheta}}(g)$, where

$$\mathcal{TD}_{e^{i\vartheta}}(g) = \sum_{g,h\neq g\in\psi} \vartheta_{\mathcal{F}(gh)} + \vartheta_A(g) = \sum_{g,h\neq g\in\psi} \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g) + \vartheta_{\mathcal{C}}(h) - \vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)} + \vartheta_A(g).$$

Example 2: From above example, we have

- *The degree of nodes in τ are as follows:*

$$\mathcal{D}_\tau(g) = 0.63e^{i2\pi(0.74)},$$
$$\mathcal{D}_\tau(h) = 0.37e^{i2\pi(0.39)},$$
$$\mathcal{D}_\tau(r) = 0.3e^{i2\pi(0.25)},$$
$$\mathcal{D}_\tau(s) = 0.3e^{i2\pi(0.5)},$$

- *The total degree of nodes in τ are as follows:*

$$\mathcal{TD}_\tau(g) = 1.13e^{i2\pi(1.14)},$$
$$\mathcal{TD}_\tau(h) = 0.77e^{i2\pi(0.69)},$$
$$\mathcal{TD}_\tau(r) = 0.75e^{i2\pi(0.75)},$$
$$\mathcal{TD}_\tau(s) = 0.6e^{i2\pi(1.1)},$$

Definition 10: Let $\tau = (\mathcal{C}, \mathcal{F})$ be a CDFG on a graph $\tau^* = (\psi, \mathcal{A})$. The complement of τ for amplitude term is determined by:

1. $\varsigma_{\bar{\mathcal{C}}}(g) = \varsigma_{\mathcal{C}}(g)$.

$$2.\ \varsigma_{\bar{\mathcal{F}}}(gh) = \begin{cases} \dfrac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}, \\ \quad \text{if} \quad \varsigma_{\mathcal{F}}(gh) = 0. \\ \left(\dfrac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} - \varsigma_{\mathcal{F}}(gh)\right) \\ \quad \text{if} \quad 0 < \varsigma_{\mathcal{F}}(gh) \leq 1. \end{cases}$$

Similarly the complement of τ for phase term is determined by:

1. $\vartheta_{\bar{\mathcal{C}}}(g) = \vartheta_{\mathcal{C}}(g)$.

$$2.\ \vartheta_{\bar{\mathcal{F}}}(gh) = \begin{cases} \dfrac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g) + \vartheta_{\mathcal{C}}(h) - \vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}, \\ \quad \text{if} \quad \vartheta_{\mathcal{F}}(gh) = 0. \\ \left(\dfrac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g) + \vartheta_{\mathcal{C}}(h) - \vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)} - \vartheta_{\mathcal{F}}(gh)\right) \\ \quad \text{if} \quad 0 < \vartheta_{\mathcal{F}}(gh) \leq 2\pi. \end{cases}$$

Further, the complement of a CDFG τ is denoted by $\bar{\tau} = (\bar{\mathcal{C}}, \bar{\mathcal{F}})$.

Definition 11: A homomorphism $\mathcal{Z} : \tau \to \mathcal{G}'$ of two CDFGs $\tau = (\mathcal{C}, \mathcal{F})$ and $\mathcal{G}' = (\mathcal{C}', \mathcal{F}')$ is a mapping $\mathcal{Z} : \psi \to \psi'$ satisfying

1. $\varsigma_{\mathcal{C}}(g) \leq \varsigma_{\mathcal{C}'}(\mathcal{Z}(g))$, $\vartheta_{\mathcal{C}}(g) \leq \vartheta_{\mathcal{C}'}(\mathcal{Z}(g))$, for all $g \in \psi$.

2. $\varsigma_{\mathcal{F}}(gh) \leq \varsigma_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))$, $\vartheta_{\mathcal{F}}(gh) \leq \vartheta_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))$, for all $gh \in \mathcal{A}$.

Definition 12: An isomorphism $\mathcal{Z} : \tau \to \mathcal{G}'$ of two CDFGs $\tau = (\mathcal{C}, \mathcal{F})$ and $\mathcal{G}' = (\mathcal{C}', \mathcal{F}')$ is a bijective mapping $\mathcal{Z} : \psi \to \psi'$ satisfying

1. $\varsigma_{\mathcal{C}}(g) = \varsigma_{\mathcal{C}'}(\mathcal{Z}(g))$, $\vartheta_{\mathcal{C}}(g) = \vartheta_{\mathcal{C}'}(\mathcal{Z}(g))$ for all $g \in \psi$.

2. $\varsigma_{\mathcal{F}}(gh) = \varsigma_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))$, $\vartheta_{\mathcal{F}}(gh) = \vartheta_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))$ for all $gh \in \mathcal{A}_1$.

Definition 13: A weak isomorphism $\mathcal{Z} : \tau \to \mathcal{G}'$ of two CDFGs $\tau = (\mathcal{C}, \mathcal{F})$ and $\mathcal{G}' = (\mathcal{C}', \mathcal{F}')$ is a bijective mapping $\mathcal{Z} : \psi \to \psi'$ satisfying

1. $\mathcal{Z}$ is a homomorphism.

2. $\varsigma_{\mathcal{C}}(g) = \varsigma_{\mathcal{C}'}(\mathcal{Z}(g))$, $\vartheta_{\mathcal{C}}(g) = \vartheta_{\mathcal{C}'}(\mathcal{Z}(g))$ for all $g \in \psi$.

Definition 14: An co-weak isomorphism $\mathcal{Z} : \tau \to \mathcal{G}'$ of two CDFGs $\tau = (\mathcal{C}, \mathcal{F})$ and $\mathcal{G}' = (\mathcal{C}', \mathcal{F}')$ is a bijective mapping $\mathcal{Z} : \psi \to \psi'$ satisfying

1. $\mathcal{Z}$ is a homomorphism.

2. $\varsigma_{\mathcal{F}}(gh) = \varsigma_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))$, $\vartheta_{\mathcal{F}}(gh) = \vartheta_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))$ for all $gh \in \mathcal{A}$.

Definition 15: A CDFG $\tau = (\mathcal{C}, \mathcal{F})$ is called self complementary if $\bar{\tau} \cong \tau$.

Proposition 1: If $\tau = (\mathcal{C}, \mathcal{F})$ is a self complementary CDFG, then

$$\sum_{g\neq h} \varsigma_{\mathcal{F}}(gh) = \frac{1}{2}\Big(\sum_{g\neq h} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}\Big),$$
$$\sum_{g\neq h} \vartheta_{\mathcal{F}}(gh) = \frac{1}{2}\Big(\sum_{g\neq h} \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g) + \vartheta_{\mathcal{C}}(h) - \vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}\Big),$$

Proof: Suppose that τ is a self complementary CDFG, then there occurs an isomorphism $\mathcal{Z} : \psi \to \psi$ such that

$$\overline{\varsigma_{\mathcal{C}}(\mathcal{Z}(g))} = \varsigma_{\mathcal{C}}(g), \overline{\vartheta_{\mathcal{C}}(\mathcal{Z}(g))} = \vartheta_{\mathcal{C}}(g) \text{ for all } g \in \psi.$$
$$\overline{\varsigma_{\mathcal{F}}(\mathcal{Z}(g)\mathcal{Z}(h))} = \varsigma_{\mathcal{F}}(gh),$$
$$\overline{\vartheta_{\mathcal{F}}(\mathcal{Z}(g)\mathcal{Z}(h))} = \vartheta_{\mathcal{F}}(gh) \text{ for all } gh \in \mathcal{A}.$$

By using definition of complement, we have

$$\overline{\varsigma_{\mathcal{F}}(\mathcal{Z}(g)\mathcal{Z}(h))} = \frac{\overline{\varsigma_{\mathcal{C}}(\mathcal{Z}(g))\varsigma_{\mathcal{C}}(\mathcal{Z}(h))}}{\overline{\varsigma_{\mathcal{C}}(\mathcal{Z}(g))} + \overline{\varsigma_{\mathcal{C}}(\mathcal{Z}(h))} - \overline{\varsigma_{\mathcal{C}}(\mathcal{Z}(g))\varsigma_{\mathcal{C}}(\mathcal{Z}(h))}} - \varsigma_{\mathcal{F}}(\mathcal{Z}(g)\mathcal{Z}(h)).$$

$$\varsigma_{\mathcal{F}}(gh) = \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} - \varsigma_{\mathcal{F}}(\mathcal{Z}(g)\mathcal{Z}(h)),$$

$$\sum_{g\neq h} \varsigma_{\mathcal{F}}(gh) = \sum_{g\neq h} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} - \sum_{g\neq h} \varsigma_{\mathcal{F}}(\mathcal{Z}(g)\mathcal{Z}(h)),$$

$$\sum_{g\neq h} \varsigma_{\mathcal{F}}(gh) + \sum_{g\neq h} \varsigma_{\mathcal{F}}(\mathcal{Z}(g)\mathcal{Z}(h)) = \sum_{g\neq h} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)},$$

$$2\sum_{g\neq h} \varsigma_{\mathcal{F}}(gh) = \sum_{g\neq h} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)},$$

$$\sum_{g\neq h} \varsigma_{\mathcal{F}}(gh) = \frac{1}{2}\sum_{g\neq h} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)},$$

Similarly, the phase term can be proved. □

Proposition 2: If a CDFG $\mathcal{G} = (\mathcal{A}, \mathcal{B})$ on an underlying graph $\tau^* = (\psi, \mathcal{A})$ satisfy the following:

$$\varsigma_{\mathcal{F}}(gh) = \frac{1}{2}\Big(\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}\Big),$$
$$\vartheta_{\mathcal{F}}(gh) = \frac{1}{2}\Big(\frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g) + \vartheta_{\mathcal{C}}(h) - \vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}\Big),$$

for all g,h$\in \psi$, then τ is self complementary.

Proof: Consider τ is CDFG that satisfies

$$\varsigma_{\mathcal{F}}(gh) = \frac{1}{2}\Big(\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}\Big),$$

for all g,h$\in\psi$, then the identity mapping $\mathcal{I} : \psi \to \psi$ is an isomorphism from τ to $\bar{\tau}$ that satisfies the following conditions:

$$\overline{\varsigma_{\mathcal{C}}(\mathcal{I}(g))} = \varsigma_{\mathcal{C}}(g), \quad \overline{\vartheta_{\mathcal{C}}(\mathcal{I}(g))} = \vartheta_{\mathcal{C}}(g), \text{ for all } g \in \psi.$$

Since the membership value of an edge set gh is given by

$$\varsigma_{\mathcal{F}}(gh) = \frac{1}{2}\Big(\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}\Big), \text{ for all } g, h \in \psi.$$

We have

$$\begin{aligned}
&\overline{\varsigma_{\mathcal{F}}(\mathcal{I}(g)\mathcal{I}(h))} \\
&= \overline{\varsigma_{\mathcal{F}}(gh)} = \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} - \varsigma_{\mathcal{F}}(gh) \\
&= \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} \\
&\quad - \frac{1}{2}\Big(\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}\Big) \\
&= \frac{1}{2}\Big(\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}\Big) \\
&= \varsigma_{\mathcal{F}}(gh).
\end{aligned}$$

Similarly the phase term condition of isomorphism

$$\overline{\vartheta_{\mathcal{F}}(\mathcal{I}(g)\mathcal{I}(h))} = \vartheta_{\mathcal{F}}(gh),$$

is satisfied by $\mathcal{I}$. Hence $\mathcal{G} = (\mathcal{A}, \mathcal{B})$ is self complementary. □

Proposition 3: Let $\tau = (\mathcal{C}, \mathcal{F})$ and $\mathcal{G}' = (\mathcal{C}', \mathcal{F}')$ be two CDFGs, then $\tau \cong \mathcal{G}'$ iff $\bar{\tau} \cong \bar{\tau}'$.

Proof: Suppose that τ and $\mathcal{G}'$ are two isomorphic CDFGs. Then by definition of isomorphism, there occur a bijective mapping $\mathcal{Z} : \psi \to \psi'$ that satisfies

$$\varsigma_{\mathcal{C}}(g) = \varsigma_{\mathcal{C}'}(\mathcal{Z}(g)), \quad \vartheta_{\mathcal{C}}(g) = \vartheta_{\mathcal{C}'}(\mathcal{Z}(g)) \text{ for all } g \in \psi_1.$$
$$\varsigma_{\mathcal{F}}(gh) = \varsigma_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h)),$$
$$\vartheta_{\mathcal{F}}(gh) = \vartheta_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h)) \text{ for all } gh \in \mathcal{A}_1.$$

By using definition of complement, the membership of an edge gh is

$$\overline{\varsigma_{\mathcal{F}}(gh)} = \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} - \varsigma_{\mathcal{F}}(gh),$$
$$\overline{\varsigma_{\mathcal{F}}(gh)} = \frac{\varsigma_{\mathcal{C}'}(\mathcal{Z}(g))\varsigma_{\mathcal{C}'}(\mathcal{Z}(h))}{\varsigma_{\mathcal{C}'}(\mathcal{Z}(g)) + \varsigma_{\mathcal{C}'}(\mathcal{Z}(h)) - \varsigma_{\mathcal{C}'}(\mathcal{Z}(g))\varsigma_{\mathcal{C}'}(\mathcal{Z}(h))}$$

$$- \varsigma_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h)),$$
$$\overline{\varsigma_{\mathcal{F}}(gh)} = \overline{\varsigma_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))}.$$

Similarly for the phase term,

$$\overline{\vartheta_{\mathcal{F}}(gh)} = \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)} - \vartheta_{\mathcal{F}}(gh),$$
$$\overline{\vartheta_{\mathcal{F}}(gh)} = \frac{\vartheta_{\mathcal{C}'}(\mathcal{Z}(g))\vartheta_{\mathcal{C}'}(\mathcal{Z}(h))}{\vartheta_{\mathcal{C}'}(\mathcal{Z}(g))+\vartheta_{\mathcal{C}'}(\mathcal{Z}(h))-\vartheta_{\mathcal{C}'}(\mathcal{Z}(g))\vartheta_{\mathcal{C}'}(\mathcal{Z}(h))}$$
$$- \vartheta_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h)),$$
$$\overline{\vartheta_{\mathcal{F}}(gh)} = \overline{\vartheta_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))}.$$

We conclude that the complement of τ is isomorphic to the complement of $\mathcal{G}'$. Similarly, we can prove its converse part. □

Proposition 4: Let $\tau = (\mathcal{C},\mathcal{F})$ and $\mathcal{G}' = (\mathcal{C}',\mathcal{F}')$ be two weak isomorphic CDFGs, then $\bar{\tau}$ and $\bar{\mathcal{G}}'$ are also weak isomorphic to each other. *Proof: Suppose that τ and $\mathcal{G}'$ are two weak isomorphic CDFGs. Then utilizing the definition of weak isomorphism, there exist a bijective mapping $\mathcal{Z}:\psi\to\psi'$ that satisfies*

$$\varsigma_{\mathcal{C}}(g) = \varsigma_{\mathcal{C}'}(\mathcal{Z}(g)), \quad \vartheta_{\mathcal{C}}(g) = \vartheta_{\mathcal{C}'}(\mathcal{Z}(g)) \text{ for all } g \in \psi_1.$$

and

$$\varsigma_{\mathcal{F}}(gh) \le \varsigma_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h)),$$
$$\vartheta_{\mathcal{F}}(gh) \le \vartheta_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h)) \text{ for all } gh \in \mathcal{A}_1.$$

For the membership value of an edge, we have

$$\varsigma_{\mathcal{F}}(gh) \le \varsigma_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))$$
$$-\varsigma_{\mathcal{F}}(gh) \ge -\varsigma_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))$$
$$\mathcal{T}(\varsigma_{\mathcal{C}}(g),\varsigma_{\mathcal{C}}(h)) - \varsigma_{\mathcal{F}}(gh)$$
$$\ge \mathcal{T}(\varsigma_{\mathcal{C}}(g),\varsigma_{\mathcal{C}}(h)) - \varsigma_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))$$
$$\mathcal{T}(\varsigma_{\mathcal{C}}(g),\varsigma_{\mathcal{C}}(h)) - \varsigma_{\mathcal{F}}(gh)$$
$$\ge \mathcal{T}(\varsigma_{\mathcal{C}'}(\mathcal{Z}(g)),\varsigma_{\mathcal{C}'}(\mathcal{Z}(h))) - \varsigma_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))$$
$$\overline{\varsigma_{\mathcal{F}}(gh)} \ge \overline{\varsigma_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))}.$$

Similarly for the phase term

$$\vartheta_{\mathcal{F}}(gh) \le \vartheta_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))$$
$$-\vartheta_{\mathcal{F}}(gh) \ge -\vartheta_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))$$
$$\mathcal{T}(\vartheta_{\mathcal{C}}(g),\vartheta_{\mathcal{C}}(h)) - \vartheta_{\mathcal{F}}(gh)$$
$$\ge \mathcal{T}(\vartheta_{\mathcal{C}}(g),\vartheta_{\mathcal{C}}(h)) - \vartheta_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))$$
$$\mathcal{T}(\vartheta_{\mathcal{C}}(g),\vartheta_{\mathcal{C}}(h)) - \vartheta_{\mathcal{F}}(gh)$$
$$\ge \mathcal{T}(\vartheta_{\mathcal{C}'}(\mathcal{Z}(g)),\vartheta_{\mathcal{C}'}(\mathcal{Z}(h))) - \vartheta_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))$$
$$\overline{\vartheta_{\mathcal{F}}(gh)} \ge \overline{\vartheta_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h))}.$$

Hence we conclude that complement of τ is weak isomorphic to the complement of $\mathcal{G}'$. □

Definition 16: A CDFG is said to be complete if

$$\varsigma_{\mathcal{F}}(gh) = \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)},$$
$$\vartheta_{\mathcal{F}}(gh) = \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)},$$

for all g,h$\in\psi$.

Definition 17: A CDFG is said to be strong if

$$\varsigma_{\mathcal{F}}(gh) = \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)},$$
$$\vartheta_{\mathcal{F}}(gh) = \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)},$$

for all gh$\in\mathcal{A}$.

Definition 18: Let $\tau = (\mathcal{C},\mathcal{F})$ be a strong CDFG on a graph $\tau^* = (\psi,\mathcal{A})$. The complement of τ for amplitude term is represented by:

1. $\varsigma_{\bar{\mathcal{C}}}(g) = \varsigma_{\mathcal{C}}(g)$.
2. $$\varsigma_{\bar{\mathcal{F}}}(gh) = \begin{cases} \dfrac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}, \\ \quad \text{if} \quad \varsigma_{\mathcal{F}}(gh)=0. \\ 0, \quad \text{if} \quad 0<\varsigma_{\mathcal{F}}(gh)\le 1. \end{cases}$$

Similarly the complement of τ for phase term is represented by:

1. $\vartheta_{\bar{\mathcal{C}}}(g) = \vartheta_{\mathcal{C}}(g)$.
2. $$\vartheta_{\bar{\mathcal{F}}}(gh) = \begin{cases} \dfrac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}, \\ \quad \text{if} \quad \vartheta_{\mathcal{F}}(gh)=0. \\ 0, \quad \text{if} \quad 0<\vartheta_{\mathcal{F}}(gh)\le 2\pi. \end{cases}$$

Further, the complement of a strong CDFG τ is expressed by $\bar{\tau} = (\bar{\mathcal{C}},\bar{\mathcal{F}})$.

Remark 1: Every complete CDFG is strong.

Definition 19: A CDFG $\tau = (\mathcal{C},\mathcal{F})$ on a graph $\tau^* = (\psi,\mathcal{A})$ is known as regular of degree $\mathcal{R}_1 e^{i\mathcal{R}_1^*}$ or $\mathcal{R}_1 e^{i\mathcal{R}_1^*}$-regular if its every node has equal degree, i.e,

$$\mathcal{D}_{\varsigma}(g) = \sum_{g,h\neq g\in\psi} \varsigma_{\mathcal{F}(gh)} = \mathcal{R}_1,$$
$$\mathcal{D}_{e^{i\vartheta}}(g) = \sum_{g,h\neq g\in\psi} \vartheta_{\mathcal{F}(gh)} = \mathcal{R}_1^*,$$
$$\text{for all } g\in\psi$$

Example 3: Let $\tau = (\mathcal{C},\mathcal{F})$ be a CDFG on $\tau^* = (\psi,\mathcal{A})$ as shown in Figure 2, where $\psi = \{g,h,r,s\}$ and $\mathcal{A} = \{gh,qr,ps,rs\}$. The set of nodes $\mathcal{C}$ and set of edges $\mathcal{F}$ of τ are defined on ψ and $\mathcal{A}$, respectively.

$$\mathcal{C} = < \frac{g}{0.5e^{i2\pi(0.4)}}, \frac{h}{0.5e^{i2\pi(0.4)}}, \frac{r}{0.5e^{i2\pi(0.4)}}, \frac{s}{0.5e^{i2\pi(0.4)}} >$$

and

$$\mathcal{F} =< \frac{gh}{0.3e^{i2\pi(0.2)}}, \frac{qr}{0.3e^{i2\pi(0.2)}}, \frac{ps}{0.3e^{i2\pi(0.2)}}, \frac{rs}{0.3e^{i2\pi(0.2)}} >$$

By calculations, one can see that $\tau = (\mathcal{C},\mathcal{F})$ is a $0.6e^{i2\pi(0.4)}$-regular CDFG.

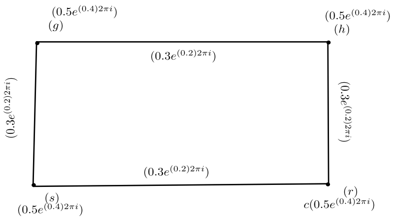

FIGURE 2. Regular-CDFG.

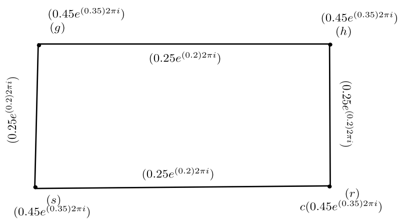

FIGURE 3. Totally Regular-CDFG.

Definition 20: A CDFG $\tau = (\mathcal{C}, \mathcal{F})$ on a graph $\tau^* = (\psi, \mathcal{A})$ is called totally regular of degree $\mathcal{T}_1 e^{i\mathcal{T}_1^*}$ or $\mathcal{T}_1 e^{i\mathcal{T}_1^*}$-totally regular if its each node has same total degree, i.e,

$$\mathcal{TD}_{\varsigma}(g) = \sum_{g,h\neq g\in\psi} \varsigma_{\mathcal{F}(gh)} + \varsigma_{\mathcal{C}}(g) = \mathcal{T}_1,$$

$$\mathcal{TD}_{e^{i\vartheta}}(g) = = \sum_{g,h\neq g\in\psi} \vartheta_{\mathcal{F}(gh)} + \vartheta_{\mathcal{C}}(g) = \mathcal{T}_1^*,$$

for all $g \in \psi$.

Example 4: Let $\tau = (\mathcal{C}, \mathcal{F})$ be a CDFG on $\tau^* = (\psi, \mathcal{A})$ as shown in Figure 3, where $\psi = \{g, h, r, s\}$ and $\mathcal{A} = \{gh, qr, ps, rs\}$. The set of nodes $\mathcal{C}$ and set of edges $\mathcal{F}$ of τ are defined on ψ and $\mathcal{A}$, respectively.

$$\mathcal{C} =< \frac{g}{0.45e^{i2\pi(0.35)}}, \frac{h}{0.45e^{i2\pi(0.35)}}, \frac{r}{0.45e^{i2\pi(0.35)}}, \frac{s}{0.45e^{i2\pi(0.35)}} >$$

and

$$\mathcal{F} =< \frac{gh}{0.25e^{i2\pi(0.2)}}, \frac{qr}{0.15e^{i2\pi(0.2)}}, \frac{ps}{0.15e^{i2\pi(0.2)}}, \frac{rs}{0.25e^{i2\pi(0.2)}} >$$

By calculations, one can see that $\tau = (\mathcal{C}, \mathcal{F})$ is a $0.85e^{i2\pi(0.75)}$-totally regular CDFG.

Theorem 1: Consider a CDFG $\tau = (\mathcal{C}, \mathcal{F})$ which is isomorphic to another CDFG $\mathcal{G}' = (\mathcal{C}', \mathcal{F}')$;

1. If τ is regular CDFG, then $\mathcal{G}'$ is also regular CDFG.

2. If τ is totally regular CDFG, then $\mathcal{G}'$ is also totally regular CDFG.

Proof: 1. Suppose that τ is isomorphic to $\mathcal{G}'$ and τ is $\mathcal{R}_1 e^{i\mathcal{R}_1^*}$-regular CDFG, therefore degree of each node of τ is given by

$$\mathcal{D}_{\tau}(g) = \mathcal{D}_{\varsigma e^{i\vartheta}}(g)$$

$$\mathcal{D}_{\tau}(g) = \sum_{gh\in\mathcal{A}} \varsigma_{\mathcal{F}}(gh) e^{i(\sum_{gh\in\mathcal{A}} \vartheta_{\mathcal{F}}(gh))}$$

$$\mathcal{D}_{\tau}(g) = \sum_{g,h\neq g\in\psi} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} \times e^{i(\sum_{g,h\neq g\in\psi} \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)})}$$
$$= \mathcal{R}_1 e^{i\mathcal{R}_1^*}.$$

Since $\tau \cong \mathcal{G}'$, we must have

$$\begin{aligned}
\mathcal{R}_1 e^{i\mathcal{R}_1^*} &= \mathcal{D}_{\varsigma e^{i\vartheta}}(g) \\
&= \sum_{gh\in\mathcal{A}} \varsigma_{\mathcal{F}}(gh) e^{i(\sum_{gh\in\mathcal{A}} \vartheta_{\mathcal{F}}(gh))} \\
&= \sum_{g,h\neq g\in\psi} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} \\
&\quad \times e^{i(\sum_{g,h\neq g\in\psi} \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)})}. \\
&= \sum_{g,h\neq g\in\psi} \frac{\varsigma_{\mathcal{C}'}(\mathcal{Z}(g))\varsigma_{\mathcal{C}'}(\mathcal{Z}(h))}{\varsigma_{\mathcal{C}'}(\mathcal{Z}(g)) + \varsigma_{\mathcal{C}'}(\mathcal{Z}(h)) - \varsigma_{\mathcal{C}'}(\mathcal{Z}(g))\varsigma_{\mathcal{C}'}(\mathcal{Z}(h))} \\
&\quad \times e^{i(\sum_{g,h\neq g\in\psi} \frac{\vartheta_{\mathcal{C}'}(\mathcal{Z}(g))\vartheta_{\mathcal{C}'}(\mathcal{Z}(h))}{\vartheta_{\mathcal{C}'}(\mathcal{Z}(g))+\vartheta_{\mathcal{C}'}(\mathcal{Z}(h))-\vartheta_{\mathcal{C}'}(\mathcal{Z}(g))\vartheta_{\mathcal{C}'}(\mathcal{Z}(h))})}, \\
&= \sum_{gh\in\mathcal{A}} \varsigma_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h)) e^{i(\sum_{gh\in\mathcal{A}} \vartheta_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h)))} \\
&= \mathcal{D}_{\varsigma' e^{i\vartheta'}}(g) \\
&= \mathcal{D}_{\mathcal{G}'}(g)
\end{aligned}$$

Thus, $\mathcal{G}'$ is a $\mathcal{R}_1 e^{i\mathcal{R}_1^*}$-regular CDFG.

2. Suppose that τ is isomorphic to $\mathcal{G}'$ and τ is $\mathcal{T}_1 e^{i\mathcal{T}_1^*}$-totally regular CDFG, therefore the total degree of each node of G is given by

$$\begin{aligned}
\mathcal{TD}_{\tau}(g) &= \mathcal{TD}_{\varsigma}(g) \\
&= (\sum_{gh\in\mathcal{A}} \varsigma_{\mathcal{F}}(gh) + \varsigma_{\mathcal{C}}(g) \\
&= (\sum_{g,h\neq g\in\psi} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} + \varsigma_{\mathcal{C}}(g), \\
&= \mathcal{T}_1.
\end{aligned}$$

Since $\tau \cong \mathcal{G}'$, we must have

$$\begin{aligned}
\mathcal{T}_1 &= \mathcal{TD}_{\varsigma}(g) \\
&= \sum_{gh\in\mathcal{A}} \varsigma_{\mathcal{F}}(gh) + \varsigma_{\mathcal{C}}(g) \\
&= \sum_{g,h\neq g\in\psi} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} + \varsigma_{\mathcal{C}}(g) \\
&= \sum_{g,h\neq g\in\psi} \frac{\varsigma_{\mathcal{C}'}(\mathcal{Z}(g))\varsigma_{\mathcal{C}'}(\mathcal{Z}(h))}{\varsigma_{\mathcal{C}'}(\mathcal{Z}(g)) + \varsigma_{\mathcal{C}'}(\mathcal{Z}(h)) - \varsigma_{\mathcal{C}'}(\mathcal{Z}(g))\varsigma_{\mathcal{C}'}(\mathcal{Z}(h))} \\
&\quad + \varsigma_{\mathcal{C}'}(\mathcal{Z}(g)), \\
&= \sum_{gh\in\mathcal{A}} \varsigma_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h)) + \varsigma_{\mathcal{C}'}(\mathcal{Z}(g)) \\
&= \mathcal{TD}_{\varsigma'}(g) \\
&= \mathcal{TD}_{\mathcal{G}'}(g).
\end{aligned}$$

Also for phase term,

$$\begin{aligned}\mathcal{TD}_{\tau}(g) &= \mathcal{TD}_{e^{i\vartheta}}(g)\\ &= \sum_{gh\in\mathcal{A}} \vartheta_{\mathcal{F}}(gh) + \vartheta_{\mathcal{C}}(g)\\ &= \sum_{g,h\neq g\in\psi} \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)} + \vartheta_{\mathcal{C}}(g)\\ &= \mathcal{T}_1^*.\end{aligned}$$

Since $\tau \cong \mathcal{G}'$, we must have

$$\begin{aligned}\mathcal{T}_1^* &= \mathcal{TD}_{e^{i\vartheta}}(g)\\ &= \sum_{gh\in\mathcal{A}} \vartheta_{\mathcal{F}}(gh) + \vartheta_{\mathcal{C}}(g)\\ &= \sum_{g,h\neq g\in\psi} \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)} + \vartheta_{\mathcal{C}}(g)\\ &= \sum_{g,h\neq g\in\psi} \frac{\vartheta_{\mathcal{C}'}(\mathcal{Z}(g))\vartheta_{\mathcal{C}'}(\mathcal{Z}(h))}{\vartheta_{\mathcal{C}'}(\mathcal{Z}(g))+\vartheta_{\mathcal{C}'}(\mathcal{Z}(h))-\vartheta_{\mathcal{C}'}(\mathcal{Z}(g))\vartheta_{\mathcal{C}'}(\mathcal{Z}(h))}\\ &\quad + \vartheta_{\mathcal{C}'}(\mathcal{Z}(g))\\ &= \sum_{gh\in\mathcal{A}} \vartheta_{\mathcal{F}'}(\mathcal{Z}(g)\mathcal{Z}(h)) + \vartheta_{\mathcal{C}'}(\mathcal{Z}(g))\\ &= \mathcal{TD}_{\vartheta'}(g)\\ &= \mathcal{TD}_{\mathcal{G}'}(g).\end{aligned}$$

□

Theorem 2: Suppose that $\tau = (\mathcal{C},\mathcal{F})$ is a CDFG on a graph $\tau^* = (\psi,\mathcal{A})$ with $\varsigma_{\mathcal{C}}e^{i\vartheta_{\mathcal{C}}}$ as a constant function, then $\tau = (\mathcal{C},\mathcal{F})$ is a regular CDFG if and only if τ is totally regular CDFG. *Proof: Suppose that $\varsigma_{\mathcal{C}}e^{i\vartheta_{\mathcal{C}}}$ is a constant function, i.e, $\varsigma_{\mathcal{C}}(g)e^{i\vartheta_{\mathcal{C}}(g)} = c_1 e^{ic_1^*}$ is a constant function for all $g\in\psi$, where $c_1e^{ic_1^*}$ is constant.*

Suppose that $\tau = (\mathcal{C},\mathcal{F})$ is $\mathcal{R}_1e^{i\mathcal{R}_1^*}$-regular CDFG, then

$$\mathcal{D}_{\varsigma}(g) = \sum_{g,h\neq g\in\psi} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} = \mathcal{R}_1,$$

$$\mathcal{D}_{e^{i\vartheta}}(g) = \sum_{g,h\neq g\in\psi} \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)} = \mathcal{R}_1^*,$$

The total degree of a node is given by

$$\begin{aligned}\mathcal{TD}_{\varsigma}(g) &= \sum_{g,h\neq g\in\psi} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}\\ &\quad + \varsigma_{\mathcal{C}}(g) = \mathcal{R}_1 + c_1,\\ \mathcal{TD}_{e^{i\vartheta}}(g) &= \sum_{g,h\neq g\in\psi} \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}\\ &\quad + \vartheta_{\mathcal{C}}(g) = \mathcal{R}_1^* + c_1^*,\end{aligned}$$

Hence, τ is a $(\mathcal{R}_1+c_1)e^{i(\mathcal{R}_1^*+c_1^*)}$-totally regular CDFG.

Conversely, suppose that $\tau = (\mathcal{C},\mathcal{F})$ is $\mathcal{T}_1e^{i\mathcal{T}_1^*}$-totally regular CDFG, then

$$\begin{aligned}\mathcal{TD}_{\varsigma}(g) &= \sum_{g,h\neq g\in\psi} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}\\ &\quad + \varsigma_{\mathcal{C}}(g) = \mathcal{T}_1,\\ \sum_{g,h\neq g\in\psi} &\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} + c_1 = \mathcal{T}_1,\\ \sum_{g,h\neq g\in\psi} &\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} = \mathcal{T}_1 - c_1,\\ \mathcal{D}_{\varsigma}(g) &= \mathcal{T}_1 - c_1 = \mathcal{R}_1.\\ \mathcal{TD}_{e^{i\vartheta}}(g) &= \sum_{g,h\neq g\in\psi} \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}\\ &\quad + \vartheta_{\mathcal{C}}(g) = \mathcal{T}_1^*,\\ \sum_{g,h\neq g\in\psi} &\frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}\\ &\quad + c_1^* = \mathcal{T}_1^*,\\ \sum_{g,h\neq g\in\psi} &\frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)} = \mathcal{T}_1^* - c_1^*,\\ \mathcal{D}_{e^{i\vartheta}}(g) &= \mathcal{T}_1^* - c_1^* = \mathcal{R}_1^*.\end{aligned}$$

So τ is a $\mathcal{R}_1e^{i\mathcal{R}_1^*}$-regular CDFG. □

Theorem 3: Suppose that $\tau = (\mathcal{C},\mathcal{F})$ is a CDFG on a graph $\tau^* = (\psi,\mathcal{A})$. If τ is both $\mathcal{R}_1e^{i\mathcal{R}_1^*}$-regular and $\mathcal{T}_1e^{i\mathcal{T}_1^*}$-totally regular CDFG, then $\varsigma_{\mathcal{C}}e^{i\vartheta_{\mathcal{C}}}$ is a constant function. *Proof: Suppose that τ is $\mathcal{R}_1e^{i\mathcal{R}_1^*}$-regular and $\mathcal{T}_1e^{i\mathcal{T}_1^*}$-totally regular CDFG. Then, then the degree of a node is given by*

$$\mathcal{D}_{\varsigma}(g) = \sum_{g,h\neq g\in\psi} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} = \mathcal{R}_1,$$

$$\mathcal{D}_{e^{i\vartheta}}(g) = \sum_{g,h\neq g\in\psi} \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)} = \mathcal{R}_1^*,$$

The total degree of a node is given by

$$\begin{aligned}\mathcal{TD}_{\varsigma}(g) &= \sum_{g,h\neq g\in\psi} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}\\ &\quad + \varsigma_{\mathcal{C}}(g) = \mathcal{T}_1,\\ \mathcal{TD}_{e^{i\vartheta}}(g) &= \sum_{g,h\neq g\in\psi} \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}\\ &\quad + \vartheta_{\mathcal{C}}(g) = \mathcal{T}_1^*,\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{TD}_{\varsigma}(a) &= \mathcal{R}_1 + \varsigma_{\mathcal{C}}(g) = \mathcal{T}_1,\\ \varsigma_{\mathcal{C}}(g) &= \mathcal{T}_1 - \mathcal{R}_1.\\ \mathcal{TD}_{e^{i\vartheta}}(a) &= \mathcal{R}_1^* + \vartheta_{\mathcal{C}}(g) = \mathcal{T}_1^*,\\ \vartheta_{\mathcal{C}}(g) &= \mathcal{T}_1^* - \mathcal{R}_1^*.\end{aligned}$$

Hence, $\varsigma_{\mathcal{C}}e^{i\vartheta_{\mathcal{C}}} = (\mathcal{T}_1-\mathcal{R}_1)e^{i(\mathcal{T}_1^*-\mathcal{R}_1^*)}$ is a constant function. □

Remark 2: Converse of above theorem need not to be true as given in the following example.

Example 5: Let $\tau = (\mathcal{C},\mathcal{F})$ be a CDFG on $\tau^* = (\psi,\mathcal{A})$ as shown in Figure 4, where $\psi = \{g,h,r,s\}$ and $\mathcal{A} =$

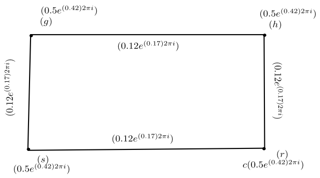

FIGURE 4. **Not Totally Regular nor Regular CDFG.**

$\{gh, ps, qr, rs\}$. *The set of nodes* $\mathcal{C}$ *and set of edges* $\mathcal{F}$ *of* τ *are defined on* ψ *and* $\mathcal{A}$, *respectively.*

$$\mathcal{C} =< \frac{g}{0.5e^{i2\pi(0.42)}}, \frac{h}{0.5e^{i2\pi(0.42)}}, \frac{r}{0.5e^{i2\pi(0.42)}}, \frac{s}{0.5e^{i2\pi(0.42)}} >$$

and

$$\mathcal{F} =< \frac{gh}{0.25e^{i2\pi(0.12)}}, \frac{ps}{0.12e^{i2\pi(0.17)}}, \frac{qr}{0.24e^{i2\pi(0.23)}}, \frac{rs}{0.23e^{i2\pi(0.24)}} >$$

Here $\varsigma_{\mathcal{C}}e^{i\vartheta_{\mathcal{C}}}$ *for g,h,r,s is a constant function. But*

$$\mathcal{D}_\tau(g) = 0.37e^{i2\pi(0.29)} \neq 0.47e^{i2\pi(0.47)} = \mathcal{D}_\tau(r)$$

and

$$\mathcal{TD}_\tau(g) = 0.87e^{i2\pi(0.71)} \neq 0.97e^{i2\pi(0.89)} = \mathcal{TD}_\tau(r).$$

Hence, $\tau = (\mathcal{C}, \mathcal{F})$ *is neither regular nor totally regular CDFG.*

Definition 21: Let $\mathcal{F} = \{(gh, \varsigma_{\mathcal{F}}(gh)e^{i\vartheta_{\mathcal{F}}(gh)}) | gh \in \mathcal{A}\}$ *be the set of edges in CDFG* τ, *then*

• *The degree of an edge* $gh \in \mathcal{A}$ *is represented by* $\mathcal{D}_\tau(gh) = \mathcal{D}_{\varsigma e^{i\vartheta}}(gh)$, *where*

$$\begin{aligned}\mathcal{D}_\varsigma(gh) &= \sum_{pr\in\mathcal{A}, r\neq h} \varsigma_{\mathcal{F}}(pr) + \sum_{qr\in\mathcal{A}, g\neq r} \varsigma_{\mathcal{F}}(qr)\\ &= \mathcal{D}_{\varsigma_{\mathcal{C}}}(g) + \mathcal{D}_{\varsigma_{\mathcal{C}}}(h) - 2\varsigma_{\mathcal{F}}(gh)\\ &= \mathcal{D}_{\varsigma_{\mathcal{C}}}(g) + \mathcal{D}_{\varsigma_{\mathcal{C}}}(h)\\ &\quad - 2(\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}).\\ \mathcal{D}_{e^{i\vartheta}}(gh) &= \sum_{pr\in\mathcal{A}, r\neq h} \vartheta_{\mathcal{F}}(pr) + \sum_{qr\in\mathcal{A}, g\neq r} \vartheta_{\mathcal{F}}(qr)\\ &= \mathcal{D}_{\vartheta_{\mathcal{C}}}(g) + \mathcal{D}_{\vartheta_{\mathcal{C}}}(h) - 2\vartheta_{\mathcal{F}}(gh)\\ &= \mathcal{D}_{\vartheta_{\mathcal{C}}}(g) + \mathcal{D}_{\vartheta_{\mathcal{C}}}(h)\\ &\quad - 2(\frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g) + \vartheta_{\mathcal{C}}(h) - \vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}).\end{aligned}$$

• *The total degree of an edge* $gh \in \mathcal{A}$ *is represented by* $\mathcal{TD}_\tau(gh) = \mathcal{TD}_{\varsigma e^{i\vartheta}}(gh)$, *where*

$$\begin{aligned}&\mathcal{TD}_\varsigma(gh)\\ &= \sum_{pr\in\mathcal{A}, r\neq h} \varsigma_{\mathcal{F}}(pr) + \sum_{qr\in\mathcal{A}, g\neq r} \varsigma_{\mathcal{F}}(qr) + \varsigma_{\mathcal{F}}(gh)\\ &= \mathcal{D}_{\varsigma_{\mathcal{C}}}(g) + \mathcal{D}_{\varsigma_{\mathcal{C}}}(h) - \varsigma_{\mathcal{F}}(gh)\\ &= \mathcal{D}_{\varsigma_{\mathcal{C}}}(g) + \mathcal{D}_{\varsigma_{\mathcal{C}}}(h) - (\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}).\\ &\mathcal{TD}_{e^{i\vartheta}}(gh)\\ &= \sum_{pr\in\mathcal{A}, r\neq h} \vartheta_{\mathcal{F}}(pr) + \sum_{qr\in\mathcal{A}, g\neq r} \vartheta_{\mathcal{F}}(qr) + \vartheta_{\mathcal{F}}(gh)\\ &= \mathcal{D}_{\vartheta_{\mathcal{C}}}(g) + \mathcal{D}_{\vartheta_{\mathcal{C}}}(h) - \vartheta_{\mathcal{F}}(gh)\\ &= \mathcal{D}_{\vartheta_{\mathcal{C}}}(g) + \mathcal{D}_{\vartheta_{\mathcal{C}}}(h) - (\frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g) + \vartheta_{\mathcal{C}}(h) - \vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}).\end{aligned}$$

Definition 22: A CDFG $\tau = (\mathcal{C}, \mathcal{F})$ *is known as an edge regular, if the degree of its each edge is equal, i.e,*

$$\begin{aligned}\mathcal{D}_\varsigma(gh) &= \mathcal{D}_{\varsigma_{\mathcal{C}}}(g) + \mathcal{D}_{\varsigma_{\mathcal{C}}}(h) - 2\varsigma_{\mathcal{F}}(gh)\\ &= \mathcal{D}_{\varsigma_{\mathcal{C}}}(g) + \mathcal{D}_{\varsigma_{\mathcal{C}}}(h)\\ &\quad - 2(\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}) = \mathcal{L}_1.\\ \mathcal{D}_{e^{i\vartheta}}(gh) &= \mathcal{D}_{\vartheta_{\mathcal{C}}}(g) + \mathcal{D}_{\vartheta_{\mathcal{C}}}(h)\\ &\quad - 2\vartheta_{\mathcal{F}}(gh)\\ &= \mathcal{D}_{\vartheta_{\mathcal{C}}}(g) + \mathcal{D}_{\vartheta_{\mathcal{C}}}(h)\\ &\quad - 2(\frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g) + \vartheta_{\mathcal{C}}(h) - \vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}) = \mathcal{L}_1^*.\end{aligned}$$

for all $gh \in \mathcal{A}$. τ *is called* $\mathcal{L}_1e^{i\mathcal{L}_1^*}$*-edge regular CDFG.*

Definition 23: A CDFG $\tau = (\mathcal{C}, \mathcal{F})$ *is known as totally edge regular, if the total degree of its each edge is equal, i.e,*

$$\begin{aligned}\mathcal{TD}_\varsigma(gh) &= \mathcal{D}_{\varsigma_{\mathcal{C}}}(g) + \mathcal{D}_{\varsigma_{\mathcal{C}}}(h) - \varsigma_{\mathcal{F}}(gh)\\ &= \mathcal{D}_{\varsigma_{\mathcal{C}}}(g) + \mathcal{D}_{\varsigma_{\mathcal{C}}}(h)\\ &\quad - (\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g) + \varsigma_{\mathcal{C}}(h) - \varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}) = \mathcal{K}_1.\\ \mathcal{TD}_{e^{i\vartheta}}(gh) &= \mathcal{D}_{\vartheta_{\mathcal{C}}}(g) + \mathcal{D}_{\vartheta_{\mathcal{C}}}(h) - \vartheta_{\mathcal{F}}(gh)\\ &= \mathcal{D}_{\vartheta_{\mathcal{C}}}(g) + \mathcal{D}_{\vartheta_{\mathcal{C}}}(h)\\ &\quad - (\frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g) + \vartheta_{\mathcal{C}}(h) - \vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}) = \mathcal{K}_1^*.\end{aligned}$$

for all $gh \in \mathcal{A}$. τ *is known as* $\mathcal{K}_1e^{i\mathcal{K}_1^*}$*-totally edge regular CDFG.*

Example 6: Let $\tau = (\mathcal{C}, \mathcal{F})$ *be a CDFG on* $\tau^* = (\psi, \mathcal{A})$ *as shown in Figure 5, where* $\psi = \{g, h, r, s\}$ *and* $\mathcal{A} = \{gh, ps, qs, pr, qr, rs\}$. *The set of nodes* $\mathcal{C}$ *and set of edges* $\mathcal{F}$ *of* τ *are defined on* ψ *and* $\mathcal{A}$, *respectively.*

$$\mathcal{C} =< \frac{g}{0.45e^{i2\pi(0.35)}}, \frac{h}{0.35e^{i2\pi(0.45)}}, \frac{r}{0.5e^{i2\pi(0.35)}}, \frac{s}{0.35e^{i2\pi(0.5)}} >$$

and

$$\mathcal{F} =< \frac{gh}{0.11e^{i2\pi(0.13)}}, \frac{ps}{0.11e^{i2\pi(0.13)}}, \frac{qs}{0.11e^{i2\pi(0.13)}}, \frac{pr}{0.11e^{i2\pi(0.13)}}, \frac{qr}{0.11e^{i2\pi(0.13)}}, \frac{rs}{0.11e^{i2\pi(0.13)}} >.$$

Since degree of each edge is $0.44e^{i2\pi(0.52)}$ *and total degree of each edge is* $0.55e^{i2\pi(0.65)}$.

So, $\tau = (\mathcal{C}, \mathcal{F})$ *is* $0.44e^{i2\pi(0.52)}$*-edge regular and* $0.55e^{i2\pi(0.65)}$*-totally edge regular CDFG.*

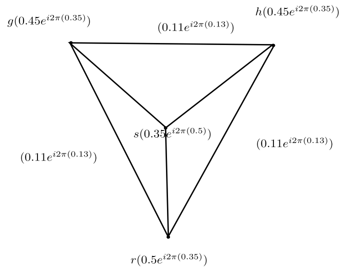

FIGURE 5. CDFG.

Theorem 4: Suppose $\tau = (\mathcal{C}, \mathcal{F})$ is $\mathcal{R}_1 e^{i\mathcal{R}_1^*}$-regular CDFG. If $\varsigma_{\mathcal{F}} e^{i\vartheta_{\mathcal{F}}}$ is a constant function, then τ is $\mathcal{L}_1 e^{i\mathcal{L}_1^*}$-edge regular CDFG. *Proof: Suppose that $\tau = (\mathcal{C}, \mathcal{F})$ is a $\mathcal{R}_1 e^{i\mathcal{R}_1^*}$-regular CDFG, then*

$$\mathcal{D}_{\varsigma e^{i\vartheta}}(g) = \sum_{g,h\neq g\in\psi} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} \times e^{i(\sum_{g,h\neq g\in\psi} \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)})} = \mathcal{R}_1 e^{i\mathcal{R}_1^*}.$$

Now $\varsigma_{\mathcal{F}} e^{i\vartheta_{\mathcal{F}}}$ is a constant function, therefore, $\varsigma_{\mathcal{F}}(gh) e^{i\vartheta_{\mathcal{F}}(gh)} = c_1 e^{ic_1^*}$ for all $gh \in \mathcal{A}$.

Since the degree of an edge $gh \in \mathcal{A}$ is given by $\mathcal{D}_\tau(gh) = \mathcal{D}_{\varsigma e^{i\vartheta}}(gh)$, where

$$\begin{aligned}\mathcal{D}_{\varsigma}(gh) &= \mathcal{D}_{\varsigma_{\mathcal{C}}}(g) + \mathcal{D}_{\varsigma_{\mathcal{C}}}(h) \\ &\quad - 2\left(\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}\right) \\ &= \mathcal{R}_1 + \mathcal{R}_1 - 2c_1 = 2(\mathcal{R}_1 - c_1) = \mathcal{L}_1. \\ \mathcal{D}_{e^{i\vartheta}}(gh) &= \mathcal{D}_{\vartheta_{\mathcal{C}}}(g) + \mathcal{D}_{\vartheta_{\mathcal{C}}}(h) \\ &\quad - 2\left(\frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}\right) \\ &= 2\mathcal{R}_1^* - 2c_1^* = 2(\mathcal{R}_1^* - c_1^*) = \mathcal{L}_1^*.\end{aligned}$$

Hence τ is $\mathcal{L}_1 e^{i\mathcal{L}_1^*}$-edge regular CDFG. □

Theorem 5: Suppose a CDFG τ is $\mathcal{L}_1 e^{i\mathcal{L}_1^*}$-edge regular and $\mathcal{K}_1 e^{i\mathcal{K}_1^*}$-totally edge regular, then $\varsigma_{\mathcal{F}} e^{i\vartheta_{\mathcal{F}}}$ is a constant function. *Proof: Suppose that τ is $\mathcal{L}_1 e^{i\mathcal{L}_1^*}$-edge regular CDFG, then the degree of its every arc is*

$$\begin{aligned}\mathcal{D}_{\varsigma}(gh) &= \mathcal{D}_{\varsigma_{\mathcal{C}}}(g) + \mathcal{D}_{\varsigma_{\mathcal{C}}}(h) \\ &\quad - 2\left(\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}\right) = \mathcal{L}_1. \\ \mathcal{D}_{e^{i\vartheta}}(gh) &= \mathcal{D}_{\vartheta_{\mathcal{C}}}(g) + \mathcal{D}_{\vartheta_{\mathcal{C}}}(h) \\ &\quad - 2\left(\frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}\right) = \mathcal{L}_1^*.\end{aligned}$$

Also τ is $\mathcal{K}_1 e^{i\mathcal{K}_1^*}$-totally edge regular CDFG, then the degree of its each edge is

$$\begin{aligned}\mathcal{TD}_{\varsigma}(gh) &= \mathcal{D}_{\varsigma_{\mathcal{C}}}(g) + \mathcal{D}_{\varsigma_{\mathcal{C}}}(h) \\ &\quad - \left(\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}\right) = \mathcal{K}_1. \\ \mathcal{TD}_{e^{i\vartheta}}(gh) &= \mathcal{D}_{\vartheta_{\mathcal{C}}}(g) + \mathcal{D}_{\vartheta_{\mathcal{C}}}(h) \\ &\quad - \left(\frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}\right) = \mathcal{K}_1^*.\end{aligned}$$

Further, it follows that

$$\begin{aligned}&\mathcal{TD}_{\varsigma}(gh) = \mathcal{K}_1 \\ &\mathcal{D}_{\varsigma_{\mathcal{C}}}(g) + \mathcal{D}_{\varsigma_{\mathcal{C}}}(h) - \left(\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}\right) = \mathcal{K}_1 \\ &\mathcal{D}_{\varsigma_{\mathcal{C}}}(g) + \mathcal{D}_{\varsigma_{\mathcal{C}}}(h) \\ &\quad - 2\left(\frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}\right) + \varsigma_{\mathcal{F}}(gh) = \mathcal{K}_1 \\ &\varsigma_{\mathcal{F}}(gh) = \mathcal{K}_1 - \mathcal{R}_1. \\ &\mathcal{TD}_{\vartheta}(gh) = \mathcal{K}_1^* \\ &\mathcal{D}_{\vartheta_{\mathcal{C}}}(g) + \mathcal{D}_{\vartheta_{\mathcal{C}}}(h) - \left(\frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}\right) = \mathcal{K}_1^* \\ &\mathcal{D}_{\vartheta_{\mathcal{C}}}(g) + \mathcal{D}_{\vartheta_{\mathcal{C}}}(h) \\ &\quad - 2\left(\frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}\right) + \vartheta_{\mathcal{F}}(gh) = \mathcal{K}_1^* \\ &\vartheta_{\mathcal{F}}(gh) = \mathcal{K}_1^* - \mathcal{R}_1^*.\end{aligned}$$

Hence, $\varsigma_{\mathcal{F}} e^{i\vartheta_{\mathcal{F}}}$ is a constant function. □

Theorem 6: Suppose $\tau = (\mathcal{C}, \mathcal{F})$ is a CDFG. Then $\varsigma_{\mathcal{F}} e^{i\vartheta_{\mathcal{F}}}$ is a constant function if and only if τ is both regular-CDFG and totally edge regular-CDFG.

Proof: Suppose that τ is a CDFG. Assume that $\varsigma_{\mathcal{F}} e^{i\vartheta_{\mathcal{F}}}$ is a constant function, therefore, $\varsigma_{\mathcal{F}}(gh) e^{i\vartheta_{\mathcal{F}}(gh)} = c_1 e^{ic_1^*}$ for all $gh \in \mathcal{A}$, where $c_1 e^{ic_1^*}$ is a constant.

Since the degree of a node $g \in \psi$ is given by $\mathcal{D}_\tau(g) = (\mathcal{D}_{\varsigma e^{i\vartheta}}(g), \mathcal{D}_{\varsigma e^{i\vartheta}}(g), \mathcal{D}_{\kappa e^{i\rho}}(g))$, where

$$\begin{aligned}\mathcal{D}_{\varsigma e^{i\vartheta}}(g) &= \sum_{gh\in\mathcal{A}} \varsigma_{\mathcal{F}}(gh) e^{i(\sum_{gh\in\mathcal{A}} \vartheta_{\mathcal{F}}(gh))} \\ \mathcal{D}_{\varsigma e^{i\vartheta}}(g) &= \sum_{g,h\neq g\in\psi} \frac{\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)}{\varsigma_{\mathcal{C}}(g)+\varsigma_{\mathcal{C}}(h)-\varsigma_{\mathcal{C}}(g)\varsigma_{\mathcal{C}}(h)} \\ &\quad e^{i(\sum_{g,h\neq g\in\psi} \frac{\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)}{\vartheta_{\mathcal{C}}(g)+\vartheta_{\mathcal{C}}(h)-\vartheta_{\mathcal{C}}(g)\vartheta_{\mathcal{C}}(h)})} \\ \mathcal{D}_{\varsigma e^{i\vartheta}}(g) &= \sum_{gh\in\mathcal{A}} c_1 e^{i(\sum_{gh\in\mathcal{A}} c_1)} \\ \mathcal{D}_{\varsigma e^{i\vartheta}}(g) &= \mathcal{R}c_1 e^{i\mathcal{R}c_1^*}.\end{aligned}$$

Thus, τ is $\mathcal{R}c_1 e^{i\mathcal{R}c_1^*}$-regular CDFG.

As the total degree of an edge $gh \in \mathcal{A}$ is given as $\mathcal{TD}_\tau(gh) = \mathcal{TD}_{\varsigma e^{i\vartheta}}(gh)$, where

$$\begin{aligned}\mathcal{TD}_{\varsigma}(gh) &= \sum_{pr\in\mathcal{A}, r\neq h} \varsigma_{\mathcal{F}}(pr) + \sum_{qr\in\mathcal{A}, g\neq r} \varsigma_{\mathcal{F}}(qr) + \varsigma_{\mathcal{F}}(gh) \\ &= \sum_{pr\in\mathcal{A}, r\neq h} c_1 + \sum_{qr\in\mathcal{A}, g\neq r} c_1 + c_1 \\ &= c_1(\mathcal{R}-1) + c_1(\mathcal{R}-1) + c_1 \\ &= c_1(2\mathcal{R}-1). \\ \mathcal{TD}_{\vartheta}(gh) &= \sum_{pr\in\mathcal{A}, r\neq h} \vartheta_{\mathcal{F}}(pr) + \sum_{qr\in\mathcal{A}, g\neq r} \vartheta_{\mathcal{F}}(qr) + \vartheta_{\mathcal{F}}(gh)\end{aligned}$$

$$= \sum_{pr\in\mathcal{A},r\neq h} c_1^* + \sum_{qr\in\mathcal{A},g\neq r} c_1^* + c_1^*$$
$$= c_1^*(\mathcal{R}-1) + c_1^*(\mathcal{R}-1) + c_1^*$$
$$= c_1^*(2\mathcal{R}-1).$$

Hence τ is $c_1(2\mathcal{R}-1)e^{i(c_1^*(2\mathcal{R}-1))}$-totally edge regular CDFG. So τ is both regular-CDFG and totally edge regular-CDFG.

Conversely, Let τ be $\mathcal{L}_1 e^{i\mathcal{L}_1^*}$-edge regular and $\mathcal{K}_1 e^{i\mathcal{K}_1^*}$-totally edge regular CDFG. Furthermore, the total degree of an edge is given by

$$\mathcal{TD}_\tau(gh) = \mathcal{TD}_{\varsigma e^{i\vartheta}}(gh), \text{ where}$$
$$\mathcal{TD}_\varsigma(gh) = \mathcal{D}_{\varsigma_\mathcal{C}}(g) + \mathcal{D}_{\varsigma_\mathcal{C}}(h) - \varsigma_\mathcal{F}(gh)$$
$$\mathcal{K}_1 = \mathcal{R}_1 + \mathcal{R}_1 - \varsigma_\mathcal{F}(gh)$$
$$\varsigma_\mathcal{F}(gh) = 2\mathcal{R}_1 - \mathcal{K}_1.$$
$$\mathcal{TD}_{e^{i\vartheta}}(gh) = \mathcal{D}_{\vartheta_\mathcal{C}}(g) + \mathcal{D}_{\vartheta_\mathcal{C}}(h) - \vartheta_\mathcal{F}(gh)$$
$$\mathcal{K}_1^* = \mathcal{R}_1^* + \mathcal{R}_1^* - \vartheta_\mathcal{F}(gh)$$
$$\vartheta_\mathcal{F}(gh) = 2\mathcal{R}_1^* - \mathcal{K}_1^*.$$
$$\text{for all } gh \in \mathcal{A}.$$

Hence, $\varsigma_\mathcal{F} e^{i\vartheta_\mathcal{F}}$ is a constant function. □

IV. APPLICATION

In this part, we present an algorithm and resolve a problem of decision making to choose the best spot to set up an internet office in a city. This situation may help us to understand the proposed methodology.

A. ALGORITHM

The algorithm to determine a suitable location or place for an internet office in a city is as follows:

INPUT: A distinct collection of suitable options $P = \{P_1, P_2, \ldots, P_n\}$ in certain conditions in order to reach the goal of and construction of complex fuzzy preference relation (CFPR) $Q = (b \times h)_{n\times n}$.

OUTPUT: The decision of an appropriate choice.

1. Consider $d_{kq} = \varsigma_{kq} e^{i\vartheta_{kq}}$ (k, h = 1, 2, ..., n) and collection of choices $P = \{P_1, P_2, \ldots, P_n\}$.

2. Aggregate all $d_{kq} = \varsigma_{kq} e^{i\vartheta_{kq}}$ (k, h = 1, 2, ..., n) corresponding to the choice P_k and obtain the complex fuzzy element (CFE) d_k of the choice P_k over all other choices by using Complex Dombi Fuzzy operator.

$$d_k = CDFoperator(d_{k1}, d_{k2}, \ldots, d_{kn})$$
$$d_k = \left(1 - \frac{1}{1 + [\sum_{h=1}^{n} \frac{1}{n}(\frac{\varsigma_{kq}}{1-\varsigma_{kq}})^\xi]^{1/\xi}}\right) \times e^{i2\pi\left(1 - \frac{1}{1+[\sum_{h=1}^{n}\frac{1}{n}(\frac{\frac{\vartheta_{kq}}{2\pi}}{1-\frac{\vartheta_{kq}}{2\pi}})^\xi]^{1/\xi}}\right)}.$$

TABLE 1. CFPR of experts.

Q	B_1	B_2	B_3
B_1	$0.6e^{i2\pi(0.5)}$	$0.7e^{i2\pi(0.3)}$	$0.4e^{i2\pi(0.2)}$
B_2	$0.5e^{i2\pi(0.6)}$	$0.6e^{i2\pi(0.5)}$	$0.3e^{i2\pi(0.4)}$
B_3	$0.7e^{i2\pi(0.8)}$	$0.5e^{i2\pi(0.6)}$	$0.6e^{i2\pi(0.5)}$

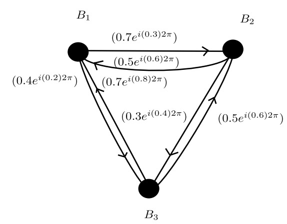

FIGURE 6. CDFG directed network.

3. The formula of score functions is given by:

$$s(d_k) = \varsigma + \frac{1}{2\pi}\vartheta$$

4. Compute the score function $s(d_k)$ of the combined overall preference value d_k (k = 1, 2, ..., n) by using the formula of score function.

5. Rank all the choices P_k(k = 1, 2, . . . , n) on the basis of score function $s(d_k)$ (k = 1, 2, ..., n).

6. Output the appropriate option based on the score functions derived in step 4 of the procedure.

B. SELECTION OF SUITABLE PLACE TO ESTABLISH AN INTERNET OFFICE

Telecommunication plays an important role in the developmental level of any country. Developed countries have strong telecommunication system. A variety of factors like social interaction, employee, economic growth, job creation, business productivity are all dependent on telecommunication system. There are many ways of telecommunication including smart phones features, skype, whatsapp, imo, snapchat, facebook etc. It plays an important role in globalization. The telecommunication through these softwares is dependent on high speed internet. A private internet company decided to build their office in a city for the convenience of its service to the public. They decided three place in a city P_k(k=1,2,3). The company make a pairwise comparison in these three places to build an internet office. Following are some parameters that are to be observed.

- a desirable location to open an office.
- Any internet office.
- Available resources.
- Expenditures and outcomes.
- Facilitation for public.

The specialists of company give their preference information in the form of CFPR $Q = (d_{kq})_{3\times3}$ as shown in Table 1, where $d_{kq} = \varsigma_{kq} e^{i\vartheta_{kq}}$ is a complex fuzzy element (CFE) preferred by the expert. Consider $0.8e^{i2\pi(0.7)}$, For the value 0.8, the amplitude term shows that eighty percent of the specialist says P_1 is best choice to establish an office over place P_2. Now the phase term 0.7 represents that seventy percent of the specialists conclude that P_1 location will create more time profit for the company over location P_2. The CFPR $Q = (d_{kq})_{3\times 3}$ is given in Table 1.

The directed network of CFPR Q represented in Table 1 and is shown as in Figure 6.

To evaluate $d_{kq} = \varsigma_{kq} e^{i\vartheta_{kq}}$ (k,h = 1,2,3) of the place P_k over all other places, we use complex dombi fuzzy operator (CDFO). We have taken $\xi = 1$. The combined overall preference value d_k (k=1,2,3) follows:

$$d_1 = 0.6e^{i2\pi(0.0533)}$$
$$d_2 = 0.4939e^{i2\pi(0.0796)}$$
$$d_3 = 0.617e^{i2\pi(0.1012)}$$

The score function $s(d_k)$ (k=1,2,3) is calculated by using $s(d_k) = \varsigma + \frac{1}{2\pi}\vartheta$ which is given below:

$$s(d_1) = 0.6085$$
$$s(d_2) = 0.5065$$
$$s(d_3) = 0.6331$$

We get the ranking order of the four terminals P_k from the score functions as follows:

$$P_3 \succ P_1 \succ P_2$$

The ranking leads to the conclusion that P_3 is best place to establish an internet office.

V. CONCLUSION

In order to represent information visually, graphs are quite useful. They are also used to model interactions between different objects. Graphical models can be found everywhere, for example, in manufacturing, communications network diagnosis, and a variety of social, biological, and physical systems, among other applications. They are extremely important since they play a critical role in changing the data received from diverse sources in order to establish the outcomes of decision-making difficulties. In this paper, the idea of CDFG is introduced. A CDFG is extension of DFG. The flexibility and comparability of CDFG is much higher. The concept of complement of CDFG is defined. The concepts of homomorphism, isomorphism, weak isomorphism, and co-weak isomorphism of two CDFGs are discussed in details with different results. Regular and totally regular CDFGs are discussed. At the end, we wrote the application of CDFG. In the future work, we will describe some operation on CDFG. Energy of CDFG may be discuss.

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COMPETING INTERESTS

All authors are here with confirm that there are no competing interests between them.

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