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RESEARCH ARTICLE

Studies of Multilevel Networks via Fault-Tolerant Metric Dimensions

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ABSTRACT A subset T of the vertex set of a network G is called a resolving set for G if each pair of vertices of G have distinct representations with respect to T. A resolving set B' among all the resolving sets of a network G is called a fault-tolerant resolving set if $B' \setminus \{t\}$ is as well a resolving set for each vertex $t \in B'$. A fault-tolerant resolving set B' of a network G which contains minimum number of vertices is called a fault-tolerant metric basis. The cardinality of a fault-tolerant metric basis is called fault-tolerant metric dimension. This concept is widely used to find the integral solution of the problems existing in different disciplines of computer science and chemistry such as linear optimization problems, robot navigation, operation research problems, sensor networking, classification of chemical compounds, drug discoveries, source localization, embedding biological sequence data, detecting network motifs, comparing the interconnected networks and image processing. In this paper, we compute the fault-tolerant metric dimensions of three wheel related networks called by r-level anti-web wheel $AWW_{(n,r)}$, r-level Helm $H_{(n,r)}$ and r-level anti-web gear $AWJ_{(2n,r)}$ networks in the form of different algebraic expressions consisting of the integral numbers n and r. At the end we discussed a simple method for finding the fault-tolerant metric dimensions and fault-tolerant resolving sets of a r-level wheel related network. We also discussed the importance of these networks in navigation.

INDEX TERMS Fault-tolerant metric dimensions, fault-tolerant resolving set, r-level anti web wheel network, r-level anti web gear network, r-level Helm network.

I. INTRODUCTION

The networks in this paper are connected and simple. A network G consists of vertices $V(G)$ and edges $E(G)$, where the vertices are connected by lines (edges). If $s, t \in V(G)$, then the distance between s and t denoted by $d(s, t)$ is equal to the number of edges in a shortest path connecting them. The number of vertices are called order of network G while edges are called size of of network G. A vertex v of network G resolves two vertices $s, t \in V(G)$, if $d(v, s) \neq d(v, t)$. For an ordered set $T = \{t_1, t_2, \cdots, t_r\} \subset V(G)$ and an arbitrary vertex a in a connected network G, the representations of a with respect to T is the ordered r-tuple $r(a/T) = \{d(a, t_1), d(a, t_2), \cdots, d(a, t_r)\}$. The set T is a resolving set or locating set for G if every two vertices of G have distinct representations. For further details (see [1], [2]). The idea of resolving set for a connected network G was found by Slater [3]. Harary and Melter [4] also independently introduced resolving sets. A resolving set B of a network G which contains a minimum number of vertices is called the metric basis of network G. The cardinality of a metric basis B is called the metric dimension of network G which is denoted by $\beta(G)$ as explained in [5]. Metric dimensions and resolving sets of networks are very efficacious in a diversity of situations in different disciplines. Due to similarity in metric dimension and trilateration in the planar networks, metric dimensions are used in robot navigation to detect their location and increase the level of communication as demonstrated in [6]. Moreover, the number of inter connected computers can be reduced in a computer networking and the chemical compounds can be predicted and classified in the general chemical structures with the help of resolving sets and their metric dimensions [7], [8]. Metric dimensions has also been used for the detection of a source in

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the spread over of a network like a device for the detection of network motifs in [9] and [10] and the base set for embedding the DNA order in a real space in [11]. For further details, we refer to [12].

Metric dimensions and resolving sets have been widely used in our daily life problems. Due to this reality of the metric dimension in our daily life, various authors researched it extensively. Various authors applied metric dimension in various fields like in application of graph theory and metric dimensions pharmaceutical chemistry has been studied in [2], sonar, facility location problem and coast guard Loran interconnected to this idea in [3], robot navigation [6], weighing problems of coins [13], diverse studies guided about combinatorial optimization in [14] and computer networks [15] are also influential to this idea. For more details about metric dimension and resolving sets one can further study the applications of this framework in [16] and [17].

In [18], elements of metric basis are considered as censors. If one of the censors does not work, then we do not know how to overcome such problems. So in the direction of these problems, Hernando *et al.* [19] presented the idea of fault-tolerant resolving sets and fault-tolerant metric dimensions. Fault tolerance is the characteristic of a system that allows it to continues working properly in the case of the failure of one or more faults within some of its components. So by the idea of fault tolerance the problems in [18] have been solved. After that various authors computed fault-tolerant metric dimensions of different networks. According to [19] a resolving set B' among all the resolving sets of a network G is called a fault-tolerant resolving set if $B' \setminus \{t\}$ is as well a resolving set for each vertex $t \in B'$. A fault-tolerant resolving set B' of a network G which contains minimum number of vertices is called a fault-tolerant metric basis denoted by B'. The cardinality of B' is called fault-tolerant metric dimension and is symboled by $\beta'(G)$. Hernando *et al.* [19] also obtained the fault-tolerant resolving sets of a tree network represented by T.

Saha *et al.* [20] computed the Fault-tolerant metric dimension of the cube of paths. Javaid *et al.* [21] obtained the fault-toerant metric dimension of a cycle C_n of order n. In past, metric and fault-tolerant metric dimensions of wheel related networks have been obtained by various authors. Buczkowski *et. al.* [22] studied the metric dimensions of a wheel network denoted by W_n and obtained $\beta(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for $n \geq 7$. Zheng *et al.* [23] computed the fault-tolerant metric dimensions of generalized wheel and convex polytopes as $\beta'(W_{(n,r)}) = r\lceil \frac{n}{2} \rceil$ for $n \geq 8$. Afzal *et al.* [24] studied the metric dimensions and resolving sets of generalized anti-web wheel and gear networks. Liu *et al.* [25] computed the fault-tolerant metric dimension of anti-web gear, gear and anti-web networks.

Slater [26] inaugurated the idea of fault-tolerant resolving sets. Fault-tolerant resolving sets are successively used in engineering, chemistry and computer sciences as explained in [27]. Raza *et al.* [28] discussed the uses of fault-tolerant resolving sets and fault-tolerant metric dimensions in different interconnection networks. In chemistry, if we remove silicon nodes from silicate network then the new network will be an oxide interconnection network represented by OX(ω). Somasundari *et al.* [29] obtained the fault-tolerant resolvability and fault-tolerant metric dimension of oxide interconnection network. In a crystal structure atoms (vertices) are arranged in a design with repetition of atoms in three dimensions. Krishnan *et al.* [30] investigated the fault-tolerant metric dimensions problems for crystal networks like lead chloride, quartz and bismuth tri-iodide. Raza *et al.* [31] investigated the uses of fault-tolerant metric dimension in convex polytopes. For more information about applications and properties of the fault-tolerance in resolving sets see [32]. In 1987, the word coronoid was conceived by [33]. The relation between coronoid and benzenoid is very close. A benzenoid which has a hole in the center is named as a coronoide. Later, in [34], multiple coronoids were found. A multiple coronoid contains more than one hole in each benzenoid. It was found that a coronid is a subset of primitive coronoide. Further primitive coronoids were classified as catacondensed. Ali *et al.* [35] studied the properties of fault-tolerant resolving sets and fault-tolerant metric dimension of hollow coronoid $HC(p, q, s)$. Javaid *et al.* [36] studied the properties of local fractional metric dimension of Prism related networks. Javaid *et al.* [37] studied the properties of sharp bounds of local fractional metric dimension of wheel related networks.

In diverse applications of the use of graph theory, the terminologies interchanged according to the situation of different problems. For example, when we transform an electrical circuit into a network then the current sources are referred to as vertices, while the voltage sources are renamed by edges. The edges become line segments and vertices are studied as principal nodes in [38]. In a wireless body area network abbreviated as WBAN independent nodes (e.g. actuators and sensors) that are located under the skin of a person, on the body or in the clothes are connected by way of a wireless communication channel. Mehmood *et al.* [39], [40] presented an energy-efficient fault-tolerant scheme to upgrade the accuracy of WBAN. Fritscher *et al.* [41] showed how a fault-aware training act on the reaction of a network changeability.

In the present article, we studied the properties of fault-tolerant resolving sets and fault-tolerant metric dimensions of three r-level wheel related networks r-level anti-web wheel, gear and Helm network.

II. PRELIMINARIES

In this section, we define some wheel related networks and three r-level anti web wheel, gear and Helm networks. Here, We also introduce some very fundamental ideas which are helpful for understanding the groundwork that has been concluded in this paper.

Definition 1: Let C_n be a cycle of order n, then a wheel network W_n can be obtained by adding the vertex v_0 to all the vertices of C_n. So $W_n \equiv C_n + v_0$, where $V(W_n) = V(C_n \cup v_0)$ and $E(W_n) = E(C_n \cup v_0c : c \in V(C_n))$. The order and size of wheel network are $n + 1$ and $2n$ respectively as mentioned in [17].

Definition 2: Let rC_n be r cycles of order n each, then a r-level wheel network $W_{(n,r)}$ can be obtained by adding the vertex v_0 to all the vertices of rC_n. So $W_{(n,r)} \equiv rC_n + v_0$, where $V(W_{(n,r)}) = V(rC_n \cup v_0)$ and $E(W_{(n,r)}) = E(rC_n \cup (v_0c : c \in V(C_n))$. The order and size of $W_{(n,r)}$ are $rn + 1$ and $2rn$ respectively (See Fig. 1).

Definition 3: Let C_n be an even cycle of order n, then anti-web network AW_n can be obtained by adding edges $\{v_iv_{i+2}\colon 1 \leq i \leq n, v_i \in C_n\}$ to C_n. In AW_n, $V(AW_n) = V(C_n)$ and $E(AW_n) = E(C_n \cup (v_iv_{i+2}\colon 1 \leq i \leq n, v_i \in C_n))$. The order and size of AW_n are n and $2n$ respectively as mentioned in [20].

Definition 4: Let W_n be a wheel network of order n, where n is even, then anti-web wheel network AWW_n can be obtained by adding edges $\{v_iv_{i+2}\colon 1 \leq i \leq n, v_i \in C_n\}$ to W_n. In AWW_n, $V(AWW_n) = V(W_n)$ and $E(AWW_n) = E(W_n \cup (v_iv_{i+2}\colon 1 \leq i \leq n, v_i \in C_n))$. The order and size of AWW_n are $n + 1$ and $3n$ respectively as mentioned in [20].

Definition 5: The anti-web gear network AWJ_{2n} can be obtained from the anti-web wheel network by deleting one after another spoke. The order and size of AWJ_{2n} are $2n + 1$ and $5n$ respectively as mentioned in [20].

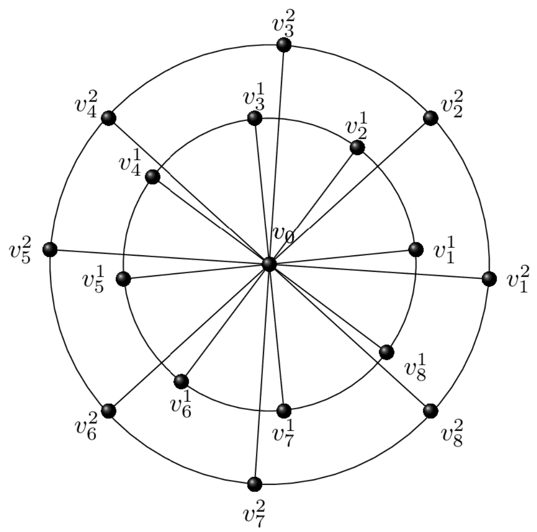

FIGURE 1. **A network of $W_{(8,2)}$.**

Definition 6: Let $rAW_n, r \geq 2$ be r anti-web networks of order n each, then a r-level anti-web wheel network $AWW_{(n,r)}$ can be obtained by adding the vertex v_0 to all the vertices of each anti-web AW_n. So $AWW_n \equiv rAW_n + v_0$, where $V(AWW_{(n,r)}) = V(rAW_n \cup v_0)$ and $E(AWW_{(n,r)}) = E(rAW_n \cup (v_0c : c \in V(rC_n))$. The order and size of $AWW_{(n,r)}$ are $rn + 1$ and $3rn$ respectively as explained in [19] (See Fig. 3).

Definition 7: A r-level anti-web gear network $AWJ_{(2n,r)}$ can be obtained from the r-level anti web wheel network by deleting one after another spokes of each cycle. The order and size of $AWJ_{(2n,r)}$ are $5rn$ and $2rn+1$ respectively as explained in [19] (See Fig. 4).

Definition 8: A Helm network denoted by H_n, $n \geq 3$ is a network obtained from a wheel network W_n by adjoining a pendant edge at each vertex on the cycle C_n. The order and size of the Helm network is $2n + 1$ and $3n$ respectively.

Definition 9: Let $rH_n, r \geq 2$ be Helm networks of order n each, then a r-level Helm network $H_{(n,r)}$ can be obtained by taking the union of r Helm networks, where the central vertex v_0 is common only. The order and size of $H_{(n,r)}$ are $2rn + 1$ and $3rn$ respectively (See Fig. 2).

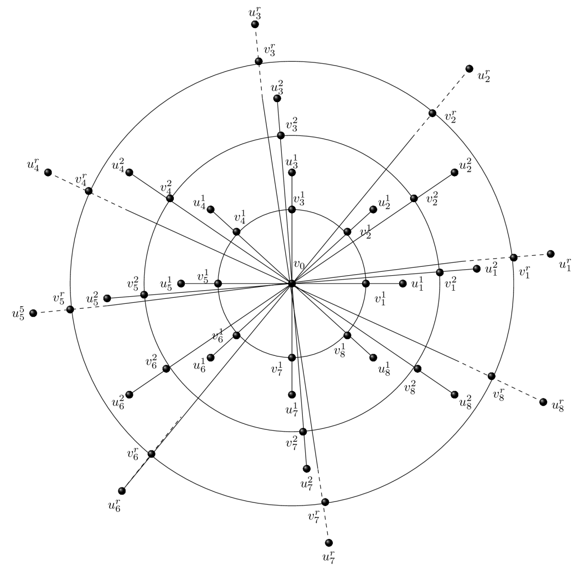

FIGURE 2. **A network of $H_{(8,r)}$.**

Now let a r-level wheel network which is obtained by the union of r isomorphic wheel networks, let S_i, $1 \leq i \leq r$, where S_i contains a fault-tolerant metric basis of G_i and $S_1 \cap S_2 \cap S_3 \cap \cdots \cap S_r = \emptyset$. A resolving set $B'_r = \{S_i, 1 \leq i \leq r\}$ of a network G is referred to as a fault-tolerant resolving set with r vertices if $B'_r \setminus \{v^i, 1 \leq i \leq r\}$ is also a resolving set for each vertex $v \in B'_r$.

Let a network G, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ are the vertices and $B' = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ be the fault-tolerant metric basis of G. Then the vertices $v_{i_x}, v_{i_{x+1}}$ for $1 \leq x \leq r$, where $v_{i_1} = v_{i_r+1}$ are referred to as adjacent vertices. The vertices in between these adjacent vertices in G are referred to as gap of fault-tolerant metric basis of B'. Now the number of vertices which a gap contains will be the cardinality of that gap. Cardinalities of these gaps are represented as $b_1, b_2, \ldots b_n$, where "b" is a whole number.For gaps b_i, b_{i+1}, b_{i+2}, the gap b_{i+1} is called the central gap while b_i and b_{i+1} are called adjacent gaps. Further the cardinalities of the gaps of fault-tolerant metric basis are referred to as maximum cardinalities. The central vertex v_0 is not included in gaps.

In Fig. 1, the vertices and fault-tolerant metric basis of $W_{(8,2)}$ are $V(G) = \{v_0^i, v_1^i, v_2^i, \ldots, v_8^i : 1 \leq i \leq 2\}$ and $B'_2 = \{v_1^1, v_3^1, v_5^1, v_7^1, v_1^2, v_3^2, v_5^2, v_7^2\}$ respectively. Now the pairs of adjacent vertices in B'_2 are (v_1^1, v_3^1), (v_3^1, v_7^1), (v_7^1, v_1^1), (v_1^2, v_3^2), (v_3^2, v_7^2), (v_7^2, v_1^2) and the vertices in gaps in $W_{(8,2)}$

are (v_2^1), (v_4^1), (v_6^1), (v_8^1), (v_2^2), (v_4^2), (v_6^2), (v_8^2). Maximum cardinalities of above gaps are 1, 1, 1, 1, 1, 1, 1, 1. The network $W_{(8,2)}$ is formed by the union of two isomorphic networks in which fault-tolerant metric dimension of $W_{(8,1)}$ is 4. Further, $\beta'(W_{(8,2)}) = 2 \cdot 4 = 8$.

In $H_{(n,r)}$ (Fig. 2), we represent the pendant vertices by u_i^j and the vertices of cycles by v_i^j, where "$1 \leq i \leq n$" and "$1 \leq j \leq r$". The central vertex is represented by v_0. We observe two types of gaps the gaps between vertices of cycles and pendent vertices. We say that if u_i^j is a pendent vertex then v_i^j will be its corresponding cycle vertex. Now we observe that if we choose some pendent vertices then we can not choose their corresponding cycle vertices for B'_r. Further, if u_i^t and v_i^{t+2} belongs to B'_r then the gap between u_i^t and v_{i+2}^t will be 1.

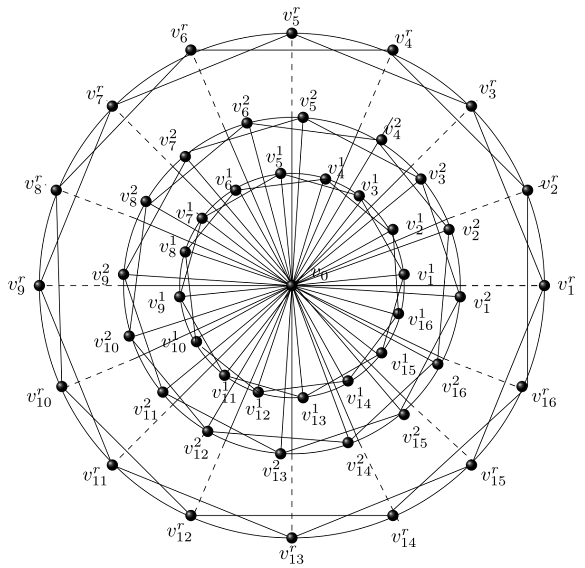

FIGURE 3. A network of $AWW_{(16,r)}$.

In $AWW_{(n,r)}$ (Fig. 3), let v_0 be the central vertex. We observed that the distance of the central vertex v_0 from all the vertices of the anti-web wheel network is 1. It follows that v_0 does not belong to the fault-tolerant metric basis of the anti-web wheel network. The cardinalities of the gaps of a cycle C_r are represented by $a_1^r, a_2^r, a_3^r \ldots a_j^r$ respectively, where $a_1, a_2, a_3, \ldots a_j$ are cardinalities of the gaps of cycle r of $AWW_{(n,r)}$. The vertices of a cycle C_r are represented by $v_1^r, v_2^r, v_3^r \ldots v_n^r$.

We refer the vertices of $AWJ_{(2n,r)}$ which have degree 4 and 5 with even numbering and odd numbering respectively (Fig. 4). In $AWJ_{(2n,r)}$, we observed that the distance of the central vertex v_0 from all the vertices of the anti-web gear network is either 1 or 2. It follows that v_0 does not belong to fault-tolerant metric basis of anti-web gear network except $AWJ_{(4,r)}$. The cardinalities of the gaps of a cycle C_r are represented by $a_1^r, a_2^r, a_3^r \ldots a_j^r$ respectively, where $a_1, a_2, a_3, \ldots a_j$ are cardinalities of the gaps of cycle r

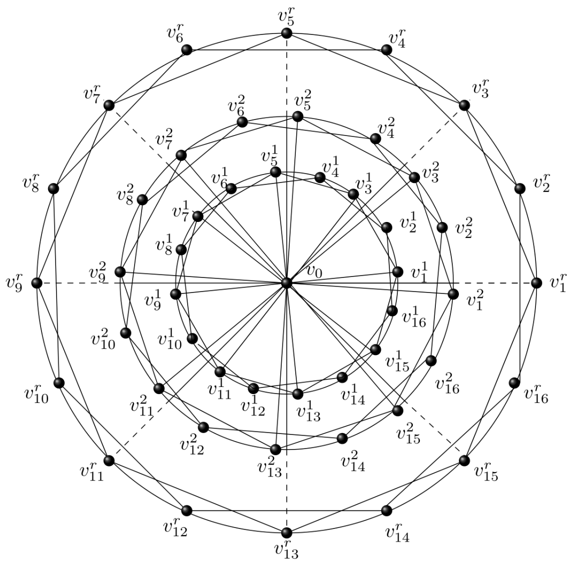

FIGURE 4. A network of $AWJ_{(16,r)}$.

of $AWJ_{(2n,r)}$. The vertices of a cycle C_r are represented by $v_1^r, v_2^r, v_3^r \ldots v_n^r$.

III. MAIN RESULTS

In this section, we compute the fault-tolerant metric dimension of r-level anti-web wheel, gear and Helm networks, where $r \geq 2$.

A. FAULT-TOLERANT METRIC DIMENSION OF r-LEVEL HELM NETWORK $H_{(n,r)}$

In this subsection of main results, we investigated the fault-tolerant resolving sets and fault-tolerant metric dimensions of r-level Helm network $H_{(n,r)}$. The fault-tolerant metric dimension of the r-level Helm network is represented by $\beta'(H_{(n,r)})$. Now we observe the following lemmas and a theorem for $n \geq 7$.

Lemma 1: For $n \geq 7$, each gap of B'_r of $H_{(n,r)}$ contains one vertex.

Proof: We observe the following cases.

Case 1 Suppose that there exists a gap in $H_{(n,r)}$ which contains 2 vertices and its adjacent gaps contain 1 and 1 vertex (maximum cardinalities are 1, 2, 1). Then there exists vertices v_i^j, v_{i+1}^j, v_{i+2}^j, v_{i+3}^j, v_{i+4}^j, v_{i+5}^j, v_{i+6}^j, v_{i+7}^j, v_{i+8}^j, where "$1 \leq j \leq r$", such that v_i^j, v_{i+2}^j, v_{i+5}^j & v_{i+7}^j belongs to B'_r. Then by $B'_r \backslash \left\{v_{i+7}^j\right\}$, we have $r(v_{i+4}^j/B'_r) = r(v_{i+6}^j/B'_r) = (3, 3, 2, 3, 3, 3, \ldots, 3)$, which is a contradiction.

Case 2 Suppose that there exists a gap in pendant vertices which contains 2 vertices and its adjacent gaps contain 1 and 1 vertex (maximum cardinalities are 1, 2, 1). Then there exists vertices u_i^j, u_{i+1}^j, u_{i+2}^j, u_{i+3}^j, u_{i+4}^j, u_{i+5}^j, u_{i+6}^j, u_{i+7}^j, u_{i+8}^j, where "$1 \leq j \leq r$", such that u_i^j, u_{i+2}^j, u_{i+5}^j & u_{i+7}^j belongs to B'_r. Then by $B'_r \backslash \left\{u_{i+7}^j\right\}$, we have

$r(u^j_{i+4}/B'_r) = r(v^j_{i+6}/B'_r) = (2, 2, 1, 2, 2, 2, \ldots, 2)$, which is a contradiction.

Case 3 Suppose that there exists a gap in between pendant vertices and cycle vertices which contains 2 vertices and its adjacent gaps contain 1 and 1 vertex (maximum cardinalities are 1, 2, 1). Then there exists vertices $v^j_i, v^j_{i+1}, v^j_{i+2}, v^j_{i+3}, v^j_{i+4}, u^j_{i+5}, u^j_{i+6}, u^j_{i+7}, u^j_{i+8}$, where "$1 \leq j \leq r$", such that $v^j_i, v^j_{i+2}, u^j_{i+5}$ & u^j_{i+7} belongs to B'_r. Then by $B'_r \backslash \left\{v^j_{i+7}\right\}$, we have $r(v^j_{i+4}/B'_r) = r(v^j_{i+6}/B'_r) = (2, 2, 2, 2, 2, 2, \ldots, 2)$, which is a contradiction.

Lemma 2: If $n \geq 7$ and n is even, then $B'_r = \{v^i_1, v^i_3, v^i_5, \ldots, v^i_{n-1} : 1 \leq i \leq r\}$ or $B'_r = \{u^i_1, u^i_3, u^i_5, \ldots, u^i_{n-1} : 1 \leq i \leq r\}$ or $B'_r = \{v^i_1, v^i_3, v^i_5, \ldots, v_l, u_{l+2}, \ldots, u^i_{n-1} : 1 \leq i \leq r\}$.

Proof: By Lemma 1 the maximum cardinality of each gap of B'_r of $H_{(n,r)}$ is exactly 1. Therefore $B'_r = \{v^i_1, v^i_3, v^i_5, \ldots, v^i_{n-1} : 1 \leq i \leq r\}$ or $B'_r = \{u^i_1, u^i_3, u^i_5, \ldots, u^i_{n-1} : 1 \leq i \leq r\}$ or $B'_r = \{v^i_1, v^i_3, v^i_5, \ldots, v_l, u_{l+2}, \ldots, u^i_{n-1} : 1 \leq i \leq r\}$ is a fault-tolerant metric basis because it satisfies Lemma 1.

Lemma 3: If $n \geq 7$ and n is odd, then $B'_r = \{v^i_1, v^i_3, v^i_5, \ldots, v^i_n : 1 \leq i \leq r\}$ or $B'_r = \{u^i_1, u^i_3, u^i_5, \ldots, u^i_n : 1 \leq i \leq r\}$ or $B'_r = \{v^i_1, v^i_3, v^i_5, \ldots, v_l, u_{l+2}, \ldots, u^i_n : 1 \leq i \leq r\}$.

Proof: By Lemma 1 the maximum cardinality of each gap of B'_p of $H_{(n,r)}$ is exactly 1. Therefore $B'_r = \{v^i_1, v^i_3, v^i_5, \ldots, v^i_n : 1 \leq i \leq r\}$ or $B'_r = \{u^i_1, u^i_3, u^i_5, \ldots, u^i_n : 1 \leq i \leq r\}$ or $B'_r = \{v^i_1, v^i_3, v^i_5, \ldots, v_l, u_{l+2}, \ldots, u^i_n : 1 \leq i \leq r\}$; is a fault-tolerant metric basis because it satisfies Lemma 1.

Theorem 1: If $n \geq 7$, then $\beta'(H_{(n,r)}) = r[\frac{n}{4}]$.

Proof: We observe that $\beta'(H_{(3,r)}) = \beta'(H_{(6,r)}) = 4r$, where $B'_r = \{v^i_1, v^i_2, v^i_3, v^i_4 : 1 \leq i \leq r\}$ and $\beta'(H_{(4,r)}) = \beta'(H_{(5,r)}) = 3r$, where $B'_r = \{u^i_1, u^i_2, u^i_3 : 1 \leq i \leq r\}$.

Now for $n \geq 7$, we observe the following two cases.

Case 1 If $n \geq 7$ and n is even, then by Lemma 2 $B'_r = \{v^i_1, v^i_3, v^i_5, \ldots, v^i_{n-1} : 1 \leq i \leq r\}$ or $B'_r = \{u^i_1, u^i_3, u^i_5, \ldots, u^i_{n-1} : 1 \leq i \leq r\}$ or $B'_r = \{v^i_1, v^i_3, v^i_5, \ldots, v_l, u_{l+2}, \ldots, u^i_{n-1} : 1 \leq i \leq r\}$. Therefore $\beta'(H_{(n,r)}) = r[\frac{n}{4}]$.

Case 2 If $n \geq 7$ and n is odd, then by Lemma 3 $B'_r = \{v^i_1, v^i_3, v^i_5, \ldots, v^i_n : 1 \leq i \leq r\}$ or $B'_r = \{u^i_1, u^i_3, u^i_5, \ldots, u^i_n : 1 \leq i \leq r\}$ or $B'_r = \{v^i_1, v^i_3, v^i_5, \ldots, v_l, u_{l+2}, \ldots, u^i_n : 1 \leq i \leq r\}$. Therefore $\beta'(H_{(n,r)}) = r[\frac{n}{4}]$.

So from above discussion, we conclude that $\beta'(H_{(n,r)}) \leq r[\frac{n}{4}]$.

Now we shall prove that $\beta'(H_{(n,r)}) \geq r[\frac{n}{4}]$.

Suppose we are taking only the vertices of cycles in B'_r then the pendant vertices will be considered as maximum cardinalities. Again we observe the following two cases.

Case a Suppose $n \geq 7$ and n is even, let $\mid B'_r \mid= rm$, then there will be m vertices in fault-tolerant metric basis in each isomorphic network H_n. Hence there exists m gaps in each cycle of $H_{(n,r)}$ and all gaps have maximum cardinalities 1. Let S_i ($1 \leq i \leq r$) represent the sum of maximum cardinalities of a H_n. If S denote the sum of maximum cardinalities of all gaps of $H_{(n,r)}$, then

$$S = \sum_{i=1}^{n} S_i$$

Or $\quad S = S_1 + S_2 + S_3 + \cdots + S_r$

Where $\quad S_1 = S_2 = S_3 = \cdots = S_r$.

So $\quad S = rS_1$

Or $\quad S = r[\frac{n}{4} + \frac{n}{2}]$

Or $\quad S = r[\frac{3n}{4}]$

So $\quad \beta'(H_{(n,r)}) \geq r[\frac{n}{4}]$.

Case b Suppose $n \geq 7$ and n is odd, let $\mid B'_r \mid= rm$, then there will be m vertices in fault-tolerant metric basis in each isomorphic network H_n. Hence there exists m gaps in each cycle of $H_{(n,r)}$ and all gaps have maximum cardinalities 1 except one gap which is empty. Let S_i ($1 \leq i \leq r$) represent the sum of maximum cardinalities of a H_n. If S denote the sum of maximum cardinalities of all gaps of $H_{(n,r)}$, then

$$S = \sum_{i=1}^{r} S_i$$

Or $\quad S = S_1 + S_2 + S_3 + \cdots + S_r$

Where $\quad S_1 = S_2 = S_3 = \cdots = S_r$.

So $\quad S = rS_1$

Or $\quad S = r[\frac{n-1}{4} + \frac{n}{2}]$

Or $\quad S = r[\frac{3n-1}{4}]$

Or $\quad \beta'(H_{(n,r)}) \geq r[\frac{n}{4}]$

Hence $\beta'(H_{(n,r)}) = r[\frac{n}{4}]$.

B. FAULT-TOLERANT METRIC DIMENSION OF r-LEVEL ANTI-WEB WHEEL NETWORK $AWW_{(n,r)}$

In this subsection, we investigated the fault-tolerant resolving sets and fault-tolerant metric dimensions of the r-level anti-web wheel network. The fault-tolerant metric dimension of r-level anti-web wheel network is represented by $\beta'(AWW_{(n,r)})$. Now we observe the following lemmas and a theorem for $n \geq 8$.

Lemma 4: For $n \geq 8$, if a gap of cardinality 2 exists, then both of its adjacent gaps of B'_r of $AWW_{(n,r)}$ can not have cardinality 1.

Proof: Suppose that there exists a gap which contains 2 vertices and its adjacent gaps contain 1 and 1 vertex (maximum cardinalities are 1, 2, 1). Then there exists vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v_{i+7}, v_{i+8}$, such that v_i, v_{i+2}, v_{i+5} & v_{i+7} belongs to B'_r. Then by $B'_r \setminus \{v_{i+2}\}$, we have $r(v_{i+3}/B'_r) = r(v_{i+4}/B'_r) = (2, 1, 2, \ldots, 2)$, which is a contradiction.

Lemma 5: If $n \geq 8$, then the maximum cardinalities of B'_r of $AWW_{(n,r)}$ are $1^i, 1^i, 1^i, \ldots, 1^i : 1 \leq i \leq r$.

Proof: We are looking for gaps with maximum cardinalities, by lemma 4 if a gap of cardinality 2 exists, then both of its adjacent gaps can not have cardinality 1. Therefore maximum cardinality of each gap of B'_r of $AWW_{(n,r)}$ is exactly 1. Hence the maximum cardinalities of B'_r of $AWW_{(n,r)}$ are $1^i, 1^i, 1^i, \ldots, 1^i : 1 \leq i \leq r$.

Lemma 6: If $n \geq 8$, then $B'_r = \{v^i_1, v^i_3, v^i_5, \ldots\ldots v^i_{n-1} : 1 \leq i \leq r\}$

Proof: By Lemma 5 the maximum cardinality of each gap of B'_r of $AWW_{(n,r)}$ is exactly 1. Therefore, we have $B'_r = \{v^i_1, v^i_3, v^i_5, \ldots\ldots v^i_{n-1} : 1 \leq i \leq r\}$ because it satisfies Lemma 4 and 5.

Theorem 2: If $n \geq 8$, then $\beta'(AWW_{(n,r)}) = r[\frac{n}{2}]$.

Proof: We observe that $\beta'(AWW_{(4,r)}) = 4r + 1$, where $B'_r = \{v_0, v^i_1, v^i_2, v^i_3, v^i_4 : 1 \leq i \leq r\}$ and $\beta'(AWW_{(6,r)}) = 6r$, where $B'_r = \{v^i_1, v^i_2, v^i_3, v^i_4, v^i_5, v^i_6 : 1 \leq i \leq r\}$.

Now for $n \geq 8$, by Lemma 6, we have $B'_r = \{v^i_1, v^i_3, v^i_5, \ldots\ldots v^i_{n-1} : 1 \leq i \leq r\}$. Therefore $\beta'(AWW_{(n,r)}) = r[\frac{n}{2}]$. So from the above discussion, we conclude that $\beta'(AWW_{(n,r)}) \leq r[\frac{n}{2}]$.

Now we shall prove that $\beta'(AWW_{(n,r)}) \geq r[\frac{n}{2}]$.

For this purpose, $n \geq 8$, let $|\ B'_r\ | = rm$, then there will be m vertices in fault-tolerant metric basis in each cycle C_r. Hence there exists m gaps in each cycle C_r and all gaps have maximum cardinalities 1. If S denote the sum of maximum cardinalities of all gaps of $AWW_{(n,r)}$, then

$$S = \sum_{i=1}^{r} S_i)$$

$$\text{Or} \quad S = S_1 + S_2 + S_3 + \cdots + S_r$$

$$\text{Where} \quad S_1 = S_2 = S_3 = \cdots = S_r.$$

$$\text{So} \quad S = rS_1$$

$$\text{Or} \quad \beta'(AWW_{(n,r)}) = r[\frac{n}{2}] \geq r[\frac{n}{2}]$$

$$\text{Hence} \quad \beta'(AWW_{(n,r)}) = r[\frac{n}{2}].$$

C. FAULT-TOLERANT METRIC DIMENSION OF r-LEVEL ANTI-WEB GEAR NETWORK $AWJ_{(2n,r)}$

In this subsection, we investigated the fault-tolerant resolving sets and fault-tolerant metric dimensions of the r-level anti-web gear network. The fault-tolerant metric dimension of $AWJ_{(2n,r)}$ is represented by $\beta'(AWJ_{(2n,r)})$. Now we observe the following lemmas and a theorem for $n \geq 9, r \geq 2$.

Lemma 7: If $n \geq 9$, then the gaps with maximum cardinalities 3, 4, 3 if exists, then that will be only 1 in $AWJ_{(2n,r)}$.

Proof: We observe the following cases.

Case 1 suppose that there exists gaps with maximum cardinalities 3, 4, 3 in $AWJ_{(2n,r)}$. Then there exists vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v_{i+7}, v_{i+8}, v_{i+9}, v_{i+10}, v_{i+11}, v_{i+12}, v_{i+13}$, where v_i has degree 4, such that $v_i, v_{i+4}, v_{i+9}, v_{i+13}$, belongs to B'_r. Then by $B'_r \setminus \{v_{i+4}\}$, we have $r(v_{i+3}/B'_r) = r(v_{i+5}/B'_r) = (1, 2, 2, 2 \text{ or } 3, 2 \text{ or } 3, 2 \text{ or } 3, \ldots, 2 \text{ or } 3)$, which is a contradiction.

Case 2 suppose that there exists gaps with maximum cardinalities 3, 4, 3, 4, 3 in $AWJ_{(2n,r)}$. Then there exists vertices $v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+17}, v_{i+18}$, where v_i has degree 5, such that $v_i, v_{i+4}, v_{i+9}, v_{i+13}, v_{i+18}, v_{i+22}$, belongs to B'_r. Then by $B'_r \setminus \{v_{i+14}\}$, we have $r(v_{i+13}/B'_r) = r(v_{i+15}/B'_r) = (2, 2, 1, 2 \text{ or } 3, 2 \text{ or } 3, 2 \text{ or } 3, \ldots, 2 \text{ or } 3)$, which is a contradiction.

Lemma 8: If $n \geq 9$, $2n = 4h + 14$, where $h \geq 1$, then $B'_r = \{v^i_1, v^i_5, v^i_9, \ldots\ldots, v^i_{n-1} : 1 \leq i \leq r\}$.

Proof: By Lemma 7 the maximum cardinality of each gap of B'_r of $AWJ_{(2n,r)}$ can not be greater than 3. Therefore $B'_r = \{v^i_1, v^i_5, v^i_9, \ldots\ldots v^i_{n-1} : 1 \leq i \leq r\}$ is a fault-tolerant resolving set because it satisfies Lemma 7.

Lemma 9: If $n \geq 9$, $2n = 4h + 16$, where $h \geq 1$, then $B'_r = \{v^i_1, v^i_5, v^i_9, \ldots\ldots, v^i_{n-3} : 1 \leq i \leq r\}$.

Proof: By Lemma 7 the maximum cardinality of each gap of B'_r of $AWJ_{(2n,r)}$ can not be greater than 3. Therefore $B'_r = \{v^i_1, v^i_5, v^i_9, \ldots\ldots v^i_{n-3} : 1 \leq i \leq r\}$ is a fault-tolerant resolving set because it satisfies Lemma 7.

Lemma 10: If $n \geq 9$, $2n = 4h + 14$, where $h \geq 1$, then the maximum cardinalities of B'_r of $AWJ_{(2n,r)}$ are $3^i, 3^i, 3^i, \ldots, 3^i, 1^i$, where "$1 \leq i \leq r$".

Proof: We are looking for gaps with maximum cardinalities, by lemma 7 the maximum cardinality of a gap can not be greater than 3. Therefore maximum cardinality of each gap of B'_r of $AWJ_{(2n,r)}$ is exactly 3 except one gap which has 1. Hence the maximum cardinalities of B'_r of $AWJ_{(2n,r)}$ are $3^i, 3^i, 3^i, \ldots, 3^i, 1^i$, where "$1 \leq i \leq r$".

Lemma 11: If $n \geq 9$, $2n = 4h + 16$, where $h \geq 1$, then the maximum cardinalities of B'_r of $AWJ_{(2n,r)}$ are $3^i, 3^i, 3^i, \ldots, 3^i, 3^i$, where "$1 \leq i \leq r$".

Proof: We are looking for gaps with maximum cardinalities, by lemma 7 the maximum cardinality of a gap can not be greater than 3. Therefore maximum cardinality of each gap of B'_r of $AWJ_{(2n,r)}$ is exactly 3. Hence the maximum cardinalities of B'_r of $AWJ_{(2n,r)}$ are $3^i, 3^i, 3^i, \ldots, 3^i, 3^i$, where "$1 \leq i \leq r$".

Theorem 3: If $n \geq 9$, then $\beta'(AWJ_{(2n,r)}) = r\lceil\frac{n}{2}\rceil$.

Proof: We observe that $\beta'(AWJ_{(4,r)}) = 4r$, $\beta'(AWJ_{(6,r)}) = 5r$, $\beta'(AWJ_{(8,r)}) = 4r$, $\beta'(AWJ_{(10,r)}) = 5r$, $\beta'(AWJ_{(12,r)}) = 6r$, $\beta'(AWJ_{(14,r)}) = 7r$ and $\beta'(AWJ_{(16,r)}) = 8r$, where $B'_r = \{v^i_1, v^i_2, v^i_3, v^i_4 : 1 \leq i \leq r\}$, $B'_r = \{v^i_1, v^i_2, v^i_3, v^i_4, v^i_5 : 1 \leq i \leq r\}$, $B'_r = \{v^i_1, v^i_3, v^i_5, v^i_7 : 1 \leq i \leq r\}$, $B'_r = \{v^i_1, v^i_3, v^i_5, v^i_7, v^i_9 : 1 \leq i \leq r\}$, $B'_r = \{v^i_1, v^i_3, v^i_5, v^i_7, v^i_9, v^i_{11} : 1 \leq i \leq r\}$, $B'_r = \{v^i_1, v^i_3, v^i_5, v^i_7, v^i_9, v^i_{11}, v^i_{13} : 1 \leq i \leq r\}$ and

$B'_r = \{v_1^i, v_3^i, v_5^i, v_7^i, v_9^i, v_{11}^i, v_{13}^i, v_{15}^i : 1 \leq i \leq r\}$ are their fault-tolerant metric basis respectively.

Now for $n \geq 9$, we observe the following two cases.

Case 1 For $n \geq 9, 2n = 4h+14$, where $h \geq 1$, by Lemma 8, we have $B' = \{v_1^i, v_5^i, v_9^i, \ldots\ldots, v_{n-1}^i : 1 \leq i \leq r\}$. Hence $\beta'(AWJ_{(2n,r)}) = r\lceil\frac{n}{2}\rceil$.

Case 2 For $n \geq 9, 2n = 4h+16$, where $h \geq 1$, by Lemma 9, we have $B' = \{v_1^i, v_5^i, v_9^i, \ldots\ldots, v_{n-3}^i : 1 \leq i \leq r\}$. Hence $\beta'(AWJ_{(2n,r)}) = r\lceil\frac{n}{2}\rceil$.

So from above two cases, we have $\beta'(AWJ_{(2n,r)}) \leq r\lceil\frac{n}{2}\rceil$.

Now we shall prove that $\beta'(AWJ_{(2n,r)}) \geq r\lceil\frac{n}{2}\rceil$.

For this purpose, again we observe the following two cases.

Case a For $n \geq 9$, $2n = 4h + 14$, where $h \geq 1$, For $n \geq 9$, let $| B'_r | = rm$, then there will be m vertices in fault-tolerant metric basis in each cycle C_r. Hence there exists m gaps in each cycle C_r and all gaps have maximum cardinalities 3 except one gap which has 1 (See Lemma 10). If S denote the sum of maximum cardinalities of all gaps of $AWJ_{(2n,r)}$, then

$$S = \sum_{i=1}^{n} S_i$$

Or $\quad S = S_1 + S_2 + S_3 + \cdots + S_r$

Where $\quad S_1 = S_2 = S_3 = \cdots = S_r.$

So $\quad S = rS_1$

Or $\quad S = r[(m-1)3 + 1]$

Or $\quad \beta'(AWJ_{(2n,r)}) = r[3m-2] \geq r[\frac{n}{2}].$

Case b For $n \geq 9$, $2n = 4h + 16$, where $h \geq 1$, For $n \geq 9$, let $| B'_r | = rm$, then there will be m vertices in fault-tolerant metric basis in each cycle C_r. Hence there exists m gaps in each cycle C_r and all gaps have maximum cardinalities 3 (See Lemma 11). If S denote the sum of maximum cardinalities of all gaps of $AWJ_{(2n,r)}$, then

$$S = \sum_{i=1}^{r} S_i$$

Or $\quad S = S_1 + S_2 + S_3 + \cdots + S_r$

Where $\quad S_1 = S_2 = S_3 = \cdots = S_r.$

So $\quad S = rS_1$

Or $\quad S = r[3m]$

Or $\quad \beta'(AWJ_{(2n,r)}) = 3rm \geq r[\frac{n}{2}]$

So from above two cases, we obtain $\beta'(AWJ_{(2n,r)}) \geq r\lceil\frac{n}{2}\rceil$

Hence $\quad \beta'(AWJ_{(2n,r)}) = r\lceil\frac{n}{2}\rceil.$

IV. CONCLUSION AND DISCUSSION

Afzal *et al.* [24] computed the metric dimension of $AWW_{(n,r)}$ and $AWJ_{(2n,r)}$ for $n \geq 9$. They obtained the following two results.

- $\beta(AWW_{(n,r)}) = r[\frac{n+2}{3}]$
- $\beta(AWJ_{(2n,r)}) = [\frac{n+1}{3}] + r[\frac{n+4}{3}]$

In this paper, the following three theorems and two inequalities are obtained for $n \geq 9$.

1) $\beta'(H_{(n,r)}) = r[\frac{n}{4}]$
2) $\beta'(AWW_{(n,r)}) = r[\frac{n}{2}]$
3) $\beta'(AWJ_{(2n,r)}) = r\lceil\frac{n}{2}\rceil$
4) $\beta'(AWW_{(n,r)}) \geq \beta'(H_{(n,r)})$.
5) $\beta'(AWJ_{(2n,r)}) \geq \beta'(H_{(n,r)})$.

From above results, we observe that the difference between metric and fault- tolerant metric dimension of $AWW_{(n,r)}$ and $AWJ_{(2n,r)}$ increases by increasing the value of n. Further from inequality 4 and 5, we see that if machines (robots) are navigating on r-level anti-web wheel, gear or Helm networks and the objective is to find a fault-tolerant system which require minimum machines then r-level Helm network is suitable.

In this paper, we concluded that if we take union of r wheel related isomorphic networks in which central vertex is common, then the fault-tolerant metric dimension of the whole network will be equal to r times the any one isomorphic network. In this study, we concluded that if we have to find the fault-tolerant metric dimension of any one of the aforesaid r-level wheel related networks then find the fault-tolerant metric dimension for $r = 1$ and multiply it with the number which is equal to total number of isomorphic networks you will find the fault-tolerant metric dimension of r- level wheel related network. Further, Afzal *et al.* [24] claimed that r- level anti-web wheel and anti-web gear networks have unbounded fault-tolerant metric dimension and in this paper we obtained the fault-tolerant metric dimensions of these networks and Helm network in the form of different algebraic expressions consisting of the integral numbers n and r. We proved that these networks have unbounded fault-tolerant metric dimensions.

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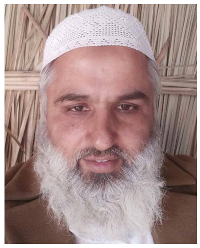

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