Received 25 August 2022, accepted 11 September 2022, date of publication 16 September 2022,
date of current version 26 September 2022.

Digital Object Identifier 10.1109/ACCESS.2022.3207468

THEORY

Stability of Cyber-Physical Systems of Numerical Methods for Stochastic Differential Equations: Integrating the Cyber and the Physical of Stochastic Systems

LIRONG HUANG
Guangzhou, Guangdong 510320, China
e-mail: lrhuang@aliyun.com

This work was supported by the National Natural Science Foundation of China under Grant 61877012.

ABSTRACT This paper presents the cyber-physical system (CPS) of a numerical method (the widely-used Euler-Maruyama method) and establishes a foundational theory of the CPSs of numerical methods for stochastic differential equations (SDEs), which transforms the way we understand the relationship between the numerical method and the underlying SDE. The CPS is a seamless integration of the SDE and the numerical method, unlike in the literature where they are treated as separate systems linked by inequalities. We formulate a new and general class of stochastic impulsive differential equations (SiDEs) that can serve as a canonic form of the CPSs and establish a Lyapunov stability theory as a theoretic foundation for our class of SiDEs. By the CPS approach, we show the equivalence and intrinsic relationship between the stability of the SDE and the stability of the numerical method. As application of our proposed results, we develop the CPS theory for linear systems and present the CPS Lyapunov inequality that is the necessary and sufficient condition for mean-square stability of the CPS of the Euler-Maruyama method for linear SDEs. Our proposed CPS theory initiates the study of systems numerics and provokes many open and interesting problems for future work.

INDEX TERMS Cyber-physical systems, exponential stability, impulsive systems, Lyapunov functions, numerical methods, stochastic differential equations.

I. INTRODUCTION

According to Newton's second law of motion, we describe a mechanical system with differential equations. A classical example is the mathematical pendulum described by a pair of differential equations [5, pp.17-18], which also constitutes a typical Hamiltonian system derived from Lagrangian mechanics, see [5, p.8] and also [16, p.5]. Usually, physical laws are expressed by means of differential equations, and so are the models of dynamical systems in many disciplines, ranging from biology to finance. Such models play a central role in all scientific and engineering disciplines [10], [35]. A model may serve many purposes. The value of a model depends on the model fidelity, where the model of a dynamical system is said to have high fidelity if it accurately describes the important properties of the system [10], [35]. Studying the model of high fidelity gives us insight into how the dynamical system will behave in the real world. Generally, a dynamical system, ranging from the motion of pollen particles to the movement of stock price, is subject to intrinsic and/or extrinsic noise in the real world [24], [26], [38]. Such randomness should/must be taken into account by a model of high fidelity if it matters, say, it affects some property of the system that is of concern to the modelling. If we allow for some noise in some coefficients of a differential equation, we often obtain a more realistic model of the situation that is able to describe the fluctuations observed in the physical system. This leads to modelling with stochastic differential

The associate editor coordinating the review of this manuscript and approving it for publication was Qichun Zhang.

equations (SDEs). The study of SDEs can be seen to have started from the classical paper of Einstein that presented a mathematical connection between microscopic Brownian motion of particles and the macroscopic diffusion equation, and the interest in SDEs has grown enormously in the last few decades [1], [32], [42], [47].

Stochastic systems described by SDEs have been intensively studied since stochastic modelling has come to play a significant role in science and engineering [2], [25], [27], [29], [30]. It is hardly possible to solve an SDE analytically and have the exact solution of the SDE. For practical purposes, numerical approximations to the exact solution are usually obtained, which, called the numerical solutions, are discrete-time stochastic processes produced by numerical methods. Such numerical schemes, in the form of stochastic difference equations, are the translations of the SDE into discretization. Practically, computers are used to excute the numerical schemes and generate the numerical solutions of the SDE, from which one could learn and/or infer some dynamical properties of the underlying physical system [4], [20], [29], [49].

As is well known, whenever a computer is used in measurement, computation, signal processing or control applications, the data, signals and systems involved are naturally described as discrete-time processes [3], [28], [29], [48]. It is worth noting that the SDE is the physical model which represents our knowledge of the physical system and a numerical method is a cyber model which is a representative of the physical model in computers, the cyber world. The physical model, namely, the SDE often refers to the phyiscal system (particularly, which is an engineered system) itself while its cyber couterpart, namely, the numerical method symbolizes it in the cyber world. In the age of networking and information technology, the cyber model plays a key role in understanding and controlling the underlying physical system, which not only envisions the approximate behaviour of the physical system [16], [20], [21], [22] but is also utilized to extract knowledge of the system from data [29], [48] and based on which control is designed and implemented [2], [3], [17], [28]. It is natural and imperative

(I) to find out the relationship between the physical model (i.e., the SDE) and its cyber counterpart (i.e., the numerical method) of a dynamical system;

(II) and to ensure that they both share some important dynamical properties such as stability, which is the concern of this study.

The principal aim of this paper is to address the problems (I) and (II) of fundamental importance in the age of networking and information technology. As a matter of fact, the fundamental importance of these problems has been recognized and they have been studied in a vast literature. Results that address the problems can be found in those many on convergence and stability of numerical methods for SDEs, where the SDE and the numerical method are treated as separate systems which are linked by inequalities in some moment sense on any finite time interval [21], [22], [31], [33], [40], [44]. The ability of a cyber system to reproduce the stability of its underlying physical system can be found in a wealth of impressive results. For example, the problem how to reproduce the stability of an SDE in its cyber counterpart, which is called the test problem, has been studied in [19], [22], [39], and [45]. The key question in a test problem is [19]

(Q1) for what stepsizes Δt does the cyber system (the numerical method) share the stability property of the underlying physical system (the SDE)?

This naturally provokes the converse question [22], [39]

(Q2) does the stability of the cyber system (the numerical method) for small stepsizes Δt imply that of the underlying physical system (the SDE)?

These questions deal with asymptotic ($t \to \infty$) properties and hence cannot be answered directly by applying traditional finite-time convergence results [22], [39]. Results that answer questions (Q1) and (Q2) can be found in the literature. For example, results for scalar linear systems were given in [19] and [45]. For multi-dimensional systems with global Lipschitz condition, Higham *et al.* [22] introduced a natural finite-time strong convergence condition, which links a cyber system with its underlying physical system by an inequality in some moment sense over any finite time interval, and proved that there is a sufficiently small positive Δt^* such that, for every $\Delta t \in (0, \Delta t^*]$, the mean-square exponential stability of the physical system is equivalent to that of its cyber counterpart. Recently, Mao [39] developed new techniques to handle the small pth moment ($p \in (0, 1)$) and showed that, under a natural finite pth moment condition and a natural finite-time convergence condition, the pth moment exponential stability of the physical system is equivalent to that of its cyber counterpart for every $\Delta t \in (0, \Delta t^*]$ with some sufficiently small positive Δt^*. As is pointed out in [39], there are many open problems in this research. For instance, although the existence of the (sufficiently small) upper bound $\Delta t^* > 0$ of stepsizes has been shown [22], [39], it is severely limited by the growth constant of the exponentially stable system, which refers to the physical system and its cyber counterpart when answering (Q1) and (Q2), respectively. Recall that, though the growth and the rate constants are related, it is only the rate constant that counts in the definition of exponential stability. It appears that, either to reproduce or to imply the exponential stability of the physical system by its cyber counterpart, the condition imposed on the stepsizes which explicitly depends on the growth constant could/should be relaxed [49]. This could significantly improve the upper bound Δt^* of stepsizes and facilitate the computation.

It is noted that the physical system (the SDE) and its cyber counterpart (the numerical method) are bound by inequalities in the literature [21], [22], [33], [40], [45], [49]. Nevertheless, they remain as two systems, largely separate. This paper constructs the cyber-physical model of a dynamical system that is a seamless, fully synergistic integration of the physical model (the SDE) and its cyber counterpart (the numerical method). Here we present a new and general class of stochastic impulsive differential equations (SiDEs) which can be used to represent the integrated dynamics of the physical system and its cyber counterpart. Impulsive differential equations, also known as impulsive systems, have been studied for several decades [9], [18], [30], [43], [46], [53]. But these impulsive systems in the literature are just the physical subsystems in our general class of SiDEs, see Section II. Our proposed SiDEs composed of the physical and the cyber subsystems are formulated as a canonic form of cyber-physical systems (CPSs), which present a systematic framework for the study of CPSs [11]. The canonic form not only provides a holistic view but also reveals the intrinsic relationship between the physical and the cyber subsystems of the CPS. In the study of numerical analysis, we present the CPS of a numerical method (the widely-used Euler-Maruyama method) for SDEs, which represents a seamless integration of the SDE and the numerical method in the form of our proposed SiDEs. The SDE and the numerical method are the physical subsystem and the cyber subsystem of the CPS, respectively. From the viewpoint of cybernetics [52], an essential problem to study is whether and how the CPS reproduces some dynamical properties such as the stability of its physical or cyber subsystem since 'the primary concern of cybernetics is on the qualitative aspects of the interrelations among the various components of a system and the synthetic behavior of the complete mechanism' [50]. Using the terminology of CPSs, we rephrase the questions (Q1) and (Q2) as follows.

(Q1) For what stepsizes Δt do the CPS and, hence, the cyber subsystem reproduce the stability property of the physical subsystem?

(Q2) Does the stability of the cyber subsystem for small stepsizes Δt imply that of the CPS and, hence, that of the physical subsystem?

This paper aims to address the fundamental problems in the age of networking and information technology. In this contribution, we shall

(i) propose a general class of SiDEs that is formulated to serve as a canonic form of the CPSs of numerical methods and construct a Lyapunov stability theory as a theoretic foundation for our proposed class of SiDEs;

(ii) present the CPS model of a numerical method for SDEs that is a seamless, fully synergistic integration of the SDE and the numerical method;

(iii) apply our established Lyapunov stability theory and prove positive results to the key questions (Q1) and (Q2), by which we expose the equivalence and inherent relationship between the stability of the SDE and the stability of the numerical method;

(iv) develop, as application of our proposed results, the CPS theory for linear systems and present the CPS Lyapunov inequality that is the necessary and sufficient condition for mean-square exponential stability of the CPSs of the Euler-Maruyama method for linear SDEs.

Our foundational theory of the CPSs for numerical methods transforms the way we understand the relationship between a numerical method and its underlying dynamical system. Moreover, we can theoretically prove that our proposed method is essentailly better than the existing results. To make the comparison, we significantly improve a key result in the literature (see Section VI and Appendix A). We also illustrate with numerical simulation the effectiveness of our theoretic results as well as those in the literature. This paper initiates the study of systems numerics and there are many interesting and/or challenging problems for future work.

II. A GENERAL CLASS OF STOCHASTIC IMPULSIVE DIFFERENTIAL EQUATIONS

Throughout this paper, unless otherwise specified, we shall employ the following notation. Let us denote by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets) and by $\mathbb{E}[\cdot]$ the expectation operator with respect to the probability measure. Let $B(t) = \begin{bmatrix} B_1(t) & \cdots & B_m(t) \end{bmatrix}^T$ be an m-dimensional Brownian motion defined on the probability space. If x, y are real numbers, then $x \vee y$ denotes the maximum of x and y, and $x \wedge y$ stands for the minimum of x and y. If A is a vector or a matrix, its transpose is denoted by A^T. If P is a square matrix, $P > 0$ (resp. $P < 0$) means that P is a symmetric positive (resp. negative) definite matrix of appropriate dimensions while $P \geq 0$ (resp. $P \leq 0$) is a symmetric positive (resp. negative) semidefinite matrix. Let $\lambda_M(\cdot)$ and $\lambda_m(\cdot)$ be a matrix's eigenvalues with maximum and minimum real parts, respectively. Denote by $|\cdot|$ the Euclidean norm of a vector and the trace (or Frobenius) norm of a matrix. Denote by I_n the $n \times n$ identity matrix and by $0_{n\times m}$ the $n \times m$ zero matrix, or, simply, by 0 the zero matrix of compatible dimensions.

Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ be the family of all nonnegative functions $V(x, t)$ on $\mathbb{R}^n \times \mathbb{R}_+$ that are continuously twice differentiable in x and once in t. Let $\mathcal{M}^p([a, b]; \mathbb{R}^n)$ be the family of $\mathbb{R}^n$-valued adapted process $\{x(t) : a \leq t \leq b\}$ such that $\mathbb{E}\int_a^b |x(t)|^p \mathrm{d}t < \infty$. Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\Xi_{\mathbb{N}}^m$ be the set of all independent and identically distributed sequences $\{\xi(k)\}_{k\in\mathbb{N}}$ with $\xi(k) = \begin{bmatrix} \xi_1(k) & \cdots & \xi_m(k) \end{bmatrix}^T$ and $\xi_j(k)$ obeying standard Gaussian distribution for $j = 1, 2, \cdots, m$. Sequence $\{t_k\}_{k\in\mathbb{N}}$ is strictly increasing and satisfies $t_0 = 0$, $0 < \underline{\Delta t} := \inf_{k\in\mathbb{N}}\{t_{k+1} - t_k\} \leq \overline{\Delta t} := \sup_{k\in\mathbb{N}}\{t_{k+1} - t_k\} < \infty$, and, hence, $t_k \to \infty$ as $k \to \infty$.

Let us consider a stochastic impulsive system described by SiDEs, which is composed of two subsystems,

$$\begin{aligned} \mathrm{d}x(t) &\\ &= f(x(t), t)\mathrm{d}t + g(x(t), t)\mathrm{d}B(t) \qquad (1a) \end{aligned}$$

$$\begin{aligned} \mathrm{d}y(t) &\\ &= \tilde{f}(x(t), y(t), t)\mathrm{d}t + \tilde{g}(x(t), y(t), t)\mathrm{d}B(t) \\ &\qquad t \in [t_k, t_{k+1}) \qquad (1b) \end{aligned}$$

$$\begin{aligned}\Delta(x(t_{k+1}^-), k+1) &:= x(t_{k+1}) - x(t_{k+1}^-)\\ &= h_f(x(t_{k+1}^-), k+1) + h_g(x(t_{k+1}^-), k+1)\bar{\xi}(k+1)\end{aligned} \tag{1c}$$

$$\begin{aligned}\tilde{\Delta}(x(t_{k+1}^-), y(t_{k+1}^-), k+1) &:= y(t_{k+1}) - y(t_{k+1}^-)\\ &= \tilde{h}_f(x(t_{k+1}^-), y(t_{k+1}^-), k+1)\\ &\quad + \bar{h}_g(x(t_{k+1}^-), y(t_{k+1}^-), k+1)\bar{\xi}(k+1)\\ &\quad + \tilde{h}_g(x(t_{k+1}^-), y(t_{k+1}^-), k+1)\xi(k+1) \quad k \in \mathbb{N}\end{aligned} \tag{1d}$$

with initial data $x(0) \in \mathbb{R}^n$ and $y(0) \in \mathbb{R}^q$, where measurement noise $\bar{\xi} \in \Xi^n_{\mathbb{N}}$ and simulation sequence $\xi \in \Xi^m_{\mathbb{N}}$ are independent of each other; $\bar{\xi}(k+1)$ and $\xi(k+1)$ are independent of $\{x(t), y(t) : 0 \le t < t_{k+1}\}$; $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}$, $h_f : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^n$, $h_g : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^{n\times n}$, $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}_+ \to \mathbb{R}^q$, $\tilde{g} : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}_+ \to \mathbb{R}^{q\times m}$, $\tilde{h}_f : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{N} \to \mathbb{R}^q$, $\bar{h}_g : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{N} \to \mathbb{R}^{q\times n}$ and $\tilde{h}_g : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{N} \to \mathbb{R}^{q\times m}$ are measurable functions. To study stability of the system, we assume that they obey $f(0, t) = 0$, $g(0, t) = 0$, $h_f(0, k) = 0$, $h_g(0, k) = 0$, $\tilde{f}(0, 0, t) = 0$, $\tilde{g}(0, 0, t) = 0$, $\tilde{h}_f(0, 0, k) = 0$, $\bar{h}_g(0, 0, k) = 0$ and $\tilde{h}_g(0, 0, k) = 0$ for all $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$ and they satisfy the global Lipschitz conditions.

Assumption 1: There is a constant $L > 0$ such that

$$\begin{aligned}|f(x, t) - f(\bar{x}, t)| \vee |g(x, t) - g(\bar{x}, t)| \vee |h_f(x, k) - h_f(\bar{x}, k)|\\ \vee |h_g(x, k) - h_g(\bar{x}, k)| \le L|x - \bar{x}|\end{aligned}$$

for all $(x, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^n$, $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$; and there is a constant $\tilde{L} > 0$ such that

$$\begin{aligned}&|\tilde{f}(x, y, t) - \tilde{f}(\tilde{x}, \tilde{y}, t)| \vee |\tilde{g}(x, y, t) - \tilde{g}(\tilde{x}, \tilde{y}, t)|\\ &\vee |\tilde{h}_f(x, y, k) - \tilde{h}_f(\tilde{x}, \tilde{y}, k)| \vee |\bar{h}_g(x, y, k) - \bar{h}_g(\tilde{x}, \tilde{y}, k)|\\ &\vee |\tilde{h}_g(x, y, k) - \tilde{h}_g(\tilde{x}, \tilde{y}, k)| \le \tilde{L}(|x - \tilde{x}| \vee |y - \tilde{y}|)\end{aligned}$$

for all $(x, y, \tilde{x}, \tilde{y}) \in \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^n \times \mathbb{R}^q$, $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$.

It is the intersection, interaction and interrelation of the physical system and its cyber counterpart [3], [10], [11], [22], [29], [34], [48] in the age of networking and information technology that motivate our study of stochastic impulsive system (1), which is formulated as a canonic form of CPSs that is a seamless, fully synergistic integration of the physical system and its cyber counterpart. It is observed in Section IV that the CPS of a numerical method for SDEs is a special case of (1) in which there is no impulse imposed on the physical subsystem (1a). We delibrately include the impulse (1c) and use the impulsive subsystem (1a,1c) in SiDE (1) to emphasize that the impulsive systems in the literature [9], [18], [30], [43], [46], [53] are just the physical subsystems (1a,1c) in our general class of SiDEs. We construct a systematic integration (1) of two impulsive subsystems in marked contrast to the impulsive systems in the literature, which highlights the distinction between our new class for the CPS models and those existing in the literature.

Remark 1: It should be noticed that, usually, the impulse interval of the subsystem $x(t)$ is a multiple of that of the subsystem $y(t)$ since the former is actually the interval between two consecutive physical impulses imposed on the physical system $x(t)$ while the latter the stepsize of the numerical method, see Section IV. In such a specific case of SiDE (1) in which $t_k = k\Delta t$ and Δt is the stepsize of the numerical method, functions $h_f(\cdot, k)$, $h_g(\cdot, k)$ and $\bar{h}_g(\cdot, \cdot, k)$ could be nonzero only if k is a multiple of integer $k_0 > 1$; otherwise, $h_f(\cdot, k) = 0$, $h_g(\cdot, k) = 0$ and $\bar{h}_g(\cdot, \cdot, k) = 0$, where $k_0\Delta t$ is the interval between two consecutive physical impulses.

Clearly, the trivial solution is the equilibrium of system (1). For a function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, the infinitesimal generator $\mathscr{L}V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ associated with system (1a) is defined as

$$\begin{aligned}\mathscr{L}V(x, t) &= V_t(x, t) + V_x(x, t)f(x, t)\\ &\quad + \frac{1}{2}trace\left[g^T(x, t)V_{xx}(x, t)g(x, t)\right],\end{aligned} \tag{2}$$

where $V_t(x, t) = \frac{\partial V(x,t)}{\partial t}$, $V_x(x, t) = \left[\frac{\partial V(x,t)}{\partial x_1} \cdots \frac{\partial V(x,t)}{\partial x_n}\right]$ and $V_{xx}(x, t) = \left[\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j}\right]_{n\times n}$. Similarly, for a function $\tilde{V} \in C^{2,1}(\mathbb{R}^q \times \mathbb{R}_+; \mathbb{R}_+)$, the generator $\tilde{\mathscr{L}}\tilde{V} : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}_+ \to \mathbb{R}$ associated with system (1b) is defined as

$$\begin{aligned}\tilde{\mathscr{L}}\tilde{V}(x, y, t) &= \tilde{V}_t(y, t) + \tilde{V}_y(y, t)\tilde{f}(x, y, t)\\ &\quad + \frac{1}{2}trace\left[\tilde{g}^T(x, y, t)\tilde{V}_{yy}(y, t)\tilde{g}(x, y, t)\right].\end{aligned} \tag{3}$$

Let $z(t) = [x^T(t) \ \ y^T(t)]^T \in \mathbb{R}^{n+q}$, $C = [I_n \ \ 0_{n\times q}]$ and $D = [0_{q\times n} \ \ I_q]$. Then $x(t) = Cz(t)$ and $y(t) = Dz(t)$ for all $t \ge 0$. The stochastic impulsive system (1) can be written in a compact form

$$\begin{aligned}&\mathrm{d}z(t)\\ &\quad = F(z(t), t)\mathrm{d}t + G(z(t), t)\mathrm{d}B(t) \qquad t \ne t_k\end{aligned} \tag{4a}$$

$$\begin{aligned}&\Delta_z(z(t_k^-), k)\\ &\quad := z(t_k) - z(t_k^-) = H_F(z(t_k^-), k)\\ &\qquad + \bar{H}_G(z(t_k^-), k)\bar{\xi}(k) + H_G(z(t_k^-), k)\xi(k) \quad k \in \mathbb{N}\end{aligned} \tag{4b}$$

with initial value $z(0) = [x(0)^T \ \ y(0)^T]^T \in \mathbb{R}^{n+q}$, where functions $F : \mathbb{R}^{n+q} \times \mathbb{R}_+ \to \mathbb{R}^{n+q}$, $G : \mathbb{R}^{n+q} \times \mathbb{R}_+ \to \mathbb{R}^{(n+q)\times m}$, $H_F : \mathbb{R}^{n+q} \times \mathbb{N} \to \mathbb{R}^{n+q}$, $\bar{H}_G : \mathbb{R}^{n+q} \times \mathbb{N} \to \mathbb{R}^{(n+q)\times n}$ and $H_G : \mathbb{R}^{n+q} \times \mathbb{N} \to \mathbb{R}^{(n+q)\times m}$ are given by

$$F(z, t) = \begin{bmatrix} f(Cz, t) \\ \tilde{f}(Cz, Dz, t)\end{bmatrix}, \quad G(z, t) = \begin{bmatrix} g(Cz, t) \\ \tilde{g}(Cz, Dz, t)\end{bmatrix},$$

$$H_F(z, k) = \begin{bmatrix} h_f(Cz, k) \\ \tilde{h}_f(Cz, Dz, k)\end{bmatrix},$$

$$\bar{H}_G(z, k) = \begin{bmatrix} h_g(Cz, k) \\ \bar{h}_g(Cz, Dz, k)\end{bmatrix},$$

$$H_G(z, k) = \begin{bmatrix} 0_{n\times m} \\ \tilde{h}_g(Cz, Dz, k)\end{bmatrix}.$$

The functions in stochastic impulsive system (4) obey $F(0, t) = 0$, $G(0, t) = 0$, $H_F(0, k) = 0$, $\bar{H}_G(0, k) = 0$ and $H_G(0, k) = 0$ for all $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$. And they satisfy the global Lipschitz condition Assumption 1, that is, there is a constant $L_z > 0$ such that

$$\begin{aligned}&|F(z,t)-F(\tilde{z},t)| \vee |G(z,t)-G(\tilde{z},t)| \\ &\quad \vee |H_F(z,k)-H_F(\tilde{z},k)| \vee |\bar{H}_G(z,k)-\bar{H}_G(\tilde{z},k))| \\ &\quad \vee |H_G(z,k)-H_G(\tilde{z},k))| \le L_z|z-\tilde{z}| \end{aligned} \tag{5}$$

for all $(z,\tilde{z}) \in \mathbb{R}^{n+q} \times \mathbb{R}^{n+q}$, $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$. It is easy to obtain the following result on existence and uniqueness of solutions for SiDE (4) (viz., system (1)).

Lemma 1: Under Assumption 1, there exists a unique (right-continuous) solution $z(t)$ to SiDE (4) on $t \ge 0$ and the solution belongs to $\mathcal{M}^2([0,T];\mathbb{R}^{n+q})$ for all $T \ge 0$.

Proof: Since system (4) satisfies the global Lipschitz condition (5), according to [38, Theorem 3.1, p.51], there exists a unique solution $z(t)$ to SiDE (4) on $t \in [t_0,t_1)$ and the solution belongs to $\mathcal{M}^2([t_0,t_1);\mathbb{R}^{n+q})$. Note that $\bar{\xi}(k+1)$ and $\xi(k+1)$ are independent of $\{z(t) : t \in [t_0,t_1)\}$. By virtue of continuity of functions $H_F(z,\cdot)$, $\bar{H}_G(z,\cdot)$ and $H_G(z,\cdot)$ with respect to z, there exists a unique solution $z(t_1)$ to (4) at $t = t_1$. Moreover, (4b) and (5) imply that the second moment of $z(t_1)$ is finite. And, again, according to [38, Theorem 3.1, p51], one has that there is a unique right-continuous solution $z(t)$ to (4) on $[t_0,t_2)$ and the solution belongs to $\mathcal{M}^2([t_0,t];\mathbb{R}^{n+q})$ for all $t \in [t_0,t_2)$. Recall that $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $t_k \to \infty$ as $k \to \infty$. By induction, one derives that there exists a unique (right-continuous) solution $z(t)$ to SiDE (4) for all $t \ge 0$ and the solution belongs to $\mathcal{M}^2([0,T];\mathbb{R}^{n+q})$ for all $T \ge 0$. □

Now that we have shown the existence and uniqueness of solutions to SiDE (4), we shall further study the stability of the solution to the SiDE. Let us introduce the definitions of exponential stability for SiDE (4).

Definition 1: [38, Definition 4.1, p.127] SiDE (4) is said to be pth ($p > 0$) moment exponentially stable if there is a pair of positive constants K and c such that

$$\mathbb{E}|z(t)|^p \le K|z(0)|^p e^{-ct}$$

for all $t \ge 0$, which leads to

$$\limsup_{t\to\infty} \frac{1}{t} \ln(\mathbb{E}|z(t)|^p) \le -c < 0$$

for all $z(0) \in \mathbb{R}^{n+q}$.

Definition 2: [38, Definition 3.1, p.119] SiDE (4) is said to be almost surely exponentially stable if

$$\limsup_{t\to\infty} \frac{1}{t} \ln|z(t)| < 0$$

for all $z(0) \in \mathbb{R}^{n+q}$.

III. LYAPUNOV STABILITY THEORY FOR THE GENERAL CLASS OF IMPULSIVE SYSTEMS

We dedicate this section to establishing by Lyapunov methods a stability theory for our proposed general class of SiDEs. The general class of SiDEs is formulated as a canonic form of CPSs and we shall develop a foundational theory for stability of CPSs, which will be applied to the CPS of a numerical method for SDEs. In the previous section, for simplicity, the compact form (4) of system (1) is employed to study the existence and uniqueness of solutions to the SiDEs. Now we exploit the structure of SiDE (4) which is composed of subsystems (1a,1c) and (1b,1d) and show the stability of the subsystems as well as that of the whole system (4).

Theorem 1: Suppose that Assumption 1 holds. Let $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+;\mathbb{R}_+)$ and $\tilde{V} \in C^{2,1}(\mathbb{R}^q \times \mathbb{R}_+;\mathbb{R}_+)$ be a pair of candidate Lyapunov functions for the subsystems (1a,1c) and (1b,1d), respectively, which satisfy

$$c_1|x|^p \le V(x,t) \le c_2|x|^p, \tag{6a}$$

$$\tilde{c}_1|y|^p \le \tilde{V}(y,t) \le \tilde{c}_2|y|^p \tag{6b}$$

for all $(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}_+$ and some positive constants $p, c_1, c_2, \tilde{c}_1, \tilde{c}_2$. Assume that there are positive constants α, $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, β, $\tilde{\beta}_1$, $\tilde{\beta}_2$ such that

$$\begin{aligned}&\mathscr{L}V(x,t) \\ &\quad \le -\alpha V(x,t),\end{aligned} \tag{7a}$$

$$\begin{aligned}&\tilde{\mathscr{L}}\tilde{V}(x,y,t) \le \tilde{\alpha}_1 V(x,t) + \tilde{\alpha}_2 \tilde{V}(y,t), \\ &\qquad t \in [t_k,t_{k+1})\end{aligned} \tag{7b}$$

$$\mathbb{E}\big[V(x+\Delta(x,k+1),t)\big|x\big] \le \beta V(x,t), \tag{7c}$$

$$\begin{aligned}&\mathbb{E}\big[\tilde{V}(y+\tilde{\Delta}(x,y,k+1),t)\big|x,y\big] \\ &\quad \le \tilde{\beta}_1 V(x,t) + \tilde{\beta}_2 \tilde{V}(y,t)\end{aligned} \tag{7d}$$

for all $(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}_+$ and $k \in \mathbb{N}$. The SiDE (4) is pth moment exponentially stable provided that the impulse time sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfies

$$\frac{\ln\beta}{\alpha} < \underline{\Delta t} \le \overline{\Delta t} < \frac{-\ln\tilde{\beta}_2}{\tilde{\alpha}_2}. \tag{8}$$

Proof: According to Lemma 1, that Assumption 1 holds implies there exists a unique solution to SiDE (1). Let us fix, for simplicity only, any $z(0) = [x(0)^T \;\; y(0)^T]^T \in \mathbb{R}^{n+q}$ and show the stability of the solution. The proof is rather technical so we devide it into five steps, in which we will show: 1) the exponential stability of $x(t)$; 2) some propeties of $y(t)$; 3) the exponential stability of $y(t)$ when $|x(0)| = 0$; 4) the exponential stability of $y(t)$ when $|x(0)| > 0$; 5) the exponential stability of $z(t)$. Some ideas and techniques in this proof are derived from our results on input-to-state stability (ISS) of SDEs [25, Theorem 3.1 and Remark 3.1], where $x(t)$ is treated as disturbance in the subsystem $y(t)$.

Step 1: Note that (6a), (7a) and (7d) as well as $\ln\beta < \alpha\underline{\Delta t}$ from (8) are a specific case of conditions (i), (ii) and (iii) of [30, Theorem 3] with $\lambda_1 = \gamma_1 = \cdots = \gamma_{\bar{m}} = 0$. This implies

$$\mathbb{E}V(x(t),t) \le V((0),0)\,e^{-\bar{\alpha}t} \quad \forall t \ge 0 \tag{9}$$

where $\bar{\alpha} \in (0,\alpha-\bar{a})$ and $\bar{a} \in (0,\alpha)$ with $\ln\beta < \bar{a}\underline{\Delta t} < \alpha\underline{\Delta t}$. By condition (6a), subsystem $x(t)$, which is part (1a,1c) of the system (1), is pth moment exponentially stable (with Lypunov exponent no larger than $-\bar{\alpha}$).

Step 2: Let us consider the dynamics of subsystem $y(t)$, which is the other part (1b,1d) of system (1). By Lemma 1 and the Itô formula, one can derive that

$$\mathbb{E}\tilde{V}(y(t),t) = \mathbb{E}\tilde{V}(y(\tilde{t}),\tilde{t}) + \int_{\tilde{t}}^{t} \mathbb{E}\tilde{\mathscr{L}}\tilde{V}(x(s),y(s),s)\mathrm{d}s \tag{10}$$

for all $t_k \le \tilde{t} \le t < t_{k+1}$ and $k \in \mathbb{N}$ while condition (7b) produces

$$\mathbb{E}\tilde{\mathscr{L}}\tilde{V}(y(t),t) \le \tilde{\alpha}_1\mathbb{E}V(x(t),t) + \tilde{\alpha}_2\mathbb{E}\tilde{V}(y(t),t) \tag{11}$$

on $[t_k, t_{k+1})$ for all $k \in \mathbb{N}$. This means that $\mathbb{E}\tilde{V}(y(t),t)$ is right-continuous on $[0,\infty)$ and could only jump at impulse times $\{t_{k+1}\}_{k\in\mathbb{N}}$. Notice condition (8) implies that $\tilde{\beta}_2 e^{\tilde{\alpha}_2\overline{\Delta t}} < 1$ and there is a pair of positives $\delta \in (0, 1-\tilde{\beta}_2)$ and $\bar{\delta} \in (0,\bar{\alpha}]$ sufficiently small for

$$(\tilde{\beta}_2+\delta)e^{(\tilde{\alpha}_2+\delta+\bar{\delta})\overline{\Delta t}} \le 1. \tag{12}$$

It is easy to observe from (11) that

$$\mathbb{E}\tilde{\mathscr{L}}\tilde{V}(y(t),t) \le (\tilde{\alpha}_2+\delta)\mathbb{E}\tilde{V}(y(t),t) \tag{13}$$

for such $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}$ that

$$\mathbb{E}\tilde{V}(y(t),t) \ge \frac{\tilde{\alpha}_1}{\delta}\mathbb{E}V(x(t),t).$$

Similarly, one can observe from (7d) that

$$\mathbb{E}\tilde{V}(y(t_{k+1}),t_{k+1}) \le (\tilde{\beta}_2+\delta)\mathbb{E}\tilde{V}(y(t_{k+1}^-),t_{k+1}^-) \tag{14}$$

whenever

$$\mathbb{E}\tilde{V}(y(t_{k+1}^-),t_{k+1}^-) \ge \frac{\tilde{\beta}_1}{\delta}\mathbb{E}V(x(t_{k+1}^-),t_{k+1}^-).$$

Step 3: If $x(0) = 0$ (namely, by (6a), $V(x(0),0) = 0$), then (9) gives $\mathbb{E}V(x(t),t) = 0$ for all $t \ge 0$. Using (7b), (10) and (11), one obtains

$$\mathbb{E}\tilde{V}(y(t),t) \le \tilde{V}(y(0),0) + \tilde{\alpha}_2\int_0^t \mathbb{E}\tilde{V}(y(s),s)\mathrm{d}s \tag{15}$$

for all $t \in [0,t_1)$. This, by the Gronwall inequality ([33, Lemma 4.5.1, p129], [38, Theorem 8.1, p45]), implies

$$\mathbb{E}\tilde{V}(y(t),t) \le \tilde{V}(y(0),0)\,e^{\tilde{\alpha}_2 t} \quad \forall t \in [0,t_1) \tag{16}$$

and, particularly, $\mathbb{E}\tilde{V}(y(t_1^-),t_1^-) \le \tilde{V}(y(0),0)\,e^{\tilde{\alpha}_2 t_1}$. Conditions (7d) and (12) produce

$$\begin{aligned}\mathbb{E}\tilde{V}(y(t_1),t_1) &\le \tilde{\beta}_2\mathbb{E}\tilde{V}(y(t_1^-),t_1^-)\\ &\le \tilde{\beta}_2\,\tilde{V}(y(0),0)\,e^{\tilde{\alpha}_2 t_1}\\ &< \tilde{V}(y(0),0)\,e^{-(\tilde{\alpha}_2+\delta+\bar{\delta})\overline{\Delta t}}\,e^{\tilde{\alpha}_2 t_1}\\ &\le \tilde{V}(y(0),0)e^{-(\delta+\bar{\delta})\overline{\Delta t}}.\end{aligned} \tag{17}$$

One can repeat the derivations (15)-(17) over the interval between any two consecutive impulse times and obtain

$$\mathbb{E}\tilde{V}(y(t),t) \le \tilde{V}(y(0),0)e^{\tilde{\alpha}_2(t-t_k)-k(\delta+\bar{\delta})\overline{\Delta t}} \tag{18}$$

for all $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}$. This implies

$$\mathbb{E}\tilde{V}(y(t),t) \le e^{(\tilde{\alpha}_2+\delta+\bar{\delta})\overline{\Delta t}}\,\tilde{V}(y(0),0)\,e^{-(\delta+\bar{\delta})t} \tag{19}$$

for all $t \ge 0$. By condition (6b), subsystem $y(t)$ is pth moment exponentially stable (with Lyapunov exponent no larger than $-(\delta+\bar{\delta})$) when $x(0) = 0$.

Step 4: Let us show the exponential stability of $y(t)$ when $|x(0)| > 0$, namely, $V(x(0),0) \ge c_1|x(0)| > 0$. Recall that both $\mathbb{E}V(x(t),t)$ and $\mathbb{E}\tilde{V}(y(t),t)$ are right-continuous on $[0,\infty)$, which could only jump at impulse times $\{t_{k+1}\}_{k\in\mathbb{N}}$. Define a function $\bar{v}: \mathbb{R}_+ \to \mathbb{R}$ as

$$\bar{v}(t) = \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\mathbb{E}V(x(t),t) - \mathbb{E}\tilde{V}(y(t),t) \tag{20}$$

for all $t \in [0,\infty)$ with initial value

$$\bar{v}(0) = \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}V(x(0),0) - \tilde{V}(y(0),0).$$

Due to the properties of $\mathbb{E}V(x(t),t)$ and $\mathbb{E}\tilde{V}(y(t),t)$, $\bar{v}(t)$ is right-continuous on $[0,\infty)$ and could only jump at impulse times $\{t_{k+1}\}_{k\in\mathbb{N}}$. Given any $t \ge 0$, either $\bar{v}(t) \ge 0$ or $\bar{v}(t) < 0$. So the interval $[0,\infty)$ is broken into a disjoint union of subsets $T_+ \cup T_-$, where

$$T_+ = \{t \ge 0 : \bar{v}(t) \ge 0\}, \quad T_- = \{t \ge 0 : \bar{v}(t) < 0\}. \tag{21}$$

From (9), it is easy to have

$$\begin{aligned}\mathbb{E}\tilde{V}(y(t),t) &\le \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\mathbb{E}V(x(t),t)\\ &\le \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\,V(x(0),0)\,e^{-\bar{\alpha}t} \quad \forall t \in T_+.\end{aligned} \tag{22}$$

Due to $V(x(0),0) > 0$, one has $\bar{v}(0) > 0$ if $\tilde{V}(y(0),0) = 0$; othewise, one can choose a sufficiently small δ such that

$$0 < \delta < (\tilde{\alpha}_1 \vee \tilde{\beta}_1)\frac{V(x(0),0)}{\tilde{V}(y(0),0)}$$

and, hence, $\bar{v}(0) > 0$. Without loss of generality, one can assume that $\bar{v}(0) > 0$. Due to the right-continuity, $\bar{v}(t) > 0$ on $[0,\epsilon)$ for some $\epsilon > 0$, i.e., $[0,\epsilon) \subset T_+$. If $T_+ = [0,\infty)$ (namely, $T_- = \emptyset$), by (22), the proof is complete. Otherwise (namely, $T_- \ne \emptyset$), let us consider the right-continuous process $\mathbb{E}\tilde{V}(y(t),t)$ on the subset T_-. Due to the right-continuity of $\bar{v}(t)$ on $[0,\infty)$, for any $\bar{t} \in T_-$, there exists an ordered pair $\tau_1(\bar{t}) < \tau_2(\bar{t})$ such that

$$\bar{t} \in \big(\tau_1(\bar{t}),\tau_2(\bar{t})\big) \subset T_-, \tag{23}$$

where $\tau_1(\bar{t}) = \inf\{\underline{\tau} \le \bar{t} : \bar{v}(\tau) < 0, \forall\tau \in [\underline{\tau},\bar{t}]\}$ and $\tau_2(\bar{t}) = \sup\{\bar{\tau} > \bar{t} : \bar{v}(\tau) < 0, \forall\tau \in [\bar{t},\bar{\tau})\}$. For convenience, we also write $\tau_1 = \tau_1(\bar{t})$ and $\tau_2 = \tau_2(\bar{t})$ where there is no ambiguity. Given any $\bar{t} \in T_-$, the interval $[\tau_1,\tau_2)$ falls into one of the three categories:

(C0) There is no impulse time on $[\tau_1,\tau_2)$.
(C1) There is exactly one impulse time on $[\tau_1,\tau_2)$.
(C2) There are at least two impulse times on $[\tau_1,\tau_2)$.

Each of them is considered as follows.

(C0) There is $k \in \mathbb{N}$ such that $t_k < \tau_1 < \tau_2 \le t_{k+1}$. Since $\bar{v}(t)$ is right-continuous and could only jump at impulse times

$\{t_{k+1}\}_{k\in\mathbb{N}}$, that $t_k < \tau_1 < \tau_2 \leq t_{k+1}$ implies that $\bar{v}(t)$ is continuous at $t = \tau_1$ and, by (23), $\bar{v}(\tau_1) = 0$. This means

$$\mathbb{E}\tilde{V}(y(\tau_1), \tau_1) = \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\mathbb{E}V(x(\tau_1), \tau_1) \leq \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta} V(x(0), 0)\, e^{-\bar{\alpha}\tau_1}. \quad (24)$$

Using the Gronwall inequality, one can derive from (10), (11) and (13) that

$$\mathbb{E}\tilde{V}(y(t), t) \leq e^{(\tilde{\alpha}_2+\delta)(t-\tau_1)}\mathbb{E}\tilde{V}(y(\tau_1), \tau_1) \quad (25)$$

for all $t \in [\tau_1, \tau_2)$. Notice that $\tau_2 - \tau_1 < t_{k+1} - t_k \leq \overline{\Delta t}$. Substitution of (24) into (25) yields, for all $t \in [\tau_1, \tau_2)$,

$$\begin{aligned}\mathbb{E}\tilde{V}(y(t), t) &< e^{(\tilde{\alpha}_2+\delta)\overline{\Delta t}}\mathbb{E}\tilde{V}(y(\tau_1), \tau_1)\\ &\leq \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta} e^{(\tilde{\alpha}_2+\delta)\overline{\Delta t}} V(x(0), 0)\, e^{-\bar{\alpha}\tau_1}\\ &\leq \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta} e^{(\bar{\alpha}+\tilde{\alpha}_2+\delta)\overline{\Delta t}} V(x(0), 0)\, e^{-\bar{\alpha}t}. \quad (26)\end{aligned}$$

(C1) There is exactly one impulse time t_k on $[\tau_1, \tau_2)$, where k is some positive integer since $[0, \epsilon) \subset T_+$. There are two cases: (C1a) $\tau_1 < t_k$ and (C1b) $\tau_1 = t_k$.

(C1a) There is $k \geq 1$ such that $t_{k-1} < \tau_1 < t_k < \tau_2 \leq t_{k+1}$. As above, this means that $\bar{v}(t)$ is continuous on $t = \tau_1$ and $\bar{v}(\tau_1) = 0$. So (24) holds at $t = \tau_1$ and (26) for all $t \in [\tau_1, t_k)$. But, from (14) and (12),

$$\begin{aligned}\mathbb{E}\tilde{V}(y(t_k), t_k) &\leq (\tilde{\beta}_2 + \delta)\mathbb{E}\tilde{V}(y(t_k^-), t_k^-)\\ &\leq (\tilde{\beta}_2 + \delta)e^{(\tilde{\alpha}_2+\delta)(t_k-\tau_1)}\mathbb{E}\tilde{V}(y(\tau_1), \tau_1)\\ &\leq (\tilde{\beta}_2 + \delta)e^{(\tilde{\alpha}_2+\delta)\overline{\Delta t}}\mathbb{E}\tilde{V}(y(\tau_1), \tau_1)\\ &\leq e^{-\bar{\delta}\,\overline{\Delta t}}\mathbb{E}\tilde{V}(y(\tau_1), \tau_1). \quad (27)\end{aligned}$$

Using the Gronwall inequality, one can derive from inequalities (10), (11), (13) and (27) that

$$\begin{aligned}\mathbb{E}\tilde{V}(y(t), t) &\leq e^{(\tilde{\alpha}_2+\delta)(t-t_k)}\mathbb{E}\tilde{V}(y(t_k), t_k)\\ &\leq e^{(\tilde{\alpha}_2+\delta)\overline{\Delta t}}\mathbb{E}\tilde{V}(y(t_k), t_k)\\ &\leq e^{(\tilde{\alpha}_2+\delta-\bar{\delta})\overline{\Delta t}}\mathbb{E}\tilde{V}(y(\tau_1), \tau_1) \quad \forall\, t \in [t_k, \tau_2). \quad (28)\end{aligned}$$

Therefore, when $\tau_1 < t_k < \tau_2$, combination of (24), (26) and (28) imply that (26) holds for all $t \in [\tau_1, \tau_2)$.

(C1b) By the definition of τ_1 as well as the right-continuity of $\bar{v}(t)$, that $\tau_1 = t_k$ implies $\bar{v}(t_k^-) \geq 0$ and hence

$$\mathbb{E}\tilde{V}(y(t_k^-), t_k^-) \leq \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\mathbb{E}V(x(t_k^-), t_k^-). \quad (29)$$

Inequalities (7d), (9) and (29) produce

$$\begin{aligned}\mathbb{E}\tilde{V}(y(\tau_1), \tau_1) &= \mathbb{E}\tilde{V}(y(t_k), t_k)\\ &\leq \Big(\tilde{\beta}_1 + \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\,\tilde{\beta}_2\Big)\,\mathbb{E}V(x(t_k^-), t_k^-)\\ &\leq \Big(\tilde{\beta}_1 + \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\,\tilde{\beta}_2\Big)\,V(x(0), 0)\, e^{-\bar{\alpha}\tau_1} \quad (30)\end{aligned}$$

and, therefore,

$$\begin{aligned}&\mathbb{E}\tilde{V}(y(t), t)\\ &\leq e^{(\tilde{\alpha}_2+\delta)(t-\tau_1)}\mathbb{E}\tilde{V}(y(\tau_1), \tau_1)\\ &\leq \Big(\tilde{\beta}_1 + \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\,\tilde{\beta}_2\Big)\, e^{(\tilde{\alpha}_2+\delta)(t-\tau_1)}\,\mathbb{E}V(x(\tau_1^-), \tau_1^-)\\ &\leq \Big(\tilde{\beta}_1 + \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\,\tilde{\beta}_2\Big)\, e^{(\tilde{\alpha}_2+\delta)\overline{\Delta t}}\, V(x(0), 0)\, e^{-\bar{\alpha}\tau_1}\\ &\leq \Big(\tilde{\beta}_1 + \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\,\tilde{\beta}_2\Big)\, e^{(\bar{\alpha}+\tilde{\alpha}_2+\delta)\overline{\Delta t}}\, V(x(0), 0)\, e^{-\bar{\alpha}t}\end{aligned}$$

for all $t \in [\tau_1, \tau_2)$. This combined with (26) yields

$$\mathbb{E}\tilde{V}(y(t), t) \leq K\, V(x(0), 0)\, e^{-\bar{\alpha}t} \quad \forall\, t \in [\tau_1, \tau_2) \quad (31)$$

when there is only one impulse time on the interval $[\tau_1, \tau_2)$, where K is a positive constant

$$K = \Big(\frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta} \vee \Big(\tilde{\beta}_1 + \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\,\tilde{\beta}_2\Big)\Big)\, e^{(\bar{\alpha}+\tilde{\alpha}_2+\delta)\overline{\Delta t}}.$$

(C2) There are at least two impulse times on $[\tau_1, \tau_2)$. For any two consecutive impulse times t_k and t_{k+1} on $[\tau_1, \tau_2)$, using the reasoning as above, one can derive that

$$\mathbb{E}\tilde{V}(y(t), t) \leq e^{(\tilde{\alpha}_2+\delta)(t-t_k)}\,\mathbb{E}\tilde{V}(y(t_k), t_k) \quad (32)$$

for all $t \in [t_k, t_{k+1})$ and then

$$\begin{aligned}\mathbb{E}\tilde{V}(y(t_{k+1}), t_{k+1}) &\leq (\tilde{\beta}_2 + \delta)\,\mathbb{E}\tilde{V}(y(t_{k+1}^-), t_{k+1}^-)\\ &\leq (\tilde{\beta}_2 + \delta)\, e^{(\tilde{\alpha}_2+\delta)(t_{k+1}-t_k)}\,\mathbb{E}\tilde{V}(y(t_k), t_k)\\ &\leq (\tilde{\beta}_2 + \delta)\, e^{(\tilde{\alpha}_2+\delta)\overline{\Delta t}}\,\mathbb{E}\tilde{V}(y(t_k), t_k)\\ &\leq e^{-\bar{\delta}\,\overline{\Delta t}}\mathbb{E}\tilde{V}(y(t_k), t_k). \quad (33)\end{aligned}$$

Denote by $t_k < \cdots < t_{\bar{k}+1} < \cdots$ the impulse times on $[\tau_1, \tau_2)$, where $\bar{k} \geq k \geq 1$. Let us consider $\mathbb{E}\tilde{V}(y(t), t)$ on the interval $[t_k, \tau_2)$. Using (32) and (33), one obtains

$$\mathbb{E}\tilde{V}(y(t), t) \leq e^{(\tilde{\alpha}_2+\delta)(t-t_{\bar{k}})-(\bar{k}-k)\,\bar{\delta}\,\overline{\Delta t}}\,\mathbb{E}\tilde{V}(y(t_k), t_k)$$

for all $t \in [t_{\bar{k}}, t_{\bar{k}+1} \wedge \tau_2)$ and, therefore,

$$\mathbb{E}\tilde{V}(y(t), t) \leq e^{(\tilde{\alpha}_2+\delta+\bar{\delta})\,\overline{\Delta t}-\bar{\delta}\,(t-t_k)}\,\mathbb{E}\tilde{V}(y(t_k), t_k) \quad (34)$$

for all $t \in [t_k, \tau_2)$. Recall that $0 < \bar{\delta} \leq \bar{\alpha}$ and $0 \leq t_k - \tau_1 \leq \overline{\Delta t}$. Again, there are two cases: $\tau_1 < t_k$ and $\tau_1 = t_k$. In the case where $\tau_1 < t_k$, from (24), (27) and (34), one has

$$\begin{aligned}&\mathbb{E}\tilde{V}(y(t), t)\\ &\leq e^{(\tilde{\alpha}_2+\delta+\bar{\delta})\,\overline{\Delta t}-\bar{\delta}\,(t-t_k)}\, e^{-\bar{\delta}\,\overline{\Delta t}}\mathbb{E}\tilde{V}(y(\tau_1), \tau_1)\\ &\leq \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\, e^{(\tilde{\alpha}_2+\delta+\bar{\delta})\,\overline{\Delta t}}\, V(x(0), 0)\, e^{-(\bar{\alpha}\tau_1+\bar{\delta}\,\overline{\Delta t}-\bar{\delta}t_k)-\bar{\delta}\,t}\\ &\leq \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\, e^{(\tilde{\alpha}_2+\delta+\bar{\delta})\,\overline{\Delta t}}\, V(x(0), 0)\, e^{-\bar{\delta}\,(\tau_1+\overline{\Delta t}-t_k)-\bar{\delta}\,t}\\ &\leq \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\, e^{(\tilde{\alpha}_2+\delta+\bar{\delta})\,\overline{\Delta t}}\, V(x(0), 0)\, e^{-\bar{\delta}\,t} \quad (35)\end{aligned}$$

for all $t \in [t_k, \tau_2)$ and then, by (26),

$$\mathbb{E}\tilde{V}(y(t), t)$$

$$\leq \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta} e^{(\tilde{\alpha}_2+\delta+\bar{\delta})\overline{\Delta t}} V(x(0),0)\, e^{-\bar{\delta}\, t} \tag{36}$$

for all $t \in [\tau_1, \tau_2)$. In the other case where $\tau_1 = t_k$, substitution of (30) into (34) gives

$$\begin{aligned}\mathbb{E}\tilde{V}(y(t),t) &\leq \big(\tilde{\beta}_1 + \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\tilde{\beta}_2\big) \\ &\cdot e^{(\tilde{\alpha}_2+\delta+\bar{\delta})\overline{\Delta t}} V(x(0),0)\, e^{-(\bar{\alpha}-\bar{\delta})\,\tau_1-\bar{\delta}\, t} \\ &\leq \big(\tilde{\beta}_1 + \frac{(\tilde{\alpha}_1 \vee \tilde{\beta}_1)}{\delta}\tilde{\beta}_2\big)\, e^{(\tilde{\alpha}_2+\delta+\bar{\delta})\overline{\Delta t}} V(x(0),0)\, e^{-\bar{\delta}\, t}\end{aligned} \tag{37}$$

on $t \in [\tau_1, \tau_2)$. Combination of (36) and (37) yields

$$\mathbb{E}\tilde{V}(y(t),t) \leq KV(x(0),0)\, e^{-\bar{\delta}\, t} \tag{38}$$

for all $t \in [\tau_1, \tau_2)$ on which there are at least two impulse times, where K is the positive constant given by (31). From inequlities (26), (31) and (38), one has

$$\mathbb{E}\tilde{V}(y(t),t) \leq KV(x(0),0)\, e^{-\bar{\delta}\, t} \quad \forall t \in T_-. \tag{39}$$

Combining (22) and (39), one can conclude that

$$\mathbb{E}\tilde{V}(y(t),t) \leq KV(x(0),0)\, e^{-\bar{\delta}\, t} \quad \forall t \geq 0. \tag{40}$$

By condition (6b), this means that subsystem $y(t)$ is pth moment exponentially stable (with Lyapunov exponent no larger than $-\bar{\delta}$) when $|x(0)| > 0$.

Step 5: We have shown the pth moment exponential stability of $x(t)$ by (9) and that of $y(t)$ by (19) and (40) when $|x(0)| = 0$ and $|x(0)| > 0$, respectively.

Note that $z(t) = [x^T(t)\ y^T(t)]^T$ and, therefore, $|z(t)|^2 = |x(t)|^2 + |y(t)|^2$ for all $t \geq 0$. It is easy to see that

$$\begin{aligned}|z(t)|^p = (|z(t)|^2)^{p/2} &= (|x(t)|^2 + |y(t)|^2)^{p/2} \\ &\leq k_p(|x(t)|^p + |y(t)|^p),\end{aligned} \tag{41}$$

where $k_p = 1$ when $0 < p < 2$ and $k_p = 2^{(p-2)/2}$ if $p \geq 2$. In the case where $|x(0)| = 0$ and $\mathbb{E}|x(t)|^p = 0$ for all $t \geq 0$, by (19) and (41) as well as $|z(0)| = |y(0)|$,

$$\begin{aligned}\mathbb{E}|z(t)|^p &\leq k_p \mathbb{E}|y(t)|^p \\ &\leq \frac{k_p}{\tilde{c}_1} e^{(\tilde{\alpha}_2+\delta+\bar{\delta})\overline{\Delta t}}\, \tilde{V}(y(0),0)\, e^{-(\delta+\bar{\delta})\, t} \\ &\leq \frac{k_p \tilde{c}_2}{\tilde{c}_1} e^{(\tilde{\alpha}_2+\delta+\bar{\delta})\overline{\Delta t}} |z(0)|^p e^{-(\delta+\bar{\delta})\, t}\end{aligned} \tag{42}$$

for all $t \geq 0$. In the general case where $|x(0)| > 0$, by (6), (9), (40) and (41),

$$\begin{aligned}\mathbb{E}|z(t)|^p &\leq \frac{k_p c_2}{c_1} |x(0)|^p \big((1 \vee \beta)\, e^{-\bar{\alpha} t} + K\, e^{-\bar{\delta}\, t}\big) \\ &\leq \bar{K}_p |z(0)|^p e^{-\bar{\delta}\, t}\end{aligned} \tag{43}$$

for all $t \geq 0$, where K is the positive constant given by (31) and $\bar{K}_p$ is a positive constant

$$\bar{K}_p = \frac{k_p\, c_2}{c_1}((1 \vee \beta) + K).$$

So (42) and (43) mean that system (4), or say, system (1) is pth moment exponentially stable (with Lyapunov exponent no larger than $-\bar{\delta}$). □

Remark 2: Notice that, in Theorem 1, the continuous dynamics of subsystem $x(t)$ stabilizes the subsystem, though the discrete one could destabilize it, while the discrete dynamics of subsystem $y(t)$ stabilizes the subsystem, though the continuous one could destabilize it, which results in the exponential stability of the both subsystems and hence that of the whole system $z(t) = [x^T(t)\ \ y^T(t)]^T$. Similarly, one can obtain a stability criterion for the case where the impulses stabilize the physical subsystem $x(t)$ as the continuous dynamics could destabilize it (see [30, Theorem 2]) while the conditions on the subsystem $y(t)$ are kept the same as those in Theorem 1.

Furthermore, under Assumption 1, we have the following result on the almost sure stability of system (1).

Theorem 2: If Assumption 1 holds, then the pth ($p > 0$) moment exponential stability of SiDE (4) (i.e., system (1)) implies that it is also almost surely exponentially stable.

The proof is similar to that of [38, Theorem 4.2, p.128] and, therefore, is omitted.

IV. THE CYBER-PHYSICAL SYSTEMS OF NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS

In this section, we address the problem (I) of fundamental importance. We compose a hybrid model in the form of our proposed SiDE (1) to represent the tight integration of the physical system (the SDE) and its cyber counterpart (the numerical method). This systematic representation is expressed by our canonic form of CPS models.

Let us consider a physical system described by the SDE

$$\mathrm{d}x(t) = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}B(t) \quad \forall t \geq 0 \tag{44}$$

with initial value $x(0) \in \mathbb{R}^n$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy the global Lipschitz condition

$$|f(x) - f(\bar{x})| \vee |g(x) - g(\bar{x})| \leq L|x - \bar{x}| \tag{45}$$

for all $(x, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^n$ as well as $f(0) = 0$ and $g(0) = 0$ for study of the stability problem. Given a fixed parameter $\theta \in [0, 1]$, the following numerical scheme for SDE (44) is called the stochastic theta method [19], [21], [39], [40]

$$\begin{aligned}X_{k+1} = X_k &+ (1-\theta)f(X_k)\Delta t + \theta f(X_{k+1})\Delta t \\ &+ g(X_k)\sqrt{\Delta t}\,\xi(k+1) \qquad \forall k \in \mathbb{N}\end{aligned} \tag{46}$$

with initial value $X_0 = x(0)$, where $\Delta t > 0$ is the constant stepsize and $\sqrt{\Delta t}\,\xi(k+1)$ is the implementation of the increment $\Delta B_k = B((k+1)\Delta t) - B(k\Delta t)$. The stochastic theta method for SDEs is a set of popular algorithms [20], [33] employed to describe and compute the physical dynamics (44) in the techniques driven by software modelling and simulation tools. The numerical method (46) is in essence a cyber model of the physical system (44), which is a translation of (44) into discretization, the language in computers, and represents the physical dynamics in the cyber world.

When $\theta = 0$, the numerical scheme (46) gives the widely-used Euler-Maruyama method. The Euler-Maruyama method applied to SDE (44) computes approximations $X_k \approx x(t_k)$ by setting $X_0 = x(0)$ and forming

$$X_{k+1} = X_k + f(X_k)\Delta t + g(X_k)\sqrt{\Delta t}\,\xi(k+1) \tag{47}$$

for all $k \in \mathbb{N}$, where $t_k = k\Delta t$. Stochastic difference equations (47), also known as discrete-time stochastic systems [28], have been intensively studied over the past a few decades in the age of computers. In practice, it is natural to form and use some continuous-time extension of the discrete approximations $\{X_k\}_{k\in\mathbb{N}}$ such as [21], [40]

$$X(t) = \sum_{k=0}^{\infty} X_k \mathbb{1}_{[t_k, t_{k+1})}(t) \quad \forall t \geq 0 \tag{48}$$

where $\mathbb{1}_T$ is the indicator function of set T. This is a simple step process of the equidistant Euler-Maruyama approximations so its sample paths are continuous on (t_k, t_{k+1}) for each $k \in \mathbb{N}$ and right-continuous on $[0, \infty)$.

This paper considers the widely-used Euler-Maruyama method (47)-(48) the cyber system, which is virtually a representative of the physical system (44) in the cyber world. Other numerical schemes, or say, other translations can also be employed to represent the physical system in the cyber world in future work. This section is to discover the inherent relationship between a physical system and its cyber couterpart. Consider the process $y(t)$ of difference between the exact solution $x(t)$ of the physical system (44) and the numerical solution $X(t)$ by its cyber counterpart (47)-(48)

$$y(t) = x(t) - X(t) \qquad \forall t \geq 0 \tag{49}$$

with initial value $y(0) = x(0) - X(0) = 0$. Notice that $x(t)$ is a process of continuous paths and $X(t)$ a simple step process. This implies that $y(t)$ is right-continuous on $[0, \infty)$ and could only jump at $\{t_{k+1}\}_{k\in\mathbb{N}}$. According to the scheme (47)-(48), the jump of $y(t)$ at each $t = t_{k+1}$ for $k \in \mathbb{N}$ gives

$$\begin{aligned} y(t_{k+1}) &- y(t_{k+1}^-) \\ &= x(t_{k+1}) - X(t_{k+1}) - \big(x(t_{k+1}^-) - X(t_{k+1}^-)\big) \\ &= X(t_{k+1}^-) - X(t_{k+1}) = X(t_k) - X(t_{k+1}) \\ &= -f(X_k)\Delta t - g(X_k)\sqrt{\Delta t}\,\xi(k+1) \\ &= -f(X(t_{k+1}^-))\Delta t - g(X(t_{k+1}^-))\sqrt{\Delta t}\,\xi(k+1) \\ &= -f(x(t_{k+1}^-) - y(t_{k+1}^-))\Delta t \\ &\quad - g(x(t_{k+1}^-) - y(t_{k+1}^-))\sqrt{\Delta t}\,\xi(k+1) \end{aligned} \tag{50}$$

since $X(t) = x(t) - y(t) = X_k$ for all $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}$. The integrative dynamics of the physical system (44) and the process (49) of difference is described by the following hybrid system in the form of SiDEs

$$\begin{aligned} &\mathrm{d}x(t) \\ &\quad = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}B(t) \end{aligned} \tag{51a}$$

$$\begin{aligned} &\mathrm{d}y(t) \\ &\quad = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}B(t) \quad t \in [t_k, t_{k+1}) \end{aligned} \tag{51b}$$

$$\begin{aligned} \tilde{\Delta}(x(t_{k+1}^-), y(t_{k+1}^-), k+1) &:= y(t_{k+1}) - y(t_{k+1}^-) \\ &= -f(x(t_{k+1}^-) - y(t_{k+1}^-))\Delta t \\ &\quad - g(x(t_{k+1}^-) - y(t_{k+1}^-))\sqrt{\Delta t}\,\xi(k+1) \quad k \in \mathbb{N} \end{aligned} \tag{51c}$$

with $x(0) \in \mathbb{R}^n$ and $y(0) = x(0) - X(0) = 0$. Clearly, the physical system (44) has no impulse and its cyber-physical model (51) of the Euler-Maruyama method is a specific case of our canonic form (1) of CPSs in which $q = n, f(x,t) = f(x)$, $g(x,t) = g(x)$, $\tilde{f}(x,y,t) = f(x)$, $\tilde{g}(x,y,t) = g(x)$, $h_f(x,k) = 0$, $h_g(x,k) = 0$, $\tilde{h}_f(x,y,k) = -f(x-y)\Delta t$, $\tilde{h}_g(x,y,k) = 0$, $\tilde{h}_g(x,y,k) = -g(x-y)\sqrt{\Delta t}$ and $t_k = k\Delta t$. Consequently, the infinitesimal generators (2) and (3) associated with (51a) and (51b) are of the specific forms

$$\begin{aligned} \mathscr{L}V(x) &= V_x(x)f(x) + \frac{1}{2}trace\left[g^T(x)\,V_{xx}(x)\,g(x)\right], \\ \mathscr{L}\tilde{V}(x,y) &= \tilde{V}_y(y)f(x) \\ &\quad + \frac{1}{2}trace\left[g^T(x)\,\tilde{V}_{yy}(y)\,g(x)\right], \end{aligned} \tag{52}$$

respectively. It is easy to see that Assumption 1 holds since both f and g satisfy the global Lipschitz condition (45). According to Lemma 1, there exists a unique (right-continuous) solution to SiDE (51) on $t \geq 0$ and the solution belongs to $\mathcal{M}^2([0,T];\mathbb{R}^{2n})$ for all $T \geq 0$. Moreover, the results of our established stability theory for the general class (1) of SiDEs, say, Theorem 1 and Theorem 2 apply to the CPS (51).

We construct the CPS (51) of the widely-used Euler-Maruyama method (47)-(48) for the SDE (44), which is a seamless, fully synergistic integration of the physical system (44) and its cyber counterpart (47)-(48). The CPS not only provides a holistic view of the physical system and its cyber counterpart but also reveals their intrinsic relationship that they are not two separate systems but the components of an integrative system. Recall that the SDE describes our knowledge of the physical dynamics while the numerical method is the cyber representive, namely, the translation of our knowledge in the cyber world. As a result, the CPS (51) is an integration of our knowledge of the physical system and the cyber representative as well as the simulation sequence $\{\xi(k)\}_{k\in\mathbb{N}}$. Moreover, the CPS clearly shows that the numerical solution is driven by the SDE and the numerical method as well as the simulation sequence while the exact solution is, of course, conducted by the SDE itself only. Usually, to control the underlying physical processes, our knowledge of both the physical and the cyber sides of the system is utilized in the synthesis of the CPS. This leads to the resulting CPS with $y(t)$ involved in the dynamics/system equation of $x(t)$ as well. Such synthesized CPSs are considered in our study of stabilization problems.

Remark 3: We have derived with details the CPS (51) of the Euler-Maruyama method (47)-(48) for the SDE (44). It is not difficult to follow the exemplary derivation and obtain the CPS of the stochastic theta method (46) for the SDE (44),

$$\mathrm{d}x(t) = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}B(t) \tag{53a}$$

$$\mathrm{d}y(t) = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}B(t) \quad t \in [t_k, t_{k+1})$$

$$\times y(t_{k+1}) + \theta f(x(t_{k+1}^-) - y(t_{k+1})) \tag{53b}$$

$$= y(t_{k+1}^-) - (1-\theta) f(x(t_{k+1}^-) - y(t_{k+1}^-))\Delta t - g(x(t_{k+1}^-) - y(t_{k+1}^-))\sqrt{\Delta t}\,\xi(k+1) \quad k \in \mathbb{N} \tag{53c}$$

where the impulse of $y(t)$ at $t = t_{k+1}$ is generated by (53c) based on $x(t_{k+1}^-)$ and $y(t_{k+1}^-)$ as well as the simulation $\sqrt{\Delta t}\,\xi(k+1)$ of $\Delta B_k = B((k+1)\Delta t) - B(k\Delta t)$. Notice that (53) is also a formal expression of impulsive systems in the literature [18]. Similarly, one can derive the CPS of some other given numerical scheme for the SDE, which is among future work suggested in Section VII.

V. EXPONENTIAL STABILITY OF THE CYBER-PHYSICAL SYSTEMS OF NUMERICAL METHODS

The CPS (51) of the Euler-Maruyama method (47)-(48) for the SDE (44) consists of the physical and the cyber subsystems. The key questions (Q1) and (Q2) naturally arise. In this section, we address the problem (II) of fundamental importance and prove positive results to the key questions (Q1) and (Q2). These fundamental results and their applicaton to linear systems comprise a foundational theory of the CPSs of numerical methods for SDEs.

Let us begin with the test problem (Q1) of the CPS (51), to which Theorem 1 and Theorem 2 can be directly applied. Under some conditions (see [15], [32]), a seminal converse Lyapunov theorem [32, Theorem 5.12, p172] states that, if the SDE (44) is pth moment exponentially stable, there is a Lypunov function that proves the exponential stability of the dynamical system. One may postulate that the Lyapunov function for the physical subsystem (51a) could help construct a candidate Lyapunov function for the subsystem (51b,51c) due to their interrelation. The direct application of Theorem 1 to the CPS (51) shows that the CPS (51) and, hence, the cyber system (47)-(48) share the exponential stability with the physical system (44).

Theorem 3: Let $V \in C^2(\mathbb{R}^n; \mathbb{R}_+)$ be a candidate Lyapunov function for both subsystems (51a) and (51b,51c) and

$$c_1|x|^p \leq V(x) \leq c_2|x|^p \qquad \forall x \in \mathbb{R}^n \tag{54}$$

for some positives p, c_1, c_2. Assume that there are positives $\alpha, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\beta}_1, \tilde{\beta}_2$ such that

$$\mathscr{L}V(x) \leq -\alpha V(x) \quad \forall x \in \mathbb{R}^n \tag{55a}$$

$$\tilde{\mathscr{L}}V(x,y) \leq \tilde{\alpha}_1 V(x) + \tilde{\alpha}_2 V(y) \quad \forall t \in [t_k, t_{k+1}) \tag{55b}$$

$$\mathbb{E}\big[V(y + \tilde{\Delta}(x,y,k+1))\big|x,y\big] \leq \tilde{\beta}_1 V(x) + \tilde{\beta}_2 V(y) \quad \forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \tag{55c}$$

for all $k \in \mathbb{N}$. If the stepsize

$$\Delta t < \frac{-\ln \tilde{\beta}_2}{\tilde{\alpha}_2}, \tag{56}$$

then the CPS (51) is pth moment exponentially stable and is also almost surely exponentially stable. Moreover, the cyber system (47)-(48) shares the pth moment exponential stability with its underlying physical system (44) and, hence, it is also almost surely exponentially stable.

Proof: From Theorem 1 and Theorem 2, it follows that CPS (51), which is a specific case of system (1) (namely, SiDE (4)), is pth moment exponentially stable and is also almost surely exponentially stable.

Notice that the state $X(t) = x(t) - y(t)$ of cyber system (47)-(48) is the difference of the subsystems (51a) and (51b,51c). Therefore,

$$|X(t)|^p \leq \bar{k}_p(|x(t)|^p + |y(t)|^p) \leq 2\bar{k}_p|z(t)|^p \tag{57}$$

for all $t \geq 0$, where $\bar{k}_p = 1$ if $0 < p < 1$, and $\bar{k}_p = 2^{p-1}$ if $p \geq 1$. This immediately implies that the cyber system (47)-(48) is pth moment exponentially stable and is also almost surely exponentially stable. □

This means that, if the underlying physical system (44) is pth moment exponentially stable, the CPS (51) and, hence, the numerical method (47)-(48) reproduce the pth moment exponential stability of the physical dynamics when the conditions in Theorem 3 hold. The ability of the cyber system (the numerical method) to reproduce the mean-square exponential stability of its underlying physical system (the SDE) has been studied in [19] and [22], In our proposed framework of CPS (51), let us consider the ability of the cyber system (47)-(48) to reproduce the mean-square exponential stability of the physical system (44). A result on mean-square exponential stability is then derived from Theorem 3 as follows, in which the Lyapunov function for mean-square exponential stability of the underlying physical system (44) also proves the mean-square exponential stability of its cyber counterpart (47)-(48) as well as that of the CPS (51).

Theorem 4: Let the candidate Lyapunov function $V \in C^2(\mathbb{R}^n; \mathbb{R}_+)$ for physical system (44) be a quadratic function

$$V(x) = x^T P x \tag{58}$$

for some positive definite matrix $P \in \mathbb{R}^{n \times n}$. Assume there are positives $\bar{\alpha}$ and $\overline{\Delta t}$ with $\bar{\alpha}\,\overline{\Delta t} < 1$ such that

$$\mathscr{L}V(x) + \overline{\Delta t}\, V(f(x)) \leq -\bar{\alpha} V(x) \tag{59}$$

for all $x \in \mathbb{R}^n$. Then the CPS (51) with $\Delta t \in (0, \overline{\Delta t}\,]$ is mean-square exponentially stable and is also almost surely exponentially stable. Moreover, the cyber system (47)-(48) with $\Delta t \in (0, \overline{\Delta t}\,]$ shares the mean-square exponential stability with its underlying physical system (44) and, hence, it is also almost surely exponentially stable.

Proof: It will follow the conclusion from Theorem 3 if one shows that conditions (54)-(56) of Theorem 3 are satisfied with $p = 2$ for the CPS (51). Since the quadratic function (58) gives $\lambda_m(P)\,|x|^2 \leq V(x) \leq \lambda_M(P)\,|x|^2$, condition (54) holds for positive constants $p = 2$, $c_1 = \lambda_m(P)$, $c_2 = \lambda_M(P)$. It is not difficult to observe that (55a) and (59) are equivalent. Obviously, (59) implies that (55a) holds for some positive constant $\alpha \geq \bar{\alpha}$. But (55a) implies that there is a pair of positive numbers $\bar{\alpha}$ and $\overline{\Delta t}$ such that (59) holds. For instance, if (55a) holds for some $\alpha > 0$, the pair of positives, $\bar{\alpha} < \alpha \wedge \frac{\lambda_M(P)\, L^2}{\alpha \lambda_m(P)}$ and $\overline{\Delta t} = \frac{(\alpha-\bar{\alpha})\lambda_m(P)}{\lambda_M(P)\, L^2}$, yields $\bar{\alpha}\, \overline{\Delta t} < (\alpha - \bar{\alpha})/\alpha < 1$, $\overline{\Delta t}\, V(f(x)) \leq \overline{\Delta t}\, \lambda_M(P)\, L^2 |x|^2 \leq (\alpha - \bar{\alpha})\lambda_m(P)\, |x|^2 \leq (\alpha - \bar{\alpha}) V(x)$ and thus (59). According to [38, Theorem 4.4, p.130] and [38, Theorem 4.2, p.128], system (44) is mean-square exponentially stable and is also almost surely exponentially stable.

By the Itô formula, [27, Lemmas 1 and 2] and global Lipschitz condition (45),

$$\begin{aligned}
&\tilde{\mathscr{L}} V(x, y) \\
&\quad = 2\, y^T P f(x) + trace\big[g^T(x)\, P g(x)\big] \\
&\quad \leq \tilde{\alpha}_2 y^T P y + \tilde{\alpha}_2^{-1} f^T(x)\, P f(x) + \lambda_M(P)\ trace\big[g^T(x) g(x)\big] \\
&\quad \leq \tilde{\alpha}_2^{-1} \lambda_M(P) |f(x)|^2 + \lambda_M(P)\ L^2 |x|^2 + \tilde{\alpha}_2 \tilde{V}(y) \\
&\quad \leq (\tilde{\alpha}_2^{-1} + 1)\lambda_M(P) L^2\, |x|^2 + \tilde{\alpha}_2 \tilde{V}(y) \\
&\quad \leq \tilde{\alpha}_1 x^T P x + \tilde{\alpha}_2 \tilde{V}(y) = \tilde{\alpha}_1 V(x) + \tilde{\alpha}_2 \tilde{V}(y), \qquad (60)
\end{aligned}$$

where

$$\tilde{\alpha}_1 = \frac{(1 + \tilde{\alpha}_2)\, \lambda_M(P) L^2}{\tilde{\alpha}_2\, \lambda_m(P)}$$

and $\tilde{\alpha}_2$ given as (66) below are both positive numbers. So condition (55b) of Theorem 3 is satisfied. Note that, $\forall\, \Delta t \in (0, \overline{\Delta t}\,]$, inequality (59) implies

$$\mathscr{L} V(x) + \Delta t\, V(f(x)) \leq -\bar{\alpha} V(x), \qquad (61)$$

and (51c) gives

$$\begin{aligned}
&y + \tilde{\Delta}(x, y, k+1) \\
&\quad = y - f(x{-}y)\, \Delta t {-} g(x - y) \sqrt{\Delta t}\ \xi(k+1) \\
&\quad = x - (x - y) - f(x{-}y)\, \Delta t {-} g(x - y) \sqrt{\Delta t}\ \xi(k+1).
\end{aligned}$$

Using inequality (61) and [27, Lemma 1], one obtains

$$\begin{aligned}
&\mathbb{E}\big[V(y + \tilde{\Delta}(x, y, k+1)) \big| x, y\big] \\
&\quad = x^T P x - 2x^T P(x - y) + (x - y)^T P(x - y) \\
&\qquad - 2\Delta t x^T P f(x - y) + \Delta t \Big\{ 2(x - y)^T P f(x - y) \\
&\qquad + trace[g^T(x - y) P g(x - y)] \\
&\qquad + \Delta t f^T(x - y) P f(x - y) \Big\} \\
&\quad \leq (1 + c^{-1}) x^T P x + (1 + c)(x - y)^T P(x - y) \\
&\qquad - 2\Delta t x^T P f(x - y) \\
&\qquad + \Delta t\big[\mathscr{L} V(x - y) + \Delta t\, V(f(x - y))\big] \\
&\quad \leq (1 + c^{-1}) V(x) + (1 + c)(x - y)^T P(x - y) \\
&\qquad - 2\Delta t x^T P f(x - y) - \bar{\alpha} \Delta t\, (x - y)^T P(x - y) \\
&\quad \leq (1 + c^{-1}) V(x) + (1 + c - \bar{\alpha}\Delta t)\, (x - y)^T P(x - y) \\
&\qquad - 2\Delta t x^T P f(x - y) \quad \forall\, k \in \mathbb{N} \qquad (62)
\end{aligned}$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, where c is a positive constant with $c < \bar{\alpha}\Delta t$. Hence, $0 < 1 + c - \bar{\alpha}\Delta t < 1$ due to $0 < \bar{\alpha}\Delta t < 1$.

By [27, Lemmas 1-2] and global Lipschitz condition (45),

$$\begin{aligned}
(x - y)^T P(x - y) &\leq x^T P x - 2\, x^T P y + y^T P y \\
&\leq (1 + b^{-1}) x^T P x + (1 + b) y^T P y, \qquad (63)
\end{aligned}$$

and, therefore,

$$\begin{aligned}
-2x^T P f(x - y) &\leq b^{-1} x^T P x + b f(x{-}y)^T P f(x - y) \\
&\leq b^{-1} x^T P x + b \lambda_M(P)\, L^2\, (x - y)^T (x - y) \\
&\leq \Big(b^{-1} + \frac{(1 + b)\lambda_M(P)\, L^2}{\lambda_m(P)}\Big) V(x) \\
&\quad + \frac{b(1 + b)\lambda_M(P)\, L^2}{\lambda_m(P)} V(y), \qquad (64)
\end{aligned}$$

where b is a positive constant sufficiently small for

$$\tilde{\beta}_2 = (1 + c - \bar{\alpha}\Delta t)(1 + b) + \Delta t\ \frac{b(1 + b)\lambda_M(P)\, L^2}{\lambda_m(P)} < 1. \qquad (65)$$

Substitution of (63) and (64) into (62) yields

$$\mathbb{E}\big[V(y + \tilde{\Delta}(x, y, k+1)) \big| x, y\big] \leq \tilde{\beta}_1 V(x) \ + \tilde{\beta}_2 V(y),$$

where

$$\begin{aligned}
\tilde{\beta}_1 &= (1 + c^{-1}) + (1 + c - \bar{\alpha}\Delta t)(1 + b^{-1}) \\
&\quad + \Delta t \left(b^{-1} + \frac{(1 + b)\lambda_M(P)\, L^2}{\lambda_m(P)} \right)
\end{aligned}$$

and $\tilde{\beta}_2$ given as (65) above are both positive constants. This is the condition (55c) of Theorem 3.

Let $\tilde{\alpha}_2$ be a positive number such that

$$\tilde{\alpha}_2 < \frac{-\ln \tilde{\beta}_2}{\overline{\Delta t}}. \qquad (66)$$

For instance, let

$$\tilde{\alpha}_2 = \frac{-\ln \tilde{\beta}_2}{2\, \overline{\Delta t}} \ \Rightarrow\ \Delta t \leq \overline{\Delta t} = \frac{-\ln \tilde{\beta}_2}{2\, \tilde{\alpha}_2} < \frac{-\ln \tilde{\beta}_2}{\tilde{\alpha}_2}.$$

So the condition (56) of Theorem 3 is also satisfied. According to Theorem 3, it follows the assertions. □

In the literature [22] and [39], to ensure that the cyber system shares the exponential stability with its underlying physical system, the stepsize Δt is explicitly and severly limited by both the growth and the rate constants of the physical system. Although the both are related, it is only the rate constant that plays a key role in the definitions of exponential stability. It is reasonable and possible to lessen the dependence of the stepsize Δt on the growth constant, which itself could be very conservative due to condition (54). In Theorem 4, we manage to remove the explicit dependence of the stepsize Δt on the growth constant $\lambda_M(P)\, /\lambda_m(P)$. Instead, we show that the growth constant $\lambda_M(P)\, /\lambda_m(P)$ has an influence on the stepsize Δt through the rate-like constant $\tilde{\beta}_2$ given by (65). This could reduce much the restriction imposed by the growth constant. As will be shown in Section VI, it improves the upper bound $\overline{\Delta t}$ of stepsizes and eases its computation for linear systems.

Recall that Theorem 3 is the direct application of Theorem 1 to the CPS (51) of the numerical method (47)-(48) for the SDE (44), from which Theorem 4 is derived for the mean-square exponential stability. Let us proceed to apply Theorem 4 and study the converse question (Q2) whether one can infer that the CPS (51) and, hence, the physical system (44) are mean-square exponentially stable if the cyber system (47)-(48) is mean-square exponentially stable for small stepsizes $\Delta t > 0$. Similarly, the converse Lyapunov theorem [32], [51] gives that, if the cyber system (47)-(48) is mean-square exponentially stable, there is a Lyapunov function that proves the exponential stability of the system. Due to the interrelation of the physical and cyber systems, one may make use of this Lyapunov function to study the stability of the physical system and that of the whole CPS. Applying Theorem 4, we find that the mean-square exponential stability of the CPS (51) and, hence, that of the physical system (44) can be inferred from the mean-square exponential stability of the cyber system (47)-(48).

Theorem 5: Assume that there is a candidate Lyapunov function $V \in C^2(\mathbb{R}^n; \mathbb{R}_+)$ of the quadratic form (58) for the cyber system (47)-(48) with $\Delta t = \overline{\Delta t} > 0$ such that

$$\mathbb{E}\big[V(X_{k+1})\big|X_k\big] \le \bar{c}\, V(X_k) \tag{67}$$

for some positive constant $\bar{c} < 1$ and all $X_k \in \mathbb{R}^n$. Then CPS (51) with $\Delta t \in (0, \overline{\Delta t}]$ is mean-square exponentially stable and also almost surely exponentially stable, which implies that physical system (44) is mean-square exponentially stable and also almost surely exponentially stable.

Proof: By the Lyapunov stability theory [6], [32], conditions (58) and (67) as well as (54) derived from (58) immediately imply that the cyber system (47)-(48) with $\Delta t = \overline{\Delta t}$ is mean-square exponentially stable. Let function $V(x)$ given by (58) also be the candidate Lyapunov function for the physical system (44). But condition (67)

$$\begin{aligned}
&\mathbb{E}\big[V(X_{k+1})\big|X_k\big]\\
&\quad = \mathbb{E}\big[X_{k+1}^T P X_{k+1}^T \big| X_k\big]\\
&\quad = \mathbb{E}\Big[\big(X_k + f(X_k)\overline{\Delta t} + g(X_k)\sqrt{\overline{\Delta t}}\,\xi(k+1)\big)^T P\\
&\qquad \cdot\big(X_k + f(X_k)\overline{\Delta t} + g(X_k)\sqrt{\overline{\Delta t}}\,\xi(k+1)\big)\big|X_k\Big]\\
&\quad = V(X_k) + \overline{\Delta t}\Big[X_k^T Pf(X_k) + f^T(X_k)PX_k\\
&\qquad + trace\big[g^T(X_k)Pg(X_k)\big] + \overline{\Delta t} f^T(X_k)Pf(X_k)\Big]\\
&\quad \le \bar{c}\, V(X_k)
\end{aligned}$$

produces that

$$\begin{aligned}
&X_k^T Pf(X_k) + f^T(X_k)PX_k\\
&\qquad + trace\big[g^T(X_k)Pg(X_k)\big] + \overline{\Delta t} f^T(X_k)Pf(X_k)\\
&\quad \le -\bar{\alpha} V(X_k)
\end{aligned}$$

for all $X_k \in \mathbb{R}^n$, where $\bar{\alpha}$ is a postive number such that

$$\bar{\alpha}\,\overline{\Delta t} = 1 - \bar{c}. \tag{68}$$

This means

$$\begin{aligned}
&\mathscr{L}V(x) + \overline{\Delta t}\, V(f(x))\\
&\quad = x^T Pf(x) + f^T(x)\, Px\\
&\qquad + trace\big[g^T(x)Pg(x)\big] + \overline{\Delta t} f^T(x)\, Pf(x)\\
&\quad \le -\bar{\alpha} V(x)
\end{aligned}$$

for all $x \in \mathbb{R}^n$, which is exactly the condition (59) of Theorem 4. According to Theorem 4, $z(t) = [x^T(t)\ \ y^T(t)]^T$ of CPS (51) with $\Delta t \in (0, \overline{\Delta t}]$ is mean-square exponentially stable and also almost surely exponentially stable. It immediately follows that, due to $|x(t)|^2 \le |z(t)|^2$, the physical system (44) is mean-square exponentially stable and also almost surely exponentially stable. Alternatively, conditions (58) and $\mathscr{L}V(x) \le \mathscr{L}V(x) + \overline{\Delta t}\, V(f(x)) \le -\bar{\alpha} V(x)$ for all $x \in \mathbb{R}^n$ as well as (54) derived from (58) imply that, by [38, Theorems 4.2-4.4, pp128-130], the physical system (44) is mean-square exponentially stable and also almost surely exponentially stable. □

Our positive results to the key questions (Q1) and (Q2) expose the equivalence and intrinsic relationship (68) between (59) and (67), which are the stability conditions for the physical system (44) and its cybercounterpart (47)-(48), respectively. For this purpose, we employ the same Lyapuov function $V(x) = \tilde{V}(x) = x^T Px$ for both the subsystems in Theorems 3-5, see also Section VI. Actually, this is also a sensible choice due to the structure of CPS (51) in which the physical subsystem plays a dominant role. Our proposed theory for synthetic CPSs should be developed by using various techniques of Lyapunov functions/functionals to exploit the structure of the resulting controlled CPS, see [13], [30], [36], [37], [41] and also Remark 5. It is worth noting that the initial condition $y(0) = x(0) - X(0) = 0$ is not required in our established stability theory and its application in Section VI. But this condition could make a difference in the study of convergence as well as some control problems, see Appendix B.

VI. THE CPS THEORY FOR LINEAR SYSTEMS

Let us consider a linear stochastic system

$$\mathrm{d}x(t) = Fx(t)\mathrm{d}t + \sum_{j=1}^{m} G_j x(t)\mathrm{d}B_j(t) \quad \forall\, t \ge 0 \tag{69}$$

with initial value $x(0) \in \mathbb{R}^n$, where $F \in \mathbb{R}^{n\times n}$ and $G_j \in \mathbb{R}^{n\times n}$, $j = 1, 2, \cdots, m$, are constant matrices. Obviously, the linear system (69) satisfies the global Lipschitz condition and has a unique solution $x(t)$ on $[0, \infty)$. It is well known that the linear stochastic system (69) is mean-square exponentially stable if and only if there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that [6], [12]

$$F^T P + PF + \sum_{j=1}^{m} G_j^T PG_j < 0. \tag{70}$$

This is the Lyapunov-Itô inequality [6], [12], the linear matrix inequality (LMI) equivalent [14] of the classical Lyapunov-Itô equation [1], [37]. By [38, Theorem 4.2, p128], the mean-square exponential stability of (69) implies that it is also almost surely exponentially stable.

The Euler-Maruyama method (47)-(48) for the linear system (69) computes approximations, for all $k \in \mathbb{N}$,

$$X_{k+1} = X_k + FX_k \Delta t + \sum_{j=1}^{m} G_j X_k \sqrt{\Delta t}\, \xi_j(k+1) \quad (71)$$

with $X_0 = x(0)$, where $\Delta t > 0$ is the constant stepsize and $\sqrt{\Delta t}\, \xi_j(k+1)$ is the implementation of $\Delta B_{j,k} = B_j((k+1)\Delta t) - B_j(k\Delta t)$. The cyber system (71) is mean-square exponentially stable if and only if there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that [6]

$$(I_n + \Delta t\, F)^T P (I_n + \Delta t\, F) + \Delta t \sum_{j=1}^{m} G_j^T P G_j < P. \quad (72)$$

Let $y(t)$ be the difference between $x(t)$ and $X(t)$ as (49) above. The CPS of the Euler-Maruyama method (71) for the linear SDE (69) is a specific case of CPS (51)

$$\mathrm{d}x(t) = Fx(t)\mathrm{d}t + \sum_{j=1}^{m} G_j x(t)\mathrm{d}B_j(t) \quad (73a)$$

$$\mathrm{d}y(t) = Fx(t)\mathrm{d}t + \sum_{j=1}^{m} G_j x(t)\mathrm{d}B_j(t) \qquad t \in [t_k, t_{k+1}) \quad (73b)$$

$$\begin{aligned}\tilde{\Delta}(x(t_{k+1}^-), y(t_{k+1}^-), k+1) &:= y(t_{k+1}) - y(t_{k+1}^-) \\ &= -F\big(x(t_{k+1}^-) - y(t_{k+1}^-)\big)\Delta t \\ &\quad - \sum_{j=1}^{m} G_j\big(x(t_{k+1}^-) - y(t_{k+1}^-)\big)\sqrt{\Delta t}\, \xi_j(k+1) \\ &\qquad k \in \mathbb{N}\end{aligned} \quad (73c)$$

with initial data $x(0) \in \mathbb{R}^n$ and $y(0) = x(0) - X(0) = 0$, where $t_k = k\Delta t$ for all $k \in \mathbb{N}$. The CPS (73) is an integration of the physical system (69) and the cyber system (71), which is in our proposed canonic form (1) and satisfies the global Lipschitz conditions Assumption 1. Our established theory immediately provides positive results to the key questions (Q1) and (Q2) for linear CPS (73), which also presents the upper bound $\overline{\Delta t}$ of stepsizes for exponential stability.

Theorem 6: The following are equivalent.

($\mathcal{A}$) There exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that the CPS Lyapunov inequality holds for some positive number $\overline{\Delta t}$, namely,

$$F^T P + PF + \sum_{j=1}^{m} G_j^T P G_j + \overline{\Delta t} F^T P F < 0. \quad (74)$$

($\mathcal{B}$) The physical system (69) is mean-square exponentially stable.

($\mathcal{C}$) The cyber system (71) with $\Delta t \in (0, \overline{\Delta t}\,]$ is mean-square exponentially stable.

($\mathcal{D}$) The CPS (73) with $\Delta t \in (0, \overline{\Delta t}\,]$ is mean-square exponentially stable.

That is, $(\mathcal{A}) \Leftrightarrow (\mathcal{B}) \Leftrightarrow (\mathcal{C}) \Leftrightarrow (\mathcal{D})$.

Proof: $(\mathcal{A}) \Leftrightarrow (\mathcal{B})$. We only need to show that the classical Lyapunov inequality (70) and the CPS Lyapunov inequality (74) are equivalent. But this is implied by the equivalence of the inequalities (55a) and (59) which has been shown in the proof of Theorem 4. Alternatively, it can be easily proved as follows. Clearly, (74) implies (70). But inequality (70) implies that there is a sufficiently small positive number $\overline{\Delta t}$ such that (74) holds. So the LMI (70) $\Leftrightarrow$ the LMI (74).

$(\mathcal{A}) \Rightarrow (\mathcal{C})$ & $(\mathcal{D})$. Let us consider the quadratic Lyapunov function $V(x) = x^T P x$ for the linear system (69). The LMI (74) implies that there is a positve number $\bar{\alpha} < 1/\overline{\Delta t}$ sufficiently small for

$$F^T P + PF + \sum_{j=1}^{m} G_j^T P G_j + \overline{\Delta t} F^T P F \le -\bar{\alpha} P, \quad (75)$$

and the condition (59) holds. It follows from Theorem 4 that the CPS (73) and, hence, the cyber system (71) with $\Delta t \in (0, \overline{\Delta t}\,]$ are mean-square exponentially stable.

$(\mathcal{D}) \Rightarrow (\mathcal{B})$ & $(\mathcal{C})$. The CPS (73) is a specific case of system (4), where $z(t) = [x^T(t) \;\; y^T(t)]^T$ in the compact form. Notice that $|x(t)|^2 \le |z(t)|^2$ and $|X(t)|^2 \le 2(|x(t)|^2 + |y(t)|^2) \le 4|z(t)|^2$ for all $t \ge 0$. If $z(t)$ of the CPS (73) is mean-square exponentially stable, then both $x(t)$ of its physical subsystem (69) and $X(t)$ of its cyber subsystem (71) are mean-square exponentially stable.

$(\mathcal{C}) \Rightarrow (\mathcal{B})$ & $(\mathcal{D})$. Let $\Delta t = \overline{\Delta t}$. Since the cyber system (71) is mean-square exponentially stable, there is a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that the Lyapunov inequality (72) holds for $\Delta t = \overline{\Delta t} > 0$. This implies that there is a positive number $\bar{c} \in (0, 1)$ sufficiently close to 1 for

$$(I_n + \overline{\Delta t}\, F)^T P (I_n + \overline{\Delta t}\, F) + \overline{\Delta t} \sum_{j=1}^{m} G_j^T P G_j \le \bar{c} P. \quad (76)$$

Let the quadratic function $V(x) = x^T P x$ be the candidate Lyapunov function for the cyber system (71) with $\Delta t = \overline{\Delta t}$. Observe that, for the linear system, (76) is exactly the condition (67) of Theorem 5. It follows from Theorem 5 that the CPS (51) with $\Delta t \in (0, \overline{\Delta t}]$ and, hence, the physical system (44) are mean-square exponentially stable. □

Note that the mean-square exponential stability of the physical system (69), the cyber system (71) and the CPS (73) imply that they are also almost surely exponentially stable, respectively. It is easy to obtain the upper bound $\overline{\Delta t}$ of stepsizes for the ability of the cyber system to reproduce the exponential stability of the underlying linear physical system by solving the CPS Lyapunov inequality (74), which can also be called the numerical Lyapunov inequality in the study of numerical methods for differential equations. Notice that the LMI (74) is a linear function with respect to $\overline{\Delta t}$ and it can be formulated as a generalized eigenvalue minimization problem. So we immediately obtain the upper bound $\overline{\Delta t} = -\lambda$ by solving

$$\min \lambda \qquad s.t. \quad P > 0, \quad F^T P F > 0,$$

$$F^TP + PF + \sum_{j=1}^{m} G_j^T PG_j < \lambda F^T PF \tag{77}$$

with some toolboxes such as [14], which is one of the advantages of our proposed method.

Remark 4: The prestigious Black-Scholes model is a special case of the linear SDE (69) with $n = m = 1$,

$$\mathrm{d}x(t) = \mu x(t)\mathrm{d}t + \sigma x(t)\mathrm{d}B(t), \quad t \geq 0, \quad x(0) \neq 0 \tag{78}$$

where μ and σ are both real constants. Thus the CPS of the Euler-Maruyama method (47)-(48) for the Black-Scholes model (78) is a specific case of (73) with $n = m = 1$

$$\begin{aligned}
&\mathrm{d}x(t) \\
&\quad = \mu x(t)\mathrm{d}t + \sigma x(t)\mathrm{d}B(t) && \text{(79a)}\\
&\mathrm{d}y(t) \\
&\quad = \mu x(t)\mathrm{d}t + \sigma x(t)\mathrm{d}B(t) \qquad t \in [t_k, t_{k+1}) && \text{(79b)}\\
&y(t_{k+1}) - y(t_{k+1}^-) \\
&\quad = -\mu\big(x(t_{k+1}^-) - y(t_{k+1}^-)\big)\Delta t \\
&\qquad -\sigma\big(x(t_{k+1}^-) - y(t_{k+1}^-)\big)\sqrt{\Delta t}\,\xi(k+1) \quad k \in \mathbb{N}. && \text{(79c)}
\end{aligned}$$

The CPS Lyapunov inequality (74) immediately gives

$$2\mu + \sigma^2 + \mu^2\overline{\Delta t} < 0 \quad \Leftrightarrow \quad \overline{\Delta t} < \frac{-(2\mu + \sigma^2)}{\mu^2}. \tag{80}$$

According to Theorem 6, this is the necessary and sufficent condition for mean-square exponential stability of the linear scalar physical system (78), the cyber counterpart of the Euler-Maruyama method and its CPS (79). Notice that (80) is exactly the inequality (4.3) in [19] with $\theta = 0$ for the Euler-Maruyama method. Obviously, condition (80) is the very scalar case of our CPS Lyapunov inequality (74) that is applicable to general multi-dimensional linear systems. It is an important and interesting problem among future work to study the almost sure stability [19], [28], [38] of the CPS (73) and its application, say, to the CPS (79) for the Black-Scholes model (78).

Recently, based on the reformulation of some well-known results, [7] developed an approach to mean-square stability analysis of numerical methods (including the widely-used Euler-Maruyama scheme) for multi-dimensional linear SDEs (viz. system (69) with $n \geq 2$), which was applied in [8] to study the mean-square numerical stability for a linear SDE of non-normal drift and skew-symmetic diffusion structures [23]. Specifically, on one hand, some well-known result ([1, Theorem 8.5.5, p142], [32, Remark 6.4, p183]) expressed in the vectorization of matrices and Kronecker product gives [7, Lemma 3.3]

$$Re_M(\mathcal{S}) < 0 \tag{81}$$

if and only if linear SDE (69) is mean-square exponentially stable, where $Re_M(\mathcal{S})$ is the real part of the eigenvalue $\lambda_M(\mathcal{S})$ of $n^2 \times n^2$ matrix

$$\mathcal{S} := I_n \otimes F + F \otimes I_n + \sum_{j=1}^{m} G_j \otimes G_j.$$

On the other hand, a stability result for discrete-time stochastic systems (see, e.g., [32, p197]) is applied to study mean-square stability of some numerical schemes for the SDE (69). For example, by [7, Lemma 3.4, Theorem 3.7], the Euler-Maruyama method (71) is mean-square exponentially stable if and only if

$$\rho(\mathcal{S}_0(\Delta t)) < 1, \tag{82}$$

where $\rho(\mathcal{S}_0(\Delta t))$ is the spectral radius of $n^2 \times n^2$ matrix

$$\mathcal{S}_0(\Delta t) := \big(\bar{A}(\Delta t) \otimes \bar{A}(\Delta t)\big) + \sum_{j=1}^{m} \big(\bar{B}_j(\Delta t) \otimes \bar{B}_j(\Delta t)\big)$$

with $\bar{A}(\Delta t) = I_n + \Delta t F$ and $\bar{B}_j(\Delta t) = \sqrt{\Delta t}\, G_j$ for $j = 1, \cdots, m$.

In [7], $\mathcal{S}$ is called the mean-square stability matrix of the SDE (69) and $\mathcal{S}_0(\Delta t)$ that of the Euler-Maruyama method (71). Notice that $\mathcal{S}_0(\Delta t)$ is a function of stepsize Δt while, obviously, $\mathcal{S}$ is not. The results in [7] and [8] provided the explicit structure of stability matrices $\mathcal{S}$ and $\mathcal{S}_0(\Delta t)$, and showed the comparative stability regions [8, Fig.2] for the SDE and the numerical method with a few numerical examples of nonnormal SDEs [23]. However, the relationship between the stability conditions (81) and (82) (for the SDE and the Euler-Maruyama method, respectively) has not been figured out. Here we prove their equivalence and reformulate the stability conditions (81) and (82) in the form of LMIs, which is relegated to Appendix A. So it is easy to handle the problems with some computing techniques and toolboxes [6], [12], [14].

It is easy to obtain the upper bound $\overline{\Delta t}$ of stepsizes for the test problem (Q1) by solving the $n \times n$-dimensional LMI (74) of our proposed method. Clearly, LMI (74) holds for all $\Delta t \in (0, \overline{\Delta t}]$ provided it is satisfied for some $\overline{\Delta t} > 0$. But, to calculate the upper bound $\widetilde{\Delta t}$ by the approach of mean-square stability matrix [7], one has to deal with the spectral radius (82) of $n^2 \times n^2$ matrix $\mathcal{S}_0(\Delta t)$ that involves a polynomial of the stepsize Δt whose order is some exponential function of n. Alternatively, one can solve the following LMI with respect to positive definite matrix $\bar{P} \in \mathbb{R}^{n^2 \times n^2}$, which we show is equivalent to (82) in Appendix A,

$$\begin{aligned}
&\mathcal{S}^T\bar{P} + \bar{P}\mathcal{S} + \Delta t\big(\mathcal{S}^T\bar{P}\mathcal{S} + \bar{F}^T\bar{P} + \bar{P}\bar{F}\big) \\
&\qquad + (\Delta t)^2\big(\mathcal{S}^T\bar{P}\bar{F} + \bar{F}^T\bar{P}\mathcal{S}\big) + (\Delta t)^3\bar{F}^T\bar{P}\bar{F} \\
&\quad = \big(\mathcal{S} + \Delta t\bar{F}\big)^T\bar{P} + \bar{P}\big(\mathcal{S} + \Delta t\bar{F}\big) \\
&\qquad + \Delta t\,\big(\mathcal{S} + \Delta t\bar{F}\big)^T\bar{P}\big(\mathcal{S} + \Delta t\bar{F}\big) < 0,
\end{aligned} \tag{83}$$

where $\bar{F} = F \otimes F$. This involves a cubic function of the prescribed parameter $\Delta t > 0$ for all n. So, unlike the CPS Lyapunov inequality (74), the LMI (83) may not be reformulated as a generalized eigenvalue minimization problem.

Note that, for a multi-dimensional SDE ($n \geq 2$), the spectral radius (82) of $n^2 \times n^2$ matrix $\mathcal{S}_0(\Delta t)$ involves a polynomial of Δt of up to (very) high order. For example, in the case $n = 2$ of non-normal SDE [8, Eq.(9)] (see also [23]), the characteristic equation of mean-square stability matrix $\mathcal{S}_0(\Delta t)$ of 4×4 dimensions for the Euler-Maruyama scheme ([8, Eq.(15)] with $\theta = 0$) involves a polynomial of Δt of up to order 8. The conditions of this approach are quite cumbersome [32]. It is easy to tackle the equivalent LMI (83) using some toolboxes such as [6], [14], which is a cubic function of the prescribed $\Delta t > 0$ for all n.

However, one should be aware that, unlike the linear inequality (74), that the inequality (82) or its equivalent LMI (83) is satisfied for some $\overline{\Delta t} > 0$ may not necessarily mean that it holds for all $\Delta t \in (0, \overline{\Delta t}]$. Thus the results such as the upper bound $\overline{\Delta t}$ obtained by approach of (82) from [7] or its equivalent (83) could be restrictive due to the highly nonlinearity of Δt involved in the computation, see Appendix A.

We can further show that our proposed method (74) gives better bound $\overline{\Delta t}$ of stepsizes than (82) from [7] or its LMI equivalent (83). This is: if (82) and (83) hold for all $\Delta t \in (0, \widetilde{\Delta t}]$ for some $\widetilde{\Delta t} > 0$, then the CPS Lyapunov inequality (74) holds for some $\overline{\Delta t} \geq \widetilde{\Delta t}$, namely, either $\overline{\Delta t} = \widetilde{\Delta t}$ or $\overline{\Delta t} > \widetilde{\Delta t}$. In short, we shall show either $\widehat{\Delta t} = \widetilde{\Delta t}$ or $\widehat{\Delta t} > \widetilde{\Delta t}$, where

$$\widetilde{\Delta t} := \sup\{\overline{\Delta t} > 0 : (83) \textit{ holds for all } \Delta t \in (0, \overline{\Delta t}\,]\} \\ \times \textit{and} \quad \widehat{\Delta t} := \sup\{\overline{\Delta t} > 0 : (74) \textit{ holds}\}. \tag{84}$$

It is observed that, due to the continuity of (83) with respect to Δt, the strict inequality (83) does not hold at $\Delta t = \widetilde{\Delta t}$ and, similarly, the strict inequality (74) does not hold at $\overline{\Delta t} = \widehat{\Delta t}$.

To show $\widehat{\Delta t} \geq \widetilde{\Delta t}$ (viz either $\widehat{\Delta t} = \widetilde{\Delta t}$ or $\widehat{\Delta t} > \widetilde{\Delta t}$), we consider a linear SDE with $\Delta t \in (0, \overline{\Delta t}\,]$ for some $\overline{\Delta t} > 0$

$$\mathrm{d}x(t) = Fx(t)\mathrm{d}t + \sum_{j=1}^{m} G_j x(t)\mathrm{d}B_j(t) \\ + \sqrt{\Delta t}\, F\, x(t)\mathrm{d}B_{m+1}(t) \quad \forall\, t \geq 0 \tag{85}$$

where $B_{m+1}(t)$ is a scalar Brownian motion. Notice that (85) is exctly (69) if $\Delta t = 0$ and, according to [7, Lemma 3.3], the SDE (85) is mean-square exponentially stable if and only if (87) holds. In fact, by the well-known results [1], [6], [7], [14], [32], the following are equivalent.

(a) The CPS Lyapunov LMI (74) holds.

(b) There is a positive definite matrix $\bar{P} \in \mathbb{R}^{n^2 \times n^2}$ such that, $\forall\, \Delta t \in (0, \overline{\Delta t}\,]$,

$$\big(\mathcal{S} + \Delta t \bar{F}\big)^T \bar{P} + \bar{P}\big(\mathcal{S} + \Delta t \bar{F}\big) \\ = \mathcal{S}^T \bar{P} + \bar{P}\mathcal{S} + \Delta t\big(\bar{F}^T \bar{P} + \bar{P}\bar{F}\big) < 0. \tag{86}$$

(c) The following inequality holds for each $\Delta t \in (0, \overline{\Delta t}\,]$

$$Re_M(\mathcal{S} + \Delta t \bar{F}) < 0. \tag{87}$$

(d) The SDE (85) is mean-square exponentially stable.

That is, (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) and they all give the same supremum $\widehat{\Delta t}$ defined as (84). Similarly, one may formulate the LMI (86) as a generalized eigenvalue minimization problem like (77) above. By [27, Lemma 1],

$$\mathcal{S}^T \bar{P}\mathcal{S} + \Delta t\big(\mathcal{S}^T \bar{P}\bar{F} + \bar{F}^T \bar{P}\mathcal{S}\big) + (\Delta t)^2 \bar{F}^T \bar{P}\bar{F} \geq 0$$

and, therefore,

$$\begin{aligned} &\mathcal{S}^T \bar{P} + \bar{P}\mathcal{S} + \Delta t\big(\mathcal{S}^T \bar{P}\mathcal{S} + \bar{F}^T \bar{P} + \bar{P}\bar{F}\big) \\ &\quad + (\Delta t)^2\big(\mathcal{S}^T \bar{P}\bar{F} + \bar{F}^T \bar{P}\mathcal{S}\big) + (\Delta t)^3 \bar{F}^T \bar{P}\bar{F} \\ &= \mathcal{S}^T \bar{P} + \bar{P}\mathcal{S} + \Delta t\big(\bar{F}^T \bar{P} + \bar{P}\bar{F}\big) \\ &\quad + \Delta t\Big[\mathcal{S}^T \bar{P}\mathcal{S} + \Delta t\big(\mathcal{S}^T \bar{P}\bar{F} + \bar{F}^T \bar{P}\mathcal{S}\big) + (\Delta t)^2 \bar{F}^T \bar{P}\bar{F}\Big] \\ &\geq \mathcal{S}^T \bar{P} + \bar{P}\mathcal{S} + \Delta t\big(\bar{F}^T \bar{P} + \bar{P}\bar{F}\big). \end{aligned}$$

Thus (83) implies (86) but not vice versa. This means that (74), (86) and (87) hold for all $\overline{\Delta t} \in (0, \widetilde{\Delta t})$, and, therefore, $\widehat{\Delta t} \geq \widetilde{\Delta t}$. Notice that (83), (86) and (87) are all continuous with respect to Δt. Due to the continuity of (83) at $\Delta t = \widetilde{\Delta t}$,

$$\begin{aligned} &\big(\mathcal{S} + \widetilde{\Delta t}\bar{F}\big)^T \bar{P} + \bar{P}\big(\mathcal{S} + \widetilde{\Delta t}\bar{F}\big) \\ &\qquad + \widetilde{\Delta t}\,\big(\mathcal{S} + \widetilde{\Delta t}\bar{F}\big)^T \bar{P}\big(\mathcal{S} + \widetilde{\Delta t}\bar{F}\big) \leq 0 \\ &\Leftrightarrow \quad \big(\mathcal{S} + \widetilde{\Delta t}\bar{F}\big)^T \bar{P} + \bar{P}\big(\mathcal{S} + \widetilde{\Delta t}\bar{F}\big) \\ &\leq -\widetilde{\Delta t}\,\big(\mathcal{S} + \widetilde{\Delta t}\bar{F}\big)^T \bar{P}\big(\mathcal{S} + \widetilde{\Delta t}\bar{F}\big). \end{aligned}$$

Unless matrix $\mathcal{S} + \widetilde{\Delta t}\bar{F}$ is singular, the LMIs (86) and its equivalent (74) hold at $\overline{\Delta t} = \widetilde{\Delta t}$ and, due to their continuity at $\Delta t = \widetilde{\Delta t}$, the LMIs (86) and (74) hold for some $\overline{\Delta t} > \widetilde{\Delta t}$, which gives $\widehat{\Delta t} > \widetilde{\Delta t}$. So we have $\widehat{\Delta t} = \widetilde{\Delta t}$ if matrix $\mathcal{S} + \widetilde{\Delta t}\bar{F}$ is singular; otherwise, $\widehat{\Delta t} > \widetilde{\Delta t}$. The latter, namely, $\widehat{\Delta t} > \widetilde{\Delta t}$ could often be the case. This clearly shows that our proposed CPS Lyapunov LMI (74) gives a better bound $\overline{\Delta t}$ of stepsizes than (82) from [7], or, its LMI equivalent (83).

It is observed that the CPS LMI (74) holds for all $\Delta t \in (0, \overline{\Delta t}]$ if it holds with some $\overline{\Delta t} > 0$. Recall that such a desired property has not been observed/shown in the mean-square stability matrix method (82). Instead, one finds that $\rho(\mathcal{S}_0(\Delta t)) \rightarrow 1$ as $\Delta t \rightarrow 0$ in the method (82). This could be a restriction in some applications. As shown above, our proposed method (74) has a number of impressive advantages, which include: the upper bound $\overline{\Delta t} > 0$ can be easily obtained by solving the generalized eigenvalue minimization problem (77); the LMI (74) holds for all $\Delta t \in (0, \overline{\Delta t}]$; and a better bound $\widehat{\Delta t} \geq \widetilde{\Delta t}$ has been theoretically proved.

Let us apply the CPS Lyapunov LMI (74) to an interesting example from [23]. A particular case of SDE (69) has been studied in [23] to show the impact of noise on a highly nonnormal system, in which $n = 2$, $m = 1$,

$$F = \begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 & b^{-\frac{1}{4}} \\ -b^{-\frac{1}{4}} & 0 \end{bmatrix} \tag{88}$$

with $b > 0$. Thus the CPS of the Euler-Maruyama method (71) for the nonnormal SDE (88) is a specific case

TABLE 1. Upper bounds $\overline{\Delta t}$ for the nonnormal SDE (88) with various b.

b	0.3	0.8	1.3	1.8	2.3	2.8
$\overline{\Delta t}$	0.1498	0.6614	0.6278	0.4842	0.2875	0.0124

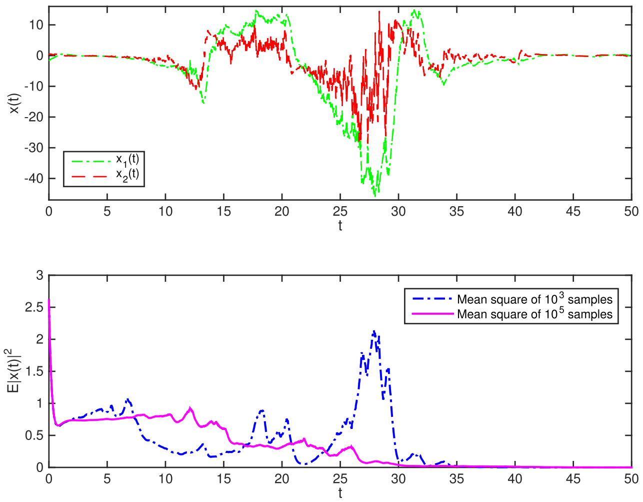

FIGURE 1. A trajectory sample (above) and the mean squares of samples (below) of the Euler-Maruyama method for the SDE (88) with $b = 2.8$.

of (73) with $n = 2$ and $m = 1$,

$$\begin{aligned} \mathrm{d}x(t) &= Fx(t)\mathrm{d}t + G_1x(t)\mathrm{d}B(t) && (89a)\\ \mathrm{d}y(t) &= Fx(t)\mathrm{d}t + G_1x(t)\mathrm{d}B(t) \qquad t \in [t_k, t_{k+1}) && (89b)\\ y(t_{k+1}) - y(t_{k+1}^-) &= -F\big(x(t_{k+1}^-) - y(t_{k+1}^-)\big)\Delta t \\ &\quad - G_1\big(x(t_{k+1}^-) - y(t_{k+1}^-)\big)\sqrt{\Delta t}\,\xi(k+1) \quad k \in \mathbb{N}. && (89c)\end{aligned}$$

According to the condition (81) as well as [8, Theorem 2.2], the SDE (88) is mean-square exponentially stable if and only if $b < 2.8181$. It is interesting to note that the noise term of the SDE (88) becomes smaller as b increases while, due to the nonnormality, the system is destablized (in mean-square sense) by smaller noise, see [23] as well as [8] for more details. It is also noticed that the mean-square stability matrix $\mathcal{S}_0(\Delta t)$ of the Euler-Maruyama method for the SDE (88) given in [8] clearly demonstrates that $\mathcal{S}_0(\Delta t) \to I_2 \otimes I_2 = I_4$ and thus $\rho(\mathcal{S}_0(\Delta t)) \to 1$ as $\Delta t \to 0$. Our CPS LMI (74) has the desired property for small stepsizes Δt. Given any $b \in (0, 2.8181)$, we immediately obtain the upper bound $\overline{\Delta t}$ of stepsizes by solving the generalized eigenvalue minimization problem (77), some of which are listed in Table 1.

As an example of numerical simulation, Figure 1 displays not only a trajectory sample but also the mean square of 10^5 samples as well as that of 10^3 samples of the Euler-Maruyama method for the nonnormal SDE (88) with $b = 2.8$, where the stepsize $\Delta t = 0.0124$ is the upper bound in Table 1 and the initial value $x(0) = [-1.5 \;\; 0.6]^T$ is from [23]. The numerical simulation verifies the effectiveness of our proposed method (74) and (77) as well as the mean-square exponential stability of the CPS (89) with $b = 2.8$ and $\Delta t = 0.0124$. Actually, the set of realizations shown in Figure 1 illustrates well our theoretic results and those in the literature [8], [23]. As it has been proved that the nonnormal SDE (88) with $b = 2.8$ is mean-square exponentially stable [8], a trajectory sample could depart far away before it eventually converges to the origin [23]. The trajectory samples with large departure may have a big effect on the mean square of 10^3 samples but a much smaller one on that of 10^5 samples while both the mean squares of samples decay towards zero. This attests the effectiveness of our proposed results.

Remark 5: The upper bound $\overline{\Delta t}$ of stepsizes in the CPS Lyapunov inequality (74) is obtained by employing the same Lyapunov function $V(x) = \tilde{V}(x) = x^TPx$ for both the subsystems, which is a special application of Theorem 3. This is reasonable since the physical subsystem plays a dominant role in the CPS (73). But the results on synthesized CPSs, in which the state $y(t)$ of cyber subsystem is utilized in some feedback mechanism to steer the physical subsystem, can be further developed by using various techniques of Lypunov functions/functionals [13], [30], [36] such as using a couple of Lyapunov functions for the subsystems and constructing a Lyapunov function/funcational for the whole CPS [36], [37], [41] to exploit the structure of the composition of the subsystems [36].

VII. CONCLUSION

In this paper, we have formulated a new and general class (1) of SiDEs that can be used to represent a seamless integration of the physical system (the SDE) and its cyber counterpart (the numerical method), which is a novel framework for numerical study of dynamical systems. Our proposed CPS of the Euler-Maruyama method for SDEs not only provides a holistic view of the physical system (the SDE) and its cyber counterpart (the numerical method) but also reveals their intrinsic relationship: they are not two separate systems but the subsystems of the CPS. By our CPS approach, we have proved positive results to the key questions (Q1) and (Q2) using the Lyapunov stability theory we establish for our general class of SiDEs. These fundamental results and their applications construct a theoretic foundation for the CPSs of numerical methods for SDEs. This foundational theory may be further developed with various techniques of Lypunov functions and functionals [13], [30], [36], [41].

In the classical numerical analysis of initial-value problems, the convergence and the stability of a numerical method are two main concerns [49]. The proposed CPS also provides a novel approach to convergence analysis of the numerical method for SDEs. As an example, we show by our CPS approach the classical finite-time convergence result

$$\mathbb{E}\Big[\sup_{0\le t\le T} |y(t)|^2\Big] = O(\Delta t)$$

for the widely-used Euler-Maruyama method. The novel proof is relegated to Appendix B, which utilizes the dyanimcs of the discretization error and is different from those in the literature [21], [33], [38], [49].

Our proposed CPS theory of numerical methods for differential equations has initiated the study of systems numerics, where there are a lot of open and interesting problems. For example, it is among future work to extend our established theory not only to many other (explicit, semi-implicit/symplectic and implicit) numerical methods [16], [19], [21], [31], [33], [40], [44], [49] but also to various dynamical systems such as SDEs with time delay [25], [30], singular SDEs with switching and stochastic partial differential equations [8], [33], [49]. It is of theoretic and practical importance as well to study a CPS that involves multi-scale processes [29] and/or stochastic stabilization [19], [28], [38], which could be one of the many challenging problems in the future development of the proposed CPS theory.

APPENDIX A. THE EQUIVALENCE OF THE STABILITY CONDITIONS (81) AND (82)

Proof: It is observed that $\mathcal{S}_0(0) = I_{n^2}$, $\dot{\mathcal{S}}_0(0) = \mathcal{S}$, $\ddot{\mathcal{S}}_0(\Delta t) = 2\,(F \otimes F) =: 2\bar{F}$, where $\dot{\mathcal{S}}_0(\Delta t)$ and $\ddot{\mathcal{S}}_0(\Delta t)$ are the first and the second derivatives of $\mathcal{S}_0$ with respect to Δt, respectively. For $\Delta t > 0$, Taylor expansion produces

$$\mathcal{S}_0(\Delta t) = I_{n^2} + \Delta t\,\mathcal{S} + (\Delta t)^2\bar{F}. \tag{90}$$

(81) $\Rightarrow$ (82). Stability condition (81) for the linear SDE equivalently means that there is a positive definite matrix $\bar{P} \in \mathbb{R}^{n^2\times n^2}$ such that [6], [32]

$$\mathcal{S}^T\bar{P} + \bar{P}\mathcal{S} < 0. \tag{91}$$

The Taylor series (90) gives

$$\begin{aligned}\mathcal{S}_0^T&(\Delta t)\bar{P}\mathcal{S}_0(\Delta t)\\ &= \bar{P} + \Delta t\big[\mathcal{S}^T\bar{P} + \bar{P}\mathcal{S} + \Delta t\big(\mathcal{S}^T\bar{P}\mathcal{S} + \bar{F}^T\bar{P} + \bar{P}\bar{F}\big)\\ &\quad + (\Delta t)^2\big(\mathcal{S}^T\bar{P}\bar{F} + \bar{F}^T\bar{P}\mathcal{S}\big) + (\Delta t)^3\bar{F}^T\bar{P}\bar{F}\big].\end{aligned} \tag{92}$$

Owing to (91), there is a pair of (sufficiently small) positive numbers $\overline{\Delta t}$ and $\bar{a} = \bar{a}(\overline{\Delta t})$ such that, $\forall\, \Delta t \in (0, \overline{\Delta t}\,]$,

$$\mathcal{S}_0^T(\Delta t)\bar{P}\mathcal{S}_0(\Delta t) \le \bar{P} + \bar{a}\,\Delta t\big(\mathcal{S}^T\bar{P} + \bar{P}\mathcal{S}\big) < \bar{P}.$$

This implies that stability condition (82) is satisfied for all $\Delta t \in (0, \overline{\Delta t}\,]$.

(82) $\Rightarrow$ (81). Notice that (92) can be rewritten as

$$\begin{aligned}\mathcal{S}_0^T&(\Delta t)\bar{P}\mathcal{S}_0(\Delta t)\\ &= \bar{P} + \Delta t\big[\mathcal{S}^T\bar{P} + \bar{P}\mathcal{S}\\ &\quad + \Delta t\big(\mathcal{S}^T\bar{P}\mathcal{S} + \bar{F}^T\bar{P} + \bar{P}\bar{F}\big)\\ &\quad + (\Delta t)^2\big(\mathcal{S}^T\bar{P}\bar{F} + \bar{F}^T\bar{P}\mathcal{S}\big) + (\Delta t)^3\bar{F}^T\bar{P}\bar{F}\big]\\ &= \bar{P} + \Delta t\Big[\big(\mathcal{S} + \Delta t\bar{F}\big)^T\bar{P} + \bar{P}\big(\mathcal{S} + \Delta t\bar{F}\big)\\ &\quad + \Delta t\,\big(\mathcal{S} + \Delta t\bar{F}\big)^T\bar{P}\big(\mathcal{S} + \Delta t\bar{F}\big)\Big].\end{aligned} \tag{93}$$

Suppose that condition (82) holds for all $\Delta t \in (0, \overline{\Delta t}]$, where $\overline{\Delta t}$ is some positive number. Then, equivalently, there is a positive definite matrix $\bar{P} = \bar{P}(\Delta t) \in \mathbb{R}^{n^2\times n^2}$ such that [6], [32]

$$\mathcal{S}_0^T(\Delta t)\bar{P}\mathcal{S}_0(\Delta t) < \bar{P} \quad \forall\, \Delta t \in (0, \overline{\Delta t}\,]. \tag{94}$$

Substitution of (93) into (94) produces (83) for all $\Delta t \in (0, \overline{\Delta t}\,]$, or, by Schur complement,

$$\begin{bmatrix}\big(\mathcal{S} + \Delta t\bar{F}\big)^T\bar{P} + \bar{P}\big(\mathcal{S} + \Delta t\bar{F}\big) & \sqrt{\Delta t}\,\big(\mathcal{S} + \Delta t\bar{F}\big)^T\bar{P}\\ \sqrt{\Delta t}\,\big(\mathcal{S} + \Delta t\bar{F}\big)^T\bar{P} & -\bar{P}\end{bmatrix} < 0 \tag{95}$$

for all $\Delta t \in (0, \overline{\Delta t}\,]$.

So (82) $\Leftrightarrow$ (83) $\Leftrightarrow$ (94) $\Leftrightarrow$ (95). But (83) implies the LMI (86). This equivalently means that matrix $\mathcal{S} + \Delta t\bar{F}$ is Hurwitz, namely, inequality (87) holds for each $\Delta t \in (0, \overline{\Delta t}\,]$. Recall that $\bar{F} = F \otimes F$. Letting $\Delta t \to 0$ in (86) and thus (87) gives stability condition (81) for the SDE (69). The proof is complete. $\square$

We remark that, by approach of mean-square stability matrices [7], the upper bound $\overline{\Delta t}$ can be calculated by solving either the spectral radius problem (82) or the LMI equivalent (83). The former involves a polynomial of the stepsize Δt whose order is some exponential function of n while the latter remains as a cubic function of Δt for all n. The highly nonlinearity would introduce not only computational complexity but also conservativeness to the results. We have reformulted the highly nonlinear problem (82) into the LMI (83). This has significantly simplified the approach of mean-square stability matrices $\mathcal{S}$ and $\mathcal{S}_0(\Delta t)$. Moreover, the LMI (83) discloses the inherent relationship between the stability conditions for the Euler-Maruyama method and the SDE,

LMI (83) $\to$ LMI (91) as $\Delta t \to 0$

while $\mathcal{S}_0(\Delta t) \to I_n \otimes I_n = I_{n^2}$ and hence $\rho(\mathcal{S}_0(\Delta t)) \to 1$ as $\Delta t \to 0$. It is also worth noting that, for a linear n-dimensional SDE, our proposed numerical Lyapunov LMI (74) of $n \times n$ dimensions is always a linear inequality of the stepsize $\overline{\Delta t}$ while the LMI problem (83) involves not only a cubic function of Δt but also matrices of $n^2 \times n^2$ dimensions.

APPENDIX B. A NOVEL PROOF OF THE CONVERGENCE OF THE EULER-MARUYAMA METHOD

Proof: For the convergence problem of the numerical method, the implimentation $\sqrt{\Delta t}\,\xi(k+1)$ should be replaced by the increment $\Delta B_k = B((k+1)\Delta t) - B(k\Delta t)$ itself in SiDE (51), that is,

$$\mathrm{d}x(t) = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}B(t) \tag{96a}$$

$$\mathrm{d}y(t) = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}B(t) \quad t \in [t_k, t_{k+1}) \tag{96b}$$

$$\begin{aligned}\tilde{\Delta}(x(t_{k+1}^-), y(t_{k+1}^-), k+1) &:= y(t_{k+1}) - y(t_{k+1}^-)\\ &= -f\big(x(t_{k+1}^-) - y(t_{k+1}^-)\big)\Delta t\\ &\quad - g\big(x(t_{k+1}^-) - y(t_{k+1}^-)\big)\Delta B_k \quad k \in \mathbb{N}\end{aligned} \tag{96c}$$

with $x(0) \in \mathbb{R}^n$ and $y(0) = 0$, where $t_k = k\Delta t$ for all $k \in \mathbb{N}$. According to the existing results ([33], [38] as well as

Lemma 1, SiDE (96) has a unique (right-continuous) solution $z(t) = [x^T(t) \;\; y^T(t)]^T$, which belongs to $\mathcal{M}^2([0, T]; \mathbb{R}^{n+q})$ for all $T \geq 0$. In particular, [38, Lemma 3.2, p51] gives

$$\mathbb{E}\Big[\sup_{0\le t\le T} |x(t)|^2\Big] \le (1 + 3|x(0)|^2)e^{3LT(T+4)} =: C_T. \quad (97)$$

On the interval $[t_k, t_{k+1}]$ for every $k \in \mathbb{N}$,

$$\begin{aligned} &y(t_{k+1}) - y(t_k) \\ &\quad = \int_{t_k}^{t_{k+1}^-} f(x(t))\mathrm{d}t + \int_{t_k}^{t_{k+1}^-} g(x(t))\mathrm{d}B(t) \\ &\quad -f(x(t_{k+1}^-) - y(t_{k+1}^-))\Delta t - g(x(t_{k+1}^-) - y(t_{k+1}^-))\Delta B_k \\ &\quad = \int_{t_k}^{t_{k+1}^-} \big[f(x(t)) - f(x(t_k) - y(t_k))\big]\mathrm{d}t \\ &\qquad + \int_{t_k}^{t_{k+1}^-} \big[g(x(t)) - g(x(t_k) - y(t_k))\big]\mathrm{d}B(t) \end{aligned}$$

and, due to $y(0) = 0$,

$$\begin{aligned} y(t_{k+1}) &= \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}^-} \big[f(x(t)) - f(x(t_j) - y(t_j))\big]\mathrm{d}t \\ &\quad + \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}^-} \big[g(x(t)) - g(x(t_j) - y(t_j))\big]\mathrm{d}B(t) \\ &= \int_0^{t_{k+1}} \big[f(x(t)) - f(x(t_*) - y(t_*))\big]\mathrm{d}t \\ &\quad + \int_0^{t_{k+1}} \big[g(x(t)) - g(x(t_*) - y(t_*))\big]\mathrm{d}B(t), \quad (98) \end{aligned}$$

where $t_* := \sup\{t_j : t_j \le t, j \in \mathbb{N}\}$ for all $t \ge 0$. By Cauchy-Schwaz inequality, (98) produces

$$\begin{aligned} &|y(t_{k+1})|^2 \\ &\quad = \Big|\int_0^{t_{k+1}} \big[f(x(t)) - f(x(t_*) - y(t_*))\big]\mathrm{d}t \\ &\qquad + \int_0^{t_{k+1}} \big[g(x(t)) - g(x(t_*) - y(t_*))\big]\mathrm{d}B(t)\Big|^2 \\ &\quad \le 2\Big[t_{k+1} \int_0^{t_{k+1}} \big|f(x(t)) - f(x(t_*) - y(t_*))\big|^2\mathrm{d}t \\ &\qquad + \Big|\int_0^{t_{k+1}} \big[g(x(t)) - g(x(t_*) - y(t_*))\big]\mathrm{d}B(t)\Big|^2\Big]. \end{aligned}$$

By the Itô isometry and the global Lipschitz condition (45),

$$\begin{aligned} &\mathbb{E}\,|y(t_{k+1})|^2 \\ &\quad \le 2t_{k+1}\,\mathbb{E}\int_0^{t_{k+1}} \big|f(x(t)) - f(x(t_*) - y(t_*))\big|^2\mathrm{d}t \\ &\qquad + 2t_{k+1}\,\mathbb{E}\int_0^{t_{k+1}} \big|g(x(t)) - g(x(t_*) - y(t_*))\big|^2\mathrm{d}t \\ &\quad \le 2L^2(t_{k+1}+1)\,\mathbb{E}\int_0^{t_{k+1}} \big|x(t) - x(t_*) + y(t_*)\big|^2\mathrm{d}t. \end{aligned}$$

Since (96a) and (96b) give $x(t) - x(t_*) = y(t) - y(t_*)$ for all $t \ge 0$, this implies

$$\mathbb{E}\,|y(t_{k+1})|^2 \le 2L^2(t_{k+1}+1)\,\mathbb{E}\int_0^{t_{k+1}} |y(t)|^2\mathrm{d}t. \quad (99)$$

For any $T \ge 0$, using the Itô formula, (45), (52) and (99), one obtains

$$\begin{aligned} &\mathbb{E}\,|y(T)|^2 \\ &\quad = \mathbb{E}\,|y(T_*)|^2 + \mathbb{E}\int_{T_*}^{T} \big[2\,y^T(s)f(x(s)) + |g(x(s)|^2\big]\mathrm{d}s \\ &\quad \le 2L^2(T_*+1)\,\mathbb{E}\int_0^{T_*} |y(s)|^2\mathrm{d}s + \mathbb{E}\int_{T_*}^{T} |y(s)|^2\mathrm{d}s \\ &\qquad + \mathbb{E}\int_{T_*}^{T} \big[|f(x(s))|^2 + |g(x(s)|^2\big]\mathrm{d}s \\ &\quad \le K_T\,\mathbb{E}\int_0^{T_*} |y(s)|^2\mathrm{d}s + 2\,L^2\,\mathbb{E}\int_{T_*}^{T} |x(s)|^2\mathrm{d}s, \end{aligned}$$

where constant $K_T = 2L^2(T_*+1) \vee 1$. This implies

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\le t\le T} |y(t)|^2\Big] \le 2\,L^2 \int_0^{\Delta t} \mathbb{E}\Big[\sup_{0\le t_j\le T_*} |x(t_j+s)|^2\Big]\mathrm{d}s \\ &\qquad + K_T \int_0^T \mathbb{E}\Big[\sup_{0\le s\le t} |y(s)|^2\Big]\mathrm{d}t \\ &\quad \le 2C_T L^2 \Delta t + K_T \int_0^T \mathbb{E}\Big[\sup_{0\le s\le t} |y(s)|^2\Big]\,\mathrm{d}t, \end{aligned}$$

where C_T is given by (97) above. In view of the Gronwall inequality (see, e.g., [33, Lemma 4.5.1, p.129] and [38, Theorem 8.1, p.45]), this yields

$$\mathbb{E}\Big[\sup_{0\le t\le T} |y(t)|^2\Big] \le 2C_T L^2 e^{K_T T}\Delta t,$$

which completes the proof. $\square$

ACKNOWLEDGMENT

The author gratefully acknowledges Prof. X. Mao (the author's Ph.D. Supervisor at the University of Strathclyde, U.K.) for his helpful comments, which improve the quality of this work. The author would like to thank the reviewers for their comments, which help improve this work. Some of this work was done during the author's visit to The Centre for Stochastic & Scientific Computations, Harbin Institute of Technology, and the author would like to thank Prof. M. Song, Prof. M. Liu, and their group for the helpful discussions. The author is also thankful to Prof. K. C. Sou for the communications in the past a few years. This paper is also dedicated to the memory of J. Long, the author's mathematics teacher at Guangzhou No.95 Senior Middle School where the author enjoyed very much his inspiring style of teaching and the beauty of mathematics in his lectures.

REFERENCES

[1] L. Arnold, *Stochastic Differential Equations: Theory and Applications*. New York, NY, USA: Wiley, 1974.
[2] K. J. Åström, *Introduction to Stochastic Control Theory*. New York, NY, USA: Academic, 1970.
[3] K. J. Åström and B. Wittenmark, *Computer-Controlled Systems: Theory and Design*, 3rd ed. Upper Saddle River, NJ, USA: Prentice-Hall, 1997.

[4] K. J. Åström and P. R. Kumar, "Control: A perspective," *Automatica*, vol. 50, no. 1, pp. 3–43, 2014.
[5] S. Blanes and F. Cacas, *A Concise Introduction to Geometric Numerical Integration*. Boca Raton, FL, USA: CRC Press, 2016.
[6] S. Boyed, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 1994.
[7] E. Buckwar and C. Kelly, "Non-normal drift structures and linear stability analysis of numerical methods for systems of stochastic differential equations," *Comput. Math. Appl.*, vol. 64, no. 7, pp. 2282–2293, 2012.
[8] E. Buckwar and T. Sickenberger, "A structural analysis of asymptotic mean-square stability for multi-dimensional linear stochastic differential systems," *Appl. Numer. Math.*, vol. 62, no. 7, pp. 842–859, 2012.
[9] T. Caraballo, M. A. Hammami, and L. Mchiri, "Practical exponential stability of impulsive stochastic functional differential equations," *Syst. Control Lett.*, vol. 109, pp. 43–48, Nov. 2017.
[10] Control Panel Members. *Control in an Information Rich World: Report of the Panel on Future Directions in Control, Dynamics, and Systems*. Accessed: Apr. 11, 2022. [Online]. Available: http://www.cds.caltech.edu/~murray/cdspanel/report/latest.pdf
[11] CPS Summit Report members. *Report: Cyber-Physical Systems Summit*. Accessed: Apr. 11, 2022. [Online]. Available: http://iccps2012.cse.wustl.edu/_doc/CPS_Summit_Report.pdf
[12] L. El Ghaouii, "State-feedback control of systems with multiplicative noise via linear matrix inequalities," *Syst. Control Lett.*, vol. 24, pp. 223–228, Feb. 1995.
[13] E. Fridman, "Tutorial on Lyapunov-based methods for time-delay systems," *Eur. J. Control*, vol. 20, no. 6, pp. 271–283, 2014.
[14] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox*. Natick, MA, USA: MathWorks, 1995.
[15] I. I. Gikhman and A. V. Skorokhod, *Introduction to the Theory of Random Processes*. Philadelphia, PA, USA: Dover, 1996.
[16] E. Hairer, C. Lubich, and G. Wanner, *Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations*, 2nd ed. Berlin, Germany: Springer-Verlag, 2006.
[17] T. A. N. Heirung, J. A. Paulson, J. O'Leary, and A. Mesbah, "Stochastic model predictive control—How does it work?" *Comput. Chem. Eng.*, vol. 114, pp. 158–170, Jun. 2018.
[18] J. P. Hespanha, D. Liberzon, and A. R. Teel, "Lyapunov conditions for input-to-state stability of impulsive systems," *Automatica*, vol. 44, no. 11, pp. 2735–2744, Nov. 2008.
[19] D. J. Higham, "Mean-square and asymptotic stability of the stochastic theta method," *SIAM J. Numer. Anal.*, vol. 38, no. 3, pp. 753–769, 2000.
[20] D. J. Higham, "An algorithmic introduction to numerical simulation of stochastic differential equations," *SIAM Rev.*, vol. 43, no. 3, pp. 525–546, Mar. 2001.
[21] D. J. Higham, X. Mao, and A. M. Stuart, "Strong convergence of Euler-type methods for nonlinear stochastic differential equations," *SIAM J. Numer. Anal.*, vol. 40, no. 3, pp. 1041–1063, 2002.
[22] D. J. Higham, X. Mao, and A. M. Stuart, "Exponential mean-square stability of numerical solutions to stochastic differential equations," *LMS J. Comput. Math.*, vol. 6, no. 6, pp. 297–313, Jan. 2003.
[23] D. J. Higham and X. Mao, "Nonnormality and stochastic differential equations," *BIT Numer. Math.*, vol. 46, no. 3, pp. 525–532, Sep. 2006.
[24] D. J. Higham, "Modeling and simulating chemical reactions," *SIAM Rev.*, vol. 50, no. 2, pp. 347–368, 2008.
[25] L. Huang and X. Mao, "On input-to-state stability of stochastic retarded systems with Markovian switching," *IEEE Trans. Autom. Control*, vol. 54, no. 8, pp. 1898–1902, Aug. 2009.
[26] L. Huang, "Stability and stabilisation of stochastic delay systems," Ph.D. thesis, Dept. Math. Statist., Univ. Strathclyde, Glasgow, U.K., 2010.
[27] L. Huang and X. Mao, "SMC design for robust H_∞ control of uncertain stochastic delay systems," *Automatica*, vol. 46, pp. 405–412, Feb. 2010.
[28] L. Huang, H. Hjalmarsson, and H. Koeppl, "Almost sure stability and stabilization of discrete-time stochastic systems," *Syst. Control Lett.*, vol. 82, pp. 26–32, Aug. 2015.
[29] L. Huang, L. Pauleve, C. Zechner, M. Unger, A. S. Hansen, and H. Koeppl, "Reconstructing dynamic molecular states from single-cell time series," *J. Roy. Soc. Interface*, vol. 13, no. 122, Sep. 2016, Art. no. 20160533.
[30] L. Huang and S. Xu, "Impulsive stabilization of systems with control delay," *IEEE Trans. Autom. Control*, early access, Mar. 3, 2022, doi: 10.1109/TAC.2022.3155666.
[31] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden, "Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients," *Ann. Appl. Probab.*, vol. 22, no. 4, pp. 1611–1641, 2012.
[32] R. Khasminskii, *Stochastic Stability of Differential Equations*, 2nd ed. Berlin, Germany: Springer-Verlag, 2012.
[33] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*. Berlin, Germany: Springer-Verlag, 1992.
[34] E. A. Lee, "CPS foundations," in *Proc. 47th IEEE/ACM Design Autom. Conf. (DAC)*, Chicago, IL, USA, 2010, pp. 737–742.
[35] E. A. Lee, "The past, present and future of cyber-physical systems: A focus on models," *Sensors*, vol. 15, no. 3, pp. 4837–4869, 2015.
[36] Y. Liu and Z. Song, *Theory and Application of Large-Scale Dynamic Systems: Decomposition, Stability and Structure*, vol. 1. Guangzhou, China: South China Univ. of Technology, 1988.
[37] Y. Liu and Z. Feng, *Theory and Application of Large-Scale Dynamic Systems: Stochastic Stability and Control*, vol. 4. Guangzhou, China: South China Univ. of Technology, 1992.
[38] X. Mao, *Stochastic Differential Equations and Applications*, 2nd ed. Chichester, U.K.: Horwood Publishing, 2007.
[39] X. Mao, "Almost sure exponential stability in the numerical simulation of stochastic differential equations," *SIAM J. Numer. Anal.*, vol. 53, no. 1, pp. 370–389, 2015.
[40] X. Mao, "The truncated Euler–Maruyama method for stochastic differential equations," *J. Comput. Appl. Math.*, vol. 290, pp. 370–384, Dec. 2015.
[41] P. Naghshtabrizi, J. P. Hespanha, and A. R. Teel, "Exponential stability of impulsive systems with application to uncertain sampled-data systems," *Syst. Control Lett.*, vol. 57, no. 5, pp. 378–385, May 2008.
[42] B. Øksendal, *Stochastic Differential Equations: An Introduction With Applications*, 5th ed. New York, NY, USA: Springer, 2000.
[43] S. Peng and Y. Zhang, "Razumikhin-type theorems on *p*th moment exponential stability of impulsive stochastic delay differential equations," *IEEE Trans. Autom. Control*, vol. 55, no. 8, pp. 1917–1922, Aug. 2010.
[44] S. Sabanis, "A note on tamed Euler approximations," *Electron. Commun. Probab.*, vol. 47, pp. 1–10, Jan. 2013.
[45] Y. Saito and T. Mitsui, "Stability analysis of numerical schemes for stochastic differential equations," *SIAM J. Numer. Anal.*, vol. 33, no. 6, pp. 2254–2267, Dec. 1996.
[46] A. M. Samoilenko and N. A. Perestyuk, *Theory of Impulsive Differential Equations*. Singapore: Word Scientific, 1995.
[47] S. Särkkä and A. Solin, *Applied Stochastic Differential Equations*. Cambridge, U.K.: Cambridge Univ. Press, 2019.
[48] F. Scotton, L. Huang, S. A. Ahmadi, and B. Wahlberg, "Physics-based modeling and identification for HVAC systems?" in *Proc. Eur. Control Conf.*, Zurich, Switzerland, 2013, pp. 1404–1409.
[49] A. M. Stuart and P. Humphries, *Dynamical Systems and Numerical Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1996.
[50] H. S. Tsien, *Engineering Cybernetics*. Shanghai, China: Shanghai Jiaotong Univ. Press, 2015.
[51] U. Vaidya, "Stochastic stability analysis of discrete-time system using Lyapunov measure," in *Proc. Amer. Control Conf.*, Chicago, IL, USA, 2015, pp. 4646–4651.
[52] N. Wiener, *Cybernetics: Or Control and Communication in the Animal and the Machine*, 2nd ed. Cambridge, MA, USA: MIT Press, 1961.
[53] T. Yang, *Impulsive Control Theory*. Berlin, Germany: Springer-Verlag, 2001.

LIRONG HUANG was born in Guangzhou, Guangdong, China. He received the B.Eng. and M.Eng. degrees from the South China University of Technology, Guangzhou, the M.Sc. degree from King's College London, and the Ph.D. degree from the University of Strathclyde, Glasgow, U.K.

From November 2010 to October 2012, he was a Postdoctoral Researcher with the Automatic Control Laboratory, KTH Royal Institute of Technology, Stockholm, Sweden. From November 2012 to October 2015, he was with the Automatic Control Laboratory and then the Institute of Molecular Systems Biology, ETH Zürich, Switzerland. From May 2018 to February 2021, he worked under a fix-term contract with the Guangdong University of Technology, Guangzhou.