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RESEARCH ARTICLE

Gaussian Filtering for Simultaneously Occurring Delayed and Missing Measurements

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ABSTRACT Approximate filtering algorithms in nonlinear systems assume Gaussian prior and predictive density and remain popular due to ease of implementation as well as acceptable performance. However, these algorithms are restricted by two major assumptions: they assume no missing or delayed measurements. However, practical measurements are frequently delayed and intermittently missing. In this paper, we introduce a new extension of the Gaussian filtering to handle the simultaneous occurrence of the delay in measurements and intermittently missing measurements. Our proposed algorithm uses a novel modified measurement model to incorporate the possibility of the delayed and intermittently missing measurements. Subsequently, it redesigns the traditional Gaussian filtering for the modified measurement model. Our algorithm is a generalized extension of the Gaussian filtering, which applies to any of the traditional Gaussian filters, such as the extended Kalman filter (EKF), unscented Kalman filter (UKF), and cubature Kalman filter (CKF). A further contribution of this paper is that we study the stochastic stability of the proposed method for its EKF-based formulation. We compared the performance of the proposed filtering method with the traditional Gaussian filtering (particularly the CKF) and three extensions of the traditional Gaussian filtering that are designed to handle the delayed and missing measurements individually or simultaneously.

INDEX TERMS Delayed measurements, Gaussian filtering, missing measurements, nonlinear Bayesian filtering.

I. INTRODUCTION

Modern scientific tools and technologies often involve sensors that give noisy data [1], [2]. In non-engineering applications, survey and experimental data are often noisy [3]. Very often, noisy data are used for determining the hidden states of a system (or process). A popular mathematical tool to handle this problem is estimation, and a recursive process of estimation is known as filtering [1], [2]. Some crucial scientific and engineering domains involving the applications of estimation and filtering are target tracking [4], biomedical modeling [5], industrial diagnosis [6], weather forecasting [3], and forecasting prices of financial derivatives [7].

A commonly accepted filtering framework is the Bayesian framework that gives a probabilistic solution in terms of the prior and posterior probability density functions (PDF). The popularly known Kalman filter [8] is an optimal linear Bayesian filter. However, no optimal nonlinear Bayesian filter in closed form is reported in the literature till today, and the practitioners rely on approximated methods. The literature witnesses two popular approximated nonlinear filtering methodologies, known as Gaussian filtering [9], [10] and particle filtering (PF) [9], [11]. Particle filters require specialized hardware in real time applications due to their high computational complexity. We focus on Gaussian filters, i.e., filters which assume Gaussian prior and posterior densities in this paper, which are far easier to implement, are very common in engineering

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applications and can also work as a proposal stage for particle filters.

Gaussian filtering [9], [10] approximates the prior and posterior PDFs as Gaussian and characterizes them with the corresponding means and covariances. However, the computation of mean and covariance involves integrals which are unavailable in close form and need to be numerically approximated during filtering. Some of the popular Gaussian filters are the extended Kalman filter (EKF) [1], the unscented Kalman filter (UKF) [12], the cubature Kalman filter (CKF) [13], the cubature quadrature Kalman filter (CQKF) [14], the Gauss-Hermite filter (GHF) [15], and the exponentially-fitted CKF (ECKF) [16]. The Gaussian filters are sub-optimal, mainly due to the Gaussian approximation of arbitrary prior and posterior PDFs. However, they are often accurate enough for practical applications under lenient environments. Here, the term 'lenient environment' imposes two major restrictions: i) the measurements are not delayed in time and ii) definite availability of measurements is ensured at every sampling instant. This presents difficulties if practical measurements are often time-delayed and intermittently missing.

It is worth mentioning that a marginal delay in measurements is inherent due to the response time of the measuring devices. Moreover, the delay may further increase and become large due to various reasons, such as network delay [17] and propagation delay [18]. In such cases, ignoring the measurement time delay may have a serious impact on the state estimation accuracy. The early literature on filtering with delayed measurements [19], [20], [21] extended the traditional Gaussian filters for delays up to one or two sampling intervals. Later, [22] extended the traditional Gaussian filtering for larger delays, considering that the delay statistics are known. This method is further extended in [23] for unknown delay statistics.

Similar to the delay, there are various practical reasons for intermittently missing measurements. A predominant reason is the inefficient network and communication channels used for measurement data transmission [24]. Some other common reasons, such as the use of time-sharing sensors and the temporal sensor failures, are reported in [25], [26], and [27]. Besides these possible reasons, the measurements may become unidentifiable in heavy clutter environment, which is also addressed as a missing measurement [28].

In an early development on filtering with missing measurements, [26] modified the traditional EKF for partial missing measurements, assuming that a fraction of each measurement is available. Later, [29], [30] investigated the UKF for nonlinear stochastic systems with missing measurements. In another development, [31] introduced a distributed filtering method for saturated systems. However, its design aspect is limited to linear dynamical models with certain modifications to handle the nonlinearity involved due to the saturation.

The above discussed filters [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31] are designed to handle either the delayed or the missing measurements. The literature [18], [32], [33], [34] on filtering with simultaneously occurring delayed and missing measurements witnesses some developments for different classes of systems. For example, [32] considered this problem in coupled neural networks and developed an estimation method. Similarly, [18] develops a filtering algorithm for the delayed and missing measurements. In this regard, it introduces separate stochastic models for incorporating the delay and missing measurements possibilities. Subsequently, it introduces a Ricatti-like equation for designing Kalman filtering method. However, [18] restricts the delay up to one sampling interval, wherein the practical delays can often be larger. Furthermore, [33] introduced unbiased finite impulse response-based filtering approach for finite-horizon case, considering the presence of delayed and missing measurements. However, it assumes that the delay is time-stamped, while delays without time-stamping are observed in many practical systems [35], [36]. Finally, [34] introduces three new designs of robust linear Kalman filtering for handling the simultaneously occurring delayed and missing measurements. However, in every design, it restricts the delay as one sampling interval, which can lead to poor accuracy if the real measurement delay is higher. Moreover, [18], [34] rely on the augmented state-space approaches, which may increase the computational complexity significantly.

Although the above-discussed filtering methods [18], [33], [34] can handle the simultaneously occurring delayed and missing measurements, they have two broad limitations: i) they are designed for linear dynamical systems, ii) they fail to handle delays of more than one sampling interval without time-stamping. For the nonlinear dynamical systems, [37] is probably the only existing method to handle the simultaneously occurring delayed and missing measurements. It integrates a likelihood-based technique with the nonlinear Gaussian filtering framework for handling the simultaneously occurring delayed and missing measurements. However, [37] uses an augmented state transition equation, with the size of the system growing by a factor of d, where d the maximum number of delays. The authors also use Gaussian mixture with d components as a likelihood function, and mention computational issues with large number of components in the likelihood function. Indeed, the increased size of the covariance matrices itself can present significant computational issues in multidimensional integration involved.

In this paper, we develop a new extension of Gaussian filtering to handle the simultaneously occurring delayed and missing measurements for nonlinear dynamical systems. In this regard, we reformulate the measurement model using a set of Bernoulli random variables to incorporate the possibilities of delayed and missing measurements. Subsequently, we re-derive the Gaussian filtering method for the modified measurement model. It should be mentioned that we modify only the measurement model. Consequently, in the redesigned Gaussian filtering, only the filtering parameters related to the measurements, such as the measurement estimate, the measurement covariance, and the state-measurement cross-covariance, are re-derived. It is worth mentioning that our contribution is on developing a generalized Gaussian filtering methodology for the problem of delayed as well as missing measurements. Thus, it can be used for extending any of the existing Gaussian filters, such as the EKF, the UKF, and the CKF, for handling the simultaneous occurrence of delayed and missing measurements. We study the stability of the proposed method by formulating it under the EKF-based design strategy. We test the performance of the proposed method under CKF-based formulation due to its popularity for high accuracy at a low computational cost. The performance analysis reveals an improved accuracy for the proposed method compared to the traditional Gaussian filtering method and its extensions in presence of delayed and missing measurements.

In view of the above discussion, we highlight the main contributions of the manuscript as follows:

- We introduce a stochastically formulated measurement model that incorporates the possibility of simultaneously occurring delayed and missing measurements.
- We redesign the traditional Gaussian filtering for the modified measurement model to handle the simultaneous occurrences of the delayed and missing measurements.
- We consider arbitrarily large delays without time-stamping for nonlinear systems, whereas the existing filters such as those reported in [34] and [18] (without time-stamping) and [33] (with time-stamping) address the delayed measurements only for linear systems. Moreover, our algorithm, in contrast to [37], avoids computationally expensive state augmentation and instead relies on analytical expressions for the necessary conditional moments (which are additive in the number of maximum delays).
- We study the stochastic stability of the proposed Gaussian filtering structure for the EKF-based formulation.
- We validate the performance of the proposed Gaussian filtering methodology by two comprehensive simulation examples.

The remaining part of the paper is organized as follows. In Section II, we mathematically formulate the problem of simultaneously occurring delayed and missing measurements, which is followed by the explanation of the proposed methodology in Section III. In Section IV, the stochastic stability of the modified Gaussian filter under the EKF-based formulation is performed. The simulation results are presented in Section V, and finally the discussion and conclusion are highlighted in Section VI.

II. PROBLEM FORMULATION

Our problem is to develop an advanced Gaussian filtering methodology to handle the simultaneous occurrence of delayed and missing measurements. The standard representation of the state-space model in a lenient environment (defined in the previous section) is as follows

$$\mathbf{x}_k = f_k(\mathbf{x}_{k-1}, k-1) + \boldsymbol{\eta}_k \tag{1}$$
$$\mathbf{z}_k = h_k(\mathbf{x}_k, k) + \boldsymbol{\nu}_k, \tag{2}$$

where $\mathbf{x}_k \in \mathbb{R}^n$ and $\mathbf{z}_k \in \mathbb{R}^q$ are state and measurement variables, respectively, at k^{th} sampling instant, $k \in \{1, 2, \dots, T_s\}$ with T_s representing the number of sampling intervals. Moreover, $f_k : \mathbf{x}_{k-1} \to \mathbf{x}_k$ and $h_k : \mathbf{x}_k \to \mathbf{z}_k$ are general nonlinear functions. Finally, $\boldsymbol{\eta}_k$ and $\boldsymbol{\nu}_k$ are zero-mean Gaussian noises representing the process and measurement noises, respectively. The covariances of $\boldsymbol{\eta}_k$ and $\boldsymbol{\nu}_k$ are denoted as $\mathbf{Q}_k$ and $\mathbf{R}_k$, respectively.

Following our problem statement, we need to reformulate the measurement model (Eq. (2)) to address the simultaneous occurrence of randomly delayed and randomly missing measurements. Our reformulation of the measurement model is based on two sets of Bernoulli random variables, denoted by α and Θ: α corresponds to the missing measurements and Θ corresponds to the delayed measurements.

The measurements are generally received from multiple sources, and they all may not be missing at the same time. Thus, we consider that the measurement at any sampling instant may be partly missing, i.e., specific elements of the measurement may be missing at any particular instant. Thus, we define a matrix of Bernoulli random variables, $\boldsymbol{\lambda}_k = \text{diag}\{\alpha_k^1, \alpha_k^2, \dots, \alpha_k^q\}$ with α_k^i $\forall i \in \{1, 2, \dots, q\}$ being q equiprobable Bernoulli random variables and $E[\boldsymbol{\lambda}_k] = \boldsymbol{\mu}_k = \text{diag}\{\mu_k^1, \mu_k^2, \dots, \mu_k^q\}$. It should be mentioned that α_k^i is either 0 or 1, with $\alpha_k^i = 0$ representing that the i^{th} element of the received measurement $\mathbf{y}_k$, denoted as $\mathbf{y}_k(i)$, is missing.

For modeling the delay portion, we restrict the maximum delay to N_{max}. N_{max} is the practitioner's choice and it can be assigned with a fairly large value if the expected delay is large. Therefore, our model and the proposed filtering technique should not be deemed to be restricted to small delays. We define $N_{max} + 1$ equiprobable Bernoulli random variables: one for each of the current and the N_{max} possible delayed instants. At k^{th} instant, we denote them as Θ_k^j $\forall j \in \{1, 2, \dots, N_{max} + 1\}$ with $\text{P}(\Theta_k^j = 1) = E[\Theta_k^j] = \delta_d$. Note that Θ_k^{j+1} corresponds to j^{th} delayed instant. We assign $\Theta_k^0 = 0$, and model the actual measurement as

$$\mathbf{y}_k = \boldsymbol{\lambda}_k \Big[(1 - \Theta_k^0)\Theta_k^1 \mathbf{z}_k + (1 - \Theta_k^0)(1 - \Theta_k^1)\Theta_k^2 \mathbf{z}_{k-1} + \dots + (1 - \Theta_k^0)(1 - \Theta_k^1)\dots(1 - \Theta_k^{(N_{max})}) \times \Theta_k^{N_{max}+1} \mathbf{z}_{k-N_{max}} \Big]. \tag{3}$$

The coefficients of $\mathbf{z}_{k-m}$ $\forall m \in \{1, 2, \dots, N_{max}\}$ govern the delay extent. For example, if the measurement is one time-step delayed, i.e., $\mathbf{y}_k = \mathbf{z}_{k-1}$, then coefficient of $\mathbf{z}_{k-1}$, i.e., $(1 - \Theta_k^0)(1 - \Theta_k^1)\Theta_k^2$ takes the value one, while the random variables associated with $\mathbf{z}_{k-m}$ $\forall m \neq 1$ remain zero. At the same time, $\boldsymbol{\lambda}_k$ regulates the missing measurement possibility. The diagonal elements of

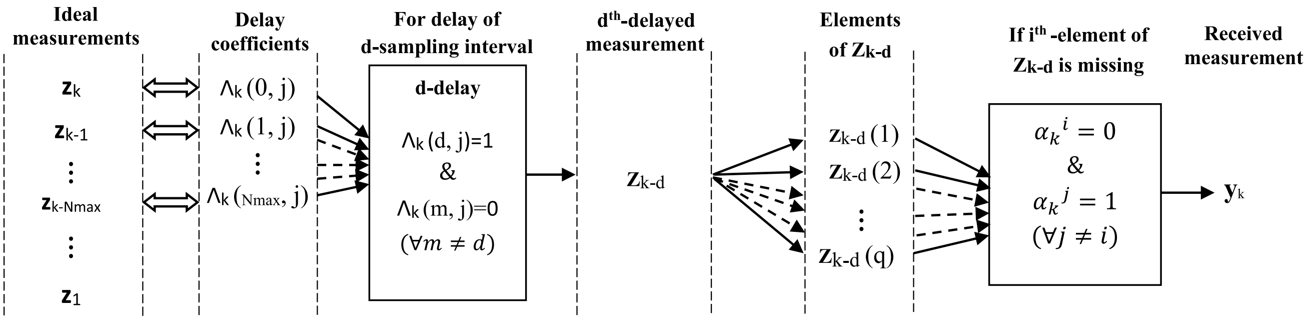


FIGURE 1. Pictorial diagram representing the sequence to be followed to obtain the received measurement y_k from the ideal $z_1, z_2, \ldots, z_k$ that would have been received in the lenient environment.

$\boldsymbol{\lambda}_k$ are Bernoulli random variables, which take the values zero or one. The value one ensures that the measurement is received, while the value zero indicates that the measurement is lost.

To this end, let us simplify the notation for the coefficients of $\mathbf{z}_{k-m}$ as

$$\Lambda_k(m, j) = \left(\prod_{j=0}^{m}(1-\Theta_k^j)\right)\Theta_k^{(m+1)}, \tag{4}$$

so that the received measurement is m-step delayed if $\Lambda_k(m, j) = 1$, which means $\Theta_k^{m+1} = 1$ and $\Theta_k^j = 0\ \forall j \leq m$. Subsequently, Eq. (3) can be represented as

$$\mathbf{y}_k = \boldsymbol{\lambda}_k\Big[\Lambda_k(0, j)\mathbf{z}_k + \Lambda_k(1, j)\mathbf{z}_{k-1} + \cdots + \Lambda_k(N_{max}, j)\mathbf{z}_{k-N_{max}}\Big]. \tag{5}$$

Thus, the measurement model can be finally given as

$$\mathbf{y}_k = \boldsymbol{\lambda}_k \sum_{m=0}^{N_{max}} \Lambda_k(m, j)\mathbf{z}_{k-m}, \tag{6}$$

where $\mathbf{y}_k$ is the actual received measurement due to delay and missing possibilities. At this end, $\mathbf{z}_k$ may be considered as an ideal measurement that might have been received in the lenient environment. A pictorial diagram representing the sequence to be followed to get $\mathbf{y}_k$ from $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k$ is shown in Fig. 1.

It is assumed that α_k^i and Θ_k^j are independent random variables $\forall\{k, i, j\}$. Furthermore, α_k^j and Θ_k^j are independent of α_k^i and Θ_k^i, respectively for $j \neq i$. Our objective in the next section is to redesign the Gaussian filtering method for the state-space model represented by Eqs. (1) and (6) so that the possibilities of delayed and missing measurements are incorporated.

The above discussions emphasize the importance of the measurement model (6) for developing the proposed filtering algorithm. As mentioned in the previous section, some of the existing filters, such as [19], [20], [21], [22], and [26], also formulated similar measurement models. However, the models in [19], [20], [21], and [22] characterize the delay possibilities only, while the same in [26] characterizes only the missing measurement possibility. Moreover, [19], [20], [21] characterize only limited and small delays, while [26] characterizes fractionally available measurements instead of being completely missing. Thus, they fail to characterize the general practical scenarios of the simultaneously occurring delayed and missing measurements. Our measurement model in Eq. (6) efficiently characterizes the simultaneously occurring delayed and missing measurement possibilities. It considers any large delays and completely missing measurements unlike [19], [20], [21], and [26], respectively.

Remark 1: Considering the above-discussed competency of our measurement model, the Gaussian filtering algorithm designed for this model should accomplish an improved accuracy for simultaneously occurring delayed and missing measurements.

III. MODIFIED GAUSSIAN FILTERING FOR DELAYED AND MISSING MEASUREMENTS

The traditional Gaussian filtering is designed with respect to the measurement $\mathbf{z}$, modeled in Eq. (2). In this section, we derive the necessary modifications to the algorithm to deal with the modified measurement $\mathbf{y}$, modeled in Eq. (6). As the measurement model is changed, we re-derive all the related expressions in the traditional Gaussian filtering to propose the advanced Gaussian filtering for $\mathbf{y}$. The traditional Gaussian filtering uses only three such expressions, namely the measurement estimate $\hat{\mathbf{z}}$, measurement error covariance $\mathbf{P}^{\mathbf{zz}}$, and the cross covariance $\mathbf{P}^{\mathbf{xz}}$, derived for $\mathbf{z}$. We re-derive all the measurement related expressions in the Gaussian filtering algorithm for the modified measurement model above. On a different note, it should be mentioned that the state dynamics remains unaffected from the simultaneous occurrence of the delayed and missing measurements. Therefore, the time update step of the proposed filtering technique remains the same as the traditional Gaussian filtering [9], [10].

A. MODIFIED GAUSSIAN FILTER

In this part, we re-derive the measurement parameters, such as $\hat{\mathbf{y}}$, $\mathbf{P}^{\mathbf{yy}}$, and $\mathbf{P}^{\mathbf{xy}}$. Before proceeding to the derivation, it should be mentioned that only one of $\Lambda_k(m, j)$ $\forall m$ is one and the others are zero at any instant t_k to ensure that only one measurement is received. Although this consideration violates the independence of $\Lambda_k(m, j)$ for different m (i.e., different delay), they will be assumed to be statistically independent in our derivation. We now derive the expressions of $\hat{\mathbf{y}}$, $\mathbf{P}^{\mathbf{yy}}$, and $\mathbf{P}^{\mathbf{xy}}$.

1) MEASUREMENT ESTIMATE FOR $\mathbf{y}_k$

For $\mathbf{y}_k$ given in Eq. (6), the measurement estimate is

$$\hat{\mathbf{y}}_{k|k-1} = E[\mathbf{y}_k] = E\left[\left(\sum_{m=0}^{N_{max}} \boldsymbol{\lambda}_k \Lambda_k(m, j)\mathbf{z}_{k-m}\right)\right]. \quad (7)$$

As the missing and delay occurrences are mutually independent events, $\boldsymbol{\lambda}_k$ and $\Lambda_k(m, j)$ are statistically independent. Moreover, $\boldsymbol{\lambda}_k$ and $\Lambda_k(m, j)$ are independent of the measurement value $\mathbf{z}_k$ also. Thus, we simplify the above equation as

$$\hat{\mathbf{y}}_{k|k-1} = \sum_{m=0}^{N_{max}} E\left[\boldsymbol{\lambda}_k\right] E\left[\Lambda_k(m, j)\right] E\left[\mathbf{z}_{k-m}\right]. \quad (8)$$

Following our previous notations, $E\left[\mathbf{z}_{k-m}\right] = \hat{\mathbf{z}}_{k-m|k-1}$. Recalling the previous discussion, we get

$$E\left[\Lambda_k(m, j)\right] = E\left[\left(\prod_{j=0}^{m}(1-\Theta_k^j)\right)\Theta_k^{(m+1)}\right] = (1-\delta_d)^m\delta_d. \quad (9)$$

Substituting $E\left[\mathbf{z}_{k-m}\right]$, $E\left[\boldsymbol{\lambda}_k\right]$, and $E\left[\Lambda_k(m, j)\right]$ in Eq. (8), we get

$$\hat{\mathbf{y}}_{k|k-1} = \sum_{m=0}^{N_{max}} \boldsymbol{\mu}_k(1-\delta_d)^m\delta_d\hat{\mathbf{z}}_{k-m|k-1}. \quad (10)$$

2) MEASUREMENT ERROR COVARIANCE FOR $\mathbf{y}_k$

The measurement error covariance is

$$\mathbf{P}^{\mathbf{yy}}_{k|k-1} = E\left[\left(\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}\right)\left(\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}\right)^T\right]. \quad (11)$$

From Eqs. (6) and (10), we get

$$\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1} = \sum_{m=0}^{N_{max}} \boldsymbol{\lambda}_k \Lambda_k(m, j)\mathbf{z}_{k-m} - \sum_{m=0}^{N_{max}} \boldsymbol{\mu}_k(1-\delta_d)^m\delta_d\hat{\mathbf{z}}_{k-m|k-1}. \quad (12)$$

We can rewrite this expression as

$$\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1} = \underbrace{\sum_{m=0}^{N_{max}} \boldsymbol{\lambda}_k \Lambda_k(m, j)(\mathbf{z}_{k-m} - \hat{\mathbf{z}}_{k-m|k-1})}_{\mathrm{A}_1} + \underbrace{\sum_{m=0}^{N_{max}} \left(\boldsymbol{\lambda}_k \Lambda_k(m, j) - \boldsymbol{\mu}_k(1-\delta_d)^m\delta_d\right)\hat{\mathbf{z}}_{k-m|k-1}}_{\mathrm{A}_2}. \quad (13)$$

From Eqs. (11) and (13), we can write

$$\mathbf{P}^{\mathbf{yy}}_{k|k-1} = E[\mathrm{A}_1\mathrm{A}_1^T] + E[\mathrm{A}_1\mathrm{A}_2^T] + E[\mathrm{A}_2\mathrm{A}_1^T] + E[\mathrm{A}_2\mathrm{A}_2^T]. \quad (14)$$

We can now compute every expectation term individually for A_1 and A_2 defined in Eq. (13) and add them to obtain $\mathbf{P}^{\mathbf{yy}}_{k|k-1}$. In this regard, for A_1 given in Eq. (13), we get

$$E[\mathrm{A}_1\mathrm{A}_1^T] = \sum_{m=0}^{N_{max}} E\left[\boldsymbol{\lambda}_k^2\right] E\left[\Lambda_k^2(m, j)\right] \times E\left[(\mathbf{z}_{k-m} - \hat{\mathbf{z}}_{k-m|k-1})(\mathbf{z}_{k-m} - \hat{\mathbf{z}}_{k-m|k-1})^T\right]. \quad (15)$$

Following previous discussions, $E[\boldsymbol{\lambda}_k^2] = E[\boldsymbol{\lambda}_k] = \boldsymbol{\mu}_k$, $E[\Lambda_k^2(m, j)] = E[(\Lambda_k(m, j))] = (1-\delta_d)^m\delta_d$, and $E[(\mathbf{z}_{k-m} - \hat{\mathbf{z}}_{k-m|k-1})(\mathbf{z}_{k-m} - \hat{\mathbf{z}}_{k-m|k-1})^T] = \mathbf{P}^{\mathbf{zz}}_{k-m|k-1}$. Thus, Eq. (15) can be simplified as

$$E[\mathrm{A}_1\mathrm{A}_1^T] = \sum_{m=0}^{N_{max}} \boldsymbol{\mu}_k(1-\delta_d)^m\delta_d\mathbf{P}^{\mathbf{zz}}_{k-m|k-1}. \quad (16)$$

Similarly, for A_1 and A_2 given in Eq. (13), we can write

$$E[\mathrm{A}_1\mathrm{A}_2^T] = E\left[\left(\sum_{s=0}^{N_{max}} \boldsymbol{\lambda}_k \Lambda_k(s, j)(\mathbf{z}_{k-s} - \hat{\mathbf{z}}_{k-s|k-1})\right) \times \left(\sum_{t=0}^{N_{max}} \left(\boldsymbol{\lambda}_k \Lambda_k(t, j) - \boldsymbol{\mu}_k(1-\delta_d)^t\delta_d\right)\hat{\mathbf{z}}_{k-t|k-1}\right)^T\right].$$

After further simplification, we get

$$E[\mathrm{A}_1\mathrm{A}_2^T] = \sum_{s=0}^{N_{max}}\sum_{t=0}^{N_{max}} E\Big[\boldsymbol{\lambda}_k \Lambda_k(s, j)\big(\boldsymbol{\lambda}_k \Lambda_k(t, j) - \boldsymbol{\mu}_k(1-\delta_d)^t\delta_d\big)^T\mathbf{z}_{k-s}\hat{\mathbf{z}}^T_{k-t|k-1} - \boldsymbol{\lambda}_k \Lambda_k(s, j) \times \big(\boldsymbol{\lambda}_k \Lambda_k(t, j) - \boldsymbol{\mu}_k(1-\delta_d)^t\delta_d\big)^T\hat{\mathbf{z}}_{k-s|k-1}\hat{\mathbf{z}}^T_{k-t|k-1}\Big]. \quad (17)$$

After substituting all the expectation terms from previous discussions and considering that $E[\mathbf{z}_{k-s}] = \hat{\mathbf{z}}_{k-s|k-1}$, we get

$$E[\mathrm{A}_1\mathrm{A}_2^T] = 0. \quad (18)$$

This also leads to

$$E[\mathrm{A}_2\mathrm{A}_1^T] = 0. \quad (19)$$

Finally, for A_2 given in Eq. (13), we have

$$E[\mathrm{A}_2\mathrm{A}_2^T] = E\left[\sum_{m=0}^{N_{max}} \left(\boldsymbol{\lambda}_k \Lambda_k(m, j) - \boldsymbol{\mu}_k(1-\delta_d)^m \times \delta_d\right)^2\hat{\mathbf{z}}_{k-m|k-1}\hat{\mathbf{z}}^T_{k-m|k-1}\right]. \quad (20)$$

Applying binomial expansion and simplifying further, we get

$$E[\mathrm{A}_2\mathrm{A}_2^T] = \sum_{m=0}^{N_{max}} \Big(E[\boldsymbol{\lambda}_k^2]E[\Lambda_k^2(m,j)] + (\boldsymbol{\mu}_k\delta_d(1-\delta_d)^m)^2 - 2E[\boldsymbol{\lambda}_k]E[\Lambda_k(m,j)]\boldsymbol{\mu}_k(1-\delta_d)^m\delta_d \Big) \times \hat{\mathbf{z}}_{k-m|k-1}\hat{\mathbf{z}}_{k-m|k-1}^T. \quad (21)$$

Substituting $E[\boldsymbol{\lambda}_k^2] = \boldsymbol{\mu}_k$ and $E\left[\Lambda_k^2(m,j)\right] = (1-\delta_d)^m\delta_d$, we obtain

$$E[\mathrm{A}_2\mathrm{A}_2^T] = \sum_{m=0}^{N_{max}} \Big(\boldsymbol{\mu}_k(1-\delta_d)^m\delta_d - \big(\boldsymbol{\mu}_k(1-\delta_d)^m\delta_d\big)^2 \Big) \times \hat{\mathbf{z}}_{k-m|k-1}\hat{\mathbf{z}}_{k-m|k-1}^T. \quad (22)$$

We now substitute $E[\mathrm{A}_1\mathrm{A}_1^T]$, $E[\mathrm{A}_1\mathrm{A}_2^T]$, $E[\mathrm{A}_2\mathrm{A}_1^T]$, and $E[\mathrm{A}_2\mathrm{A}_2^T]$ from Eqs. (16), (18), (19), and (22), respectively, in Eq. (14) to obtain $\mathbf{P}_{k|k-1}^{\mathbf{yy}}$ as

$$\mathbf{P}_{k|k-1}^{\mathbf{yy}} = \sum_{m=0}^{N_{max}} \boldsymbol{\mu}_k(1-\delta_d)^m\delta_d\mathbf{P}_{k-m|k-1}^{\mathbf{zz}} + \sum_{m=0}^{N_{max}} (\boldsymbol{\mu}_k \times (1-\delta_d)^m\delta_d - \big(\boldsymbol{\mu}_k(1-\delta_d)^m\delta_d\big)^2) \times \hat{\mathbf{z}}_{k-m|k-1}\hat{\mathbf{z}}_{k-m|k-1}^T. \quad (23)$$

3) CROSS-COVARIANCE FOR $\mathbf{y}_k$

The cross-covariance between the state and measurement is

$$\mathbf{P}_{k|k-1}^{\mathbf{xy}} = E\left[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})(\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1})^T\right]. \quad (24)$$

Substituting $\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}$ from Eq. (13), we get

$$\mathbf{P}_{k|k-1}^{\mathbf{xy}} = E\left[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})\mathrm{A}_1^T\right] + E\left[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})\mathrm{A}_2^T\right]. \quad (25)$$

For A_1 given in Eq. (13), we get

$$E\left[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})\mathrm{A}_1^T\right] = \sum_{m=0}^{N_{max}} \Big(E[\boldsymbol{\lambda}_k]E[\Lambda_k(m,j)] \times E\left[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})(\mathbf{z}_{k-m} - \hat{\mathbf{z}}_{k-m|k-1})^T\right] \Big). \quad (26)$$

Substituting $E[\boldsymbol{\lambda}_k]$ and $E[\Lambda_k(m,j)]$, we obtain

$$E\left[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})\mathrm{A}_1^T\right] = \sum_{m=0}^{N_{max}} (1-\delta_d)^m\delta_d\mathbf{P}_{k-m|k-1}^{\mathbf{xz}}\boldsymbol{\mu}_k. \quad (27)$$

Similarly, for A_2 given in Eq. (13), we get

$$E\left[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})\mathrm{A}_2^T\right] = \sum_{m=0}^{N_{max}} E\Big[(\boldsymbol{\lambda}_k\Lambda_k(m,j) - \boldsymbol{\mu}_k \times (1-\delta_d)^m\delta_d)\Big] E\left[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})\hat{\mathbf{z}}_{k-m|k-1}^T\right]. \quad (28)$$

As λ_k and $\Lambda_k(m,j)$ are independent, $E[\boldsymbol{\lambda}_k\Lambda_k(m,j)] = E[\boldsymbol{\lambda}_k]E[\Lambda_k(m,j)] = \boldsymbol{\mu}_k(1-\delta_d)^m\delta_d$. Thus, we can write

$$E\left[\left(\boldsymbol{\lambda}_k\Lambda_k(m,j) - \boldsymbol{\mu}_k(1-\delta_d)^m\delta_d\right)\right] = 0. \quad (29)$$

Substituting this into Eq. (28), we get

$$E\left[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})\mathrm{A}_2^T\right] = 0. \quad (30)$$

Substituting $E\left[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})\mathrm{A}_1^T\right]$ and $E\left[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})\mathrm{A}_2^T\right]$ from Eqs. (27) and (30), respectively, into Eq. (25), we get

$$\mathbf{P}_{k|k-1}^{\mathbf{xy}} = \sum_{m=0}^{N_{max}} \delta_d(1-\delta_d)^m\mathbf{P}_{k-m|k-1}^{\mathbf{xz}}\boldsymbol{\mu}_k. \quad (31)$$

As discussed at the beginning of this section, the proposed filtering method modifies the traditional Gaussian filtering by re-deriving the expressions of measurement estimate, measurement covariance, and state-measurement cross-covariance (Eqs. (10), (23), and (31), respectively). Please follow [9], [10] for a detailed discussion on the traditional Gaussian filtering. The proposed filtering algorithm also follows the same filtering strategy by replacing the expressions of $\hat{\mathbf{z}}$, $\mathbf{P}^{\mathbf{zz}}$, and $\mathbf{P}^{\mathbf{xz}}$ with the re-derived expressions of $\hat{\mathbf{y}}$, $\mathbf{P}^{\mathbf{yy}}$, and $\mathbf{P}^{\mathbf{xy}}$, respectively. We provide the pseudo-code for implementing the proposed filtering method in Algorithm 1.

In advancing the traditional Gaussian filtering for handling various measurement irregularities, such as the delayed and missing measurements, the major difficulty appears in incorporating those irregularities through mathematical models. The problem becomes yet more challenging if the irregularities are uncertain to appear at any particular sampling instant, as considered in this paper. We handled this problem by mathematically characterizing such irregularities, particularly the delayed and missing measurements, by formulating a stochastic model, as in Eq. (6).

Remark 2: Our measurement model utilizes a sequence of Bernoulli random variables to characterize the possibility of a measurement coming from various possible past instants. A future research problem may be to introduce a more convenient model by reducing the required number of random variables.

Remark 3: Our filter design strategy concludes that handling the measurement irregularities becomes convenient if an efficient mathematical model for characterizing the concerned irregularities is formulated.

Remark 4: The proposed method fetches some information, such as the measurement estimate, measurement covariance, and state-measurement cross-covariance, from past instants, which causes additional storage capacity requirement. Similar additional storage requirements also occur in existing delay filters, e.g., see [9].

Remark 5: The proposed filtering methodology simplifies to the traditional Gaussian filtering methodology for zero probabilities of delay and missing measurements ($\mu^i = \delta = 1$) and $N_{max} = 0$, if we use the convention $0^0 = 1$.

Algorithm 1 Pseudo-Code for Extending the Sigma-Point Based Gaussian Filters Under the Proposed Filtering Technique

Input: $\mathbf{Q}_k$, $\mathbf{R}_k$, T_s, $\boldsymbol{\mu}$, δ_d, filter-specific sigma points, and weights.
Output: $\hat{\mathbf{x}}_{k|k}$.
Initialisation: $\hat{\mathbf{x}}_{0|0}$, $\hat{\mathbf{P}}_{0|0}$, $k = 1$.
1: **while** $k \leq T_s$ **do**
2: Compute the predicted estimate and covariance of $\mathbf{x}_k$: $\hat{\mathbf{x}}_{k|k-1}$ and $\hat{\mathbf{P}}_{k|k-1}$ (see, e.g., References [9], [10]).
3: Compute the estimate and covariance of the ideal measurement ($\mathbf{z}_k$): $\hat{\mathbf{z}}_{k|k-1}$, $\mathbf{P}^{\mathbf{zz}}_{k|k-1}$ (see, e.g., References [9], [10]).
4: Compute the cross-covariance between state and ideal measurement $\mathbf{z}_k$: $\mathbf{P}^{\mathbf{xz}}_{k|k-1}$ (see, e.g., References [9], [10]).
5: Compute the estimate and covariance of the received measurement $\mathbf{y}_k$: $\hat{\mathbf{y}}_{k|k-1}$ (Eq. (10)) and $\mathbf{P}^{\mathbf{yy}}_{k|k-1}$ (Eq. (23)).
6: Compute the cross-covariance between $\mathbf{x}_k$ and received measurements $\mathbf{y}_k$: $\mathbf{P}^{\mathbf{xy}}_{k|k-1}$ (Eq. (31)).
7: Kalman gain: $\mathbf{K} = \mathbf{P}^{\mathbf{xy}}_{k|k-1}(\mathbf{P}^{\mathbf{yy}}_{k|k-1})^{-1}$.
8: Updated estimate: $\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}(\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1})$.
9: Updated covariance: $\hat{\mathbf{P}}_{k|k} = \hat{\mathbf{P}}_{k|k-1} - \mathbf{K}\mathbf{P}^{\mathbf{yy}}_{k|k-1}\mathbf{K}^T$.
10: **return** $\hat{\mathbf{x}}_{k|k}$
11: **end while**

IV. STOCHASTIC STABILITY OF MODIFIED GAUSSIAN FILTERING

In this section, we analyze the stability of the proposed method for the EKF-based formulation. It should be mentioned that the proposed filtering method is generic and applicable to any of the existing Gaussian filters, such as the EKF, UKF, and CKF. The EKF propagates the locally linearized system models during the filtering, which makes the stability analysis conveniently realizable. However, other Gaussian filters, such as the UKF and CKF, directly propagate the nonlinear systems models, which makes the stability analysis partially unrealizable with the existing theories of nonlinear dynamics. Thus, our stability analysis is limited to the EKF-based formulation of the proposed method only. In the non-linear filtering literature, it is a common practice to analyze the stability for the EKF-based formulations only [38].

In our stability analysis, we formulate a stochastic model for the estimation error of the EKF-based formulation of the proposed method. Subsequently, we show that the estimation error of the EKF-based formulation of the proposed method is exponentially bounded in a mean square if the filter, noise, and system parameters satisfy a set of presumed conditions. In the remaining part of this manuscript, the EKF-based formulation of the proposed method is abbreviated as MDEKF.

A. STOCHASTIC MODELING OF THE ESTIMATION ERROR

Here, our objective is to formulate the stochastic model representing the dynamics of the MDEKF's estimation error. Before proceeding further, we would like to introduce the time update and measurement update steps of the ordinary EKF that is designed for $\mathbf{z}$. The EKF determines the time update parameters as

$$\begin{cases} \hat{\mathbf{x}}_{k|k-1} = f(\hat{\mathbf{x}}_{k-1|k-1}) \\ \mathbf{P}_{k|k-1} = \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}^T_{k-1} + \mathbf{Q}_k, \end{cases} \tag{32}$$

where $\hat{\mathbf{x}}_{k|k-1}$ and $\mathbf{P}_{k|k-1}$ are the prior estimate and error covariance, respectively, at k^{th} sampling instant. Moreover, $\mathbf{F}_{k-1} = \frac{\partial f(\mathbf{x}_{k-1})}{\partial \mathbf{x}_{k-1}}\big|_{\mathbf{x}_{k-1}=\hat{\mathbf{x}}_{k-1|k-1}}$ represents the Jacobian matrix of $f(\mathbf{x}_{k-1})$ computed at $\hat{\mathbf{x}}_{k-1|k-1}$.

Furthermore, the EKF determines the measurement update parameters as

$$\begin{cases} \hat{\mathbf{z}}_{k|k-1} = h(\hat{\mathbf{x}}_{k|k-1}) \\ \mathbf{P}^{\mathbf{zz}}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}^T_k + \mathbf{R}_k \\ \mathbf{P}^{\mathbf{xz}}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}^T_k \\ \hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}\left(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1}\right), \end{cases} \tag{33}$$

where $\mathbf{H}_k = \frac{\partial h(\mathbf{x}_k)}{\partial \mathbf{x}_k}\big|_{\mathbf{x}_k=\hat{\mathbf{x}}_{k|k-1}}$ denotes the Jacobian of $h(\mathbf{x}_k)$ computed at $\hat{\mathbf{x}}_{k|k-1}$.

Assuming that $f(\cdot)$ is a sufficiently smooth function, let us expand $f(\mathbf{x}_k)$, using the Taylor series expansion around $\hat{\mathbf{x}}_{k|k}$, as

$$f(\mathbf{x}_k) = f(\hat{\mathbf{x}}_{k|k}) + \mathbf{F}_k\mathbf{e}_{k|k} + \Psi_f(\mathbf{x}_k, \hat{\mathbf{x}}_{k|k}), \tag{34}$$

where $\mathbf{e}_{k|k} = \mathbf{x}_k - \hat{\mathbf{x}}_{k|k}$ represents the estimation error and $\Psi_f(\mathbf{x}_k, \hat{\mathbf{x}}_{k|k})$ is the remainder term.
A similar expansion of $h(\mathbf{x}_k)$ around $\hat{\mathbf{x}}_{k|k-1}$ gives

$$h(\mathbf{x}_k) = h(\hat{\mathbf{x}}_{k|k-1}) + \mathbf{H}_k\mathbf{e}_{k|k-1} + \Psi_h(\mathbf{x}_k, \hat{\mathbf{x}}_{k|k-1}), \tag{35}$$

where $\mathbf{e}_{k|k-1} = \mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}$ is the prediction error and $\Psi_h(\mathbf{x}_k, \hat{\mathbf{x}}_{k|k-1})$ has an explanation similar to $\Psi_f(\mathbf{x}_k, \hat{\mathbf{x}}_{k|k})$. Substituting $\mathbf{x}_k$ and $\hat{\mathbf{x}}_{k|k-1}$ from Eqs. (1) and (32), respectively, we obtain

$$\mathbf{e}_{k|k-1} = f(\mathbf{x}_{k-1}) - f(\hat{\mathbf{x}}_{k-1|k-1}) + \boldsymbol{\eta}_k. \tag{36}$$

Using Eq. (34), the above equation can be written as

$$\mathbf{e}_{k|k-1} = \mathbf{F}_{k-1}\mathbf{e}_{k-1|k-1} + \Psi_f(\mathbf{x}_{k-1}, \hat{\mathbf{x}}_{k-1|k-1}) + \boldsymbol{\eta}_k. \tag{37}$$

Similarly, substituting $\hat{\mathbf{x}}_{k|k}$ from Eq. (33), but for $\mathbf{y}_k$, in $\mathbf{e}_{k|k} = \mathbf{x}_k - \hat{\mathbf{x}}_{k|k}$, we get $\mathbf{e}_{k|k} = \mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1} - \mathbf{K}(\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1})$. Then, substituting $\mathbf{x}_k$, $\mathbf{y}_k$, $\hat{\mathbf{y}}_{k|k-1}$, and $\hat{\mathbf{x}}_{k|k-1}$ from Eqs. (1), (6), (10), and (32), respectively, we get

$$\mathbf{e}_{k|k} = f(\mathbf{x}_{k-1}) - f(\hat{\mathbf{x}}_{k-1|k-1}) + \boldsymbol{\eta}_k - \mathbf{K}\bigg(\lambda_k \sum_{m=0}^{N_{max}} \Lambda_k(m, j) \times \mathbf{z}_{k-m} - \boldsymbol{\mu}_k \sum_{m=0}^{N_{max}} \delta_d(1-\delta_d)^m \hat{\mathbf{z}}_{k-m|k-1}\bigg).$$

Let us do the following substitutions: i) $\mathbf{z}_{k-m} = h(\mathbf{x}_{k-m}) + \boldsymbol{\nu}_{k-m}$, ii) $\hat{\mathbf{z}}_{k-m|k-1} = h(\hat{\mathbf{x}}_{k-m|k-1})$, and iii) $f(\mathbf{x}_k)$ from Eq. (34). In the resulting expression, let us substitute $h(\mathbf{x}_{k-m})$ expanded by Eq. (35). Subsequently, we obtain

$$\begin{aligned}\mathbf{e}_{k|k} &= \mathbf{F}_{k-1}\mathbf{e}_{k-1|k-1} + \Psi_f(\mathbf{x}_{k-1}, \hat{\mathbf{x}}_{k-1|k-1}) + \boldsymbol{\eta}_k - \mathbf{K}\boldsymbol{\lambda}_k \\ &\quad \times \sum_{m=0}^{N_{max}} \Lambda_k(m,j)\boldsymbol{\nu}_{k-m} \\ &\quad - \mathbf{K}\Bigg[\sum_{m=0}^{N_{max}} (\boldsymbol{\lambda}_k\Lambda_k(m,j) - \boldsymbol{\mu}_k\delta_d(1-\delta_d)^m) \\ &\quad \times \big(h(\hat{\mathbf{x}}_{k-m|k-1}) + \mathbf{H}_{k-m}\mathbf{e}_{k-m|k-1}\big) + \boldsymbol{\lambda}_k \sum_{m=0}^{N_{max}} \Lambda_k(m,j) \\ &\quad \times \Psi_h(\mathbf{x}_{k-m}, \hat{\mathbf{x}}_{k-m|k-1}) \\ &\quad + \sum_{m=0}^{N_{max}} \boldsymbol{\mu}_k\delta_d(1-\delta_d)^m\mathbf{H}_{k-m}\mathbf{e}_{k-m|k-1}\Bigg].\end{aligned}$$

We expand $\mathbf{e}_{k-m|k-1}$ using Eq. (37) and substitute the expanded $\mathbf{e}_{k-m|k-1}$ in the above equation. Subsequently, after some rearrangements, we obtain the desired stochastic model of the error dynamics in the form of Eq. (38), as shown at the bottom of the page, where $\mathbf{I}$ represents an identity matrix of the appropriate dimension.

It is worth mentioning that Eq. (38) represents the error dynamics in terms of stochastic difference equation. Interestingly, the state of this difference equation is the error of the MDEKF. Thus, we can consider Eq. (38) as a time-series representation of a hypothetical system with the state being the error of the MDEKF. Subsequently, the stability of this hypothetical system ensures the stability of the MDEKF.

To this end, it should be mentioned that the literature contains several notions of stability for nonlinear systems. The readers may please refer to [39] for a detailed discussion. In this paper, we particularly use exponential stability, where the stability is ensured if the system's convergence is bounded with an exponential envelope.

B. EXPONENTIAL STABILITY

Before proceeding forward to prove that the error dynamics presented in Eq. (38) is exponentially bounded, we mathematically define the exponentially bounded process, as follows.

Definition 1: Let us consider that $\mathbf{e}_k$ denotes a stochastic process and $\kappa' > 0$, $\xi > 0$, and $0 < \beta < 1$ are real numbers. Then, $\mathbf{e}_k$ is said to be exponentially bounded in mean square if it satisfies

$$E\left[\|\mathbf{e}_k\|^2\right] \leq \beta^k\kappa' E\left[\|\mathbf{e}_0\|^2\right] + \xi \quad \forall k \in \{1, 2, \dots\}, \tag{39}$$

where $\|\cdot\|$ represents the spectral norm for matrices and Euclidean norm for vectors.

It should be mentioned that the above definition is general and does not limit $\mathbf{e}_k$ to be the estimation error only. However, our notations and discussions will be focused on the estimation error.

In our stability analysis, we approach Eq. (39) in a different way. In this regard, please refer to the subsequent discussion.

Remark 6: Let us consider that $\tau_1 > 0$, $\tau_2 > 0$, $\gamma' > 0$, and $0 < \phi < 1$ denote real numbers, and $V(\mathbf{e}_k)$ represents a scalar-valued stochastic process, which satisfies

$$\tau_1 \|\mathbf{e}_k\|^2 \leq V(\mathbf{e}_k) \leq \tau_2 \|\mathbf{e}_k\|^2 \tag{40}$$

and

$$E\left[V(\mathbf{e}_k)|\mathbf{e}_{k-1}\right] - V(\mathbf{e}_{k-1}) \leq \gamma' - \phi V(\mathbf{e}_{k-1}) \leq 0. \tag{41}$$

Then, the stochastic process $\mathbf{e}_k$ satisfies

$$E\left[\|\mathbf{e}_k\|^2\right] \leq \frac{\tau_2}{\tau_1}E\left[\|\mathbf{e}_0\|^2\right](1-\phi)^k + \frac{\gamma'}{\tau_1}\sum_{i=0}^{k-1}(1-\phi)^i. \tag{42}$$

Please refer to [38] and [40], for a detailed discussion.

In the subsequent discussion, we conclude that this remark is another way of defining the exponential bound in mean square.

Since $\sum_{i=0}^{k-1}(1-\phi)^i \leq \sum_{i=0}^{\infty}(1-\phi)^i = 1/\phi$, Eq. (42) can be written as

$$E\left[\|\mathbf{e}_k\|^2\right] \leq \frac{\tau_2}{\tau_1}E\left[\|\mathbf{e}_0\|^2\right](1-\phi)^k + \frac{\gamma'}{\tau_1\phi}. \tag{43}$$

$$\begin{aligned}\mathbf{e}_{k|k} &= \underbrace{\big(\mathbf{I} - \delta_d\mathbf{K}\boldsymbol{\mu}_k\mathbf{H}_k\big)\mathbf{F}_{k-1}}_{\mathrm{A}_k}\mathbf{e}_{k-1|k-1} + \underbrace{\boldsymbol{\eta}_k - \mathbf{K}\sum_{m=0}^{N_{max}}\big(\boldsymbol{\mu}_k\delta_d(1-\delta_d)^m\mathbf{H}_{k-m}\boldsymbol{\eta}_{k-m} + \boldsymbol{\lambda}_k\Lambda_k(m,j)\boldsymbol{\nu}_{k-m}\big)}_{\mathrm{C}_k} \\ &\quad \underbrace{+\, \Psi_f(\mathbf{x}_{k-1}, \hat{\mathbf{x}}_{k-1|k-1}) - \mathbf{K}\sum_{m=0}^{N_{max}}\big(\boldsymbol{\lambda}_k\Lambda_k(m,j)\Psi_h(\mathbf{x}_{k-m}, \hat{\mathbf{x}}_{k-m|k-1}) + \boldsymbol{\mu}_k\delta_d(1-\delta_d)^m\mathbf{H}_{k-m}\Psi_f(\mathbf{x}_{k-1-m}, \hat{\mathbf{x}}_{k-1-m|k-1})\big)}_{\mathrm{B}_k} \\ &\quad \times \underbrace{-\mathbf{K}\Bigg[\sum_{m=0}^{N_{max}}(\boldsymbol{\lambda}_k\Lambda_k(m,j) - \boldsymbol{\mu}_k\delta_d(1-\delta_d)^m)(h(\hat{\mathbf{x}}_{k-m|k-1}) + \mathbf{H}_{k-m}\mathbf{e}_{k-m|k-1}) + \sum_{m=1}^{N_{max}}\boldsymbol{\mu}_k\delta_d(1-\delta_d)^m\mathbf{H}_{k-m}\mathbf{F}_{k-m}\mathbf{e}_{k-1-m|k-1}\Bigg]}_{\mathrm{D}_k}.\end{aligned} \tag{38}$$

It should be mentioned that with $\tau_2/\tau_1 = \kappa'$, $1-\phi = \beta$, and $\gamma'/(\tau_1\phi) = \xi$, Eq. (43) is the same as Eq. (39).

Remark 7: In conclusion to the above discussion, we state that satisfying Eq. (43) for a stochastic process $\mathbf{e}_k$ ensures that $\mathbf{e}_k$ is exponentially bounded.

Remark 8: Since Eq. (43) is concluded from Eqs. (40) and (41), we can further state that if the stochastic process $\mathbf{e}_k$ satisfies Eqs. (40) and (41) for any scalar-valued stochastic process $V(\mathbf{e}_k)$, then the stochastic process $\mathbf{e}_k$ is exponentially bounded.

We will use Remark 8 as our stability criterion. Alternatively, we will conclude the stability of the MDEKF by inferring that the estimation error of the MDEKF satisfies Eqs. (40) and (41). The proof is based on several assumptions, as follows.

- $\mathbf{F}_k$ is a non-singular matrix.
- The system, noise, and filter parameters satisfy the following bounds:

$$\|\eta_k\| \leq \omega \text{ and } \|\nu_k\| \leq \nu \tag{44}$$
$$\|\Psi_f(\mathbf{x}_{k-1}, \hat{\mathbf{x}}_{k-1|k-1})\| \leq c_1\|\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1}\|^2 \tag{45}$$
$$\|\Psi_h(\mathbf{x}_k, \hat{\mathbf{x}}_{k|k-1})\| \leq c_2\|\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}\|^2 \tag{46}$$
$$\|\mathbf{F}_k\| \leq f \text{ and } \|\mathbf{H}_k\| \leq h \tag{47}$$
$$\rho_1\mathbf{I} \leq \mathbf{P}_{k|k} \leq \mathbf{P}_{k|k-1} \leq \rho_2\mathbf{I} \tag{48}$$
$$\left\|\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1}\right\| = \left\|\mathbf{e}_{k-1|k-1}\right\| \leq \epsilon' \tag{49}$$
$$\left\|\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}\right\| = \left\|\mathbf{e}_{k|k-1}\right\| \leq \epsilon' \tag{50}$$
$$\left\|h(\hat{\mathbf{x}}_{k|k-1})\right\| \leq h_m \tag{51}$$
$$q\mathbf{I} \leq \mathbf{Q}_k \leq q'\mathbf{I} \text{ and } r\mathbf{I} \leq \mathbf{R}_k \leq r'\mathbf{I}, \tag{52}$$

with ω, ν, c_1, c_2, f, h, ρ_1, ρ_2, ϵ', h_m, q, q', r, and r' representing positive real numbers.

Before proving that the MDEKF's estimation error satisfies Eqs. (40) and (41), we derive a series of inequalities through the subsequent lemmas. These inequalities will be used in the final proof.

Lemma 1: For the inequalities presented in Eqs. (47), (48), and (52), the Kalman gain $\mathbf{K}$ satisfies

$$\|\mathbf{K}\| \leq \frac{\rho_2 h}{r}. \tag{53}$$

Proof: Let us expand $\mathbf{P}^{\mathbf{yy}}_{k|k-1}$ and $\mathbf{P}^{\mathbf{xy}}_{k|k-1}$ as follows: i) substitute $\hat{\mathbf{z}}_{k-m|k-1}$ and $\mathbf{P}^{\mathbf{zz}}_{k-m|k-1}$ obtained using Eqs. (32) and (33), respectively in $\mathbf{P}^{\mathbf{yy}}_{k|k-1}$ given in Eq. (23), and ii) substitute $\mathbf{P}^{\mathbf{xz}}_{k-m|k-1}$ from Eq. (33) into $\mathbf{P}^{\mathbf{xy}}_{k|k-1}$ given in Eq. (31). Substituting the expanded $\mathbf{P}^{\mathbf{yy}}_{k|k-1}$ and $\mathbf{P}^{\mathbf{xy}}_{k|k-1}$ in Kalman gain expression $\mathbf{K} = \mathbf{P}^{\mathbf{xy}}_{k|k-1}(\mathbf{P}^{\mathbf{yy}}_{k|k-1})^{-1}$, we obtain

$$\mathbf{K} = \sum_{m=0}^{N_{max}} \delta_d(1-\delta_d)^m \mathbf{P}_{k-m|k-1}\mathbf{H}_k^T\boldsymbol{\mu}_k \Bigg[\sum_{m=0}^{N_{max}} \delta_d(1-\delta_d)^m \times \boldsymbol{\mu}_k\left(\mathbf{H}_k\mathbf{P}_{k-m|k-1}\mathbf{H}_k^T + \mathbf{R}_k\right) + \sum_{m=0}^{N_{max}}\left(\boldsymbol{\mu}_k\delta_d(1-\delta_d)^m - \left(\boldsymbol{\mu}_k\delta_d(1-\delta_d)^m\right)^2\right)h(\hat{\mathbf{x}}_{k-m|k-1})h(\hat{\mathbf{x}}_{k-m|k-1})^T\Bigg]^{-1}.$$

It is worth mentioning that $\mathbf{H}_k\mathbf{P}_{k-m}\mathbf{H}_k^T \geq 0$ because $\mathbf{P}_{k-m|k-1}$ is a positive definite matrix. Subsequently, applying the norm property, we can write

$$\|\mathbf{K}\| \leq \delta_d\|\boldsymbol{\mu}_k\|\left\|\sum_{m=0}^{N_{max}}(1-\delta_d)^m\mathbf{P}_{k-m}\mathbf{H}_k^T\right\|\left\|\Bigg[\sum_{m=0}^{N_{max}}\delta_d(1-\delta_d)^m \times \boldsymbol{\mu}_k\left(\mathbf{R}_k + \left(\mathbf{I} - \boldsymbol{\mu}_k\delta_d(1-\delta_d)^m\right)h(\hat{\mathbf{x}}_{k-m|k-1}) \times h(\hat{\mathbf{x}}_{k-m|k-1})^T\right)\Bigg]^{-1}\right\|.$$

For any invertible matrix $\mathbf{M}$, please note that $\sigma^+(\mathbf{M}^{-1}) = (\sigma^-(\mathbf{M}))^{-1}$, where $\sigma^+(\cdot)$ and $\sigma^-(\cdot)$ represent the largest and smallest singular values, respectively. Thus, substituting $\|\boldsymbol{\mu}_k\| = \mu$ and using the inequalities presented in Eqs. (47) and (48), we get

$$\|\mathbf{K}\| \leq \left(\mu\rho_2 h\left(1-(1-\delta_d)^{N_{max}+1}\right)\right)\Bigg[\sigma^-\Bigg(\sum_{m=0}^{N_{max}}\delta_d(1-\delta_d)^m \times \boldsymbol{\mu}_k\left(\mathbf{R}_k + \left(\mathbf{I} - \boldsymbol{\mu}_k\delta_d(1-\delta_d)^m\right)h(\hat{\mathbf{x}}_{k-m|k-1}) \times h(\hat{\mathbf{x}}_{k-m|k-1})^T\right)\Bigg)\Bigg]^{-1}. \tag{54}$$

The bound presented in Eq. (52) assumes that $\mathbf{R}_k$ is a positive definite matrix. Moreover, $h(\hat{\mathbf{x}}_{k-m|k-1})h(\hat{\mathbf{x}}_{k-m|k-1})^T$ is a positive semidefinite matrix. Thus, the matrix in the second factor of the above equation is also positive definite. Subsequently, it follows that $\sigma^-(\cdot) = \lambda^-(\cdot)$, with $\lambda^-(\cdot)$ representing the smallest eigenvalue. We now calculate the smallest eigenvalue of the second factor by using the Rayleigh-Ritz characterization [41] in Eq. (55), as shown at the bottom of the page.

$$\lambda^-\left(\sum_{m=0}^{N_{max}}\delta_d(1-\delta_d)^m\boldsymbol{\mu}_k(\mathbf{R}_k + \left(\mathbf{I} - \delta_d(1-\delta_d)^m\boldsymbol{\mu}_k\right)h(\hat{\mathbf{x}}_{k-m|k-1})h(\hat{\mathbf{x}}_{k-m|k-1})^T)\right)$$
$$\geq \min_{\|\mathbf{x}\|=1}\left(\mathbf{x}^T\sum_{m=0}^{N_{max}}\delta_d(1-\delta_d)^m\boldsymbol{\mu}_k\mathbf{R}_k\mathbf{x}\right) + \min_{\|\mathbf{x}\|=1}\left(\mathbf{x}^T\sum_{m=0}^{N_{max}}\left(\delta_d(1-\delta_d)^m\boldsymbol{\mu}_k - \left(\delta_d(1-\delta_d)^m\boldsymbol{\mu}_k\right)^2\right)h(\hat{\mathbf{x}}_{k-m|k-1})h(\hat{\mathbf{x}}_{k-m|k-1})^T\mathbf{x}\right). \tag{55}$$

As $h(\hat{\mathbf{x}}_{k-m|k-1})h(\hat{\mathbf{x}}_{k-m|k-1})^T$ is positive semidefinite, the second term on the right side of Eq. (55) is zero. Subsequently, using the bound given in Eq. (52), we obtain

$$\lambda^{-}\bigg(\sum_{m=0}^{N_{max}} \delta_d(1-\delta_d)^m \boldsymbol{\mu}_k(\mathbf{R}_k + \big(\mathbf{I} - \delta_d(1-\delta_d)^m\boldsymbol{\mu}_k\big) \times h(\hat{\mathbf{x}}_{k-m|k-1})h(\hat{\mathbf{x}}_{k-m|k-1})^T)\bigg) \geq \mu\left(1-(1-\delta_d)^{N_{max}+1}\right) r. \tag{56}$$

As discussed previously, $\sigma^{-}(\cdot) = \lambda^{-}(\cdot)$ for positive definite matrix. Thus, substituting Eq. (56) into Eq. (54) gives the bound of $\mathbf{K}$ as given in Eq. (53).

Lemma 2: For non-singular matrix $\mathbf{F}_k$ and $0 < \phi < 1$, the following inequality holds

$$\mathbf{F}_{k-1}^T\left(\mathbf{I} - \delta_d\mathbf{K}\boldsymbol{\mu}_k\mathbf{H}_k\right)^T\mathbf{P}_{k|k}^{-1}\left(\mathbf{I} - \delta_d\mathbf{K}\boldsymbol{\mu}_k\mathbf{H}_k\right)\mathbf{F}_{k-1} \leq (1-\phi)\,\mathbf{P}_{k-1|k-1}^{-1}. \tag{57}$$

Proof: Please note that $\mathbf{P}_{k|k} = E[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})^T]$. Substituting $\mathbf{x}_k$ from Eq. (1) and $\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}(\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1})$, we get

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{P}_{k|k-1}^{\mathbf{xy}}\mathbf{K}^T - \mathbf{K}(\mathbf{P}_{k|k-1}^{\mathbf{xy}})^T + \mathbf{K}\mathbf{P}_{k|k-1}^{\mathbf{yy}}\mathbf{K}^T.$$

Substituting $\mathbf{P}_{k|k-1}^{\mathbf{yy}}$ and $\mathbf{P}_{k|k-1}^{\mathbf{xy}}$ from Eqs. (23) and (31), respectively, we obtain

$$\begin{aligned}\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} &- \sum_{m=0}^{N_{max}} \delta_d(1-\delta_d)^m\mathbf{P}_{k-m|k-1}^{\mathbf{xz}}\boldsymbol{\mu}_k\mathbf{K}^T - \mathbf{K}\sum_{m=0}^{N_{max}}\big(\delta_d \\ &\times (1-\delta_d)^m\boldsymbol{\mu}_k(\mathbf{P}_{k-m|k-1}^{\mathbf{xz}})^T\big) \\ &+ \mathbf{K}\bigg[\sum_{m=0}^{N_{max}} \boldsymbol{\mu}_k\delta_d(1-\delta_d)^m\mathbf{P}_{k-m|k-1}^{\mathbf{zz}} \\ &+ \sum_{m=0}^{N_{max}}\left(\boldsymbol{\mu}_k\delta_d(1-\delta_d)^m - \big(\boldsymbol{\mu}_k\delta_d(1-\delta_d)^m\big)^2\right) \\ &\times \hat{\mathbf{z}}_{k-m|k-1}\hat{\mathbf{z}}_{k-m|k-1}^T\bigg]\mathbf{K}^T.\end{aligned}$$

It is worth mentioning that $\hat{\mathbf{z}}_{k-m|k-1}$, $\mathbf{P}_{k-m|k-1}^{\mathbf{zz}}$, and $\mathbf{P}_{k-m|k-1}^{\mathbf{xz}}$ can be determined using Eq. (33). Substituting these parameters in the above expression and simplifying further, we get $\mathbf{P}_{k|k}$ in the form of Eq. (58), as shown at the bottom of the page.

Please note that $\mathbf{R}_k$ is a positive definite matrix in Eq. (58). Similarly, the positive definiteness of $\mathbf{P}_{k-m|k-1}$ ensures that $\delta_d(1-\delta_d)^m\boldsymbol{\mu}_k\mathbf{H}_k\mathbf{P}_{k-m|k-1}\mathbf{H}_k^T(\mathbf{I}-\delta_d(1-\delta_d)^m\boldsymbol{\mu}_k)$ is positive semidefinite. Moreover, $h(\hat{\mathbf{x}}_{k-m|k-1})h(\hat{\mathbf{x}}_{k-m|k-1})^T$ is a positive semidefinite matrix. Since the sum of the positive definite and positive semidefinite matrices is positive definite, we further conclude

$$\begin{aligned}\mathbf{K}\bigg[&\sum_{m=0}^{N_{max}} \delta_d(1-\delta_d)^m\boldsymbol{\mu}_k\mathbf{H}_k\mathbf{P}_{k-m|k-1}\mathbf{H}_k^T\left(\mathbf{I} - \delta_d(1-\delta_d)^m\boldsymbol{\mu}_k\right) \\ &+ \sum_{m=0}^{N_{max}}\left(\boldsymbol{\mu}_k\delta_d(1-\delta_d)^m - \big(\boldsymbol{\mu}_k\delta_d(1-\delta_d)^m\big)^2\right) \\ &\times h(\hat{\mathbf{x}}_{k-m|k-1})h(\hat{\mathbf{x}}_{k-m|k-1})^T \\ &+ \sum_{m=0}^{N_{max}}\delta_d(1-\delta_d)^m\boldsymbol{\mu}_k\mathbf{R}_k\bigg]\mathbf{K}^T \geq 0.\end{aligned}$$

Consequently, the following inequality can be deduced from Eq. (58):

$$\mathbf{P}_{k|k} \geq \left(\mathbf{I} - \delta_d\mathbf{K}\boldsymbol{\mu}_k\mathbf{H}_k\right)\mathbf{P}_{k|k-1}\left(\mathbf{I} - \delta_d\mathbf{K}\boldsymbol{\mu}_k\mathbf{H}_k\right)^T - \sum_{m=1}^{N_{max}}\mathbf{P}_{k-m|k-1}. \tag{59}$$

Substituting Eq. (32), we can rearrange the above inequality as

$$\begin{aligned}\mathbf{P}_{k|k} \geq \left(\mathbf{I}-\delta_d\mathbf{K}\boldsymbol{\mu}_k\mathbf{H}_k\right)\mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}\Big[&\mathbf{I}+\mathbf{P}_{k-1|k-1}^{-1}\mathbf{F}_{k-1}^{-1}\big(\mathbf{Q}_k \\ &- \left(\mathbf{I} - \delta_d\mathbf{K}\boldsymbol{\mu}_k\mathbf{H}_k\right)^{-1} \\ &\times \sum_{m=1}^{N_{max}}\mathbf{P}_{k-m|k-1}\left(\mathbf{I} - \delta_d\mathbf{K}\boldsymbol{\mu}_k\mathbf{H}_k\right)^{-T}\big)\mathbf{F}_{k-1}^{-T}\Big] \\ &\times \mathbf{F}_{k-1}^T\left(\mathbf{I} - \delta_d\mathbf{K}\boldsymbol{\mu}_k\mathbf{H}_k\right)^T.\end{aligned}$$

Applying the bounds of Eqs. (47), (48), and (52), and taking the inverse, the above inequality can be rearranged as

$$\begin{aligned}&\mathbf{F}_{k-1}^T(\mathbf{I} - \delta_d\mathbf{K}\boldsymbol{\mu}_k\mathbf{H}_k)^T\mathbf{P}_{k|k}^{-1}\left(\mathbf{I} - \delta_d\mathbf{K}\boldsymbol{\mu}_k\mathbf{H}_k\right)\mathbf{F}_{k-1} \\ &\quad\leq \left(1 + \frac{1}{\rho_2 f^2}\left(q - \zeta^2 N_{max}\rho_2\right)\right)^{-1}\mathbf{P}_{k-1|k-1}^{-1},\end{aligned}$$

where $\zeta = \sigma^{+}((\mathbf{I}-\delta_d\mathbf{K}\boldsymbol{\mu}_k\mathbf{H}_k)^{-1})$. We choose q, f, and ρ_2 for which $0 < (1 + 1/(\rho_2 f^2)(q - \zeta^2 N_{max}\rho_2))^{-1} = 1 - \phi < 1$ is

$$\begin{aligned}\mathbf{P}_{k|k} = &\sum_{m=0}^{N_{max}}\left(\mathbf{I} - \delta_d(1-\delta_d)^m\mathbf{K}\boldsymbol{\mu}_k\mathbf{H}_k\right)\mathbf{P}_{k-m|k-1}\left(\mathbf{I} - \delta_d(1-\delta_d)^m\mathbf{K}\boldsymbol{\mu}_k\mathbf{H}_k\right)^T + \mathbf{K}\bigg[\sum_{m=0}^{N_{max}}\delta_d(1-\delta_d)^m\boldsymbol{\mu}_k\big(\mathbf{H}_k\mathbf{P}_{k-m|k-1}\mathbf{H}_k^T \\ &\times \left(\mathbf{I} - \delta_d(1-\delta_d)^m\boldsymbol{\mu}_k\right) + \sum_{m=0}^{N_{max}}\mathbf{R}_k\big) + \sum_{m=0}^{N_{max}}\left(\boldsymbol{\mu}_k\delta_d(1-\delta_d)^m - \big(\boldsymbol{\mu}_k\delta_d(1-\delta_d)^m\big)^2\right)h(\hat{\mathbf{x}}_{k-m|k-1})h(\hat{\mathbf{x}}_{k-m|k-1})^T\bigg]\mathbf{K}^T \\ &- \sum_{m=1}^{N_{max}}\mathbf{P}_{k-m|k-1}.\end{aligned} \tag{58}$$

satisfied. Subsequently, the above inequality is simplified to Eq. (57).

Theorem 1: Let us consider the stochastic error model given in Eq. (38) and EKF parameters through Eqs. (32) and (33). Furthermore, let us assume that $\mathbf{F}_k$ is non-singular and the conditions given through (45) to (52) hold $\forall k \geq 0$. Then, the stochastic process $\mathbf{e}_{k|k}$ is exponentially bounded in mean square, i.e.,

$$E\left[\left\|\mathbf{e}_{k|k}\right\|^2\right] \leq \frac{\tau_2}{\tau_1} E\left[\left\|\mathbf{e}_{0|0}\right\|^2\right](1-\phi)^k + \frac{\gamma'}{\tau_1\phi}. \quad (60)$$

Proof: Throughout the theorem, we adopt the following simplified notations: $\hat{\mathbf{x}}_{k-m}$, $\hat{\mathbf{x}}_{k-1-m}$, $\mathbf{e}_{k-m}$, and $\mathbf{e}_{k-1-m}$ for $\hat{\mathbf{x}}_{k-m|k-1}$, $\hat{\mathbf{x}}_{k-1-m|k-1}$, $\mathbf{e}_{k-m|k-1}$, and $\mathbf{e}_{k-1-m|k-1}$, respectively. Then, Eq. (38) can be expressed as

$$\mathbf{e}_{k|k} = \mathbf{A}_k \mathbf{e}_{k-1|k-1} + \mathbf{B}_k + \mathbf{C}_k + \mathbf{D}_k. \quad (61)$$

We now define a scalar-valued stochastic process $V : \mathbb{R}^n \to \mathbb{R}$ as $\mathbf{e}_{k|k}^T \mathbf{P}_{k|k}^{-1} \mathbf{e}_{k|k}$. It should be mentioned that V is a positive definite function. Substituting $\mathbf{e}_k$ from Eq. (61) in the expression of V and simplifying further, we obtain

$$\begin{aligned} V(\mathbf{e}_{k|k}) = {} & \mathbf{e}_{k-1|k-1}^T \mathbf{A}_k^T \mathbf{P}_{k|k}^{-1} \mathbf{A}_k \mathbf{e}_{k-1|k-1} + \mathbf{B}_k^T \mathbf{P}_{k|k}^{-1} (2\mathbf{A}_k \\ & \times \mathbf{e}_{k-1|k-1} + \mathbf{B}_k) + 2\mathbf{C}_k^T \mathbf{P}_{k|k}^{-1} \\ & \times \left(\mathbf{A}_k \mathbf{e}_{k-1|k-1} + \mathbf{B}_k + \mathbf{D}_k\right) \\ & + \mathbf{C}_k^T \mathbf{P}_{k|k}^{-1} \mathbf{C}_k + 2\mathbf{D}_k^T \mathbf{P}_{k|k}^{-1} \left(\mathbf{A}_k \mathbf{e}_{k-1|k-1} + \mathbf{B}_k\right) \\ & + \mathbf{D}_k^T \mathbf{P}_{k|k}^{-1} \mathbf{D}_k. \end{aligned} \quad (62)$$

It should be mentioned that the above equation is scalar. Thus, applying norm property to the second expression on the right side, we get

$$\begin{aligned} & \left\|\mathbf{B}_k^T \mathbf{P}_{k|k}^{-1} \left(2\mathbf{A}_k \mathbf{e}_{k-1|k-1} + \mathbf{B}_k\right)\right\| \\ & \quad \leq \|\mathbf{B}_k\| \left\|\mathbf{P}_{k|k}^{-1}\right\| \left(2\|\mathbf{A}_k\|\|\mathbf{e}_{k-1|k-1}\| + \|\mathbf{B}_k\|\right). \end{aligned} \quad (63)$$

We will now calculate the bound of each term in the right side individually. In this regard, let us recall $\mathbf{B}_k$ defined in Eq. (38) and use norm property to get

$$\begin{aligned} & \|\mathbf{B}_k\| \\ & \leq \|\Psi_f(\mathbf{x}_{k-1}, \hat{\mathbf{x}}_{k-1})\| \\ & \quad + \|\mathbf{K}\| \|\sum_{m=0}^{N_{max}} \boldsymbol{\lambda}_k \Lambda_k(m,j) \Psi_h(\mathbf{x}_{k-m}, \hat{\mathbf{x}}_{k-m})\| \\ & \quad + \|\mathbf{K}\| \left\| \sum_{m=0}^{N_{max}} \delta_d (1-\delta_d)^m \boldsymbol{\mu}_k \mathbf{H}_{k-m} \Psi_f(\mathbf{x}_{k-1-m}, \hat{\mathbf{x}}_{k-1-m}) \right\|. \end{aligned}$$

It should be mentioned that the elements of $\boldsymbol{\lambda}_k$ are Bernoulli random variables, which take on values zero or one. Thus, $\|\boldsymbol{\lambda}_k\| \leq 1$. Furthermore, for only a particular value of m, we get $\Lambda_k(m,j) = 1$, otherwise $\Lambda_k(m,j) = 0$. Subsequently, applying the bounds presented in Eqs. (45), (46), and (47), and from Lemma 1, we obtain

$$\|\mathbf{B}_k\| \leq c_1 \epsilon'^2 \left(1 + \frac{\rho_2 h^2 \mu (1-(1-\delta_d)^{N_{max}+1})}{r}\right) + \frac{\rho_2 h c_2 \epsilon'^2}{r}. \quad (64)$$

Similarly, for $\mathbf{A}_k$ given in Eq. (38), we get

$$\|\mathbf{A}_k\| \leq \|\big(\mathbf{I} + \delta_d \mathbf{K} \boldsymbol{\mu}_k \mathbf{H}_k\big)\| \|\mathbf{F}_{k-1}\| = (1 + \frac{\rho_2 h^2 \mu \delta_d}{r}) f. \quad (65)$$

Applying the bound given in Eq. (48), we have $\|\mathbf{P}_{k|k}^{-1}\| \leq 1/\rho_1$. Thus, substituting Eqs. (49), (64), and (65) into Eq. (63), it is simplified as

$$\left\|\mathbf{B}_k^T \mathbf{P}_{k|k}^{-1} \left(2\mathbf{A}_k \mathbf{e}_{k-1|k-1} + \mathbf{B}_k\right)\right\| \leq \chi_1 \epsilon'^3, \quad (66)$$

where

$$\begin{aligned} \chi_1 = \frac{1}{\rho_1} \Bigg[& c_1 \left(1 + \frac{\rho_2 h^2 \mu (1-(1-\delta_d)^{N_{max}+1})}{r}\right) \\ & \times \Bigg(2 + \frac{2\rho_2 h^2 f \mu \delta_d}{r} \\ & + c_1 \epsilon' \left(1 + \frac{\rho_2 h^2 \mu (1-(1-\delta_d)^{N_{max}+1})}{r}\right)\Bigg)\Bigg]. \end{aligned}$$

Let us now consider the fourth expression on the right side of Eq. (62). For $\mathbf{C}_k$ defined in Eq. (38), applying $\|\mathbf{C}_k^T\| = \|\mathbf{C}_k\|$ gives

$$\begin{aligned} & \left\|\mathbf{C}_k^T \mathbf{P}_{k|k}^{-1} \mathbf{C}_k\right\| \\ & \leq \Bigg(\|\boldsymbol{\eta}_k\| + \|\mathbf{K}\| \Bigg\| \sum_{m=0}^{N_{max}} \big(\boldsymbol{\mu}_k \delta_d (1-\delta_d)^m \\ & \quad \times \mathbf{H}_{k-m} \boldsymbol{\eta}_{k-m} + \boldsymbol{\lambda}_k \Lambda_k(m,j) \boldsymbol{\nu}_{k-m}\big) \Bigg\| \Bigg)^2 \left\|\mathbf{P}_{k|k}^{-1}\right\|. \end{aligned}$$

We now substitute $\|\boldsymbol{\mu}_k\| = \mu$, $\|\boldsymbol{\lambda}_k\| \leq 1$, $\sum_{m=0}^{N_{max}} \Lambda_k(m,j) = 1$, and $\|\mathbf{P}_{k|k}^{-1}\| \leq 1/\rho_1$. Then, using the inequalities presented in Eqs. (44) and (47), and substituting the bound of $\mathbf{K}$ from Lemma 1, we finally get

$$\left\|\mathbf{C}_k^T \mathbf{P}_{k|k}^{-1} \mathbf{C}_k\right\| \leq \chi_2, \quad (67)$$

with χ_2 be a constant, given as

$$\chi_2 = \frac{1}{\rho_1}\left(\omega + \frac{\rho_2 h}{r}\left(\mu \omega h (1-(1-\delta_d)^{N_{max}+1}) + v\right)\right)^2. \quad (68)$$

For $\mathbf{D}_k$ defined in Eq. (38), applying the matrix norm property, the fifth expression on the right side of Eq. (62)

can be expressed as

$$\|\mathbf{D}_k\| \leq \|\mathbf{K}\| \Bigg[\Bigg\| \sum_{m=0}^{N_{max}} \boldsymbol{\lambda}_k \Lambda_k(m,j)(h(\hat{\mathbf{x}}_{k-m}) + \mathbf{H}_{k-m}\mathbf{e}_{k-m}) \Bigg\| + \Bigg\| \sum_{m=0}^{N_{max}} \delta_d(1-\delta_d)^m \boldsymbol{\mu}_k (h(\hat{\mathbf{x}}_{k-m}) + \mathbf{H}_{k-m}\mathbf{e}_{k-m}) \Bigg\| + \Bigg\| \sum_{m=1}^{N_{max}} \delta_d(1-\delta_d)^m \boldsymbol{\mu}_k \mathbf{H}_{k-m}\mathbf{F}_{k-m}\mathbf{e}_{k-1-m} \Bigg\| \Bigg].$$

We now substitute $\|\boldsymbol{\mu}_k\| = \mu$, $\|\boldsymbol{\lambda}_k\| \leq 1$, and $\sum_{m=0}^{N_{max}} \Lambda_k(m,j) = 1$. Furthermore, using Eqs. (47), (50), and (51), and applying the bound of $\mathbf{K}$ from Lemma 1, the above equation can be expressed as

$$\|\mathbf{D}_k\| \leq \frac{\rho_2 h}{r} \Big((h_m + h\epsilon') \left(1 + \mu(1-(1-\delta_d)^{N_{max}+1})\right) + hf\epsilon' \mu(1-\delta_d)(1-(1-\delta_d)^{N_{max}}) \Big). \tag{69}$$

Please note that $\|2\mathbf{D}_k^T \mathbf{P}_{k|k}^{-1}(\mathbf{A}_k \mathbf{e}_{k-1|k-1} + \mathbf{B}_k)\| \leq \|2\mathbf{D}_k^T\| \, \|\mathbf{P}_{k|k}^{-1}\| \, \|(\mathbf{A}_k\mathbf{e}_{k-1|k-1} + \mathbf{B}_k)\|$. Substituting the bounds of $\|\mathbf{B}_k\|$, $\|\mathbf{A}_k\|$, and $\|\mathbf{D}_k\|$ from Eqs. (64), (65), and (69), respectively, into this expression, we get

$$\left\| 2\mathbf{D}_k^T \mathbf{P}_{k|k}^{-1} \left(\mathbf{A}_k \mathbf{e}_{k-1|k-1} + \mathbf{B}_k\right) \right\| \leq \chi_3 \epsilon', \tag{70}$$

where

$$\chi_3 = \frac{2}{\rho_1} \Bigg[\frac{\rho_2 h}{r} \Big((h_m + h\epsilon') \left(1 + \mu(1-(1-\delta_d)^{N_{max}+1})\right) + hf\epsilon' \mu(1-\delta_d)(1-(1-\delta_d)^{N_{max}}) \Big) \Big(1 + \frac{\rho_2 h^2 f \mu \delta_d}{r} + c_1\epsilon' \Big(1 + \frac{\rho_2 h^2 \mu (1-(1-\delta_d)^{N_{max}+1})}{r}\Big) + \frac{\rho_2 h c_2 \epsilon'}{r}\Big)\Bigg]. \tag{71}$$

Let us now consider the sixth term in the summation on the right side of Eq. (62). Applying the norm property, we get $\|\mathbf{D}_k^T \mathbf{P}_{k|k}^{-1} \mathbf{D}_k\| \leq \|\mathbf{D}_k\|^2 \|\mathbf{P}_{k|k}^{-1}\|$. Subsequently, substituting $\|\mathbf{D}_k\|$ from Eq. (69), we obtain

$$\left\| \mathbf{D}_k^T \mathbf{P}_{k|k}^{-1} \mathbf{D}_k \right\| \leq \chi_4, \tag{72}$$

where χ_4 is a positive real number given as

$$\chi_4 = \frac{1}{\rho_1} \Big(\frac{\rho_2 h}{r} \Big((h_m + h\epsilon') \left(1 + \mu(1-(1-\delta_d)^{N_{max}+1})\right) + hf\epsilon' \mu(1-\delta_d)(1-(1-\delta_d)^{N_{max}}) \Big) \Big)^2. \tag{73}$$

Please note that $\left(\mathbf{I} - \delta_d \mathbf{K} \boldsymbol{\mu}_k \mathbf{H}_k\right) \mathbf{F}_{k-1} = \mathbf{A}_k$. We now substitute Eqs. (66), (67), (70), and (72) into Eq. (62), and apply Lemma 2. Then, taking conditional expectation, we obtain

$$E\left[V(\mathbf{e}_{k|k})|\mathbf{e}_{k-1|k-1}\right] \leq (1-\phi)\, \mathbf{e}_{k-1|k-1}^T \mathbf{P}_{k-1|k-1}^{-1} \mathbf{e}_{k-1|k-1} + E\left[2\mathbf{C}_k^T \mathbf{P}_{k|k}^{-1} \left(\mathbf{A}_k \mathbf{e}_{k-1|k-1} + \mathbf{B}_k + \mathbf{D}_k\right) \Big| \mathbf{e}_{k-1|k-1} \right] + \left(\chi_1 \epsilon'^3 + \chi_2 + \chi_3\epsilon' + \chi_4\right). \tag{74}$$

For $\mathbf{C}_k$ defined in Eq. (38), we conclude that $E[2\mathbf{C}_k^T \mathbf{P}_{k|k}^{-1} \left(\mathbf{A}_k \mathbf{e}_{k-1|k-1} + \mathbf{B}_k + \mathbf{D}_k\right) | \mathbf{e}_{k-1|k-1}] = 0$. Then, using the definition of $V(\mathbf{e}_k)$ and substituting $\chi_1\epsilon'^3 + \chi_2 + \chi_3\epsilon' + \chi_4 = \gamma'$, Eq. (74) can be expressed as

$$E\left[V(\mathbf{e}_{k|k})|\mathbf{e}_{k-1|k-1}\right] - V(\mathbf{e}_{k-1|k-1}) \leq \gamma' - \phi V(\mathbf{e}_{k-1|k-1}). \tag{75}$$

We select the parameters γ' and ϕ such that $V(\mathbf{e}_{k-1|k-1}) \geq \gamma'/\phi$ satisfies. Then, the above equation is the same as Eq. (41).

We now consider the inequality presented in Eq. (48). Taking inverse, then multiplying $\mathbf{e}_{k|k}^T$ and $\mathbf{e}_{k|k}$ from left and right side, respectively, we get

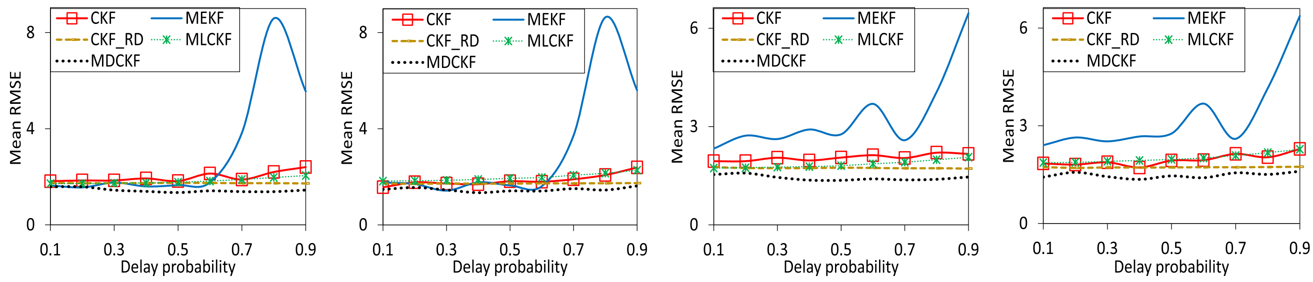
$$\frac{1}{\rho_2}\|\mathbf{e}_{k|k}\|^2 \leq V(\mathbf{e}_{k|k}) \leq \frac{1}{\rho_1}\|\mathbf{e}_{k|k}\|^2. \tag{76}$$

Please note that with $1/\rho_2 = \tau_1$ and $1/\rho_1 = \tau_2$, the above equation is same as Eq. (40). Therefore, we conclude that Eqs. (75) and (76) satisfy Eqs. (41) and (40), respectively. It further infers that, for chosen $V(\mathbf{e}_k)$, $\mathbf{e}_k$ satisfies Eq. (42) and hence Eq. (60). Thus, we conclude that MDEKF's estimation error remains exponentially bounded in mean square.

In this section, we proved the stochastic stability of the proposed method for its EKF-based formulation. The essential requirements of the proof involve a set of bounds on the system, noise, and filter parameters. Moreover, the stability analysis requires the initial estimation error to be bounded. Note that *boundedness* of noise and system parameters does not automatically imply exponential stability. The stability analysis of the proposed MDEKF reduces to the stability analysis of the traditional EKF for the delay and missing measurements probabilities being zero ($\mu^i = \delta = 1$) and $N_{max} = 0$, considering the convention $0^0 = 1$.

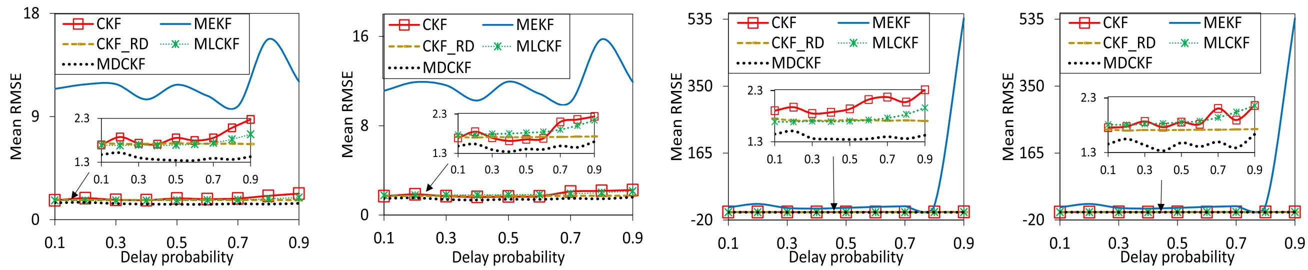
V. SIMULATION RESULTS

In real-life problems, the measurement systems (including the measuring devices and the supplementary units) are usually designed to efficiently capture the measurements. Therefore, they may not be expected to miss many measurements. Subsequently, the missing measurement probability is usually small. Thus, we consider the missing measurement probability up to 0.2 for the simulation, which means around 20% of the measurements are missing. On the other hand, the delay inherently appears in the measurements. Therefore, we consider a sufficiently large range of the delay probability ($0.1 \leq 1 - \delta_d \leq 0.9$). Moreover, it should be mentioned that the practical measurement systems are designed for small delays. Hence, we restrict the maximum possible delay to one or two time-steps (denoted as 1-delay and 2-delay scenarios), such as the delay up to one or two sampling intervals.



(a) State-1 ($\mu_m = 0.1$) (b) State-2 ($\mu_m = 0.1$) (c) State-1 ($\mu_m = 0.2$) (d) State-2 ($\mu_m = 0.2$)

FIGURE 2. Problem 1, one-delay scenario: Mean RMSE plots of all filters for varying delay probabilities, considering the missing measurement probability μ_m as 0.1 and 0.2.



(a) State-1 ($\mu_m = 0.1$) (b) State-2 ($\mu_m = 0.1$) (c) State-1 ($\mu_m = 0.2$) (d) State-2 ($\mu_m = 0.2$)

FIGURE 3. Problem 1, two-delay scenario: Mean RMSE plots of all filters for varying delay probabilities, considering the missing measurement probability μ_m as 0.1 and 0.2.

For the performance analysis, we considered three popular and advanced Gaussian filters, namely, the CKF [13], CQKF [14], and GHF [15]. With their extensions under the modified filtering method, which are abbreviated as MDCKF, MDCQKF, and MDGHF, respectively. We use the root mean square error (RMSE) as our performance metrics. Please note that we will frequently use the notation $\mu_m = 1-\mu$ to denote the missing measurements probability.

We compare the MDCKF with the following filters: i) traditional CKF [13], ii) the CKF-based formulation of [22], which extends the Gaussian filtering technique for arbitrary delays, iii) [26], wherein the EKF is modified for missing measurements, and iv) the CKF-based formulation of [37], which considers simultaneously occurring delay and missing measurements. We abbreviate the CKF-based formulations of [22], and [37] as CKF_RD, and MLCKF, respectively, while the EKF-based formulation of [26] is abbreviated as MEKF. Please note that we use the EKF-based design of [26] unlike the CKF-based designs for other filters, as [26] is particularly designed for the EKF and becomes inapplicable to other filters.

A. PROBLEM 1

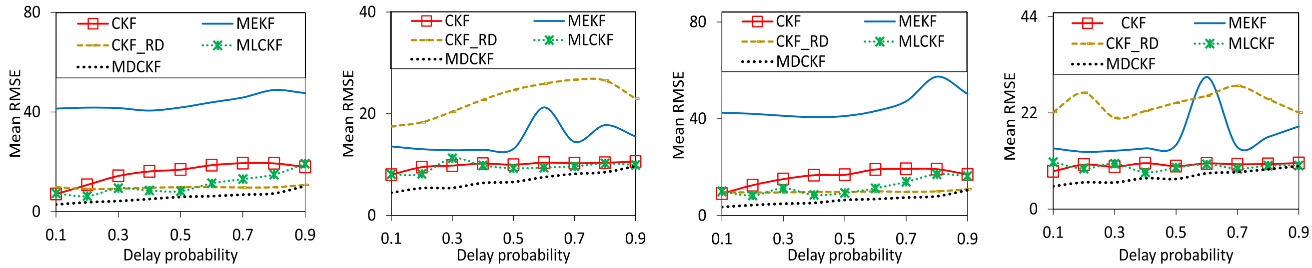
In the first problem, we consider a two-dimensional nonlinear dynamical system with the state-space model given as $\mathbf{x}_k = 2\cos(\mathbf{x}_{k-1}) + \boldsymbol{\eta}_k$ and $\mathbf{z}_k = \sqrt{1+\mathbf{x}_k^T\mathbf{x}_k} + \boldsymbol{\nu}_k$ [22].

The true data of the state and measurement are generated by considering the initial state as $\mathbf{x}_0 = [0.1 \quad 0.1]^T$. The filter is initialized with the initial estimate $\hat{\mathbf{x}}_{0|0} = 0.9\mathbf{x}_0$ and $\mathbf{P}_{0|0} = 7\mathbf{I}_2$. The noise covariances are assigned as $\mathbf{Q}_k = 0.1\mathbf{I}_2$ and $\mathbf{R}_k = 0.1$. The simulation is performed for 200 time-steps and the RMSEs are obtained by implementing 500 Monte-Carlo simulations.

Figs. 2 and 3 show the mean RMSE plots for varying delay probability under different scenarios formed by changing the maximum delay possibility and the missing measurements probability. It should be mentioned that the mean of the RMSEs is obtained over the 200 time-steps. The mean RMSE plots show a reduced RMSE for the MDCKF compared to the ordinary CKF, CKF_RD, MEKF, and MLCKF, which concludes that the proposed filtering method outperforms the ordinary Gaussian filtering as well as the existing filters for handling the delay and missing measurements. The relative computational times of the CKF, MEKF, CKF_RD, MLCKF, and MDCKF are obtained as 1, 2.86, 3.01, 5.78, and 3.04, respectively. It concludes that the computational time of the proposed method is marginally increased in comparison to some of the existing filters, while it remains marginally lower than others.

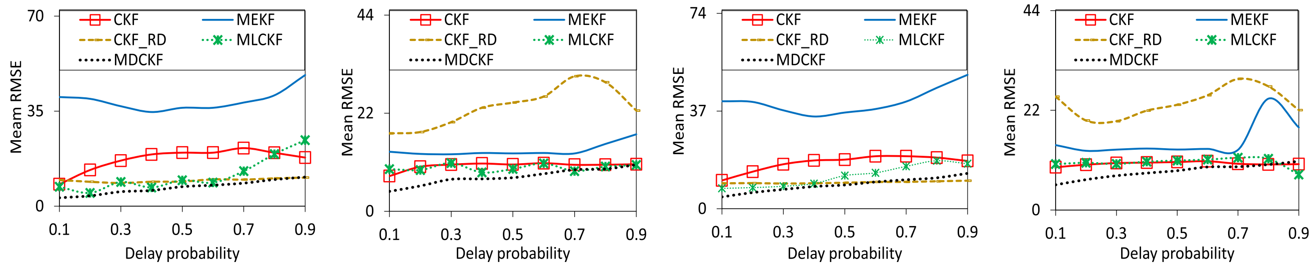
B. PROBLEM 2

In the second simulation problem, we consider an identification problem of individual sinusoids from the measurements of multiple superimposed sinusoids [9], [22]. We consider that the superimposed signal consists of three sinusoids. Identification of individual sinusoids is equivalent to estimating



(a) Frequency ($\mu_m = 0.1$) (b) Amplitude ($\mu_m = 0.1$) (c) Frequency ($\mu_m = 0.2$) (d) Amplitude ($\mu_m = 0.2$)

FIGURE 4. Problem 2, one-delay scenario: Mean RMSE plots of all filters for varying delay probabilities, considering the missing measurement probability μ_m as 0.1 and 0.2.



(a) Frequency ($\mu_m = 0.1$) (b) Amplitude ($\mu_m = 0.1$) (c) Frequency ($\mu_m = 0.2$) (d) Amplitude ($\mu_m = 0.2$)

FIGURE 5. Problem 2, two-delay scenario: Mean RMSE plots of all filters for varying delay probabilities, considering the missing measurement probability μ_m as 0.1 and 0.2.

their amplitudes and frequencies from the measurements of the superimposed signal.

In conclusion to the above discussion, the state-dynamics is formed as $\mathbf{x}_k = \mathbf{I}\mathbf{x}_{k-1} + \boldsymbol{\eta}_k$, where $\mathbf{x}_k = [f_{1,k}\ f_{2,k}\ f_{3,k}\ a_{1,k}\ a_{2,k}\ a_{3,k}]^T$, with a_n and f_n representing the amplitude and frequency, respectively, for the n^{th} sinusoid. Moreover, the measurement is a two-dimensional vector representing the superposition of the real and imaginary parts of the superimposed sinusoids. Please refer to [22] and [9] for a detailed discussion on the two models. The initial true and estimated states are taken as $\mathbf{x}_0 = [200\ 800\ 1000\ 2\ 3\ 5]^T$ and $\hat{\mathbf{x}}_{0|0} = [205\ 785\ 990\ 4\ 2\ 3]^T$, respectively, while the initial covariance is taken as $\mathbf{P}_{0|0} = diag([25\ \ 50\ \ 20\ \ 4\ \ 1\ \ 4])$. The noise covariances are taken as $\mathbf{Q}_k = diag([0.01\ \ 0.01\ \ 0.04\ \ 0.25\ \ 0.25\ \ 0.25])$ and $\mathbf{R}_k = diag([0.9\ 0.9])$.

The simulation is performed for 800 time-steps and 200 Monte-Carlo runs with a sampling interval of 0.25 milliseconds. Please note that the number of Monte-Carlo runs is reduced to 200 (in comparison to 500 runs used in the first problem), as the existing MEKF and MLCKF failed for higher number of Monte-Carlo runs. We obtain the RMSEs for the amplitude and frequency by taking the square root of the average of the mean square errors of the three amplitudes and frequencies, respectively.

We plot the mean RMSE (obtained over the time-steps) for varying delay probabilities in Figs. 4 and 5 for 1-delay and 2-delay, respectively. The mean RMSE plots show a reduced RMSE for the proposed MDCKF compared to the ordinary CKF and considered delay and missing filters. It concludes that the proposed filtering method has improved accuracy compared to these filters. The relative computational times for the MEKF, CKF, CKF_RD, MLCKF, and MDCKF are observed as 1, 1.56, 1.69, 3.24, and 1.72, respectively. It gives a similar conclusion as discussed in the previous problem.

TABLE 1. Problem 1, one-delay scenario: Average RMSEs obtained by MDCQKF, MDGHF, and their counterparts for different delay probabilities.

States	Filters	$\mu_m = 0.1$			$\mu_m = 0.2$		
		0.2	0.5	0.8	0.2	0.5	0.8
state-1	CQKF	1.65	1.44	1.62	1.73	1.47	1.54
	MDCQKF	1.59	1.33	1.36	1.59	1.34	1.36
	GHF	1.74	1.67	2.17	1.89	1.69	2.24
	MDGHF	1.68	1.36	1.35	1.69	1.36	1.37
state-2	CQKF	1.56	1.50	1.47	1.58	1.52	1.46
	MDCQKF	1.56	1.43	1.33	1.56	1.43	1.33
	GHF	1.75	1.84	2.04	1.85	1.90	2.07
	MDGHF	1.67	1.49	1.33	1.65	1.48	1.34

C. PERFORMANCE VALIDATION FOR OTHER GAUSSIAN FILTERS

To compare the estimation accuracy of the proposed method and the ordinary Gaussian filtering method, we present the mean RMSEs of CKF-based plots only in Figs. 2-5.

TABLE 2. Problem 2, one-delay scenario: Average RMSEs obtained by MDCQKF, MDGHF, and their counterparts for different delay probabilities.

States	Filters	$\mu_m = 0.1$			$\mu_m = 0.2$		
		0.2	0.5	0.8	0.2	0.5	0.8
Amplitude	CQKF	8.78	10.24	10.54	9.52	10.33	10.51
	MDCQKF	5.55	7.36	8.70	6.10	7.61	9.10
	GHF	7.25	8.57	9.57	7.36	8.82	9.96
	MDGHF	5.01	5.48	6.79	4.15	5.06	6.62
Frequency	CQKF	11.70	17.10	18.55	13.89	18.34	18.09
	MDCQKF	3.78	5.92	8.71	4.40	6.61	9.02
	GHF	9.27	13.60	18.66	10.60	12.86	17.57
	MDGHF	3.46	5.89	8.20	4.49	6.69	8.41

TABLE 3. Problem 1, one-delay scenario: Relative computational time comparison of MDCQKF, MDGHF, and their counterparts for 0.3 delay probability.

μ_m	Filters			
	CQKF	MDCQKF	GHF	MDGHF
0.1	1	1.091	0.812	1.083
0.2	1	1.1	0.813	0.908

However, we further extend the comparative analysis for the other advanced and popular Gaussian filters, such as the CQKF [14] and GHF [15], in Tables 1 and 2. The tables present the mean RMSEs obtained using the CQKF, GHF, MDCQKF, and MDGHF for various delay and missing measurement probabilities. From the tables, we conclude that RMSE is reduced for the MDCQKF and MDGHF compared with their traditional counterparts CQKF and GHF, respectively. It is worth mentioning that the results for 2-delay scenarios are qualitatively very similar and are omitted for brevity. The computational times remain similar for the proposed and the existing Gaussian filtering methods (Table 3).

VI. DISCUSSION AND CONCLUSION

The manuscript introduces a new extension of Gaussian filtering to efficiently handle the simultaneously occurring delayed and missing measurements. The proposed method reformulates the measurement model stochastically to introduce the possibility of simultaneously occurring delayed and missing measurements. Subsequently, the proposed filtering method is designed by re-deriving the traditional Gaussian filtering method for the modified measurement model. We compare the proposed filter with the CKF and three well-known filters which handle the delay and missing measurements individually or simultaneously. The performance of the proposed method is validated for two simulation problems. We also studied the exponential stability of the proposed method for its EKF-based design. It is worth mentioning that the computational time of the proposed method remains similar to traditional Gaussian filtering.

DATA ACCESS STATEMENT

This research did not use any experimentally generated data or data from any publicly available dataset. Model definitions (including the specified probability distributions) and parameter values (including the initialization parameters) provided in the paper are adequate for reproducing the exact qualitative behavior of the algorithms illustrated in the paper.

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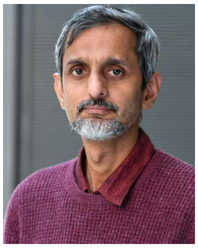

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