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RESEARCH ARTICLE

Aggregation and Interaction Aggregation Soft Operators on Interval-Valued q-Rung Orthopair Fuzzy Soft Environment and Application in Automation Company Evaluation

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ABSTRACT In several practical decision procedures, it is not accessible to denote assessments by a single crisp number due to a lack of information. However, representing information by an interval number within [0, 1] is a more credible approach. In multi-criteria decision making (MCDM) such an interval number can significantly catch information. In addition, the combination of soft sets with interval-valued q-rung orthopair fuzzy sets can be viewed as interval-valued q-rung orthopair fuzzy soft sets (IVq-ROFSs). It can be a reliable tool to cope with uncertainties. Usually, aggregation operators are functional in MCDM techniques; therefore, aggregation operators on IVq-ROFSs can significantly aggregate pieces of information in intervals with IVq-ROFSs. In this paper, we investigated some crucial properties of interval valued q-rung orthopair fuzzy soft sets (IVq-ROFSSs) and expressed a different representation of IVq-ROFSS in the form of IVq-ROFS number. Based on this representation, we investigated IVq-ROF weighted averaging, IVq-ROF weighted geometric operators and given their basic properties. Moreover, we consider interactions between non-memberships and memberships of different interval-valued q-rung orthopair fuzzy values and defined IVq-ROF weighted interaction averaging, IVq-ROF weighted interaction geometric aggregation operators in IVq-ROFS environments. A decision-making process is given, and an illustration is provided by tackling application in automation company evaluation.

INDEX TERMS Soft set, fuzzy set, q-ROFSs, aggregation operators, interaction aggregation operators, MCDM.

I. INTRODUCTION

Till 1965, only probability theory and error calculus were partly able to satisfy the need to handle a special kind of uncertainty, namely, randomness. The probability theory has no composition for describing fuzzy predicates such as small, young, much larger etc. In 1965, Zadeh [58] launched his seminal paper "Fuzzy sets". Fuzzy sets (FSs) are more conceivable in seeking for scalable knowledge among uncertainties and evolve as vital fundamentals of pattern recognition and multi criteria decision making. However, when assessing real situations in daily life, things are acquiring more complexity, which suggests that a single membership value can not reflect the essence of the objects. Thus, due to such difficulties with FSs, a more effective concept based on the FS is concluded whose name is intuitionistic fuzzy set(IFS)and it was introduced by Atanassove [8].

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IFS provide a membership value γ and non-membership value η such that their range of sum lies in inequality $\gamma + \eta \leq 1$. Yager and Xu developed some fundamental operators and their underlying properties [50], [51]. IFSs plot cognitive aspects of complex information in specific domains where uncertainty grasps outcomes. In some complex problems when $\gamma + \eta \geq 1$ then IFS become insufficient. This insufficiency of IFS is tackled when Yager proposed Pythagorean FS (PFS) [53]. The space is extended in PFS as $\gamma^2 + \eta^2 \leq 1$, and it tackled several incompleteness of an information. Although, PFSs works in their specifications but a more general form of extended space cover up in q-rung orthopair FSs (q-ROFSs) [54]. Several discussions on q-ROFSs has been made and it applied in wide rage of domains including, [3], [10], [28], [29], [30], [31], [40], [47], [52], and [10]. Some interaction aggregation operators (IAOs) on PFSs by considering interaction laws proposed by Wei [48]. Zhang [59] developed Hamacher IAOs on PFSs and introduced a method of multiple criteria decision-making (MCDM).

The q-ROFSs particularized memberships and no-memberships grades, but usually such assessed grades appear in intervals. An interval form of q-ROFSs was required, therefore, interval valued q-ROFS (IVq-ROFS) introduced by Joshi [22]. After the arrival of this essential idea, researchers around the world were attracted by IVq-ROFSs [14], [44], [49]. Moreover, IVq-ROF weighted averaging (IVq-ROFWA), IVq-ROF ordered weighted averaging (IVq-ROFOWA) operators have been introduced by Ju *et al.* [23]. Although, IVq-ROFWA or IVq-ROFOWA aggregate data but in certain situations it is required to combine parametrization concept with IVq-ROFS.

A comprehensive parameterized and theoretical structure which represent and appraise uncertainty, known as soft sets (SSs) [38], arise from distant circumstances concerning parametrization. Different from stranded sets, SS comprises mapping where meanings of objects in domain appraised. In recent decade, the findings on SS theory has emerged quickly [1], [2], [4], [5], [11], [19], [26], [37]. Maji *et al.* [39] and Ali *et al.* [6] make larger the scope of SSs by defining useful operations. In certain situations fuzziness prevails in approximations of SS, thus Maji *et al.* [33] explore fuzzy SSs (FSSs) which assess both parametrization and indistinctness. In recent years, several hybrid models of FSSs and intuitionistic fuzzy SSs have been emerged [12], [16], [17], [24], [25], [34], [35], [36], [43].

Recently, q-rung orthopair fuzzy SSs (q-ROFSSs) and q-ROFSSs based MCDM methods are proposed by Riaz *et al.* [20]. The q-ROFSS is a classical settings of q-ROFS to deal with uncertainties in the framework and descriptions of parameters in contrast of reliable and un-reliable informations in the larger space [42]. Hussain *et al.* [21] proposed MCDM frameworks by making fissile averaging operators and order averaging operators on q-ROFSSs. Another model of averaging operators over q-ROFSSs [18] has been developed by a different mean of generalizations parameter. Recently, Zulqarnain *et al.* [55] proposed IAOs to solve MCDM problem under pythagorean fuzzy soft (PFS) environment. Moreover, geometric interaction averaging operators in IFSs environments has been introduced by He *et al.* [15]. Although many methods on q-ROFS can be found but in combination with soft sets it gives a classical notion to handle complex information. Soft representation of q-ROFS makes linearity of different components of an environment more viable. A large q-ROFS data might difficult to compute in many cases but q-ROFSSs do not have such inadequacy.

In some practical decision procedures, due to lack of availability of information, it is not accessible to denote assessments by a single crisp number, however representation of information by an interval number within [0, 1] is a fair choice [9], [13], [27], [41], [45], [46]. As interval based memberships are relay on composure of assessment when it is difficult to choose a single value in [0, 1]. Therefore, it was required that the classical model of q-ROFSSs can further elaborate on interval of membership and interval of non-membership. Recently, Ali *et al.* [7] introduced interval valued q-ROFSSs (IVq-ROFSSs) and applications in attributes reduction. As discussed earlier benefits of interval of membership and interval of non-membership in MCDM, further, existence of parametrization in IVq-ROFSS can manage larger information. Especially, aggregation operators on environments of IVq-ROFSS can overcome drawbacks which exists in operators in q-ROFS or q-ROFSS. Moreover, interactions between non-memberships and memberships of different interval-valued q-rung orthopair fuzzy values can be substantial to aggregate intervals of memberships and intervals of non-memberships in IVq-ROFSS. In recent years MCDM approaches have been adopted with extensive success to support decision making in a wide range of complex MCDM real-world problems. The integration of IVq-ROFSS, MCDM specifies new potentialities relating to the modeling of MCDM problems in complex environments. By the motivations of such interval capturing prospect of IVq-ROFSS and IOAs [15], [55], it is required to study aggregations operators on IVq-ROFSSs. Therefore in this work we proposed group MCDM on IVq-ROFSS by mean of new introduced IVq-ROFS aggregations (averaging, geometric, interaction averaging, interaction geometric) operators in this work.

II. PRELIMINARIES

This part of paper expresses some fundamental rudiments of q-ROFSs and soft sets. Throughout this section $\mathscr{U}$ denotes universe of discourse. A fuzzy set (FS) $\mathcal{A}$ is defined as;

$$\mathcal{A} = \{(x, \rho_{\mathcal{A}}(x)) \mid x \in \mathscr{U}\},$$

where $\rho_{\mathcal{A}}$ is known as fuzzy membership grade of x in $\mathcal{A}$ [58].

In several complex problems FSs are little tawdry to deal with credibility and incredibility of an information. In such a situation IFSs are implemented.

Definition 1 [8]: An IFS over $\mathscr{U}$ is indicated as

$$\mathcal{A} = \{(x, \rho_{\mathcal{A}}(x), \varrho_{\mathcal{A}}(x)) \mid x \in \mathscr{U}\},$$

where the functions $\rho_{\mathcal{A}} : \mathscr{U} \to [0, 1]$ and $\varrho_{\mathcal{A}}(x) : \mathscr{U} \to [0, 1]$ assign the membership grade and non-membership grade of an object x in $\mathscr{U}$. Mainly, it is required that $0 \leq \rho_{\mathcal{A}}(x) + \varrho_{\mathcal{A}}(x) \leq 1 \ \forall x \in \mathscr{U}$.

Yager investigated Pythagorean fuzzy sets (PFSs) [53] and q-rung orthopair fuzzy sets (q-ROFSs) [54], which are vital generalizations of IFSs. The q-ROFSs comprise an overall scenarios with arrangement of membership grade and non-membership grade in the larger space.

Definition 2 [54]: A q-rung orthopair fuzzy set (q-ROFS) in a universe $\mathscr{U}$ is defined as

$$\mathcal{P} = \{(x, \mu_{\mathcal{P}}(x), \nu_{\mathcal{P}}(x)) \mid x \in \mathscr{U}\},$$

where the functions $\mu_{\mathcal{P}} : \mathscr{U} \to [0, 1]$ and $\nu_{\mathcal{P}} : \mathscr{U} \to [0, 1]$ respectively assign the degree of membership and non-membership grades of the element $x \in \mathscr{U}$. Further, it is required that $0 \leq (\mu_{\mathcal{P}}(x))^q + (\nu_{\mathcal{P}}(x))^q \leq 1 \ \forall x \in \mathscr{U}$, where $q \geq 1$. The hesitancy degree of q-ROFS is indicated as $\pi_{\mathcal{P}}(x) = \sqrt[q]{1 - (\mu_{\mathcal{P}}(x))^q - (\nu_{\mathcal{P}}(x))^q}$.

The set of all q-ROFSs over $\mathscr{U}$ is denoted by $q-ROFS^{\mathscr{U}}$.

In most real-life complex MCDM problems under the q-ROF environment, it is not suitable for decision-makers to give accurate values of both grades (membership grade and non-membership grade) in the situation of hesitation or oscillation of judgments. It is more appropriate for decision-makers to provide their assessments in the closed interval subset when situation is unclear by providing one grade.

Definition 3 [22]: A IVq-ROFS in a universe $\mathscr{U}$ is expressed as

$$\overline{\mathcal{P}} = \{(x, \overline{\rho}_{\overline{\mathcal{P}}}(x), \overline{\varrho}_{\overline{\mathcal{P}}}(x)) \mid x \in \mathscr{U}\},$$

where the functions $\overline{\rho}_{\overline{\mathcal{P}}} : \mathscr{U} \to int[0, 1]$ and $\overline{\varrho}_{\overline{\mathcal{P}}} : \mathscr{U} \to int[0, 1]$, that is, $\overline{\rho}_{\overline{\mathcal{P}}} = [\rho^-_{\overline{\mathcal{P}}}, \rho^+_{\overline{\mathcal{P}}}]$ and $\overline{\varrho}_{\overline{\mathcal{P}}} = [\varrho^-_{\overline{\mathcal{P}}}, \varrho^+_{\overline{\mathcal{P}}}]$ respectively, known as interval degree of membership grade and non-membership grade. Further, it is required that $0 \leq (\rho^+_{\overline{\mathcal{P}}}(x))^q + (\varrho^+_{\overline{\mathcal{P}}}(x))^q \leq 1 \ \forall x \in \mathscr{U}$, where $q \geq 1$. For $x \in \mathscr{U}$ the nondeterminacy index is expressed as $\overline{\pi}_{\overline{\mathcal{P}}} = [\pi^-_{\overline{\mathcal{P}}}, \pi^+_{\overline{\mathcal{P}}}] = [\sqrt[q]{1 - (\rho^+_{\overline{\mathcal{P}}}(x))^q - (\varrho^+_{\overline{\mathcal{P}}}(x))^q}, \sqrt[q]{1 - (\rho^-_{\overline{\mathcal{P}}}(x))^q - (\varrho^-_{\overline{\mathcal{P}}}(x))^q}]$.

The set of all IVq-ROFSs over $\mathscr{U}$ is denoted by $IVq-ROFS^{\mathscr{U}}$. Let $\overline{\mathcal{P}}_1 = \{(x, [\rho^-_{\overline{\mathcal{P}}_1}(x), \rho^+_{\overline{\mathcal{P}}_1}(x)], [\varrho^-_{\overline{\mathcal{P}}_1}(x), \varrho^+_{\overline{\mathcal{P}}_1}(x)]) \mid x \in \mathscr{U}\}$ and $\overline{\mathcal{P}}_2 = \{(x, [\rho^-_{\overline{\mathcal{P}}_2}(x), \rho^+_{\overline{\mathcal{P}}_2}(x)], [\varrho^-_{\overline{\mathcal{P}}_2}(x), \varrho^+_{\overline{\mathcal{P}}_2}(x)]) \mid x \in \mathscr{U}\}$ be two IVq-ROFSs over $\mathscr{U}$. Then, as shown in the equation at the bottom of the page, And $(\overline{\mathcal{P}}_1)^c = \{(x, \overline{\varrho}_{\overline{\mathcal{P}}_1}(x), \overline{\rho}_{\overline{\mathcal{P}}_1}(x)) \mid x \in \mathscr{U}\}$, $\overline{\mathcal{P}}_1 \subseteq \overline{\mathcal{P}}_2$ if and only if $\rho^-_{\overline{\mathcal{P}}_1}(x) \leq \rho^-_{\overline{\mathcal{P}}_2}(x), \rho^+_{\overline{\mathcal{P}}_1}(x) \leq \rho^+_{\overline{\mathcal{P}}_2}(x), \varrho^-_{\overline{\mathcal{P}}_1}(x) \geq \varrho^-_{\overline{\mathcal{P}}_2}(x), \varrho^+_{\overline{\mathcal{P}}_1}(x) \geq \varrho^+_{\overline{\mathcal{P}}_2}(x)$.

Definition 4 [23]: Assume $\mathcal{F} = ([\rho^-, \rho^+], [\varrho^-, \varrho^+])$ be an IVq-ROFV, a score function $\mathscr{S}$ can be defined as

$$\mathscr{S}(\mathcal{F}) = \frac{1}{4}\begin{bmatrix} \left(1 + (\rho^-)^q - (\varrho^-)^q\right) + \\ \left(1 + (\rho^+)^q - (\varrho^+)^q\right) \end{bmatrix}, \quad \mathscr{S}(\mathcal{F}) \in [0, 1].$$

Definition 5 [22]: Assume $\mathcal{F}_1 = \langle[\rho_1^-, \rho_1^+][\varrho_1^-, \varrho_2^+]\rangle$ and $\mathcal{F}_2 = \langle[\rho_2^-, \rho_2^+][\varrho_2^-, \varrho_2^+]\rangle$ be IVq-ROFNs, then the operational axioms are defined as below

(i) $\mathcal{F}_1 \oplus \mathcal{F}_2 = \langle[\left((\rho_1^-)^q + (\rho_2^-)^q - (\rho_1^-)^q \times (\rho_2^-)^q\right)^{\frac{1}{q}}, \left((\rho_1^+)^q + (\rho_2^+)^q - (\rho_1^+)^q \times (\rho_2^+)^q\right)^{\frac{1}{q}}], [(\varrho_1^-) \times (\varrho_2^-), (\varrho_1^+) \times (\varrho_2^+)]\rangle$.

(ii) $\mathcal{F}_1 \otimes \mathcal{F}_2 = \langle[(\rho_1^-) \times (\rho_2^-), (\rho_1^+) \times (\rho_2^+)], [\left((\varrho_1^-)^q + (\varrho_2^-)^q - (\varrho_1^-)^q \times (\varrho_2^-)^q\right)^{\frac{1}{q}}, \left((\varrho_1^+)^q + (\varrho_2^+)^q - (\varrho_1^+)^q \times (\varrho_2^+)^q\right)^{\frac{1}{q}}]\rangle$.

(iii) $\epsilon\mathcal{F}_1 = \left\langle\left[\left((1 - (\rho_1^-)^q)^\epsilon)\right)^{\frac{1}{q}}, \left((1 - (\rho_1^+)^q)^\epsilon)\right)^{\frac{1}{q}}\right], \left[(\varrho_1^-)^\epsilon, (\varrho_1^+)^\epsilon\right]\right\rangle, \epsilon > 0.$

(iv) $\mathcal{F}_1^\epsilon = \left\langle\left[(\rho_1^-)^\epsilon, (\rho_1^+)^\epsilon\right], \left[\left((1 - (\varrho_1^-)^q)^\epsilon)\right)^{\frac{1}{q}}, \left((1 - (\varrho_1^+)^q)^\epsilon)\right)^{\frac{1}{q}}\right]\right\rangle, \epsilon > 0.$

Definition 6 [38]: Assume E be the set of parameters, $M \subseteq E$. A pair (S, M) is called a soft set over $\mathscr{U}$, where S is a mapping given by $S : M \to P(\mathscr{U})$. It is indicated as

$$(S, M) = \{(a, S(a)) \mid a \in M, S(a) \in P(\mathscr{U})\}.$$

Note that $P(\mathscr{U})$ is the set of all subsets of $\mathscr{U}$. The set of all soft sets over $\mathscr{U}$, with respect to subset of E is denoted by $SS^E(\mathscr{U})$.

Now, we provide a definition of IVq-ROFSS as follows.

Definition 7 [7]: Assume $\mathscr{U}$ be the universal set and $M \subseteq E$ be set of attributes. Let $IVq-ROFS(\mathscr{U})$ be the set of all IVq-ROFS over $\mathscr{U}$. An IVq-ROFSS over $\mathscr{U}$ is expressed as $(\mathcal{F}, M)$ or $\mathcal{F}_M$ where $\mathcal{F} : M \to IVq-ROFS(\mathscr{U})$. It is defined as

$$(\mathcal{F}, M) = \left\{\left(e, \{\frac{x}{\langle\rho_{\mathcal{F}_M}(x), \varrho_{\mathcal{F}_M}(x)\rangle}\}\right) \mid e \in M, x \in \mathscr{U}\right\}$$

$$\overline{\mathcal{P}}_1 \cup \overline{\mathcal{P}}_2 = \left\{\begin{pmatrix} x, [\max\{\rho^-_{\overline{\mathcal{P}}_1}(x), \rho^-_{\overline{\mathcal{P}}_2}(x)\}, \max\{\rho^+_{\overline{\mathcal{P}}_1}(x), \rho^+_{\overline{\mathcal{P}}_2}(x)\}], \\ [\min\{\varrho^-_{\overline{\mathcal{P}}_1}(x), \varrho^-_{\overline{\mathcal{P}}_2}(x)\}, \min\{\varrho^+_{\overline{\mathcal{P}}_1}(x), \varrho^+_{\overline{\mathcal{P}}_2}(x)\}] \end{pmatrix} \mid x \in \mathscr{U}\right\}$$

$$\overline{\mathcal{P}}_1 \cap \overline{\mathcal{P}}_2 = \left\{\begin{pmatrix} x, [\min\{\rho^-_{\overline{\mathcal{P}}_1}(x), \rho^-_{\overline{\mathcal{P}}_2}(x)\}, \min\{\rho^+_{\overline{\mathcal{P}}_1}(x), \rho^+_{\overline{\mathcal{P}}_2}(x)\}], \\ [\max\{\varrho^-_{\overline{\mathcal{P}}_1}(x), \varrho^-_{\overline{\mathcal{P}}_2}(x)\}, \max\{\varrho^+_{\overline{\mathcal{P}}_1}(x), \varrho^+_{\overline{\mathcal{P}}_2}(x)\}] \end{pmatrix} \mid x \in \mathscr{U}\right\}$$

TABLE 1. IVq-ROFSS = $(\overline{\mathcal{F}}, M)$.

$\mathscr{U} \mid \mathcal{M}$	e_1	e_2	e_3	$\cdots$	e_m
x_1	$\langle[\rho_{11}^-,\rho_{11}^+],[\varrho_{11}^-,\varrho_{11}^+]\rangle$	$\langle[\rho_{12}^-,\rho_{12}^+],[\varrho_{12}^-,\varrho_{12}^+]\rangle$	$\langle[\rho_{13}^-,\rho_{13}^+],[\varrho_{13}^-,\varrho_{13}^+]\rangle$	$\cdots$	$\langle[\rho_{1m}^-,\rho_{1m}^+],[\varrho_{1m}^-,\varrho_{1m}^+]\rangle$
x_2	$\langle[\rho_{21}^-,\rho_{21}^+],[\varrho_{21}^-,\varrho_{21}^+]\rangle$	$\langle[\rho_{22}^-,\rho_{22}^+],[\varrho_{22}^-,\varrho_{22}^+]\rangle$	$\langle[\rho_{23}^-,\rho_{23}^+],[\varrho_{23}^-,\varrho_{23}^+]\rangle$	$\cdots$	$\langle[\rho_{2m}^-,\rho_{2m}^+],[\varrho_{2m}^-,\varrho_{2m}^+]\rangle$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_s	$\langle[\rho_{s1}^-,\rho_{s1}^+],[\varrho_{s1}^-,\varrho_{s1}^+]\rangle$	$\langle[\rho_{s2}^-,\rho_{s2}^+],[\varrho_{s2}^-,\varrho_{s2}^+]\rangle$	$\langle[\rho_{s3}^-,\rho_{s3}^+],[\varrho_{s3}^-,\varrho_{s3}^+]\rangle$	$\cdots$	$\langle[\rho_{sm}^-,\rho_{sm}^+],[\varrho_{sm}^-,\varrho_{sm}^+]\rangle$

where function $\rho_{\mathcal{F}_M} : \mathscr{U} \to int[0,1]$ and $\varrho_{\mathcal{F}_M} : \mathscr{U} \to int[0,1]$ are the interval degree of membership and non-membership grades, respectively. These grades can be indicated as $\rho_{\mathcal{F}_M}(x) = [\rho^-_{\mathcal{F}_M}(x), \rho^+_{\mathcal{F}_M}(x)]$ and $\varrho_{\mathcal{F}_M}(x) = [\varrho^-_{\mathcal{F}_M}(x), \varrho^+_{\mathcal{F}_M}(x)]$. Note that $0 \leq (\rho^+_{\mathcal{F}_M}(x))^q + (\rho^+_{\mathcal{F}_M}(x))^q \leq 1$ is a condition on intervals of membership grade and non-membership grade. The set of all IVq-ROFSS over $\mathscr{U}$ is denoted by $IVq - ROFSS^{\mathscr{U}}$.

Example 8: Let $\mathscr{U} = \{x_1, x_2, x_3, x_4, x_5\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$. Suppose that $M = \{e_1, e_3, e_4, \}$ such that $M \subset E$.

For $q = 3$, we defined IVq-ROFSS as follows;

$$(\mathcal{G}, M) = \left\{ \begin{array}{l} \left(\begin{array}{l} e_1, \{x_1, [0.05, 0.08], [0.23, 0.33]\}, \\ \{x_2, [0.04, 0.07], [0.26, 0.39]\}, \\ \{x_3, [0.13, 0.63], [0.31, 0.53]\}, \\ \{x_4, [0.12, 0.42], [0.43, 0.49]\}, \\ \{x_5, [0.25, 0.45], [0.42, 0.54]\} \end{array} \right), \\ \left(\begin{array}{l} e_3, \{x_1, [0.15, 0.55], [0.42, 0.46]\}, \\ \{x_2, [0.13, 0.52], [0.32, 0.64]\}, \\ \{x_3, [0.26, 0.54], [0.34, 0.58]\}, \\ \{x_4, [0.16, 0.72], [0.24, 0.62]\}, \\ \{x_5, [0.21, 0.65], [0.29, 0.52]\} \end{array} \right) \\ \left(\begin{array}{l} e_4, \{x_1, [0.17, 0.68], [0.27, 0.52]\}, \\ \{x_2, [0.37, 0.53], [0.36, 0.48]\}, \\ \{x_3, [0.29, 0.64], [0.41, 0.52]\}, \\ \{x_4, [0.26, 0.72], [0.43, 0.57]\}, \\ \{x_5, [0.28, 0.62], [0.44, 0.60]\} \end{array} \right) \end{array} \right\}$$

III. AGGREGATION OPERATORS ON IVQ-ROFSS

The Definition 7 of IVq-ROFSS can be viewed in another shape.

Definition 9: Let a soft universe $(\mathscr{U}, E)$ and $M \subseteq E$. Let $M = \{e_1, e_2, \cdots e_m\}$ be the set of attributes for the a set of alternatives $\mathscr{U} = \{x_1, x_2, \cdots x_s\}$. A pair $(\overline{\mathcal{F}}, M)$ is said to be IVq-ROFSS over $\mathscr{U}$, where $\overline{\mathcal{F}}$ is a function given by $\overline{\mathcal{F}} : M \to IVq - ROFS^{\mathscr{U}}$, it is defined as

$$\overline{\mathcal{F}}_{e_j}(x_i) = \left\{ \left\langle x_i, \begin{array}{l} \left[\rho_j^-(x_i), \rho_j^+(x_i)\right], \\ \left[\varrho_j^-(x_i), \varrho_j^+(x_i)\right] \end{array} \right\rangle_q \mid \begin{array}{l} x_i \in \mathscr{U} \ \& \\ q \geq 1 \end{array} \right\}$$

where $IVq - ROFS^{\mathscr{U}}$ represents the collection of all IVq-ROFSs of $\mathscr{U}$. Here, $[\rho_j^-(x_i), \rho_j^+(x_i)]$ is the interval degree of membership grade of an objects $x_i \in \mathscr{U}$ and $[\varrho_j^-(x_i), \varrho_j^+(x_i)]$ is the interval degree of non-membership grade of an objects $x_i \in \mathscr{U}$. Note that $0 \leq (\rho_j^+(x_i))^q + (\varrho_j^+(x_i))^q \leq 1$ is a condition on intervals of membership and non-membership grades. The set of all IVq-ROFSS over $\mathscr{U}$ is denoted by $IVq - ROFSS^{\mathscr{U}}$. A more short the notion $\overline{\mathcal{F}}_{e_j}(x_i)$ is expressed by $\overline{\mathcal{F}}_{e_{ij}} = ([\rho_{ij}^-, \rho_{ij}^+], [\varrho_{ij}^-, \varrho_{ij}^+])$ and it is known as IVq-ROFS value (IVq-ROFSV). Moreover, the interval degree of hesitancy for IVq-ROFSS is defined as

$$\pi_{\overline{\mathcal{F}}_{e_{ij}}} = [\pi^-_{e_{ij}}, \pi^+_{e_{ij}}] = \left[\begin{array}{l} \sqrt[q]{1 - (\rho_j^+)^q - (\varrho_j^+)^q}, \\ \sqrt[q]{1 - (\rho_j^-)^q - (\varrho_j^-)^q} \end{array} \right]$$

Let $M = \{e_1, e_2, \cdots e_m\}$ be the set of attributes for the a set of alternatives $\mathscr{U} = \{x_1, x_2, \cdots x_s\}$. Then a general form of IVq-ROFSS is given in Table 1. Assume $\overline{\mathcal{F}}_{e_{i2}} = ([\rho_{i2}^-, \rho_{i2}^+], [\varrho_{i2}^-, \varrho_{i2}^+])$, $(i = 1, 2)$ and $\overline{\mathcal{F}} = ([\rho^-, \rho^+], [\varrho^-, \varrho^+])$ be any three IVq-ROFSVs and $\epsilon, \epsilon_1, \epsilon_2 > 0$. Thus some basic operation on IVq-ROFSVs are given follows:

(i) $\overline{\mathcal{F}}_{e_{12}} \cup \overline{\mathcal{F}}_{e_{22}} = \left\{ \begin{array}{l} \left[\max\{\rho_{12}^-, \rho_{22}^-\}, \max\{\rho_{12}^+, \rho_{22}^+\}\right], \\ \left[\min\{\varrho_{12}^-, \varrho_{22}^-\}, \min\{\varrho_{12}^+, \varrho_{22}^+\}\right] \end{array} \right\}.$

(ii) $\overline{\mathcal{F}}_{e_{12}} \cap \overline{\mathcal{F}}_{e_{22}} = \left\{ \begin{array}{l} \left[\min\{\rho_{12}^-, \rho_{22}^-\}, \min\{\rho_{12}^+, \rho_{22}^+\}\right], \\ \left[\max\{\varrho_{12}^-, \varrho_{22}^-\}, \max\{\varrho_{12}^+, \varrho_{22}^+\}\right] \end{array} \right\}.$

(iii) $\overline{\mathcal{F}}^c = (\varrho, \rho)$ where $\overline{\mathcal{F}}^c$ denotes the complement of $\overline{\mathcal{F}}$.

(iv) $\overline{\mathcal{F}}_{e_{12}} \subseteq \overline{\mathcal{F}}_{e_{22}} \iff \rho_{12}^- \leq \rho_{22}^-, \rho_{12}^+ \leq \rho_{22}^+, \varrho_{12}^- \geq \varrho_{22}^-, \varrho_{12}^+ \geq \varrho_{22}^+$.

(v) $\overline{\mathcal{F}}_{e_{12}} \oplus \overline{\mathcal{F}}_{e_{22}}$

$$= \left\{ \begin{array}{c} \left[\begin{array}{l} \sqrt[q]{(\rho_{12}^-)^q + (\rho_{22}^-)^q - (\rho_{12}^-)^q(\rho_{22}^-)^q}, \\ \sqrt[q]{(\rho_{12}^+)^q + (\rho_{22}^+)^q - (\rho_{12}^+)^q(\rho_{22}^+)^q} \end{array} \right], \\ \left[(\varrho_{12}^-)^q(\varrho_{22}^-)^q, (\varrho_{12}^+)^q(\varrho_{22}^+)^q\right] \end{array} \right\}$$

(vi) $\overline{\mathcal{F}}_{e_{12}} \otimes \overline{\mathcal{F}}_{e_{22}}$

$$= \left\{ \begin{array}{c} \left[(\rho_{12}^-)^q(\rho_{22}^-)^q, (\rho_{12}^+)^q(\rho_{22}^+)^q\right], \\ \left[\begin{array}{l} \sqrt[q]{(\varrho_{12}^-)^q + (\varrho_{22}^-)^q - (\varrho_{12}^-)^q(\varrho_{22}^-)^q}, \\ \sqrt[q]{(\varrho_{12}^+)^q + (\varrho_{22}^+)^q - (\varrho_{12}^+)^q(\varrho_{22}^+)^q} \end{array} \right] \end{array} \right\}$$

(vii) $\overline{\mathcal{F}}\epsilon = \left(\begin{array}{l} \left[\sqrt[q]{1-(1-(\rho^-)^q)^\epsilon}, \sqrt[q]{1-(1-(\rho^+)^q)^\epsilon}\right], \\ \left[(\varrho^-)^\epsilon, (\varrho^+)^\epsilon\right] \end{array} \right).$

(viii) $\overline{\mathcal{F}}^\epsilon = \left(\begin{array}{l} \left[(\rho^-)^\epsilon, (\rho^+)^\epsilon\right], \\ \left[\sqrt[q]{1-(1-(\varrho^-)^q)^\epsilon}, \sqrt[q]{1-(1-(\varrho^+)^q)^\epsilon}\right] \end{array} \right).$

TABLE 2. *IVq – ROFSWA* $(\overline{\mathcal{F}}, M)$ for $q \geq 3$.

$\mathscr{U} \mid \mathcal{M}$	e_1	e_2	e_3	e_4	e_5
x_1	$\langle[0.12, 0.22], [0.14, 0.26]\rangle$	$\langle[0.32, 0.35], [0.42, 0.46]\rangle$	$\langle[0.12, 0.63], [0.13, 0.17]\rangle$	$\langle[0.25, 0.28], [0.16, 0.19]\rangle$	$\langle[0.24, 0.40], [0.11, 0.20]\rangle$
x_2	$\langle[0.20, 0.40], [0.10, 0.50]\rangle$	$\langle[0.44, 0.49], [0.15, 0.17]\rangle$	$\langle[0.12, 0.75], [0.12, 0.22]\rangle$	$\langle[0.21, 0.32], [0.40, 0.60]\rangle$	$\langle[0.41, 0.59], [0.12, 0.60]\rangle$
x_3	$\langle[0.21, 0.26], [0.30, 0.32]\rangle$	$\langle[0.31, 0.39], [0.32, 0.36]\rangle$	$\langle[0.17, 0.72], [0.13, 0.23]\rangle$	$\langle[0.22, 0.34], [0.32, 0.53]\rangle$	$\langle[0.34, 0.44], [0.26, 0.36]\rangle$
x_4	$\langle[0.41, 0.45], [0.25, 0.30]\rangle$	$\langle[0.22, 0.29], [0.22, 0.64]\rangle$	$\langle[0.30, 0.40], [0.14, 0.24]\rangle$	$\langle[0.23, 0.36], [0.13, 0.62]\rangle$	$\langle[0.11, 0.24], [0.41, 0.51]\rangle$
x_5	$\langle[0.26, 0.42], [0.20, 0.23]\rangle$	$\langle[0.34, 0.37], [0.33, 0.38]\rangle$	$\langle[0.20, 0.60], [0.15, 0.25]\rangle$	$\langle[0.24, 0.37], [0.14, 0.63]\rangle$	$\langle[0.14, 0.52], [0.28, 0.48]\rangle$

A. IVQ-ROFS WEIGHTED AVERAGING AGGREGATION OPERATORS ON IVQ-ROFSSS

Let $\mathcal{Q}$ be any collection of IVq-ROFV and $\mathcal{Q}^{s\times m}$ set of IVq-ROFSV in a IVq-ROFSS $(\overline{\mathcal{F}}, M)$. Then we define (IVq-ROFS weighted averaging aggregation operators) IVq-ROFSWA as follows;

Definition 10: Let $M = \{e_1, e_2, \cdots e_m\}$ be the set of attributes for a set of "s" number of alternatives $\mathscr{U} = \{x_1, x_2, \cdots x_s\}$. Then a general form of IVq-ROFSS $(\overline{\mathcal{F}}, M)$ is given in Table 1. Assume that $\overline{\mathcal{F}}_{e_{ij}} = ([\rho^-_{ij}, \rho^+_{ij}], [\varrho^-_{ij}, \varrho^+_{ij}])$, $(i = 1, 2, \cdots s, j = 1, 2, \cdots m)$ be the collection of IVq-ROFSVs in $(\overline{\mathcal{F}}, M)$. Let $W = [w_1, w_2, \cdots w_m]$ and $\Delta = [\delta_1, \delta_2, \cdots \delta_s]$ be the weighted vectors over M and $\mathscr{U}$, respectively, such that $\sum_{j=1}^m w_j = 1$, $\sum_{i=1}^s \delta_i = 1$ and $w_j, \delta_i \in [0, 1]$. Define a mapping $IVq - ROFSWA : \mathcal{Q}^{s\times m} \rightarrow \mathcal{Q}$, that is, $IVq - ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{sm}}) = \oplus_{j=1}^m(\oplus_{i=1}^s \overline{\mathcal{F}}_{e_{ij}}\delta_i)w_j$.

Theorem 11: Assume that $\overline{\mathcal{F}}_{e_{ij}} = ([\rho^-_{ij}, \rho^+_{ij}], [\varrho^-_{ij}, \varrho^+_{ij}])$, $(i = 1, 2, \cdots s, j = 1, 2, \cdots m)$ *be the collection of IVq-ROFSVs in* $(\overline{\mathcal{F}}, M)$. *Let* $W = [w_1, w_2, \cdots w_m]$ *and* $\Delta = [\delta_1, \delta_2, \cdots \delta_s]$ *be weighted vectors over* M *and* $\mathscr{U}$, *respectively, such that* $\sum_{j=1}^m w_j = 1$, $\sum_{i=1}^s \delta_i = 1$ *and* $w_j, \delta_i \in [0, 1]$. *Then,*

$$IVq - ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{sm}}) = \oplus_{j=1}^m(\oplus_{i=1}^s \overline{\mathcal{F}}_{e_{ij}}\delta_i)w_j = \begin{pmatrix} [\sqrt[q]{1 - \prod_{j=1}^m \left(\prod_{i=1}^s \left(1 - \left(\rho^-_{ij}\right)^q\right)^{\delta_i}\right)^{w_j}}, \\ \sqrt[q]{1 - \prod_{j=1}^m \left(\prod_{i=1}^s \left(1 - \left(\rho^+_{ij}\right)^q\right)^{\delta_i}\right)^{w_j}}], \\ [\prod_{j=1}^m \left(\prod_{i=1}^s \left(\varrho^-_{ij}\right)^{\delta_i}\right)^{w_j}, \\ \prod_{j=1}^m \left(\prod_{i=1}^s \left(\varrho^+_{ij}\right)^{\delta_i}\right)^{w_j}] \end{pmatrix}.$$

Proof: The proof of the Theorem is given in APPENDIX 1. □

Example 12: Let an insurance company wants to recruit a sale consultant from a set of five applicants $\mathscr{U} = \{x_1, x_2, x_3, x_4, x_5\}$. For selection of suitable candidate, five most relevant attributes are chosen as in the set $E = \{e_1, e_2, e_3, e_4, e_5\}$, that is, $e_i (i = 1, 2, 3, 4, 5)$ stand for $e_1 =$ finance and insurance professional, $e_2 = self - confidence$, $e_3 = past\ experience$, $e_4 = interpersonal\ skills$, $e_5 = score\ in\ universty\ degree$, respectively. Let the weighted vectors $W = \{0.13, 0.21, 0.31, 0.24, 0.11\}$ and $\Delta = \{0.15, 0.23, 0.05, 0.28, 0.29\}$ over for M and $\mathscr{U}$ respectively. The evaluation data obtained is given in the Table 2. Let $q = 3$ for this example. Now by using Theorem 4.1, we calculate operator as follow

$$IVq - ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \overline{\mathcal{F}}_{e_{13}}, \ldots, \overline{\mathcal{F}}_{e_{55}}) = \begin{pmatrix} [\sqrt[q]{1 - \prod_{j=1}^5 \left(\prod_{i=1}^5 \left(1 - \left(\rho^-_{ij}\right)^q\right)^{\delta_i}\right)^{w_j}}, \\ \sqrt[q]{1 - \prod_{j=1}^5 \left(\prod_{i=1}^5 \left(1 - \left(\rho^+_{ij}\right)^q\right)^{\delta_i}\right)^{w_j}}], \\ [\prod_{j=1}^5 \left(\prod_{i=1}^5 \left(\varrho^-_{ij}\right)^{\delta_i}\right)^{w_j}, \\ \prod_{j=1}^5 \left(\prod_{i=1}^5 \left(\varrho^+_{ij}\right)^{\delta_i}\right)^{w_j}] \end{pmatrix}$$

where calculation as shown at the bottom of the next page. Therefore $IVq - ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \overline{\mathcal{F}}_{e_{13}}, \cdots, \overline{\mathcal{F}}_{e_{55}}) = ([0.291494, 0.426198], [0.221314, 0.399961])$.

Lemma 13: Idempotency: If $\overline{\mathcal{F}}_{e_{ij}} = \overline{\mathcal{F}}_e$, $(\forall i = 1, 2, \cdots, s$ *and* $j = 1, 2, \cdots, m)$, *where* $\overline{\mathcal{F}}_e = (\rho, \varrho)$, *then* $IVq - ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots, \overline{\mathcal{F}}_{e_{sm}}) = \overline{\mathcal{F}}_e$.

Proof: Assume $\overline{\mathcal{F}}_{e_{ij}} = \overline{\mathcal{F}}_e = (\rho, \varrho)$ $(\forall i = 1, 2, \cdots, s$ and $j = 1, 2, \ldots m)$. Now in terms of Theorem 4.1, we have

$$(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots, \overline{\mathcal{F}}_{e_{sm}}) = \begin{pmatrix} , [\sqrt[q]{1 - \prod_{j=1}^m \left(\prod_{i=1}^s \left(1 - \left(\rho^-_{ij}\right)^q\right)^{\delta_i}\right)^{w_j}}, \\ \sqrt[q]{1 - \prod_{j=1}^m \left(\prod_{i=1}^s \left(1 - \left(\rho^+_{ij}\right)^q\right)^{\delta_i}\right)^{w_j}}], \\ [\prod_{j=1}^m \left(\prod_{i=1}^s \left(\varrho^-_{ij}\right)^{\delta_i}\right)^{w_j}, \\ \prod_{j=1}^m \left(\prod_{i=1}^s \left(\varrho^+_{ij}\right)^{\delta_i}\right)^{w_j}] \end{pmatrix}$$
$$= ([\sqrt[q]{1 - (1 - (\rho^-)^q)}, \sqrt[q]{1 - (1 - (\rho^+)^q)}], [(\varrho^-), (\varrho^+)]) = (\rho, \varrho) = \overline{\mathcal{F}}_e.$$

Therefore $(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \ldots, \overline{\mathcal{F}}_{e_{sm}}) = \overline{\mathcal{F}}_e$ □

Lemma 14: Boundedness: If

$$\overline{\mathcal{F}}^-_{e_{ij}} = \begin{pmatrix} \left[\min_j \min_i\{\rho^-_{ij}\}, \min_j \min_i\{\rho^+_{ij}\}\right], \\ \left[\max_j \max_i\{\varrho^-_{ij}\}, \max_j \max_i\{\varrho^+_{ij}\}\right] \end{pmatrix}$$

and

$$\overline{\mathcal{F}}^{+}_{e_{ij}} = \left(\begin{array}{l}\left[max_j max_i\{\rho^{-}_{ij}\}, max_j max_i\{\rho^{+}_{ij}\}\right], \\ \left[min_j min_i\{\varrho^{-}_{ij}\}, min_j min_i\{\varrho^{+}_{ij}\}\right]\end{array}\right).$$

Then

$$\overline{\mathcal{F}}^{-}_{e_{ij}} \leq IVq-ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots, \overline{\mathcal{F}}_{e_{sm}}) \leq \overline{\mathcal{F}}^{+}_{e_{ij}}.$$

Proof: As

$$\overline{\mathcal{F}}^{-}_{e_{ij}} = \left(\begin{array}{l}\left[\min_j \min_i\{\rho^{-}_{ij}\}, \min_j \min_i\{\rho^{+}_{ij}\}\right], \\ \left[\max_j \max_i\{\varrho^{-}_{ij}\}, \max_j \max_i\{\varrho^{+}_{ij}\}\right]\end{array}\right),$$

$$\overline{\mathcal{F}}^{+}_{e_{ij}} = \left(\begin{array}{l}\left[max_j max_i\{\rho^{-}_{ij}\}, max_j max_i\{\rho^{+}_{ij}\}\right], \\ \left[min_j min_i\{\varrho^{-}_{ij}\}, min_j min_i\{\varrho^{+}_{ij}\}\right]\end{array}\right).$$

We have to show that $\overline{\mathcal{F}}^{-}_{e_{ij}} \leq IVq\text{–}ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots, \overline{\mathcal{F}}_{e_{sm}}) \leq \overline{\mathcal{F}}^{+}_{e_{ij}}$. Now for each $i = 1, 2, \cdots, s$ and $j = 1, 2, \cdots, m$, we have

$$min_j min_i\{\rho^{-}_{ij}\} \leq \rho^{-}_{ij} \leq max_j max_i\{\rho^{-}_{ij}\}$$
$$\leftrightarrow 1 - max_j max_i\{(\rho^{-}_{ij})^q\} \leq 1 - (\rho^{-}_{ij})^q$$
$$\leq 1 - min_j min_i\{(\rho^{-}_{ij})^q\}$$

$$\leftrightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1 - max_j max_i\left(\rho^{-}_{ij}\right)^q\right)^{\delta_i}\right)^{w_j}$$
$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1 - \left(\rho^{-}_{ij}\right)^q\right)^{\delta_i}\right)^{w_j}$$
$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1 - min_j min_i\left(\rho^{-}_{ij}\right)^q\right)^{\delta_i}\right)^{w_j}$$
$$\leftrightarrow \left(\left(1 - min_j min_i\left(\rho^{-}_{ij}\right)^q\right)^{\sum_{i=1}^{s}\delta_i}\right)^{\sum_{j=1}^{m} w_j}$$
$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1 - \left(\rho^{-}_{ij}\right)^q\right)^{\delta_i}\right)^{w_j}$$
$$\leq \left(\left(1 - min_j min_i\left(\rho^{-}_{ij}\right)^q\right)^{\sum_{i=1}^{s}\delta_i}\right)^{\sum_{j=1}^{m} w_j}$$
$$\leftrightarrow (1 - max_j max_i(\rho^{-}_{ij})^q)$$
$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1 - \left(\rho^{-}_{ij}\right)^q\right)^{\delta_i}\right)^{w_j} \leq (1 - min_j min_i(\rho^{-}_{ij})^q)$$
$$\leftrightarrow 1 - (1 - min_j min_i(\rho^{-}_{ij})^q)$$
$$\leq 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1 - \left(\rho^{-}_{ij}\right)^q\right)^{\delta_i}\right)^{w_j}$$
$$\leq 1 - (1 - max_j max_i(\rho^{-}_{ij})^q)$$

$$\sqrt[q]{1 - \prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(1 - \left(\rho^{-}_{ij}\right)^q\right)^{\delta_i}\right)^{w_j}}$$
$$= \sqrt[3]{\begin{array}{l}1 - \left\{\left(1-(0.12)^3\right)^{0.13}\left(1-(0.20)^3\right)^{0.21}\left(1-(0.21)^3\right)^{0.31}\left(1-(0.41)^3\right)^{0.24}\left(1-(0.26)^3\right)^{0.11}\right\}^{0.15} \\ \left\{\left(1-(0.32)^3\right)^{0.13}\left(1-(0.44)^3\right)^{0.21}\left(1-(0.31)^3\right)^{0.31}\left(1-(0.22)^3\right)^{0.24}\left(1-(0.34)^3\right)^{0.11}\right\}^{0.23} \\ \left\{\left(1-(0.12)^3\right)^{0.13}\left(1-(0.12)^3\right)^{0.21}\left(1-(0.17)^3\right)^{0.31}\left(1-(0.30)^3\right)^{0.24}\left(1-(0.20)^3\right)^{0.11}\right\}^{0.05} \\ \left\{\left(1-(0.25)^3\right)^{0.13}\left(1-(0.21)^3\right)^{0.21}\left(1-(0.22)^3\right)^{0.31}\left(1-(0.23)^3\right)^{0.24}\left(1-(0.24)^3\right)^{0.11}\right\}^{0.28} \\ \left\{\left(1-(0.24)^3\right)^{0.13}\left(1-(0.41)^3\right)^{0.21}\left(1-(0.34)^3\right)^{0.31}\left(1-(0.11)^3\right)^{0.24}\left(1-(0.14)^3\right)^{0.11}\right\}^{0.29}\end{array}}$$
$$= 0.291494,$$

$$\sqrt[q]{1 - \prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(1 - \left(\rho^{+}_{ij}\right)^q\right)^{\delta_i}\right)^{w_j}} = 0.426198,$$

$$\prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(\varrho^{-}_{ij}\right)^{\delta_i}\right)^{w_j} = \left\{\{(0.14)^{0.13}(0.10)^{0.21}(0.30)^{0.31}(0.25)^{0.24}(0.20)^{0.11}\}^{0.15}\right.$$
$$\{(0.42)^{0.13}(0.15)^{0.21}(0.32)^{0.31}(0.22)^{0.24}(0.33)^{0.11}\}^{0.23}\{(0.13)^{0.13}(0.12)^{0.21}(0.13)^{0.31}(0.14)^{0.24}(0.15)^{0.11}\}^{0.05}$$
$$\left.\{(0.16)^{0.13}(0.40)^{0.21}(0.32)^{0.31}(0.13)^{0.24}(0.14)^{0.11}\}^{0.28}\{(0.11)^{0.13}(0.12)^{0.21}(0.26)^{0.31}(0.41)^{0.24}(0.28)^{0.11}\}^{0.29}\right\} = 0.221314,$$

$$\prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(\varrho^{+}_{ij}\right)^{\delta_i}\right)^{w_j} = 0.399961.$$

Hence

$$min_j min_i\{\rho_{ij}^-\} \leq \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left(\rho_{ij}^-\right)^q\right)^{\delta_i}\right)^{w_j}} \leq max_j max_i\{\rho_{ij}^-\}. \quad (1)$$

Similarly, we can obtain

$$min_j min_i\{\rho_{ij}^+\} \leq \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left(\rho_{ij}^+\right)^q\right)^{\delta_i}\right)^{w_j}} \leq max_j max_i\{\rho_{ij}^+\}. \quad (2)$$

Next for each $i = 1, 2, \cdots, s$ and $j = 1, 2, \cdots, m$, we have

$$\begin{aligned} min_j min_i\{\varrho_{ij}^-\} &\leq \varrho_{ij}^- \leq max_j max_i\{\varrho_{ij}^-\} \\ \leftrightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(min_j min_i\{\varrho_{ij}^-\}\right)^{\delta_i}\right)^{w_j} &\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(\varrho_{ij}^-\right)^{\delta_i}\right)^{w_j} \\ &\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(max_j max_i\{\varrho_{ij}^-\}\right)^{\delta_i}\right)^{w_j} \\ \leftrightarrow \left(\left(min_j min_i\{\varrho_{ij}^-\}\right)^{\sum_{i=1}^{s}\delta_i}\right)^{\sum_{j=1}^{m} w_j} &\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(\{\varrho_{ij}^-\}\right)^{\delta_i}\right)^{w_j} \\ &\leq \left(\left(max_j max_i\{\varrho_{ij}^-\}\right)^{\sum_{i=1}^{s}\delta_i}\right)^{\sum_{j=1}^{m} w_j} \end{aligned}$$

This implies that

$$min_j min_i\{\varrho_{ij}^-\} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left\{\varrho_{ij}^-\right\}^{\delta_i}\right)^{w_j} \leq max_j max_i\{\varrho_{ij}^-\}. \quad (3)$$

Similar way we get;

$$min_j min_i\{\varrho_{ij}^+\} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left\{\varrho_{ij}^+\right\}^{\delta_i}\right)^{w_j} \leq max_j max_i\{\varrho_{ij}^+\}. \quad (4)$$

Therefore, from Equations (1),(2),(3) and (4) we can write, $\overline{\mathcal{F}}_{e_{ij}}^- \leq IVq-ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \ldots, \overline{\mathcal{F}}_{e_{sm}}) \leq \overline{\mathcal{F}}_{e_{ij}}^+$. □

B. IVQ-ROFS WEIGHTED GEOMETRIC AGGREGATION OPERATORS ON IVQ-ROFSSS

Let $\mathcal{Q}$ be any collection of IVq-ROFV and $\mathcal{Q}^{s\times m}$ set of IVq-ROFSV in a IVq-ROFSS $(\overline{\mathcal{F}}, M)$. Then we define IVq-ROFSWG as follows;

Definition 15: Let $M = \{e_1, e_2, \cdots e_m\}$ be the set of attributes for a set of "s'' number of alternatives $\mathscr{U} = \{x_1, x_2, \cdots x_s\}$. Then a general form of IVq-ROFSS $(\overline{\mathcal{F}}, M)$ is given in Table 3.1. Assume that $\overline{\mathcal{F}}_{e_{ij}} = ([\rho_{ij}^-, \rho_{ij}^+], [\varrho_{ij}^-, \varrho_{ij}^+])$, $(i = 1, 2, \cdots s, j = 1, 2, \cdots m)$ be the collection of IVq-ROFSVs in $(\overline{\mathcal{F}}, M)$. Let $W = [w_1, w_2, \cdots w_m]$ and $\Delta = [\delta_1, \delta_2, \cdots \delta_s]$ be the weighted vectors over M and $\mathscr{U}$, respectively, such that $\sum_{j=1}^{m} w_j = 1$, $\sum_{i=1}^{s}\delta_i = 1$ and $w_j, \delta_i \in [0, 1]$. Define a mapping $IVq-ROFSWG : \mathcal{Q}^{s\times m} \to \mathcal{Q}$, that is, $IVq-ROFSWG(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{sm}}) = \otimes_{j=1}^{m}(\otimes_{i=1}^{s}\overline{\mathcal{F}}_{e_{ij}}^{\delta_i})^{w_j}$.

Theorem 16: Assume that $\overline{\mathcal{F}}_{e_{ij}} = ([\rho_{ij}^-, \rho_{ij}^+], [\varrho_{ij}^-, \varrho_{ij}^+])$, $(i = 1, 2, \cdots s, j = 1, 2, \cdots m)$ be the collection of IVq-ROFSVs in $(\overline{\mathcal{F}}, M)$. Let $W = [w_1, w_2, \cdots w_m]$ and $\Delta = [\delta_1, \delta_2, \cdots \delta_s]$ be weighted vectors over M and $\mathscr{U}$, respectively, such that $\sum_{j=1}^{m} w_j = 1$, $\sum_{i=1}^{s}\delta_i = 1$ and $w_j, \delta_i \in [0, 1]$. Then,

$$\begin{aligned} &IVq-ROFSWG(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{sm}}) \\ &= \otimes_{j=1}^{m}(\otimes_{i=1}^{s}\overline{\mathcal{F}}_{e_{ij}}^{\delta_i})^{w_j} \\ &= \left(\begin{array}{c} \left[\begin{array}{c} \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(\rho_{ij}^-\right)^{\delta_i}\right)^{w_j}, \\ \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(\rho_{ij}^+\right)^{\delta_i}\right)^{w_j} \end{array}\right], \\ \left[\begin{array}{c} \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left(\varrho_{ij}^-\right)^q\right)^{\delta_i}\right)^{w_j}}, \\ \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left(\varrho_{ij}^+\right)^q\right)^{\delta_i}\right)^{w_j}} \end{array}\right] \end{array}\right) \end{aligned}$$

Proof: Same as the proof of Theorem 11. □

Example 17: Consider Example 12 where IVq-ROFSS is given in Table 2. Let $q = 3$ for this example. Now by using Theorem 16, we calculate operator as follow;

$$\begin{aligned} &IVq-ROFSWG(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \overline{\mathcal{F}}_{e_{13}}, \ldots, \overline{\mathcal{F}}_{e_{55}}) \\ &= \left(\begin{array}{c} \left[\prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(\rho_{ij}^-\right)^{\delta_i}\right)^{w_j}, \prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(\rho_{ij}^+\right)^{\delta_i}\right)^{w_j}\right], \\ \left[\sqrt[q]{1-\prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(1-\left(\varrho_{ij}^-\right)^q\right)^{\delta_i}\right)^{w_j}}, \sqrt[q]{1-\prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(1-\left(\varrho_{ij}^+\right)^q\right)^{\delta_i}\right)^{w_j}}\right] \end{array}\right) \end{aligned}$$

where

$$\prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(\rho_{ij}^-\right)^{\delta_i}\right)^{w_j} = 0.237893,$$

$$\prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(\rho_{ij}^+\right)^{\delta_i}\right)^{w_j} = 0.374455,$$

$$\sqrt[q]{1-\prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(1-\left(\varrho_{ij}^-\right)^q\right)^{\delta_i}\right)^{w_j}} = 0.283800,$$

$$\sqrt[q]{1-\prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(1-\left(\varrho_{ij}^+\right)^q\right)^{\delta_i}\right)^{w_j}} = 0.485539.$$

Therefore $IVq-ROFSWG(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \overline{\mathcal{F}}_{e_{13}}, \ldots, \overline{\mathcal{F}}_{e_{55}}) = ([0.237893, 0.374455], [0.283800, 0.485539])$.

Lemma 18: Idempotency: If $\overline{\mathcal{F}}_{e_{ij}} = \overline{\mathcal{F}}_{e}$, $(\forall i = 1, 2, .., s$ *and* $j = 1, 2, \ldots m)$, *where* $\overline{\mathcal{F}}_{e} = (\rho, \varrho)$, *then* $IVq - ROFSWG(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \ldots, \overline{\mathcal{F}}_{e_{sm}}) = \overline{\mathcal{F}}_{e}$.

Proof: Same as the proof of the Lemma 13. □

Lemma 19: Boundedness: If

$$\overline{\mathcal{F}}^{-}_{e_{ij}} = \left(\begin{array}{c} \left[min_j min_i \{\rho^{-}_{ij}\}, min_j min_i \{\rho^{+}_{ij}\} \right], \\ \left[max_j max_i \{\varrho^{-}_{ij}\}, max_j max_i \{\varrho^{+}_{ij}\} \right] \end{array} \right),$$

$$\overline{\mathcal{F}}^{+}_{e_{ij}} = \left(\begin{array}{c} \left[max_j max_i \{\rho^{-}_{ij}\}, max_j max_i \{\rho^{+}_{ij}\} \right], \\ \left[min_j min_i \{\varrho^{-}_{ij}\}, min_j min_i \{\varrho^{+}_{ij}\} \right] \end{array} \right).$$

Then

$$\overline{\mathcal{F}}^{-}_{e_{ij}} \leq IVq - ROFSWG(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \ldots, \overline{\mathcal{F}}_{e_{sm}}) \leq \overline{\mathcal{F}}^{+}_{e_{ij}}.$$

Proof: Same as the proof of the Lemma 14. □

IV. INTERACTION AGGREGATION OPERATORS ON IVQ-ROFSS

Consider Definition 9 where IVq-ROFSS is given in Table 1. Some interaction operations are expressed as follows;

Definition 20: Assume $\overline{\mathcal{F}}_{e_{i2}} = ([\rho^{-}_{i2}, \rho^{+}_{i2}], [\varrho^{-}_{i2}, \varrho^{+}_{i2}])$, $(i = 1, 2)$ and $\overline{\mathcal{F}} = ([\rho^{-}, \rho^{+}], [\varrho^{-}, \varrho^{+}])$ be any three IVq-ROFSVs and $\lambda, \lambda_1, \lambda_2 > 0$. Thus some basic interactive operation on IVq-ROFSVs are given follows:

(i) $\overline{\mathcal{F}}_{e_{12}} \tilde{\oplus} \overline{\mathcal{F}}_{e_{22}}$

$$= \left\{ \left(\begin{array}{c} [\sqrt[q]{(\rho^{-}_{12})^q + (\rho^{-}_{22})^q - (\rho^{-}_{12})^q (\rho^{-}_{22})^q}, \\ \sqrt[q]{(\rho^{-}_{12})^q + (\rho^{+}_{22})^q - (\rho^{+}_{12})^q (\rho^{+}_{22})^q}], \\ [\sqrt[q]{\begin{array}{c}(\varrho^{-}_{12})^q + (\varrho^{-}_{22})^q - (\varrho^{-}_{12})^q (\varrho^{-}_{22})^q - \\ (\varrho^{-}_{12})^q (\rho^{-}_{22})^q - (\rho^{-}_{12})^q (\varrho^{-}_{22})^q \end{array}}, \\ \sqrt[q]{\begin{array}{c}(\varrho^{+}_{12})^q + (\varrho^{+}_{22})^q - (\varrho^{+}_{12})^q (\varrho^{+}_{22})^q - \\ (\varrho^{+}_{12})^q (\rho^{+}_{22})^q - (\rho^{+}_{12})^q (\varrho^{+}_{22})^q \end{array}}] \end{array} \right) \right\}.$$

(ii) $\overline{\mathcal{F}}_{e_{12}} \tilde{\otimes} \overline{\mathcal{F}}_{e_{22}}$

$$= \left\{ \left(\begin{array}{c} [\sqrt[q]{\begin{array}{c}(\rho^{-}_{12})^q + (\rho^{-}_{22})^q - (\rho^{-}_{12})^q (\rho^{-}_{22})^q - \\ (\rho^{-}_{12})^q (\varrho^{-}_{22})^q - (\varrho^{-}_{12})^q (\rho^{-}_{22})^q \end{array}}, \\ \sqrt[q]{\begin{array}{c}(\rho^{+}_{12})^q + (\rho^{+}_{22})^q - (\rho^{+}_{12})^q (\rho^{+}_{22})^q - \\ (\rho^{+}_{12})^q (\varrho^{+}_{22})^q - (\varrho^{+}_{12})^q (\rho^{+}_{22})^q \end{array}}], \\ [\sqrt[q]{(\rho^{-}_{12})^q + (\varrho^{-}_{22})^q - (\varrho^{-}_{12})^q (\varrho^{-}_{22})^q}, \\ \sqrt[q]{(\varrho^{-}_{12})^q + (\varrho^{+}_{22})^q - (\varrho^{+}_{12})^q (\varrho^{+}_{22})^q}] \end{array} \right) \right\}.$$

(iii) $$\overline{\mathcal{F}}\lambda = \left\{ \left(\begin{array}{c} [\sqrt[q]{1 - (1 - (\rho^{-})^q)^{\lambda}}, \\ \sqrt[q]{1 - (1 - (\rho^{+})^q)^{\lambda}}], \\ [\sqrt[q]{(1 - (\rho^{-})^q)^{\lambda} - (1 - (\rho^{-})^q + (\varrho^{-})^q)^{\lambda}}, \\ \sqrt[q]{(1 - (\rho^{+})^q)^{\lambda} - (1 - (\rho^{+})^q + (\varrho^{+})^q)^{\lambda}}] \end{array} \right) \right\}.$$

(iv) $$\overline{\mathcal{F}}^{\lambda} = \left\{ \left(\begin{array}{c} [\sqrt[q]{(1 - (\varrho^{-})^q)^{\lambda} - (1 - (\rho^{-})^q + (\varrho^{-})^q)^{\lambda}}, \\ \sqrt[q]{(1 - (\varrho^{+})^q)^{\lambda} - (1 - (\rho^{+})^q + (\varrho^{+})^q)^{\lambda}}], \\ [\sqrt[q]{1 - (1 - (\varrho^{-})^q)^{\lambda}}, \\ \sqrt[q]{1 - (1 - (\varrho^{+})^q)^{\lambda}}], \end{array} \right) \right\}.$$

A. IVQ-ROFS INTERACTION WEIGHTED AVERAGING AGGREGATION OPERATORS ON IVQ-ROFSSS

Based on above interaction operation, we express following notion.

Definition 21: Let $M = \{e_1, e_2, \cdots e_m\}$ be the set of attributes for a set of "s'' number of alternatives $\mathscr{U} = \{x_1, x_2, \cdots x_s\}$. Then a general form of IVq-ROFSS $(\mathcal{F}, M)$ is given in Table 1. Assume $\overline{\mathcal{F}}_{e_{ij}} = ([\rho^{-}_{ij}, \rho^{+}_{ij}], [\varrho^{-}_{ij}, \varrho^{+}_{ij}])$, be a collection of IVq-ROFSSs, where $(i = 1, 2 \cdots, s$ and $j = 1, 2, \cdots, m)$. Assume $W = [w_1, w_2, \cdots w_m]$ and $\Delta = [\delta_1, \delta_2, \cdots \delta_s]$ be the weighted vectors over M and $\mathscr{U}$, respectively, such that $\sum_{j=1}^{m} w_j = 1$, $\sum_{i=1}^{s} \delta_i = 1$ and $w_j, \delta_i > 0$. Define a mapping $IVq - ROFSIWA : \mathcal{Q}^{s \times m} \to \mathcal{Q}$, that is, $IVq - ROFSIWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{sm}}) = \tilde{\oplus}_{j=1}^{m}(\tilde{\oplus}_{i=1}^{s} \overline{\mathcal{F}}_{e_{ij}} \delta_i) w_j$, where IVq-ROFSIWA expresses IVq-ROF soft interactive weighted averaging operator.

Theorem 22: Assume $\overline{\mathcal{F}}_{e_{ij}} = ([\rho^{-}_{ij}, \rho^{+}_{ij}], [\varrho^{-}_{ij}, \varrho^{+}_{ij}])$, *be a collection of IVq-ROFSVs in* $(\mathcal{F}, M)$, *where* $(i = 1, 2 \cdots, s$ *and* $j = 1, 2, \cdots, m)$. *So,*

$$IVq - ROFSIWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{sm}})$$
$$= \tilde{\oplus}_{j=1}^{m}(\tilde{\oplus}_{i=1}^{s} \overline{\mathcal{F}}_{e_{ij}} \delta_i) w_j.$$

$$= \left(\begin{array}{c} \left[\begin{array}{c} \sqrt[q]{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{s} \left(1 - (\rho^{-}_{ij})^q \right)^{\delta_i} \right)^{w_j}}, \\ \sqrt[q]{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{s} \left(1 - (\rho^{+}_{ij})^q \right)^{\delta_i} \right)^{w_j}} \end{array} \right], \\ \left[\begin{array}{c} \sqrt[q]{\begin{array}{c} \prod_{j=1}^{m} \left(\prod_{i=1}^{s} \left(1 - (\rho^{-}_{ij})^q \right)^{\delta_i} \right)^{w_j} \\ - \prod_{j=1}^{m} \left(\prod_{i=1}^{s} \left(1 - \left((\rho^{-}_{ij})^q + (\varrho^{-}_{ij})^q \right) \right)^{\delta_i} \right)^{w_j} \end{array}}, \\ \sqrt[q]{\begin{array}{c} \prod_{j=1}^{m} \left(\prod_{i=1}^{s} \left(1 - (\rho^{+}_{ij})^q \right)^{\delta_i} \right)^{w_j} \\ - \prod_{j=1}^{m} \left(\prod_{i=1}^{s} \left(1 - \left((\rho^{+}_{ij})^q + (\varrho^{+}_{ij})^q \right) \right)^{\delta_i} \right)^{w_j} \end{array}} \end{array} \right] \end{array} \right).$$

such that, $\sum_{j=1}^{m} w_j = 1$, $\sum_{i=1}^{s} \delta_i = 1$ *and* $w_j, \delta_i > 0$.

Proof: The proof of the theorem is given in APPENDIX 2. □

Example 23: Consider Example 12 where IVq-ROFSS is given in Table 2. Let $q = 3$ for this example. Now by using Theorem 16, we calculate following;

$$\sqrt[q]{1 - \prod_{j=1}^{5} \left(\prod_{i=1}^{5} \left(1 - (\rho^{-}_{ij})^q \right)^{\delta_i} \right)^{w_j}} = 0.291494,$$

$$\sqrt[q]{1 - \prod_{j=1}^{5} \left(\prod_{i=1}^{5} \left(1 - (\rho^{+}_{ij})^q \right)^{\delta_i} \right)^{w_j}} = 0.426198,$$

$$\sqrt[q]{\begin{matrix}\prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(1-(\rho_{ij}^{-})^q\right)^{\delta_i}\right)^{w_j}- \\ \prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(1-\left((\rho_{ij}^{-})^q+(\varrho_{ij}^{-})^q\right)\right)^{\delta_i}\right)^{w_j}\end{matrix}} = 0.283161,$$

$$\sqrt[q]{\begin{matrix}\prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(1-(\rho_{ij}^{+})^q\right)^{\delta_i}\right)^{w_j}- \\ \prod_{j=1}^{5}\left(\prod_{i=1}^{5}\left(1-\left((\rho_{ij}^{+})^q+(\varrho_{ij}^{+})^q\right)\right)^{\delta_i}\right)^{w_j}\end{matrix}} = 0.482599.$$

Therefore $(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{55}})$
$= ([0.291494, 0.426198], [0.283161, 0.482599])$.

Lemma 24: Idempotency: If $\overline{\mathcal{F}}_{e_{ij}} = \overline{\mathcal{F}}_{e}$, $(\forall i = 1, 2, \cdots, s$ *and* $j = 1, 2, \cdots, m)$, *where* $\overline{\mathcal{F}}_e = (\rho, \varrho)$,*then* $IVq-ROFSIWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots, \overline{\mathcal{F}}_{e_{sm}}) = \overline{\mathcal{F}}_e$.

Proof: Assume $\overline{\mathcal{F}}_{e_{ij}} = \overline{\mathcal{F}}_e = (\rho, \varrho)$ $(\forall i = 1, 2, \cdots, s$ *and* $j = 1, 2, \cdots, m)$. Now in terms of Theorem 22, we have

$$\begin{aligned}
&(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots, \overline{\mathcal{F}}_{e_{sm}}) \\
&= \left(\begin{matrix}\left[\begin{matrix}\sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{-})^q\right)^{\delta_i}\right)^{w_j}}, \\ \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{+})^q\right)^{\delta_i}\right)^{w_j}}\end{matrix}\right], \\
\left[\begin{matrix}\sqrt[q]{\begin{matrix}\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{-})^q\right)^{\delta_i}\right)^{w_j}- \\ \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left((\rho_{ij}^{-})^q+(\varrho_{ij}^{-})^q\right)\right)^{\delta_i}\right)^{w_j}\end{matrix}}, \\
\sqrt[q]{\begin{matrix}\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{+})^q\right)^{\delta_i}\right)^{w_j}- \\ \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left((\rho_{ij}^{+})^q+(\varrho_{ij}^{+})^q\right)\right)^{\delta_i}\right)^{w_j}\end{matrix}}\end{matrix}\right]\end{matrix}\right) \\
&= \left(\begin{matrix}\left[\begin{matrix}\sqrt[q]{1-\left(\left(1-(\rho_{ij}^{-})^q\right)^{\sum_{i=1}^{s}\delta_i}\right)^{\sum_{j=1}^{s}w_j}}, \\ \sqrt[q]{1-\left(\left(1-(\rho_{ij}^{+})^q\right)^{\sum_{i=1}^{s}\delta_i}\right)^{\sum_{j=1}^{m}w_j}}\end{matrix}\right], \\
\left[\begin{matrix}\sqrt[q]{\begin{matrix}\left(\left(1-(\rho_{ij}^{-})^q\right)^{\sum_{i=1}^{s}\delta_i}\right)^{\sum_{j=1}^{m}w_j}- \\ \left(\left(1-\left((\rho_{ij}^{-})^q+(\varrho_{ij}^{-})^q\right)\right)^{\sum_{i=1}^{s}\delta_i}\right)^{\sum_{j=1}^{m}w_j}\end{matrix}}, \\
\sqrt[q]{\begin{matrix}\left(\left(1-(\rho_{ij}^{+})^q\right)^{\sum_{i=1}^{s}\delta_i}\right)^{\sum_{j=1}^{m}w_j}- \\ \left(\left(1-\left((\rho_{ij}^{+})^q+(\varrho_{ij}^{+})^q\right)\right)^{\sum_{i=1}^{s}\delta_i}\right)^{\sum_{j=1}^{m}w_j}\end{matrix}}\end{matrix}\right]\end{matrix}\right) \\
&= \left(\begin{matrix}\left[\sqrt[q]{1-\left(1-(\rho_{ij}^{-})^q\right)}, \sqrt[q]{1-\left(1-(\rho_{ij}^{+})^q\right)}\right], \\
\left[\sqrt[q]{\left(1-(\rho_{ij}^{-})^q\right)-\left(1-\left((\rho_{ij}^{-})^q+(\varrho_{ij}^{-})^q\right)\right)}, \right. \\
\left.\sqrt[q]{\left(1-(\rho_{ij}^{+})^q\right)-\left(1-\left((\rho_{ij}^{+})^q+(\varrho_{ij}^{+})^q\right)\right)}\right]\end{matrix}\right) \\
&= ([\rho_{ij}^{-}, \rho_{ij}^{+}], [\varrho_{ij}^{-}, \varrho_{ij}^{+}]) = (\rho, \varrho) = \overline{\mathcal{F}}_e
\end{aligned}$$

Therefore $(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots, \overline{\mathcal{F}}_{e_{sm}}) = \overline{\mathcal{F}}_e$. □

Lemma 25: Boundedness: If

$$\overline{\mathcal{F}}_{e_{ij}}^{-} = \left(\begin{matrix}\left[min_j min_i\{\rho_{ij}^{-}\}, min_j min_i\{\rho_{ij}^{+}\}\right], \\ \left[max_j max_i\{\varrho_{ij}^{-}\}, max_j max_i\{\varrho_{ij}^{+}\}\right]\end{matrix}\right)$$

and

$$\overline{\mathcal{F}}_{e_{ij}}^{+} = \left(\begin{matrix}\left[max_j max_i\{\rho_{ij}^{-}\}, max_j max_i\{\rho_{ij}^{+}\}\right], \\ \left[min_j min_i\{\varrho_{ij}^{-}\}, min_j min_i\{\varrho_{ij}^{+}\}\right]\end{matrix}\right)$$

Then,

$$\overline{\mathcal{F}}_{e_{ij}}^{-} \leq IVq-ROFSIWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots, \overline{\mathcal{F}}_{e_{sm}}) \leq \overline{\mathcal{F}}_{e_{ij}}^{+}.$$

Proof: As

$$\overline{\mathcal{F}}_{e_{ij}}^{-} = \left(\begin{matrix}\left[min_j min_i\{\rho_{ij}^{-}\}, min_j min_i\{\rho_{ij}^{+}\}\right], \\ \left[max_j max_i\{\varrho_{ij}^{-}\}, max_j max_i\{\varrho_{ij}^{+}\}\right]\end{matrix}\right)$$

and

$$\overline{\mathcal{F}}_{e_{ij}}^{+} = \left(\begin{matrix}\left[max_j max_i\{\rho_{ij}^{-}\}, max_j max_i\{\rho_{ij}^{+}\}\right], \\ \left[min_j min_i\{\varrho_{ij}^{-}\}, min_j min_i\{\varrho_{ij}^{+}\}\right]\end{matrix}\right).$$

We have to show that $\overline{\mathcal{F}}_{e_{ij}}^{-} \leq IVq-ROFSIWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots, \overline{\mathcal{F}}_{e_{rs}}) \leq \overline{\mathcal{F}}_{e_{ij}}^{+}$. Now for each $i = 1, 2, \cdots, s$ *and* $j = 1, 2, \cdots, m$, we have

$$\begin{aligned}
&min_j min_i\{\rho_{ij}^{-}\} \leq \rho_{ij}^{-} \leq max_j max_i\{\rho_{ij}^{-}\} \\
&\Rightarrow 1 - max_j max_i\{(\rho_{ij}^{-})^q\} \\
&\leq 1-(\rho_{ij}^{-})^q \leq 1 - min_j min_i\{(\rho_{ij}^{-})^q\} \\
&\Leftrightarrow \left(1 - max_j max_i \left(\rho_{ij}^{-}\right)^q\right)^{\delta_i} \\
&\leq \left(1-\left(\rho_{ij}^{-}\right)^q\right)^{\delta_i} \leq \left(1 - min_j min_i \left(\rho_{ij}^{-}\right)^q\right)^{\delta_i} \\
&\Leftrightarrow \left(1 - max_j max_i \left(\rho_{ij}^{-}\right)^q\right)^{\sum_{i=1}^{s}\delta_i} \\
&\leq \prod_{i=1}^{s}\left(1-\left(\rho_{ij}^{-}\right)^q\right)^{\delta_i} \leq \left(1 - min_j min_i \left(\rho_{ij}^{-}\right)^q\right)^{\sum_{i=1}^{s}\delta_i} \\
&\Leftrightarrow \left(1 - max_j max_i(\rho_{ij}^{-})^q\right)^{\sum_{j=1}^{m}w_j} \\
&\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left(\rho_{ij}^{-}\right)^q\right)^{\delta_i}\right)^{w_j} \\
&\leq \left(1 - min_j min_i \left(\rho_{ij}^{-}\right)^q\right)^{\sum_{j=1}^{m}w_j} \Leftrightarrow (1 - max_j max_i(\rho_{ij}^{-})^q) \\
&\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left(\rho_{ij}^{-}\right)^q\right)^{\delta_i}\right)^{w_j} \leq (1 - min_j min_i(\rho_{ij}^{-})^q) \\
&\Leftrightarrow (1 - min_j min_i(\rho_{ij}^{-})^q)
\end{aligned}$$

$$\leq 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left(\rho_{ij}^{-}\right)^{q}\right)^{\delta_i}\right)^{w_j}$$
$$\leq (1-max_j max_i(\rho_{ij}^{-})^q)$$

Hence

$$min_j min_i(\rho_{ij}^{-}) \leq \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left(\rho_{ij}^{-}\right)^{q}\right)^{\delta_i}\right)^{w_j}} \leq max_j max_i(\rho_{ij}^{-}) \quad (5)$$

also,

$$min_j min_i\{\rho_{ij}^{+}\} \leq \rho_{ij}^{+} \leq max_j max_i\{\rho_{ij}^{+}\}$$
$$\Rightarrow 1-max_j max_i\{(\rho_{ij}^{+})^q\} \leq 1-(\rho_{ij}^{+})^q$$
$$\leq 1-min_j min_i\{(\rho_{ij}^{+})^q\} \Leftrightarrow \left(1-max_j max_i\left(\rho_{ij}^{+}\right)^{q}\right)^{\delta_i}$$
$$\leq \left(1-\left(\rho_{ij}^{+}\right)^{q}\right)^{\delta_i} \leq \left(1-min_j min_i\left(\rho_{ij}^{+}\right)^{q}\right)^{\delta_i}$$
$$\Leftrightarrow \left(1-max_j max_i\left(\rho_{ij}^{+}\right)^{q}\right)^{\sum_{i=1}^{s}\delta_i}$$
$$\leq \prod_{i=1}^{s}\left(1-\left(\rho_{ij}^{+}\right)^{q}\right)^{\delta_i} \leq \left(1-min_j min_i\left(\rho_{ij}^{+}\right)^{q}\right)^{\sum_{i=1}^{s}\delta_i}$$
$$\Leftrightarrow \left(1-max_j max_i(\rho_{ij}^{+})^q\right)^{\sum_{j=1}^{m}w_j}$$
$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left(\rho_{ij}^{+}\right)^{q}\right)^{\delta_i}\right)^{w_j}$$
$$\leq \left(1-min_j min_i\left(\rho_{ij}^{+}\right)^{q}\right)^{\sum_{j=1}^{m}w_j}$$
$$\Leftrightarrow (1-max_j max_i(\rho_{ij}^{+})^q) \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left(\rho_{ij}^{+}\right)^{q}\right)^{\delta_i}\right)^{w_j}$$
$$\leq (1-min_j min_i(\rho_{ij}^{+})^q)$$
$$\Leftrightarrow (1-min_j min_i(\rho_{ij}^{+})^q) \leq 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left(\rho_{ij}^{+}\right)^{q}\right)^{\delta_i}\right)^{w_j}$$
$$\leq (1-max_j max_i(\rho_{ij}^{+})^q).$$

Thus

$$min_j min_i(\rho_{ij}^{+}) \leq \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left(\rho_{ij}^{+}\right)^{q}\right)^{\delta_i}\right)^{w_j}} \leq max_j max_i(\rho_{ij}^{+}) \quad (6)$$

Similarly, we have

$$min_j min_i(\varrho_{ij}^{-}) \leq \sqrt[q]{\begin{array}{l}\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{-})^q\right)^{\delta_i}\right)^{w_j}- \\ \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left((\rho_{ij}^{-})^q+(\varrho_{ij}^{-})^q\right)\right)^{\delta_i}\right)^{w_j}\end{array}} \leq max_j max_i(\varrho_{ij}^{-}) \quad (7)$$

$$min_j min_i(\varrho_{ij}^{+}) \leq \sqrt[q]{\begin{array}{l}\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{+})^q\right)^{\delta_i}\right)^{w_j}- \\ \prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left((\rho_{ij}^{+})^q+(\varrho_{ij}^{+})^q\right)\right)^{\delta_i}\right)^{w_j}\end{array}} \leq max_j max_i(\rho_{ij}^{+}) \quad (8)$$

On conclusion of inequalities (i), (ii), (iii) and (iv) $\overline{\mathcal{F}}_{e_{ij}}^{-} \leq IVq-ROFSIWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots, \overline{\mathcal{F}}_{e_{sm}}) \leq \overline{\mathcal{F}}_{e_{ij}}^{+}$. □

Lemma 26: Show that $IVq-ROFSIWA(\lambda\overline{\mathcal{F}}_{e_{11}}, \lambda\overline{\mathcal{F}}_{e_{12}}, \cdots \lambda\overline{\mathcal{F}}_{e_{sm}}) = IVq-ROFSIWA\lambda(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{sm}})$ for any positive real number λ.

Proof: Assume $\overline{\mathcal{F}}_{e_{11}}$ be a IVq-ROFSN and ≥ 0, we have

$$\lambda\overline{\mathcal{F}}_{e_{11}} = \left(\begin{array}{c}\left[\sqrt[q]{1-\left(1-(\rho_{ij}^{-})^q\right)^{\lambda}}, \sqrt[q]{1-\left(1-(\rho_{ij}^{+})^q\right)^{\lambda}}\right], \\ \left[\begin{array}{c}\sqrt[q]{1-\left(1-(\rho_{ij}^{-})^q\right)^{\lambda}-\left(1-\left((\rho_{ij}^{-})^q+(\varrho_{ij}^{-})^q\right)\right)^{\lambda}}, \\ \sqrt[q]{1-\left(1-(\rho_{ij}^{-})^q\right)^{\lambda}-\left(1-\left((\rho_{ij}^{-})^q+(\varrho_{ij}^{-})^q\right)\right)^{\lambda}}\end{array}\right]\end{array}\right).$$

So,

$$IVq-ROFSIWA(\lambda\overline{\mathcal{F}}_{e_{11}}, \lambda\overline{\mathcal{F}}_{e_{12}}, \cdots \lambda\overline{\mathcal{F}}_{e_{sm}})$$
$$= \left(\begin{array}{c}\left[\begin{array}{c}\sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{-})^q\right)^{\lambda\delta_i}\right)^{w_j}}, \\ \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{+})^q\right)^{\lambda\delta_i}\right)^{w_j}}\end{array}\right], \\ \left[\begin{array}{c}\sqrt[q]{\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{-})^q\right)^{\lambda\delta_i}\right)^{w_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left((\rho_{ij}^{-})^q+(\varrho_{ij}^{-})^q\right)\right)^{\lambda\delta_i}\right)^{w_j}}, \\ \sqrt[q]{\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{+})^q\right)^{\lambda\delta_i}\right)^{w_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left((\rho_{ij}^{+})^q+(\varrho_{ij}^{+})^q\right)\right)^{\lambda\delta_i}\right)^{w_j}}\end{array}\right]\end{array}\right)$$

$$= \left(\begin{array}{c}\left[\begin{array}{c}\sqrt[q]{1-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{-})^q\right)^{\delta_i}\right)^{w_j}\right)^{\lambda}}, \\ \sqrt[q]{1-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{+})^q\right)^{\delta_i}\right)^{w_j}\right)^{\lambda}}\end{array}\right], \\ \left[\begin{array}{c}\sqrt[q]{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{-})^q\right)^{\delta_i}\right)^{w_j}\right)^{\lambda}-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left((\rho_{ij}^{-})^q+(\varrho_{ij}^{-})^q\right)\right)^{\delta_i}\right)^{w_j}\right)^{\lambda}}, \\ \sqrt[q]{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{+})^q\right)^{\delta_i}\right)^{w_j}\right)^{\lambda}-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{s}\left(1-\left((\rho_{ij}^{+})^q+(\varrho_{ij}^{+})^q\right)\right)^{\delta_i}\right)^{w_j}\right)^{\lambda}}\end{array}\right]\end{array}\right)$$

$= \text{IVq-ROFSIWA}\lambda(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{sm}}).$

Hence proved. □

B. IVQ-ROFS INTERACTION WEIGHTED GEOMETRIC AGGREGATION OPERATORS ON IVQ-ROFSSS

Definition 27: Let $M = \{e_1, e_2, \cdots e_m\}$ be the set of attributes for a set of "s'' number of alternatives $\mathscr{U} = \{x_1, x_2, \cdots x_s\}$. Then a general form of IVq-ROFSS $(\overline{\mathcal{F}}, M)$ is given in Table 1. Assume $\overline{\mathcal{F}}_{e_{ij}} = ([\rho_{ij}^-, \rho_{ij}^+], [\varrho_{ij}^-, \varrho_{ij}^+])$, be a collection of IVq-ROFSVs in $(\overline{\mathcal{F}}, M)$, where $(i = 1, 2 \cdots, s$ and $j = 1, 2, \cdots, m)$. Assume $W = [w_1, w_2, \cdots w_m]$ and $\Delta = [\delta_1, \delta_2, \cdots \delta_s]$ be the weighted vectors over M and $\mathscr{U}$, respectively, such that $\sum_{j=1}^m w_j = 1$, $\sum_{i=1}^s \delta_i = 1$ and $w_j, \delta_i > 0$. Define a mapping $IVq-ROFSIWG : \mathcal{Q}^{s\times m} \rightarrow \mathcal{Q}$, that is, $IVq-ROFSIWG(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{sm}}) = \tilde{\otimes}_{j=1}^m (\tilde{\otimes}_{i=1}^s \overline{\mathcal{F}}_{e_{ij}}^{\delta_i})^{w_j}$, where IVq-ROFSIWG expresses IVq-ROF soft interactive weighted geometric operator.

Theorem 28: Assume $\overline{\mathcal{F}}_{e_{ij}} = ([\rho_{ij}^-, \rho_{ij}^+], [\varrho_{ij}^-, \varrho_{ij}^+])$, be a collection of IVq-ROFSVs in $(\overline{\mathcal{F}}, M)$, where $(i = 1, 2 \cdots, s$ and $j = 1, 2, \cdots, m)$. So,

$$IVq-ROFSIWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{sm}})$$
$$= \tilde{\otimes}_{j=1}^m (\tilde{\otimes}_{i=1}^s \overline{\mathcal{F}}_{e_{ij}}^{\delta_i})^{w_j}$$
$$= \left(\begin{matrix} \left[\sqrt[q]{\begin{matrix} \prod_{j=1}^m \left(\prod_{i=1}^s \left(1 - (\varrho_{ij}^-)^q\right)^{\delta_i}\right)^{w_j} - \\ \prod_{j=1}^m \left(\prod_{i=1}^s \left(1 - \left((\varrho_{ij}^-)^q + (\rho_{ij}^-)^q\right)\right)^{\delta_i}\right)^{w_j} \end{matrix}}, \right. \\ \left. \sqrt[q]{\begin{matrix} \prod_{j=1}^m \left(\prod_{i=1}^s \left(1 - (\varrho_{ij}^+)^q\right)^{\delta_i}\right)^{w_j} - \\ \prod_{j=1}^m \left(\prod_{i=1}^s \left(1 - \left((\varrho_{ij}^+)^q + (\rho_{ij}^+)^q\right)\right)^{\delta_i}\right)^{w_j} \end{matrix}} \right], \\ \left[\sqrt[q]{1 - \prod_{j=1}^m \left(\prod_{i=1}^s \left(1 - (\varrho_{ij}^-)^q\right)^{\delta_i}\right)^{w_j}}, \right. \\ \left. \sqrt[q]{1 - \prod_{j=1}^m \left(\prod_{i=1}^s \left(1 - (\varrho_{ij}^+)^q\right)^{\delta_i}\right)^{w_j}} \right] \end{matrix} \right).$$

such that, $\sum_{j=1}^m w_j = 1$, $\sum_{i=1}^s \delta_i = 1$ and $w_j, \delta_i > 0$.

Proof: Same as the proof of Theorem 22. □

Lemma 29: Idempotency: If $\overline{\mathcal{F}}_{e_{ij}} = \overline{\mathcal{F}}_e$, $(\forall i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, s)$, where $\overline{\mathcal{F}}_e = (\rho, \varrho)$, then $IVq-ROFSIWG(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots, \overline{\mathcal{F}}_{e_{rs}}) = \overline{\mathcal{F}}_e$.

Proof: Same as the proof of Lemma 24. □

Lemma 30: Boundedness: If

$$\overline{\mathcal{F}}_{e_{ij}}^- = \left(\begin{matrix} \left[min_j min_i\{\rho_{ij}^-\}, min_j min_i\{\rho_{ij}^+\}\right], \\ \left[max_j max_i\{\varrho_{ij}^-\}, max_j max_i\{\varrho_{ij}^+\}\right] \end{matrix} \right)$$

and

$$\overline{\mathcal{F}}_{e_{ij}}^+ = \left(\begin{matrix} \left[max_j max_i\{\rho_{ij}^-\}, max_j max_i\{\rho_{ij}^+\}\right], \\ \left[min_j min_i\{\varrho_{ij}^-\}, min_j min_i\{\varrho_{ij}^+\}\right] \end{matrix} \right)$$

Then $\overline{\mathcal{F}}_{e_{ij}}^- \leq IVq-ROFSIWG(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \ldots, \overline{\mathcal{F}}_{e_{sm}}) \leq \overline{\mathcal{F}}_{e_{ij}}^+$.

Proof: Same as the proof of Lemma25. □

Lemma 31: Show that $IVq-ROFSIWG(\lambda\overline{\mathcal{F}}_{e_{11}}, \lambda\overline{\mathcal{F}}_{e_{12}}, \cdots \lambda\overline{\mathcal{F}}_{e_{sm}}) = IVq-ROFSIWG\lambda(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{sm}})$ for any positive real number λ.

Proof: Same as the proof of Lemma 26. □

V. MULTI CRITERIA DECISION MAKING BASED ON IVQ-ROFSS ENVIRONMENT

MCDM played vital role in complex real life situation where it is difficult to obtain a choice with respect to certain types of criteria. It is a process to select the a logically appropriate choice among several objects. A systematic MCDM process can handle all aspects where different competitor have their own choices. In order to define a well recognized MCDM method, we explained some basic points method as in the following;

1) Assume $X = \{X_1, X_2, \cdots X_n\}$ be the discrete set of n number of various alternatives, and the related set of parameters is $E = \{e_1, e_2, \cdots e_m\}$.
2) Let experts $\Re_1, \Re_2, \cdots \Re_s$ provide assessments over E for each $X_k (k = 1, 2, \cdots, n)$ in the form of IVq-ROFSVs $(\overline{\mathcal{F}}_k)_{e_{ij}} = ([(\rho_k)_{ij}^-, (\rho_k)_{ij}^+], [(\varrho_k)_{ij}^-, (\varrho_k)_{ij}^+])$ $k = 1, 2, \cdots, n$. Denote $\mathscr{U} = \{\Re_1, \Re_2, \cdots \Re_s\}$. In another word, we obtain n numbers of IVq-ROFSSs $(\overline{\mathcal{F}}_1)_{e_{ij}}, (\overline{\mathcal{F}}_2)_{e_{ij}}, \cdots, (\overline{\mathcal{F}}_n)_{e_{ij}}$ over $\mathscr{U}$ for each $X_k (k = 1, 2, \cdots, n)$ respectively. For each $k = 1, 2, \cdots, n$ soft matrix, as shown at the bottom of the next page.
3) For $k = 1, 2, \cdots, n$ normalize SM_k in terms of parameters as follows;

$$(\overline{\mathcal{P}}_k)_{e_{ij}} = \begin{cases} \left\langle \begin{matrix} \left[(\varrho_k)_{e_{ij}}^-, (\varrho_k)_{e_{ij}}^+\right], \\ \left[(\rho_k)_{e_{ij}}^-, (\rho_k)_{e_{ij}}^+\right] \end{matrix} \right\rangle; \text{ for cost type parameter}, \\ \left\langle \begin{matrix} \left[(\rho_k)_{e_{ij}}^-, (\rho_k)_{e_{ij}}^+\right], \\ \left[(\varrho_k)_{e_{ij}}^-, (\varrho_k)_{e_{ij}}^+\right] \end{matrix} \right\rangle; \text{ for profit type parameter}, \end{cases}$$

Then obtain normalized soft matrices SM_k' for each $k = 1, 2, \cdots, n$.
4) By applying the proposed aggregation operators on IVq-ROFSVs $(\overline{\mathcal{F}}_k)_{e_{ij}} = ([(\rho_k)_{ij}^-, (\rho_k)_{ij}^+], [(\varrho_k)_{ij}^-, (\varrho_k)_{ij}^+])$ for each $k = 1, 2, \cdots, n$, obtain IVq-ROFVs

$$\ell_1 = ([(\rho_1)^-, (\rho_1)^+], [(\varrho_1)^-, (\varrho_1)^+]),$$
$$\ell_2 = ([(\rho_2)^-, (\rho_2)^+], [(\varrho_2)^-, (\varrho_2)^+]), \cdots$$
$$\ell_n = ([(\rho_n)^-, (\rho_n)^+], [(\varrho_n)^-, (\varrho_n)^+]).$$

5) Calculate the score function $\mathscr{S}$ using Definition 4 on each ℓ_k, that is, $\mathscr{S}(\ell_k)$ $k = 1, 2, \cdots, n$.
6) The best optimal result $X_{k'}$ $k' \in (1, 2, \cdots n)$ can be obtained on larger value of $\mathscr{S}$ on ℓ_k, $k = 1, 2, \cdots, n$.

The projected MCDM method is reckoned in Figure 1.

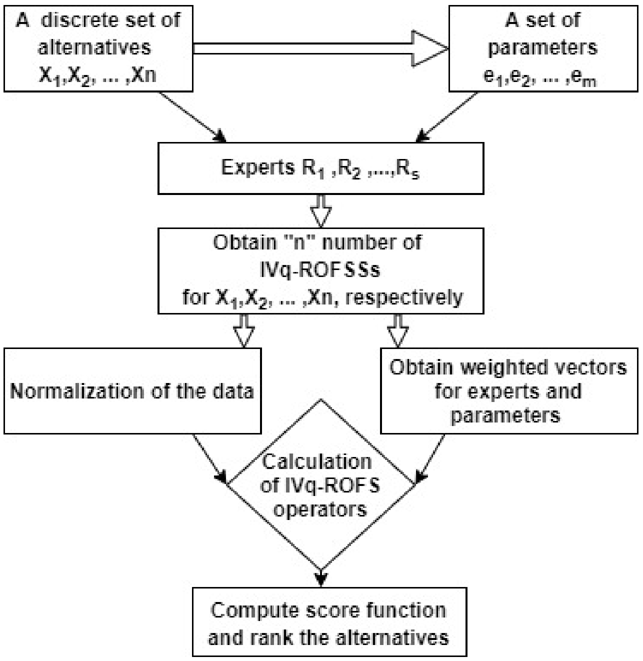

FIGURE 1. MCDM on IVq-ROFSSs.

VI. AN APPLICATION OF PROPOSED MCDM METHOD

This section brings demonstration of effectiveness and reliability of introduced results in real life complex problem with IVq-ROFS environment.

Nowadays, automation companies worldwide provide their specific work environment for automation engineers. A healthy environment in terms of leadership can boost the artistry of automation engineers. An automation company can achieve this aspiration through well-qualified automation engineers. On the other hand, automation engineers seek an aspiring workplace to excel in their creative abilities. Assume five automation companies X_1, X_2, X_3, X_4, X_5 where we have assess good environment for automation engineers. Consider a set of five senior experts $\mathscr{U} = \{\Re_1, \Re_2, \Re_3, \Re_4, \Re_5\}$ who have to assess job environment in given automation companies. The weighted vector for experts is $W = (0.18/\Re_1, 0.23/\Re_2, 0.16/\Re_3, 0.26/\Re_4, 0.17/\Re_5)^T$. The parameters for which we have to choose the best automation company for automation engineers are given as a set;

$$E = \begin{pmatrix} e_1 = Leadership\ is\ involved\ and\ engaged, \\ e_2 = Communication, \\ e_3 = healthy\ company\ culture, \\ e_4 = innovation, \\ e_5 = individuals\ are\ empowered\ to\ grow \end{pmatrix}.$$

The weighted vector over parameters is

$$\Delta = (0.42/e_1, 0.11/e_2, 0.19/e_3, 0.21/e_4, 0.07/e_5)^T.$$

In order to evaluate suitable automation company for engineers, following steps are used.

1) The senior experts give productive assessments over parameters for each company in the form of IVq-ROFSS. IVq-ROFSSs for companies X_1, X_2, X_3, X_4, X_5 are given in Tables 3-7 respectively.
2) Normalized the data in Tables 3-7. As there does not exist any cost type parameter, thus, normalization concludes same data as in the Tables 3-7.
3) Applying IVq-ROFS aggregation operators from the Definition III-2. For $q = 3$, we receive IVq-ROFVs

$$\ell_1 = \langle[0.199607, 0.512417], [0.290453, 0.453487]\rangle,$$
$$\ell_2 = \langle[0.241580, 0.414274], [0.247639, 0.459448]\rangle,$$
$$\ell_3 = \langle[0.241541, 0.808656], [0.248643, 0.657005]\rangle,$$
$$\ell_4 = \langle[0.282778, 0.441976], [0.252393, 0.352567]\rangle,$$
$$\ell_5 = \langle[0.253570, 0.484110], [0.105438, 0.241946]\rangle.$$

4) By using Definition II-4, we compute score function $\mathscr{S}(\ell_k)(k = 1, 2, 3, 4, 5)$ as follows;

$$\mathscr{S}(\ell_1) = 0.506184, \quad \mathscr{S}(\ell_2) = 0.493645,$$
$$\mathscr{S}(\ell_3) = 0.560980, \quad \mathscr{S}(\ell_4) = 0.512261,$$
$$\mathscr{S}(\ell_5) = 0.528606$$

5) Now rank the results in descending order as $\mathscr{S}(\ell_3) > \mathscr{S}(\ell_5) > \mathscr{S}(\ell_4) > \mathscr{S}(\ell_1) > \mathscr{S}(\ell_2)$. Therefore the ranking of automation companies is given by $X_3 > X_5 > X_4 > X_1 > X_2$. It observed company X_3 is very dynamic for automation engineers for services.

Similarly, we apply IVq-ROPFSIWA as follows;

1) similarly calculate IVq-ROPFSIWA are

$$\ell_1 = \langle[0.199607, 0.512417], [0.311701, 0.727133]\rangle,$$

Soft Matrix

$$SM_k = \begin{bmatrix} \begin{pmatrix} [(\rho_k)^-_{11}, (\rho_k)^+_{11}], \\ [(\varrho_k)^-_{11}, (\varrho_k)^+_{11}] \end{pmatrix} & \begin{pmatrix} [(\rho_k)^-_{12}, (\rho_k)^+_{12}], \\ [(\varrho_k)^-_{12}, (\varrho_k)^+_{12}] \end{pmatrix} & \cdots & \begin{pmatrix} [(\rho_k)^-_{1m}, (\rho_k)^+_{1m}], \\ [(\varrho_k)^-_{1m}, (\varrho_k)^+_{1m}] \end{pmatrix} \\ \begin{pmatrix} [(\rho_k)^-_{21}, (\rho_k)^+_{21}], \\ [(\varrho_k)^-_{21}, (\varrho_k)^+_{21}] \end{pmatrix} & \begin{pmatrix} [(\rho_k)^-_{22}, (\rho_k)^+_{22}], \\ [(\varrho_k)^-_{22}, (\varrho_k)^+_{22}] \end{pmatrix} & \cdots & \begin{pmatrix} [(\rho_k)^-_{2m}, (\rho_k)^+_{2m}], \\ [(\varrho_k)^-_{2m}, (\varrho_k)^+_{2m}] \end{pmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{pmatrix} [(\rho_k)^-_{s1}, (\rho_k)^+_{s1}], \\ [(\varrho_k)^-_{s1}, (\varrho_k)^+_{s1}] \end{pmatrix} & \begin{pmatrix} [(\rho_k)^-_{s2}, (\rho_k)^+_{s2}], \\ [(\varrho_k)^-_{s2}, (\varrho_k)^+_{s2}] \end{pmatrix} & \cdots & \begin{pmatrix} [(\rho_k)^-_{sm}, (\rho_k)^+_{sm}], \\ [(\varrho_k)^-_{sm}, (\varrho_k)^+_{sm}] \end{pmatrix} \end{bmatrix}$$

TABLE 3. IVq-ROFSS for automation company X_1.

	e_1	e_2	e_3	e_4	e_5
$\Re_1$	$\langle[0.43, 0.54], [0.42, 0.52]\rangle$	$\langle[0.11, 0.17], [0.22, 0.42]\rangle$	$\langle[0.01, 0.71], [0.32, 0.44]\rangle$	$\langle[0.12, 0.40], [0.31, 0.50]\rangle$	$\langle[0.34, 0.50], [0.20, 0.40]\rangle$
$\Re_2$	$\langle[0.35, 0.55], [0.32.0.42]\rangle$	$\langle[0.12, 0.18], [0.42, 0.27]\rangle$	$\langle[0.02, 0.90], [0.36, 0.62]\rangle$	$\langle[0.24, 0.64], [0.41, 0.57]\rangle$	$\langle[0.15, 0.17], [0.16, 0.75]\rangle$
$\Re_3$	$\langle[0.42, 0.58], [0.22, 0.25]\rangle$	$\langle[0.13, 0.16], [0.23, 0.47]\rangle$	$\langle[0.04, 0.64], [0.22, 0.72]\rangle$	$\langle[0.17, 0.72], [0.14, 0.76]\rangle$	$\langle[0.08, 0.72], [0.42, 0.48]\rangle$
$\Re_4$	$\langle[0.28, 0.29], [0.12, 0.21]\rangle$	$\langle[0.14, 0.19], [0.13, 0.62]\rangle$	$\langle[0.05, 0.62], [0.35, 0.55]\rangle$	$\langle[0.16, 0.62], [0.34, 0.46]\rangle$	$\langle[0.18, 0.72], [0.25, 0.65]\rangle$
$\Re_5$	$\langle[0.31, 0.39], [0.18, 0.20]\rangle$	$\langle[0.07, 0.79], [0.43, 0.46]\rangle$	$\langle[0.13, 0.40], [0.14, 0.50]\rangle$	$\langle[0.39, 0.42], [0.23, 0.65]\rangle$	$\langle[0.06, 0.90], [0.01, 0.91]\rangle$

TABLE 4. IVq-ROFSS for automation company X_2.

	e_1	e_2	e_3	e_4	e_5
$\Re_1$	$\langle[0.10, 0.20], [0.30, 0.40]\rangle$	$\langle[0.22, 0.42], [0.33, 0.47]\rangle$	$\langle[0.20, 0.41], [0.21, 0.42]\rangle$	$\langle[0.40, 0.59], [0.20, 0.40]\rangle$	$\langle[0.25, 0.75], [0.10, 0.60]\rangle$
$\Re_2$	$\langle[0.35, 0.45], [0.15, 0.25]\rangle$	$\langle[0.12, 0.14], [0.12, 0.73]\rangle$	$\langle[0.32, 0.42], [0.24, 0.26]\rangle$	$\langle[0.01, 0.04], [0.43, 0.46]\rangle$	$\langle[0.23, 0.67], [0.25, 0.35]\rangle$
$\Re_3$	$\langle[0.29, 0.31], [0.20, 0.40]\rangle$	$\langle[0.21, 0.61], [0.28, 0.42]\rangle$	$\langle[0.13, 0.17], [0.42, 0.52]\rangle$	$\langle[0.08, 0.10], [0.23, 0.42]\rangle$	$\langle[0.41, 0.49], [0.22, 0.62]\rangle$
$\Re_4$	$\langle[0.34, 0.36], [0.30, 0.50]\rangle$	$\langle[0.40, 0.60], [0.32, 0.64]\rangle$	$\langle[0.29, 0.35], [0.42, 0.44]\rangle$	$\langle[0.03, 0.30], [0.44, 0.46]\rangle$	$\langle[0.22, 0.64], [0.22, 0.42]\rangle$
$\Re_5$	$\langle[0.28, 0.56], [0.24, 0.53]\rangle$	$\langle[0.29, 0.69], [0.14, 0.44]\rangle$	$\langle[0.27, 0.52], [0.23, 0.65]\rangle$	$\langle[0.42, 0.50], [0.30, 0.60]\rangle$	$\langle[0.20, 0.30], [0.40, 0.50]\rangle$

$$\ell_2 = \langle[0.241580, 0.414274], [0.334460, 0.578685]\rangle,$$
$$\ell_3 = \langle[0.241541, 0.808656], [0.311947, 0.468633]\rangle,$$
$$\ell_4 = \langle[0.282778, 0.441976], [0.304150, 0.487587]\rangle,$$
$$\ell_5 = \langle[0.253570, 0.484110], [0.246577, 0.446510]\rangle.$$

2) By using Definition II-4, we compute score function $\mathscr{S}(\ell_k)(k = 1, 2, 3, 4, 5)$ as follows;

$$\mathscr{S}(\ell_1) = 0.466403, \quad \mathscr{S}(\ell_2) = 0.463999,$$
$$\mathscr{S}(\ell_3) = 0.626941, \quad \mathscr{S}(\ell_4) = 0.491223,$$
$$\mathscr{S}(\ell_5) = 0.506437$$

3) Now rank the results in descending order as $\mathscr{S}(\ell_3) > \mathscr{S}(\ell_5) > \mathscr{S}(\ell_4) > \mathscr{S}(\ell_1) > \mathscr{S}(\ell_2)$. Therefore the ranking of automation companies is given by $X_3 > X_5 > X_4 > X_1 > X_2$. It observed company X_3 is very dynamic for automation engineers for services.

A. COMPARATIVE ANALYSIS

First, we compare the proposed results with Hussain *et al.* [21] and Zulqarnain *et al.* [57]. For this, we consider the Example in Section VI. Consider only lower membership and lower non-membership for each IVq-ROFV in Tables 3-7, and we

TABLE 5. IVq-ROFSS for automation company X_3.

	e_1	e_2	e_3	e_4	e_5
$\Re_1$	$\langle[0.04, 0.06], [0.17, 0.64]\rangle$	$\langle[0.32, 0.42], [0.26, 0.44]\rangle$	$\langle[0.40, 0.94], [0.07, 0.91]\rangle$	$\langle[0.23, 0.73], [0.21, 0.73]\rangle$	$\langle[0.46, 0.53], [0.20, 0.70]\rangle$
$\Re_2$	$\langle[0.37, 0.62], [0.16, 0.72]\rangle$	$\langle[0.46, 0.48], [0.36, 0.47]\rangle$	$\langle[0.02, 0.82], [0.08, 0.72]\rangle$	$\langle[0.12, 0.80], [0.23, 0.72]\rangle$	$\langle[0.32, 0.47], [0.14, 0.62]\rangle$
$\Re_3$	$\langle[0.29, 0.54], [0.12, 0.73]\rangle$	$\langle[0.09, 0.41], [0.22, 0.49]\rangle$	$\langle[0.03, 0.83], [0.12, 0.72]\rangle$	$\langle[0.13, 0.82], [0.49, 0.51]\rangle$	$\langle[0.04, 0.92], [0.13, 0.45]\rangle$
$\Re_4$	$\langle[0.33, 0.39], [0.19, 0.75]\rangle$	$\langle[0.06, 0.42], [0.27, 0.37]\rangle$	$\langle[0.01, 0.90], [0.14, 0.31]\rangle$	$\langle[0.14, 0.84], [0.33, 0.66]\rangle$	$\langle[0.19, 0.71], [0.43, 0.44]\rangle$
$\Re_5$	$\langle[0.32, 0.37], [0.34, 0.39]\rangle$	$\langle[0.24, 0.27], [0.38, 0.54]\rangle$	$\langle[0.17, 0.70], [0.12, 0.80]\rangle$	$\langle[0.03, 0.85], [0.25, 0.35]\rangle$	$\langle[0.35, 0.45], [0.14, 0.15]\rangle$

TABLE 6. IVq-ROFSS for automation company X_4.

	e_1	e_2	e_3	e_4	e_5
$\Re_1$	$\langle[0.03, 0.07], [0.20, 0.30]\rangle$	$\langle[0.12, 0.15], [0.34, 0.36]\rangle$	$\langle[0.24, 0.26], [0.14, 0.16]\rangle$	$\langle[0.43, 0.50], [0.11, 0.19]\rangle$	$\langle[0.17, 0.19], [0.28, 0.29]\rangle$
$\Re_2$	$\langle[0.13, 0.16], [0.19, 0.22]\rangle$	$\langle[0.22, 0.26], [0.20, 0.42]\rangle$	$\langle[0.35, 0.47], [0.21, 0.41]\rangle$	$\langle[0.12, 0.49], [0.35, 0.40]\rangle$	$\langle[0.40, 0.60], [0.13, 0.53]\rangle$
$\Re_3$	$\langle[0.15, 0.17], [0.21, 0.23]\rangle$	$\langle[0.34, 0.43], [0.32, 0.41]\rangle$	$\langle[0.36, 0.47], [0.15, 0.51]\rangle$	$\langle[0.09, 0.71], [0.42, 0.43]\rangle$	$\langle[0.38, 0.42], [0.27, 0.72]\rangle$
$\Re_4$	$\langle[0.16, 0.18], [0.24, 0.26]\rangle$	$\langle[0.31, 0.32], [0.30, 0.60]\rangle$	$\langle[0.37, 0.44], [0.26, 0.62]\rangle$	$\langle[0.01, 0.02], [0.40, 0.60]\rangle$	$\langle[0.03, 0.08], [0.31, 0.41]\rangle$
$\Re_5$	$\langle[0.36, 0.43], [0.13, 0.23]\rangle$	$\langle[0.29, 0.69], [0.28, 0.63]\rangle$	$\langle[0.14, 0.83], [0.47, 0.52]\rangle$	$\langle[0.22, 0.74], [0.17, 0.73]\rangle$	$\langle[0.14, 0.28], [0.10, 0.88]\rangle$

get new tables as Tables 8-12. There are five q-ROFSSs given in Tables 8-12, respectively.

- Now by using the method of Hussain *et al.* [21] for q-ROFSSs ($q = 3$), we calculate scores as $\mathcal{S}(X_1) = -0.020553$, $\mathcal{S}(X_2) = -0.001352$,$\mathcal{S}(X_3) = -0.001589$,$\mathcal{S}(X_4) = 0.008104$ and $\mathcal{S}(X_5) = 0.018848$. We obtain ranking $X_5 > X_4 > X_2 > X_3 > X_1$.
- And by using the method of Zulqarnain *et al.* [57] for q-ROFSSs ($q = 3$), we calculate scores as $\mathcal{S}(X_1) = -0.027700$, $\mathcal{S}(X_2) = -0.028843$, $\mathcal{S}(X_3) = -0.020149$, $\mathcal{S}(X_4) = -0.006835$ and $\mathcal{S}(X_5) = -0.001629$. We obtain ranking $X_5 > X_4 > X_2 > X_1 > X_3$.

If we consider upper membership and upper non-membership from Tables 3-7, then;

- By using the method of Hussain *et al.* [21], we obtain scores as follows $\mathcal{S}(X_1) = 0.059255$, $\mathcal{S}(X_2) = -0.031268$,$\mathcal{S}(X_3) = 0.256643$,$\mathcal{S}(X_4) = 0.051755$ and $\mathcal{S}(X_5) = 0.120932$. We obtain ranking $X_3 > X_5 > X_1 > X_4 > X_2$.

TABLE 7. IVq-ROFSS for automation company X_5.

	e_1	e_2	e_3	e_4	e_5
$\Re_1$	⟨[0.12, 0.21], [0.03, 0.30]⟩	⟨[0.32, 0.44], [0.21, 0.31]⟩	⟨[0.24, 0.54], [0.42, 0.43]⟩	⟨[0.31, 0.41], [0.01, 0.02]⟩	⟨[0.13, 0.27], [0.15, 0.42]⟩
$\Re_2$	⟨[0.10, 0.14], [0.06, 0.33]⟩	⟨[0.23, 0.46], [0.41, 0.51]⟩	⟨[0.30, 0.60], [0.44, 0.47]⟩	⟨[0.20, 0.38], [0.02, 0.04]⟩	⟨[0.42, 0.55], [0.31, 0.54]⟩
$\Re_3$	⟨[0.30, 0.40], [0.09, 0.36]⟩	⟨[0.14, 0.42], [0.13, 0.63]⟩	⟨[0.27, 0.45], [0.32, 0.51]⟩	⟨[0.11, 0.62], [0.07, 0.09]⟩	⟨[0.21, 0.63], [0.29, 0.41]⟩
$\Re_4$	⟨[0.23, 0.60], [0.12, 0.39]⟩	⟨[0.16, 0.40], [0.11, 0.62]⟩	⟨[0.33, 0.47], [0.39, 0.60]⟩	⟨[0.40, 0.52], [0.08, 0.10]⟩	⟨[0.23, 0.44], [0.10, 0.28]⟩
$\Re_5$	⟨[0.13, 0.40], [0.16, 0.22]⟩	⟨[0.12, 0.61], [0.14, 0.43]⟩	⟨[0.11, 0.51], [0.41, 0.51]⟩	⟨[0.19, 0.56], [0.15, 0.51]⟩	⟨[0.30, 0.67], [0.13, 0.17]⟩

- And by using the method of Zulqarnain *et al.* [57] for q-ROFSSs ($q = 3$), we calculate scores as $\mathcal{S}(X_1) = -0.0279801$, $\mathcal{S}(X_2) = -0.145208$, $\mathcal{S}(X_3) = 0.464509$, $\mathcal{S}(X_4) = -0.035482$ and $\mathcal{S}(X_5) = 0.029308$. We obtain ranking $X_3 > X_5 > X_4 > X_2 > X_1$.

It can be analyzed that the above rankings are different from the ranking obtained from the proposed method by considering the Example in Section VI. The reason is that Hussain *et al.* [21] and Zulqarnain *et al.* [57] do not consider intervals of membership grades and non-membership grades. The proposed method overcomes those inadequacies, where it is difficult to take a membership and a non-membership

TABLE 8. q-ROFSS for automation company X_1.

	e_1	e_2	e_3	e_4	e_5
$\Re_1$	⟨0.43, 0.42⟩	⟨0.11, 0.22⟩	⟨0.01, 0.32⟩	⟨0.12, 0.31⟩	⟨0.34, 0.20⟩
$\Re_2$	⟨0.35, 0.32⟩	⟨0.12, 0.42⟩	⟨0.02, 0.36⟩	⟨0.24, 0.41⟩	⟨0.15, 0.16⟩
$\Re_3$	⟨0.42, 0.22⟩	⟨0.13, 0.23⟩	⟨0.04, 0.22⟩	⟨0.17, 0.14⟩	⟨0.08, 0.42⟩
$\Re_4$	⟨0.28, 0.12⟩	⟨0.14, 0.13⟩	⟨0.05, 0.35⟩	⟨0.16, 0.34⟩	⟨0.18, 0.25⟩
$\Re_5$	⟨0.31, 0.18⟩	⟨0.07, 0.43⟩	⟨0.13, 0.14⟩	⟨0.39, 0.23⟩	⟨0.06, 0.01⟩

TABLE 9. q-ROFSS for automation company X_2.

	e_1	e_2	e_3	e_4	e_5
$\Re_1$	⟨0.10, 0.30⟩	⟨0.22, 0.33⟩	⟨0.20, 0.21⟩	⟨0.40, 0.20⟩	⟨0.25, 0.10⟩
$\Re_2$	⟨0.35, 0.15⟩	⟨0.12, 0.12⟩	⟨0.32, 0.24⟩	⟨0.01, 0.43⟩	⟨0.23, 0.25⟩
$\Re_3$	⟨0.29, 0.20⟩	⟨0.21, 0.28⟩	⟨0.13, 0.42⟩	⟨0.08, 0.23⟩	⟨0.41, 0.22⟩
$\Re_4$	⟨0.34, 0.30⟩	⟨0.40, 0.32⟩	⟨0.29, 0.42⟩	⟨0.03, 0.44⟩	⟨0.22, 0.22⟩
$\Re_5$	⟨0.28, 0.24⟩	⟨0.29, 0.14⟩	⟨0.27, 0.23⟩	⟨0.42, 0.30⟩	⟨0.20, 0.40⟩

TABLE 10. q-ROFSS for automation company X_3.

	e_1	e_2	e_3	e_4	e_5
$\Re_1$	⟨0.04, 0.17⟩	⟨0.32, 0.26⟩	⟨0.40, 0.07⟩	⟨0.23, 0.21⟩	⟨0.46, 0.20⟩
$\Re_2$	⟨0.37, 0.16⟩	⟨0.46, 0.36⟩	⟨0.02, 0.08⟩	⟨0.12, 0.23⟩	⟨0.32, 0.14⟩
$\Re_3$	⟨0.29, 0.12⟩	⟨0.09, 0.22⟩	⟨0.03, 0.12⟩	⟨0.13, 0.49⟩	⟨0.04, 0.13⟩
$\Re_4$	⟨0.33, 0.19⟩	⟨0.6, 0.27⟩	⟨0.01, 0.14⟩	⟨0.14, 0.33⟩	⟨0.19, 0.43⟩
$\Re_5$	⟨0.32, 0.34⟩	⟨0.24, 0.38⟩	⟨0.17, 0.12⟩	⟨0.03, 0.25⟩	⟨0.35, 0.14⟩

TABLE 11. q-ROFSS for automation company X_4.

	e_1	e_2	e_3	e_4	e_5
$\Re_1$	⟨0.03, 0.20⟩	⟨0.12, 0.34⟩	⟨0.24, 0.14⟩	⟨0.43, 0.11⟩	⟨0.17, 0.28⟩
$\Re_2$	⟨[0.13, 0.19⟩	⟨0.22, 0.20⟩	⟨0.35, 0.21⟩	⟨0.12, 0.35⟩	⟨0.40, 0.13⟩
$\Re_3$	⟨0.15, 0.21⟩	⟨0.34, 0.32⟩	⟨0.36, 0.15⟩	⟨0.09, 0.42⟩	⟨0.38, 0.27⟩
$\Re_4$	⟨0.16, 0.24⟩	⟨0.31, 0.30⟩	⟨0.37, 0.26⟩	⟨0.01, 0.40⟩	⟨0.03, 0.31⟩
$\Re_5$	⟨0.36, 0.13⟩	⟨0.29, 0.28⟩	⟨0.14, 0.47⟩	⟨0.22, 0.17⟩	⟨0.14, 0.10⟩

TABLE 12. q-ROFSS for automation company X_5.

	e_1	e_2	e_3	e_4	e_5
$\Re_1$	⟨0.12, 0.03⟩	⟨0.32, 0.21⟩	⟨0.24, 0.42⟩	⟨0.31, 0.01⟩	⟨0.13, 0.15⟩
$\Re_2$	⟨0.10, 0.06⟩	⟨0.23, 0.41⟩	⟨0.30, 0.44⟩	⟨0.20, 0.02⟩	⟨0.42, 0.31⟩
$\Re_3$	⟨0.30, 0.09⟩	⟨0.14, 0.13⟩	⟨0.27, 0.32⟩	⟨0.11, 0.07⟩	⟨0.21, 0.29⟩
$\Re_4$	⟨0.23, 0.12⟩	⟨0.16, 0.11⟩	⟨0.33, 0.39⟩	⟨0.40, 0.08⟩	⟨0.23, 0.10⟩
$\Re_5$	⟨0.13, 0.16⟩	⟨0.12, 0.14⟩	⟨0.11, 0.41⟩	⟨0.19, 0.15⟩	⟨0.30, 0.13⟩

grade, but both membership and non-membership grades may exist in intervals. Therefore, introduced results are better for dealing with uncertainties in many complex problems.

Consider the work of Zulqarnain *et al.* [56], where correlation co-efficient of IVIFSSs are given and their application in MCDM is discussed. Consider the Example given in section 4.2 of Zulqarnain *et al.* [56], where IVIFSSs is depicted in Table 1-4. By their method the ranking of alternatives in $\mathscr{R}_4 > \mathscr{R}_3 > \mathscr{R}_1 > \mathscr{R}_2$. In order to compare results with Zulqarnain *et al.* [56], we take $q = 1$. Then using the proposed method we obtain $S_1 = 0.4606$, $S_2 = 0.4274$, $S_3 = 0.5127$ and $S_4 = 0.5420$. Thus, $\mathscr{R}_4 > \mathscr{R}_3 > \mathscr{R}_1 > \mathscr{R}_2$. Similarly we can obtain same ranking on $q > 1$. The fundamental capability of given model of MCDM is to solve real life problems by utilizing parameterizations and intervals.

VII. CONCLUSION

In this paper, we have introduced some crucial properties of IVq-ROFSSs and we have investigated IVq-ROF weighted averaging, IVq-ROF weighted geometric, IVq-ROF weighted interaction averaging, IVq-ROF weighted interaction geometric aggregation operators and given a MCDM method.

The integration of IVq-ROFS, soft sets and MCDM specifies new potentialities relating to the modeling of complex MCDM problems in distributed environments. In addition, models of IVq-ROFSSs equipped for supervising of massive data sets and manipulation of ill-structured knowledge.

In several prospectives the introduced operators are useful to cope uncertainties. The results are given on a representation of information by an interval number within [0, 1] is a fair choice. It can upgraded in different directions, because in several real life problem usually informations exist in intervals. The IVq-ROFSSs can be useful in MCDM where several prospects in terms of alternatives and parameters involves. Aggregation operators on IVq-ROFSSs are investigated with

$$
\begin{aligned}
&IVq-ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \overline{\mathcal{F}}_{e_{21}}\overline{\mathcal{F}}_{e_{22}})\\
&\quad = \oplus_{j=1}^{2}(\oplus_{i=1}^{2}\overline{\mathcal{F}}_{e_{ij}}\delta_i)w_j\\
&\quad = \oplus_{j=1}^{2}\left(\overline{\mathcal{F}}_{e_{1j}}\delta_1 \oplus \overline{\mathcal{F}}_{e_{2j}}\delta_2\right)w_j\\
&\quad = \oplus_{j=1}^{2}(([\sqrt[q]{1-(1-(\rho_{1j}^{-})^q)^{\delta_1}}, \sqrt[q]{1-(1-(\rho_{1j}^{+})^q)^{\delta_1}}], [(\varrho_{1j}^{-})^{\delta_1}, (\varrho_{1j}^{+})^{\delta_1}])\\
&\qquad \oplus ([\sqrt[q]{1-(1-(\rho_{2j}^{-})^q)^{\delta_2}}, \sqrt[q]{1-(1-(\rho_{2j}^{+})^q)^{\delta_2}}], [(\varrho_{2j}^{-})^{\delta_2}, (\varrho_{2j}^{+})^{\delta_2}]))w_j\\
&\quad = (([\sqrt[q]{1-(1-(\rho_{11}^{-})^q)^{\delta_1}}, \sqrt[q]{1-(1-(\rho_{11}^{+})^q)^{\delta_1}}], [(\varrho_{11}^{-})^{\delta_1}, (\varrho_{11}^{+})^{\delta_1}])\\
&\qquad \oplus ([\sqrt[q]{1-(1-(\rho_{21}^{-})^q)^{\delta_2}}, \sqrt[q]{1-(1-(\rho_{21}^{+})^q)^{\delta_2}}], [(\varrho_{21}^{-})^{\delta_2}, (\varrho_{21}^{+})^{\delta_2}]))w_1\\
&\qquad \oplus (([\sqrt[q]{1-(1-(\rho_{12}^{-})^q)^{\delta_1}}, \sqrt[q]{1-(1-(\rho_{12}^{+})^q)^{\delta_1}}], [(\varrho_{12}^{-})^{\delta_1}, (\varrho_{12}^{+})^{\delta_1}])\\
&\qquad \oplus ([\sqrt[q]{1-(1-(\rho_{22}^{-})^q)^{\delta_2}}, \sqrt[q]{1-(1-(\rho_{22}^{+})^q)^{\delta_2}}], [(\varrho_{22}^{-})^{\delta_2}, (\varrho_{22}^{+})^{\delta_2}]))w_2\\
&\quad = \left(\begin{array}{l}\left[\sqrt[q]{1-\prod_{i=1}^{2}(1-(\rho_{i1}^{-})^q)^{\delta_i}}, \sqrt[q]{1-\prod_{i=1}^{2}(1-(\rho_{i1}^{+})^q)^{\delta_i}}\right],\\ \left[\prod_{i=1}^{2}(\varrho_{i1}^{-})^{\delta_i}, \prod_{i=1}^{2}(\varrho_{i1}^{+})^{\delta_i}\right]\end{array}\right)w_1\\
&\qquad \oplus \left(\begin{array}{l}\left[\sqrt[q]{1-\prod_{i=1}^{2}(1-(\rho_{i2}^{-})^q)^{\delta_i}}, \sqrt[q]{1-\prod_{i=1}^{2}(1-(\rho_{i2}^{+})^q)^{\delta_i}}\right],\\ \left[\prod_{j=1}^{2}(\varrho_{i2}^{-})^{\delta_i}, \prod_{i=1}^{2}(\varrho_{i2}^{+})^{\delta_i}\right]\end{array}\right)w_2\\
&\quad = \left(\begin{array}{l}\left[\sqrt[q]{1-\prod_{j=1}^{2}\left(\prod_{i=1}^{2}\left(1-\left(\rho_{ij}^{-}\right)^q\right)^{\delta_i}\right)w_j}, \sqrt[q]{1-\prod_{j=1}^{2}\left(\prod_{i=1}^{2}\left(1-\left(\rho_{ij}^{+}\right)^q\right)^{\delta_i}\right)w_j}\right],\\ \left[\prod_{j=1}^{2}\prod_{i=1}^{2}\left(\varrho_{ij}^{-}\right)^{\delta_i}w_j, \prod_{j=1}^{2}\prod_{i=1}^{2}\left(\varrho_{ij}^{+}\right)^{\delta_i}w_j\right]\end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
&IVq-ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots, \overline{\mathcal{F}}_{e_{s_1}}, \overline{\mathcal{F}}_{e_{m_1}})\\
&\quad = \oplus_{j=1}^{m_1}(\oplus_{i=1}^{s_1}\overline{\mathcal{F}}_{e_{ij}}\delta_i)w_j\\
&\quad = \left(\begin{array}{l}\left[\sqrt[q]{1-\prod_{j=1}^{m_1}\left(\prod_{i=1}^{s_1}\left(1-\left(\rho_{ij}^{-}\right)^q\right)^{\delta_i}\right)w_j}, \sqrt[q]{1-\prod_{j=1}^{m_1}\left(\prod_{i=1}^{s_1}\left(1-\left(\rho_{ij}^{+}\right)^q\right)^{\delta_i}\right)w_j}\right],\\ \left[\prod_{j=1}^{m_1}\left(\prod_{i=1}^{s_1}\left(\varrho_{ij}^{-}\right)^{\delta_i}\right)w_j, \prod_{j=1}^{m_1}\left(\prod_{i=1}^{s_1}\left(\varrho_{ij}^{+}\right)^{\delta_i}\right)w_j\right]\end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
&IVq-ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots, \overline{\mathcal{F}}_{e_{s_1}}, \overline{\mathcal{F}}_{e_{m_1}}, \overline{\mathcal{F}}_{e_{(s_1+1)(m_1+1)}})\\
&\quad = \oplus_{j=1}^{m_1+1}(\oplus_{i=1}^{s_1+1}\overline{\mathcal{F}}_{e_{ij}}\delta_i)w_j \oplus (\overline{\mathcal{F}}_{e_{(s_1+1)(m_1+1)}}\delta_i)w_j\\
&\quad = \left(\begin{array}{l}\left[\sqrt[q]{1-\prod_{j=1}^{m_1+1}\left(\prod_{i=1}^{s_1+1}\left(1-\left(\rho_{ij}^{-}\right)^q\right)^{\delta_i}\right)w_j}, \sqrt[q]{1-\prod_{j=1}^{m_1+1}\left(\prod_{i=1}^{s_1+1}\left(1-\left(\rho_{ij}^{+}\right)^q\right)^{\delta_i}\right)w_j}\right],\\ \left[\prod_{j=1}^{m_1+1}\left(\prod_{i=1}^{s_1+1}\left(\varrho_{ij}^{-}\right)^{\delta_i}\right)w_j, \prod_{j=1}^{m_1+1}\left(\prod_{i=1}^{s_1+1}\left(\varrho_{ij}^{+}\right)^{\delta_i}\right)w_j\right]\end{array}\right)
\end{aligned}
$$

their axioms which can be helpful to complex MCDM problems. A procedure is interrupted with smooth and validate steps, which can be root mechanism in MCDM. An illustration is considered; that is, to select appropriate automation company for automation engineers where they feel comfortable to excel their abilities. Although introduced results reprieve in complex scenarios of MCDM, in future works we will focus on advanced forms of IVq-ROFSS, that is, generalized IVq-ROFSS and give an insight in to the impact of IVq-ROFSS in graphs and networks. One can develop complexity analysis of the given algorithm. Furthermore, the developed aggregation operators can be extended to T-spherical fuzzy soft environments with decision making approaches.

APPENDIX I. THE PROOF OF THE THEOREM III.1

The result can be demonstrated by the mathematical induction. Take $s = 2$ and $m = 2$, we have $IVq-ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \overline{\mathcal{F}}_{e_{21}}\overline{\mathcal{F}}_{e_{22}})$, as shown at the top of the previous page.

Hence the result is true for $s = 2$ and $m = 2$. Now we take $s = s_1$ and $m = m_1$ ($IVq-ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots, \overline{\mathcal{F}}_{e_{s_1}}, \overline{\mathcal{F}}_{e_{m_1}})$, as shown at the top of the previous page.)

Further we take $s = s_1 + 1$ and $m = m_1 + 1$; $IVq-ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots, \overline{\mathcal{F}}_{e_{s_1}}, \overline{\mathcal{F}}_{e_{m_1}}, \overline{\mathcal{F}}_{e_{(s_1+1)(m_1+1)}})$, as shown at the top of the previous page.

Hence the result is true for $s = s_1 + 1$ and $m = m_1 + 1$. Therefore by mathematical induction it is correct for all $m, s \geq 1$.

APPENDIX II. THE PROOF OF THE THEOREM IV.1

The result can be demonstrated by the mathematical induction. Take $s = 2$ and $m = 2$, we have $IVq-ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{1m}})$, as shown at the bottom of the page.

$$
\begin{aligned}
&IVq-ROFSWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{1m}}) \\
&= \tilde{\oplus}_{j=1}^{m} w_j \overline{\mathcal{F}}_{e_{1j}} \\
&= \begin{pmatrix} \left[\sqrt[q]{1 - \prod_{j=1}^{m}\left(1-(\rho_{1j}^{-})^q\right)^{w_j}}, \sqrt[q]{1 - \prod_{j=1}^{m}\left(1-(\rho_{1j}^{+})^q\right)^{w_j}} \right], \\ \left[\sqrt[q]{1 - \prod_{j=1}^{m}\left(1-(\rho_{1j}^{-})^q\right)^{w_j} - \prod_{j=1}^{m}\left(1-\left((\rho_{1j}^{-})^q + (\varrho_{1j}^{-})^q\right)\right)^{w_j}}, \right. \\ \left. \sqrt[q]{1 - \prod_{j=1}^{m}\left(1-(\rho_{1j}^{+})^q\right)^{w_j} - \prod_{j=1}^{s}\left(1-\left((\rho_{1j}^{+})^q + (\varrho_{1j}^{+})^q\right)\right)^{w_j}} \right] \end{pmatrix} \\
&= \begin{pmatrix} \left[\sqrt[q]{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-(\rho_{ij}^{-})^q\right)^{\delta_i}\right)^{w_j}}, \sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-(\rho_{ij}^{+})^q\right)^{\delta_i}\right)^{w_j}} \right], \\ \left[\sqrt[q]{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-(\rho_{ij}^{-})^q\right)^{\delta_i}\right)^{w_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-\left((\rho_{ij}^{-})^q + (\varrho_{ij}^{-})^q\right)\right)^{\delta_i}\right)^{w_j}}, \right. \\ \left. \sqrt[q]{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-(\rho_{ij}^{+})^q\right)^{\delta_i}\right)^{w_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-\left((\rho_{ij}^{+})^q + (\varrho_{ij}^{+})^q\right)\right)^{\delta_i}\right)^{w_j}} \right] \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&IVq-ROFSIWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{1s}}) \\
&= \tilde{\oplus}_{i=1}^{s} \delta_i \overline{\mathcal{F}}_{e_{i1}} \\
&= \begin{pmatrix} \left[\sqrt[q]{1 - \prod_{i=1}^{s}(1-(\rho_{i1}^{-})^q)^{\delta_i}}, \sqrt[q]{\prod_{i=1}^{s}(1-(\rho_{i1}^{+})^q)^{\delta_i}} \right], \\ \left[\sqrt[q]{1 - \prod_{i=1}^{s}(1-(\rho_{i1}^{-})^q)^{\delta_i} - \prod_{i=1}^{s}(1-((\rho_{i1}^{-})^q + (\varrho_{i1}^{-})^q))^{\delta_i}}, \right. \\ \left. \sqrt[q]{\prod_{i=1}^{s}(1-(\rho_{i1}^{+})^q)^{\delta_i} - \prod_{i=1}^{s}(1-((\rho_{i1}^{+})^q + (\varrho_{i1}^{+})^q))^{\delta_i}} \right] \end{pmatrix} \\
&= \begin{pmatrix} \left[\sqrt[q]{1 - \prod_{j=1}^{1}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{-})^q\right)^{\delta_i}\right)^{w_j}}, \sqrt[q]{1 - \prod_{j=1}^{1}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{+})^q\right)^{\delta_i}\right)^{w_j}} \right], \\ \left[\sqrt[q]{\prod_{j=1}^{1}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{-})^q\right)^{\delta_i}\right)^{w_j} - \prod_{j=1}^{1}\left(\prod_{i=1}^{s}\left(1-\left((\rho_{ij}^{-})^q + (\varrho_{ij}^{-})^q\right)\right)^{\delta_i}\right)^{w_j}}, \right. \\ \left. \sqrt[q]{\prod_{j=1}^{1}\left(\prod_{i=1}^{s}\left(1-(\rho_{ij}^{+})^q\right)^{\delta_i}\right)^{w_j} - \prod_{j=1}^{1}\left(\prod_{i=1}^{s}\left(1-\left((\rho_{ij}^{+})^q + (\varrho_{ij}^{+})^q\right)\right)^{\delta_i}\right)^{w_j}} \right] \end{pmatrix}
\end{aligned}
$$

$$\tilde{\oplus}_{j=1}^{\alpha_1+1}(\tilde{\oplus}_{i=1}^{\alpha_2}\overline{\mathcal{F}}_{e_{ij}}\delta_i)w_j$$
$$= \tilde{\oplus}_{j=1}^{\alpha_1}(\tilde{\oplus}_{i=1}^{\alpha_2}\overline{\mathcal{F}}_{e_{ij}}\delta_i)w_j\tilde{\oplus}(\tilde{\oplus}_{i=1}^{\alpha_2}\overline{\mathcal{F}}_{e_{i(\alpha_1+1)}}\delta_i)w_{(\alpha_1+1)}$$
$$= \begin{pmatrix} \left[\sqrt[q]{1-\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{ij}^-)^q\right)^{\delta_i}\right)^{w_j}}, \sqrt[q]{1-\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{ij}^+)^q\right)^{\delta_i}\right)^{w_j}}\right], \\ \left[\sqrt[q]{\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{ij}^-)^q\right)^{\delta_i}\right)^{w_j}-\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left((\rho_{ij}^-)^q+(\varrho_{ij}^-)^q\right)\right)^{\delta_i}\right)^{w_j}}, \right. \\ \left. \sqrt[q]{\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{ij}^+)^q\right)^{\delta_i}\right)^{w_j}-\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left((\rho_{ij}^+)^q+(\varrho_{ij}^+)^q\right)\right)^{\delta_i}\right)^{w_j}}\right] \end{pmatrix}$$
$$\tilde{\oplus} \begin{pmatrix} \left[\sqrt[q]{1-\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{i(\alpha_1+1)}^-)^q\right)^{\delta_i}\right)^{w(\alpha_1+1)}}, \sqrt[q]{1-\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{i(\alpha_1+1)}^+)^q\right)^{\delta_i}\right)^{w(\alpha_1+1)}}\right], \\ \left[\sqrt[q]{\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{i(\alpha_1+1)}^-)^q\right)^{\delta_i}\right)^{w(\alpha_1+1)}-\left(\prod_{i=1}^{\alpha_2}\left(1-\left((\rho_{i(\alpha_1+1)}^-)^q+(\varrho_{i(\alpha_1+1)}^-)^q\right)\right)^{\delta_i}\right)^{w(\alpha_1+1)}}, \right. \\ \left. \sqrt[q]{\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{i(\alpha_1+1)}^+)^q\right)^{\delta_i}\right)^{w(\alpha_1+1)}-\left(\prod_{i=1}^{\alpha_2}\left(1-\left((\rho_{i(\alpha_1+1)}^+)^q+(\varrho_{i(\alpha_1+1)}^+)^q\right)\right)^{\delta_i}\right)^{w(\alpha_1+1)}}\right] \end{pmatrix}$$
$$= \begin{pmatrix} \left[\sqrt[q]{1-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{ij}^-)^q\right)^{\delta_i}\right)^{w_j}}, \sqrt[q]{1-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{ij}^+)^q\right)^{\delta_i}\right)^{w_j}}\right], \\ \left[\sqrt[q]{\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{ij}^-)^q\right)^{\delta_i}\right)^{w_j}-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left((\rho_{ij}^-)^q+(\varrho_{ij}^-)^q\right)\right)^{\delta_i}\right)^{w_j}}, \right. \\ \left. \sqrt[q]{\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{ij}^+)^q\right)^{\delta_i}\right)^{w_j}-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left((\rho_{ij}^+)^q+(\varrho_{ij}^+)^q\right)\right)^{\delta_i}\right)^{w_j}}\right] \end{pmatrix}$$

$$\tilde{\oplus}_{j=1}^{\alpha_1}(\tilde{\oplus}_{i=1}^{\alpha_2+1}\overline{\mathcal{F}}_{e_{ij}}\delta_i)w_j$$
$$= \tilde{\oplus}_{j=1}^{\alpha_1}(\tilde{\oplus}_{i=1}^{\alpha_2}\overline{\mathcal{F}}_{e_{ij}}\delta_i)w_j\tilde{\oplus}(\tilde{\oplus}_{j=1}^{\alpha_1}(\overline{\mathcal{F}}_{e_{(\alpha_2+1)j}}\delta_{(\alpha_2+1)})w_j)$$
$$= \begin{pmatrix} \left[\sqrt[q]{1-\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{ij}^-)^q\right)^{\delta_i}\right)^{w_j}}, \sqrt[q]{1-\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{ij}^+)^q\right)^{\delta_i}\right)^{w_j}}\right], \\ \left[\sqrt[q]{\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{ij}^-)^q\right)^{\delta_i}\right)^{w_j}-\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left((\rho_{ij}^-)^q+(\varrho_{ij}^-)^q\right)\right)^{\delta_i}\right)^{w_j}}, \right. \\ \left. \sqrt[q]{\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{ij}^+)^q\right)^{\delta_i}\right)^{w_j}-\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left((\rho_{ij}^+)^q+(\varrho_{ij}^+)^q\right)\right)^{\delta_i}\right)^{w_j}}\right] \end{pmatrix}$$
$$\tilde{\oplus} \begin{pmatrix} \left[\sqrt[q]{1-\prod_{j=1}^{\alpha_1}\left(\left(1-(\rho_{(\alpha_2+1)j}^-)^q\right)^{\delta_{(\alpha_2+1)}}\right)^{w_j}}, \sqrt[q]{1-\prod_{j=1}^{\alpha_1}\left(\left(1-(\rho_{(\alpha_2+1)j}^+)^q\right)^{\delta_{(\alpha_2+1)}}\right)^{w_j}}\right], \\ \left[\sqrt[q]{\prod_{j=1}^{\alpha_1}\left(\left(1-(\rho_{(\alpha_2+1)j}^-)^q\right)^{\delta_{(\alpha_2+1)}}\right)^{w_j}-\prod_{j=1}^{\alpha_1}\left(\left(1-\left((\rho_{(\alpha_2+1)j}^-)^q+(\varrho_{(\alpha_2+1)j}^-)^q\right)\right)^{\delta_{(\alpha_2+1)}}\right)^{w_j}}, \right. \\ \left. \sqrt[q]{\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2}\left(1-(\rho_{ij}^+)^q\right)^{\delta_i}\right)^{w_j}-\prod_{j=1}^{\alpha_1}\left(\left(1-\left((\rho_{(\alpha_2+1)j}^+)^q+(\varrho_{(\alpha_2+1)j}^+)^q\right)\right)^{\delta_{(\alpha_2+1)}}\right)^{w_j}}\right] \end{pmatrix}$$
$$= \begin{pmatrix} \left[\sqrt[q]{1-\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-(\rho_{ij}^-)^q\right)^{\delta_i}\right)^{w_j}}, \sqrt[q]{1-\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-(\rho_{ij}^+)^q\right)^{\delta_i}\right)^{w_j}}\right], \\ \left[\sqrt[q]{\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-(\rho_{ij}^-)^q\right)^{\delta_i}\right)^{w_j}-\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-\left((\rho_{ij}^-)^q+(\varrho_{ij}^-)^q\right)\right)^{\delta_i}\right)^{w_j}}, \right. \\ \left. \sqrt[q]{\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-(\rho_{ij}^+)^q\right)^{\delta_i}\right)^{w_j}-\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-\left((\rho_{ij}^+)^q+(\varrho_{ij}^+)^q\right)\right)^{\delta_i}\right)^{w_j}}\right] \end{pmatrix}$$

$$\tilde{\bigoplus}_{j=1}^{\alpha_1+1}(\tilde{\bigoplus}_{i=1}^{\alpha_2+1}\overline{\mathcal{F}}_{e_{ij}}\delta_i)w_j$$
$$= \begin{pmatrix} \left[\sqrt[q]{1-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-(\rho_{ij}^{-})^q\right)^{\delta_i}\right)^{w_j}}, \sqrt[q]{1-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-(\rho_{ij}^{+})^q\right)^{\delta_i}\right)^{w_j}}\right], \\ \left[\sqrt[q]{\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-(\rho_{ij}^{-})^q\right)^{\delta_i}\right)^{w_j}-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-\left((\rho_{ij}^{-})^q+(\varrho_{ij}^{-})^q\right)\right)^{\delta_i}\right)^{w_j}},\right. \\ \left.\sqrt[q]{\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-(\rho_{ij}^{+})^q\right)^{\delta_i}\right)^{w_j}-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-\left((\rho_{ij}^{+})^q+(\varrho_{ij}^{+})^q\right)\right)^{\delta_i}\right)^{w_j}}\right] \end{pmatrix}.$$

For $m = 1$ and we get $w_1 = 1$. So, we have $IVq-ROFSIWA(\overline{\mathcal{F}}_{e_{11}}, \overline{\mathcal{F}}_{e_{12}}, \cdots \overline{\mathcal{F}}_{e_{1s}})$, as shown at the bottom of page 17.

Hence the result is true for $m = 1$ and $s = 1$. Therefore the Theorem holds for $m = \alpha_1$ and $s = \alpha_2$. Now we have to check three cases;

Case-1: When $s = \alpha_2, m = \alpha_1 + 1$ ($\tilde{\bigoplus}_{j=1}^{\alpha_1+1}(\tilde{\bigoplus}_{i=1}^{\alpha_2}\overline{\mathcal{F}}_{e_{ij}}\delta_i)w_j$, as shown at the previous page.)

Case-2: When $s = \alpha_2 + 1, m = \alpha_1$ ($\tilde{\bigoplus}_{j=1}^{\alpha_1}(\tilde{\bigoplus}_{i=1}^{\alpha_2+1}\overline{\mathcal{F}}_{e_{ij}}\delta_i)w_j$, as shown at the previous page.)

Case-3: When $s = \alpha_2 + 1$ and $m = \alpha_1 + 1$ by (i) and (ii), $\tilde{\bigoplus}_{j=1}^{\alpha_1+1}(\tilde{\bigoplus}_{i=1}^{\alpha_2+1}\overline{\mathcal{F}}_{e_{ij}}\delta_i)w_j$, as shown at the top of the page.

Hence the result is true for $s = \alpha_2 + 1$ and $m = \alpha_1 + 1$. Therefore by mathematical indiction it is correct for all $s, m \geq 1$.

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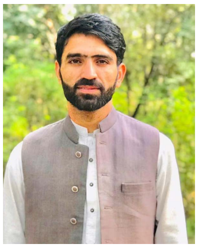

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