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RESEARCH ARTICLE

Complex q-Rung Orthopair Fuzzy Aczel–Alsina Aggregation Operators and Its Application to Multiple Criteria Decision-Making With Unknown Weight Information

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ABSTRACT In decision making problems, complex q-rung orthopair fuzzy set is regarded as a more practical tool than complex intuitionistic fuzzy set and q-rung orthopair fuzzy set. This paper proposes several aggregation operators based on the Aczel-Alsina t-norm and t-conorm for aggregating complex q-rung orthopair fuzzy data. The suggested operators are then used to establish a multiple criteria decision-making (MCDM) method. The Aczel-Alsina operations t-norm and t-conorm can have the advantage of good flexibility with the operational parameter. In this regard, we expand the notions of the Aczel-Alsina t-norm and t-conorm to the complex q-rung orthopair fuzzy environment and provide certain aggregation operators in this study. Furthermore, we show the compatible features of the suggested operators. To overcome the defects of the existing entropy measures, a novel complex q-rung orthopair fuzzy entropy approach is put forward for acquiring the unknown criteria weights objectively. Following that, we describe an MCDM technique with unknown criterion weights in a complex q-rung orthopair fuzzy environment grounded on the originated operators. Then, to demonstrate the model's flexibility and validity, we analyze and solve a problem concerning with the selection of the sector that had the most impact on the Pakistan Stock Exchange. Subsequently, we demonstrate how the parameter's inclusion in our proposed model influences decision-making outcomes. At last, the generated outcomes are compared to the past approaches to demonstrate our suggested technique's superiority.

INDEX TERMS Aczel–Alsina t-norms, complex q-rung orthopair fuzzy set, complex q-rung orthopair fuzzy Aczel–Alsina aggregation operators, entropy measure, MCDM.

I. INTRODUCTION

The technique of multi-criteria decision-making (MCDM) is a skilled way to handle complex and challenging data in real-world scenarios. MCDM is a technique that can produce ranking grades for finite alternatives based on the distinctive objects of different options; it is a crucial component of the decision making sciences [1], [2], [3], [4], [5], [6]. In daily life decision making scenarios, when analyzing data, uncertainty and incompleteness are always an issue. According to the idea of crisp sets, an item either belongs to or does not belong to a certain class. But in real world, many events cannot be represented on such a scale. For that point, the doctrine of fuzzy set (FS) was familiarized by Zadeh [7], in which only the membership grade is constrained inside the unit interval. The FS theory has gotten a great deal of attention from notable scholars and has been implemented in several scenarios and areas. However, the FS idea has often failed to operate accurately. For instance, it is challenging to use FS to get information in the form of membership and non-membership

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grades. Atanassov [8] propounded the idea of intuitionistic FS (IFS), which is a modified version of the FS to successfully handle awkward and inaccurate information, to address such issues. IFS covers the grades of membership μ and non-membership ν with the rule $0 \leq \mu^1 + \nu^1 \leq 1$. The IFS theory has been the subject of substantial investigation and has been applied in several publications [9], [10], [11]. Nonetheless, the range of IFS data is limited and relies on the unnecessary requirement that the sum of μ and ν should be kept inside the unit interval. To address this issue, Yager devised the idea of Pythagorean fuzzy sets (PyFS), a modified version of IFS for tackling complex and inaccurate information. PyFS covers the grades of membership μ and non-membership ν with the rule $0 \leq \mu^2 + \nu^2 \leq 1$. The topic of PyFS has gotten a great deal of interest from a wide range of researchers [12], [13], [14]. For instance, Ren *et al.* [12] studied the TODIM method in a Pythagorean fuzzy setting to tackle the MCDM problems. Garg [14] introduced new trigonometric sine operating rules for PyFS. Kumar and Kumar [15] addressed a decision making problem for optimal antivirus mask selection under pythagorean fuzzy data. Likewise IFS, the range of PyFS is also narrow and follows a criterion stating that the total of the squares of μ and ν should be contained within the unit interval; however, this is not essential. If certain information's sum of the squares of both grades exceeded the unit interval. For instance, if we assigned a grade of 0.8 for membership and a grade of 0.9 for non-membership, then the PyFS criteria is violated, since $0.8^2 + 0.9^2 = 1.45 > 1$. To circumvent this issue, Yager [16] extended the PyFS to q-rung orthopair fuzzy set (q-ROFS) to better handle complicated and unreliable situations including the aforesaid problems. q-ROFS covers the grades of membership μ and non-membership ν with the rule $0 \leq \mu^q + \nu^q \leq 1$ $(q \geq 1)$. Using q-ROFS, it is simple to solve the given information, as $0.9^5 + 0.8^5 = 0.91817 < 1$. The q-ROFS premise differs significantly from prevalent theories such as IFSs and PFSs. Under these characteristics, eminent scholars have examined and applied a huge study to several disciplines [17], [18], [19]. For example, several types of aggregation operators have been developed by scholars such as Hamacher norm-based [20], neutrality operational laws [21], Heronian mean [19], normalized bidirectional projection [17], exponential operation laws [22], trigonometric operations [23] to solve the decision making problems.

In our real activities, ambiguity and fuzziness in the data coincide with variations in the phase (periodicity) of the data. Thus, the present theories are inadequate to account for this information, resulting in a loss of knowledge. Ramot *et al.* [24] devised the notion of complex fuzzy set (CFS) to circumvent this issue. The idea of CFS has garnered considerable attention from eminent academics and has been the subject of multiple investigations [25], [26], [27]. Since CFS provides no information regarding the disagrees of the object, the approaches based on it are of a restrictive nature. Subsequently, Alkouri and Salleh [28] extended CFS into the complex IFS (CIFS) paradigm. CIFS addresses the grades of membership and non-membership using the rules $0 \leq \mu^1 + \nu^1 \leq 1$ and $0 \leq \left(\eth_\mu\right)^1 + (\eth_\nu)^1 \leq 1$, and has achieved widespread acceptance [23], [29], [30], [31]. Nonetheless, the range of CIFS data is notably limited by the noticeable rule $0 \leq \mu^1 + \nu^1 \leq 1$ and $0 \leq \left(\eth_\mu\right)^1 + (\eth_\nu)^1 \leq 1$, which is unnecessary. Ullah *et al.* [32] proposed the notion of complex PyFS (CPyFS) to handle unreliable and awkward information more efficiently than CIFS in order to overcome such issues. The CPyFS rules for the membership and non-membership grades are $0 \leq \mu^2 + \nu^2 \leq 1$ and $0 \leq \left(\eth_\mu\right)^2 + (\eth_\nu)^2 \leq 1$. But, the range of CPyFS data is still limited and constrained by rule $0 \leq \mu^2 + \nu^2 \leq 1$ and $0 \leq \left(\eth_\mu\right)^2 + (\eth_\nu)^2 \leq 1$. However, it his is not essential, if a person gives information in which the sum of the squares of the real parts (and imaginary parts) of both grades exceeds the unit interval. For instance, if we take $0.9\mathbf{e}^{i2\pi(0.8)}$ for membership grade and $0.8\mathbf{e}^{i2\pi(0.9)}$ for non-membership grade, then by utilizing the constraints of the CIFS and CPyFS, $0.9^2 + 0.8^2 = 1.45 > 1$ and $0.8^2 + 0.9^2 = 1.45 > 1$ indicate that the CIFS and complex CPyFS have failed. The authors of Reference [33] proposed the doctrine of complex q-ROFS (Cq-ROFS) as an updated tool of CPFS to handle unreliable information in order to address this issue. Cq-ROFS addresses the grades of membership and non-membership according to the rules $0 \leq \mu^q + \nu^q \leq 1$ and $0 \leq \left(\eth_\mu\right)^q + (\eth_\nu)^q \leq 1$. Evidently, Cq-ROFSs can readily solve the aforementioned information, since $0.9^5 + 0.8^5 = 0.91817 < 1$ and $0.8^5 + 0.9^5 = 0.91817 < 1$. Consequently, Cq-ROFS differ significantly from prevalent concepts such as CIFS and CPyFS. For detailed information on Cq-ROFS theory, please refer to the articles [34], [35], [36].

In the literature, many aggregation operators based on t-norm (TN) and t-conorm (TCN) have been developed for aggregating assessments knowledge in a fuzzy environment. Xia *et al.* [37] put forward intuitionistic fuzzy aggregation operators with Archimedean TN and TCN. Wei and Zhao [38] presented induced interval-valued hesitant aggregation operators based on Einstein TN and TCN. Seikh and Mandal [39] introduced an MCDM approach based on the q-rung orthopair fuzzy Frank aggregation operators. Ali *et al.* [40] created weighted interval-valued dual hesitant fuzzy aggregation operators using Archimedean t-norm and t-conorm. The authors of Ref. [20] studied EDAS method based on their designed q-rung orthopair fuzzy Hamacher aggregation operators. Liu *et al.* [41] studied a series of complex q-rung orthopair fuzzy Muirhead mean operators based on Schweizer–Sklar operations. Further, they discussed their relationships and special cases in length. In order to limit the impact of individual preference on decision-making, Liu and Li [42] devised dependent Hamacher aggregation operators for Cq-ROFSs. These TNs and TCNs had played a crucial role in the implementation of FSs in decision-making. Aczel and Alsina [43] initially presented a novel pair of TN and TCN that is more flexible than the previously described TN and TCN. Senapati *et al.* [44], [45] constructed

Aczel-Alsina aggregation operators for a framework of IFSs and also for interval-valued IFSs (IVIFSs), and then used them to address MCDM issues, while Senapati [46] examined Aczel-Alsina aggregation operators based on picture fuzzy sets with application to MCDM. Recently, Hussain *et al.* [47] defined the fundamental T-spherical fuzzy aggregation operators in terms of Aczel-Alsina operations. Simultaneously, Naeem and Ali [48] investigated spherical fuzzy Aczel-Alsina aggregation operators and their application to solar energy cells. To our knowledge, no research has been discovered that combines the concepts of Aczel-Alsina t-norm and t-conorm under the background of Cq-ROFS. By leveraging the benefits of Aczel-Alsina t-norm and t-conorm for offering flexibility in decision-making processes, it would be conceivable to construct an effective tool for managing a higher level of imprecision in MCDM challenges in a complex q-rung orthopair fuzzy environment.

As each criteria possesses some different aspects, all of them cannot be levelled with the same weight. Consequently, determining the proper weight for each criterion is the most important component of an MCDM problem. In most MCDM situations, the weights of the criteria are assumed to be totally known. However, this is not the case with actual MCDM applications. To determine criteria weights for complex q-rung orthopair fuzzy MCDM problems Mahmood and Ali [49] have proposed the entropy measures. But their proposed entropy measures do not consider the degree of hesitation, and also have certain counterintuitive cases as mentioned in Section []. Therefore, it is necessary to construct a new entropy measure which not only consider the degree of hesitation but also have the capability of handling the counterintuitive cases.

Motivated by the above two paragraphs, the main contributions of this study are outlined as:

1. To investigate Aczel-Alsina t-norm and t-conorm operational laws of Cq-ROFS and their characteristics.
2. To develop various complex q-rung orthopair fuzzy aggregation operators in terms of the proposed operations.
3. To introduce a novel complex q-rung orthopair fuzzy entropy measure and its significant characteristics.
4. To build an MCDM approach with Cq-ROFS based on the defined operators with unknown weight information.
5. To demonstrate the applicability and advantages of the presented approach.

This paper's structure is as follows: Section II will go through the fundamental ideas of Cq-ROFSs and Aczel–Alsina triangular norms. Section III describes the Aczel–Alsina operational laws for Cq-ROFNs. Section IV puts forward complex q-rung orthopair fuzzy Aczel–Alsina averaging and geometric operators and proves some of their desirable properties and special cases. In Section V, we give the concept of a new entropy of C-qROFS to overcome the shortcomings of the existing complex q-rung orthopair fuzzy entropies. Section VI builds a decision-making framework for dealing with MCDM problems employing Cq-ROFNs as characteristic values using the provided operators. Section VII includes a case study to show how the proposed technique may be applied. This section also delves at how a parameter affects decision-making outcomes. Section VIII provides a comparative examination of various acceptable approaches to demonstrate the adequacy of the provided technique. Section IX concludes the entire article.

II. SOME BASIC CONCEPTS

In this part, we will present t-norm, t-conorm, Aczel–Alsina t-norm, Aczel–Alsina t-conorms, and some core concepts of q-ROFSs to help readers comprehend the work.

Definition 1 ([50]): A t-norm is a function $T:[0,1]^2 \longrightarrow [0,1]$ that meets

T1. $T(\hbar_1,\hbar_2)=T(\hbar_2,\hbar_1)\ \forall\,\hbar_1,\hbar_2\in[0,1]$;
T2. $T(\hbar_1,\hbar_2)\le T(\hbar_3,\hbar_4)$ if $\hbar_1\le\hbar_3,\hbar_2\le\hbar_4\ \forall\,\hbar_1,\hbar_2,\hbar_3,\hbar_4\in[0,1]$;
T3. $T(\hbar,1)=\hbar\ \forall\,\hbar\in[0,1]$;
T4. $T(\hbar_1,T(\hbar_2,\hbar_3))=T(T(\hbar_1,\hbar_2),\hbar_3)$.

Some examples of t-norms

1). $T_P(\hbar_1,\hbar_2)=\hbar_1\hbar_2$ (product t-norm),
2). $T_M(\hbar_1,\hbar_2)=\min(\hbar_1,\hbar_2)$ (minimum t-norm),
3). $T_L(\hbar_1,\hbar_2)=\max(\hbar_1+\hbar_2-1,0)$ (Lukasiewicz t-norm),
4). $T_D(\hbar_1,\hbar_2)=\begin{cases}\hbar_1, & \text{if } \hbar_2=1\\ \hbar_2, & \text{if } \hbar_1=1\\ 0, & \text{otherwise.}\end{cases}$ (drastic t-norm)
$\forall\,\hbar_1,\hbar_2\in[0,1]$.

Definition 2 ([51]): A t-conorm is a function $S:[0,1]^2\longrightarrow[0,1]$ that meets

S1. $S(\hbar_1,\hbar_2)=S(\hbar_2,\hbar_1)\ \forall\,\hbar_1,\hbar_2\in[0,1]$;
S2. $S(\hbar_1,\hbar_2)\le S(\hbar_3,\hbar_4)$ if $\hbar_1\le\hbar_3,\hbar_2\le\hbar_4\ \forall\,\hbar_1,\hbar_2,\hbar_3,\hbar_4\in[0,1]$;
S3. $S(\hbar,0)=\hbar\ \forall\,\hbar\in[0,1]$;
S4. $S(\hbar_1,S(\hbar_2,\hbar_3))=S(S(\hbar_1,\hbar_2),\hbar_3)$.

Some examples of t-conorms

1). $S_P(\hbar_1,\hbar_2)=\hbar_1+\hbar_2-\hbar_1\hbar_2$ (probabilistic sum),
2). $S_M(\hbar_1,\hbar_2)=\max(\hbar_1,\hbar_2)$ (maximum t-conorm),
3). $S_L(\hbar_1,\hbar_2)=\min(\hbar_1+\hbar_2,1)$ (Lukasiewicz t-conorm),
4). $S_D(\hbar_1,\hbar_2)=\begin{cases}\hbar_1, & \text{if } \hbar_2=0\\ \hbar_2, & \text{if } \hbar_1=0\\ 1, & \text{otherwise.}\end{cases}$ (drastic t-conorm)
$\forall\,\hbar_1,\hbar_2\in[0,1]$.

Additionally, it established the fact [51] that when T is a t-norm and S is a t-conorm, then $T(\hbar_1,\hbar_2)\le\min\{\hbar_1,\hbar_2\}$ and $S(\hbar_1,\hbar_2)\ge\max\{\hbar_1,\hbar_2\}\ \forall\,\hbar_1,\hbar_2\in[0,1]$.

Definition 3 ([43]): The Aczel–Alsina t-norm $\left(T_A^{\zeta}\right)_{\zeta\in[0,\infty]}$ is postulated as (1), as shown at the bottom of the next page.

Some special cases: $T_A^{\infty} = \min$, $T_A^0 = T_D$, $T_A^1 = T_P$.

Definition 4 ([52]): The Aczel–Alsina t-conorm $\left(S_A^{\zeta}\right)_{\zeta\in[0,\infty]}$ is postulated as (2), as shown at the bottom of page 5.

Some special cases: $S_A^{\infty} = \max$, $S_A^0 = S_D$, $S_A^1 = S_P$.

The t-norm T_A^{ζ} and t-conorm S_A^{ζ} are dual with respect to each other $\forall\ \zeta \in [0,\infty]$. Further, T_A^{ζ} and S_A^{ζ} are strictly increasing and strictly decreasing, respectively.

It is worthy to note that the Aszel-Alsina category of t-norms are the only ones that meet the equivalence $T_A^{\zeta}\left(\hbar_1^{\lambda}, \hbar_2^{\lambda}\right) = T_A^{\zeta}\left(\hbar_1, \hbar_2\right)^{\lambda}\ \forall\ \lambda > 0$ and $\hbar_1, \hbar_2 \in [0,1]$.

In this section, we present a concise overview of q-ROFSs.

Definition 5 ([16]): Let X be a fixed set. A q-ROFS $\mathcal{Q}$ on X is described as

$$\mathcal{Q} = \left\{(\hbar, \mu(\hbar), \nu(\hbar))\,|\hbar \in X\right\}, \quad q \geq 1, \tag{3}$$

where $\mu(\hbar)\ \nu(\hbar) \in [0,1]$ denote the membership and non-membership grades of $\hbar \in X$, respectively, accorded that $0 \leq (\mu(\hbar))^q + (\nu(\hbar))^q \leq 1$. The degree of indeterminacy is $(\pi(\hbar))^q = 1 - ((\mu(\hbar))^q + (\nu(\hbar))^q)$. For convince, Yager [16] termed $\mathcal{Q} = (\mu, \nu)$ a q-rung orthopair fuzzy number (q-ROFN).

Definition 6 ([16]): Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be any two q-ROFNs and $\eta > 0$, then the basic rules of operation on them are listed as

1) $\mathcal{Q}_1 \oplus \mathcal{Q}_2 = \left(\left(\mu_1^q + \mu_2^q - \mu_1^q\mu_2^q\right)^{1/q}, \nu_1\nu_2\right)$;
2) $\mathcal{Q}_1 \otimes \mathcal{Q}_2 = \left(\mu_1\mu_2, \left(\nu_1^q + \nu_2^q - \nu_1^q\nu_2^q\right)^{1/q}\right)$;
3) $\mathcal{Q}_1^{\eta} = \left(\mu_{\mathcal{Q}_1}^{\eta}, \left(1 - \left(1 - \nu_1^q\right)^{\eta}\right)^{1/q}\right)$;
4) $\eta\mathcal{Q}_1 = \left(\left(1 - \left(1 - \mu_1^q\right)^{\eta}\right)^{1/q}, \nu_1^{\eta}\right)$;
5) $\mathcal{Q}_1^c = \left(\nu_1, \mu_{\mathcal{Q}_1}\right)$.

Definition 7 ([53]): Let $\mathcal{Q}$ be a q-ROFN, then the score function is characterized by:

$$S\left(\mathcal{Q}\right) = \mu^q - \nu^q, \tag{4}$$

where $q \in [1,\infty)$, $S\left(\mathcal{Q}\right) \in [-1,1]$. The larger the value of $S\left(\mathcal{Q}\right)$, the larger the q-ROFN $\mathcal{Q}$.

Definition 8 ([54]): Let $\mathcal{Q}$ be a q-ROFN, then the degree of accuracy is defined in the following manner:

$$A\left(\mathcal{Q}\right) = (\mu)^q + (\nu)^q; \quad A\left(\mathcal{Q}\right) \in [0,1]. \tag{5}$$

When the computed score values are similar, the larger the degree of accuracy $A\left(\mathcal{Q}\right)$, the larger the q-ROFN.

Definition 9: A Cq-ROFS $\mathbb{Z}$ on a fixed set X is given by

$$\mathbb{Z} = \left\{(\hbar, \ddot{\mu}(\hbar), \ddot{\nu}(\hbar))\,|\hbar \in X\right\}, \quad q \geq 1, \tag{6}$$

where $\ddot{\mu}(\hbar) = \mu(\hbar)\mathbf{e}^{i2\pi\left(\eth_{\mu}\right)}$, $\ddot{\nu}(\hbar) = \nu(\hbar)\mathbf{e}^{i2\pi\left(\eth_{\nu}\right)} \in [0,1]$ symbolizes the complex-valued truth and complex-valued falsity grades of $\hbar \in X$, respectively, accorded that $0 \leq (\mu(\hbar))^q + (\nu(\hbar))^q \leq 1$, $0 \leq \left(\eth_{\mu}\right)^q + \left(\eth_{\nu}\right)^q \leq 1$, where $\mu, \nu, \eth_{\mu}, \eth_{\nu} \in [0,1]$. The degree of hesitation is $\pi\left(\hbar\right)^q = \left(1 - ((\mu(\hbar))^q + (\nu(\hbar))^q)\right)\left(1 - \left(\left(\eth_{\mu}\right)^q + \left(\eth_{\nu}\right)^q\right)\right)$. For convince, we termed $\mathbb{Z} = \left(\mu\mathbf{e}^{i2\pi\left(\eth_{\mu}\right)}, \nu\mathbf{e}^{i2\pi\left(\eth_{\mu}\right)}\right)$ a complex q-rung orthopair fuzzy number (Cq-ROFN).

Definition 10 ([55]): Let $\mathbb{Z}$ be a Cq-ROFN, then the score function is characterized by:

$$S\left(\mathbb{Z}\right) = \frac{1}{4}\cdot\left(2 + \left(\mu^q - \nu^q\right) + \left(\eth_{\mu}^q - \eth_{\nu}^q\right)\right), \tag{7}$$

where $q \in [1,\infty)$, $S\left(\mathbb{Z}\right) \in [-1,1]$. The larger the value of $S\left(\mathbb{Z}\right)$, the larger the Cq-ROFN $\mathcal{Q}$.

Definition 11 ([55]): Let $\mathbb{Z}$ be a Cq-ROFN, then the degree of accuracy is defined in the following manner:

$$A\left(\mathbb{Z}\right) = \mu^q + \nu^q + \eth_{\mu}^q + \eth_{\mu}^q, \tag{8}$$

where $A\left(\mathbb{Z}\right) \in [0,1]$. When the computed score values are similar, the larger the degree of accuracy $A\left(\mathbb{Z}\right)$, the larger the Cq-ROFN.

Definition 12: Let $\mathbb{Z}_1$ and $\mathbb{Z}_2$ be any two Cq-ROFNs and $\eta > 0$, then the basic rules of operation on them are listed as (1)–(5), shown at the bottom of the next page.

III. COMPLEX Q-RUNG ORTHOPAIR FUZZY ACZEL-ALSINA OPERATIONAL LAWS

In view of the Definitions 3 and 4, in what follows, we put forward some generalized operational rules of Cq-ROFNs and their relevant characteristics.

Definition 13: Let $\mathbb{Z}_1$ and $\mathbb{Z}_2$ be any two Cq-ROFNs, $\zeta \geq 1$ and $\lambda > 0$, then the Aczel–Alsina t-norm and t-conorm operations on them are given by (1)–(4), shown at the bottom of page 6.

Theorem 1: Let $\mathbb{Z}$, $\mathbb{Z}_1$ and $\mathbb{Z}_2$ be any three Cq-ROFNs, then we have

1) $\mathbb{Z}_1 \oplus \mathbb{Z}_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_1$;
2) $\mathbb{Z}_1 \otimes \mathbb{Z}_2 = \mathbb{Z}_2 \otimes \mathbb{Z}_1$;
3) $\lambda\left(\mathbb{Z}_1 \oplus \mathbb{Z}_2\right) = \lambda\mathbb{Z}_1 \oplus \lambda\mathbb{Z}_2,\ \lambda > 0$;
4) $\left(\lambda_1 + \lambda_2\right)\mathbb{Z} = \lambda_1\mathbb{Z} \oplus \lambda_2\mathbb{Z},\ \lambda_1, \lambda_2 > 0$;
5) $\left(\mathbb{Z}_1 \otimes \mathbb{Z}_2\right)^{\lambda} = \mathbb{Z}_1^{\lambda} \otimes \mathbb{Z}_2^{\lambda},\ \lambda > 0$;
6) $\mathbb{Z}^{\lambda_1} \otimes \mathbb{Z}^{\lambda_2} = \mathbb{Z}^{(\lambda_1+\lambda_2)},\ \lambda_1, \lambda_2 > 0$.

Proof: See (1)–(6), shown at the bottom of pages 7 and 8. ∎

$$T_A^{\zeta}\left(\hbar_1, \hbar_2\right) = \begin{cases} T_D\left(\hbar_1, \hbar_2\right), & \text{if } \zeta = 0 \\ \min\left\{\hbar_1, \hbar_2\right\}, & \text{if } \zeta = \infty\ \forall\ \hbar_1, \hbar_2 \in [0,1] \\ \exp^{-\left((-\ln\hbar_1)^{\zeta} + (-\ln\hbar_2)^{\zeta}\right)^{1/\zeta}}, & \text{otherwise} \end{cases} \tag{1}$$

IV. COMPLEX Q-RUNG ORTHOPAIR FUZZY ASZEL-ALSINA AGGREGATION OPERATORS

This segment presents various complex q-rung orthopair fuzzy Aszel-Alsina aggregation operators based on the arithmetic average operator and the geometric average operator.

A. COMPLEX Q-RUNG ORTHOPAIR FUZZY ASZEL-ALSINA AVERAGING AGGREGATION OPERATORS

Based on the proposed operations, in this section, we introduce some novel averaging aggregation operators, including complex q-rung orthopair fuzzy Aszel-Alsina average (Cq-ROFAAA) operator, complex q-rung orthopair fuzzy Aczel-Alsina weighted averaging (Cq-ROFAAWA) operator, complex q-rung orthopair fuzzy Aczel–Alsina ordered weighted averaging (Cq-ROFAAOWA) operator, complex q-rung orthopair fuzzy Aczel–Alsina ordered weighted averaging (Cq-ROFAAOWA) operator, and complex q-rung orthopair fuzzy Aczel–Alsina hybrid averaging (Cq-ROFAAHA) operator. In addition, we investigate some special cases and properties of these operators.

Definition 14: Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{i2\pi(\eth_{\mu_\iota})}, \nu_\iota \mathbf{e}^{i2\pi(\eth_{\nu_\iota})}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, then the Cq-ROFAAWA operator is:

$$Cq-ROFAAWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) = \oplus_{\iota=1}^{\flat}\left(w_\iota \mathbb{Z}_\iota\right), \quad (9)$$

where $w = \left(w_1, w_2, \ldots, w_\flat\right)^T$ is the weight vector of $\mathbb{Z}_\iota$ $(\iota = 1, 2, \ldots, \flat)$ such that $w_\iota > 0$ and $\sum_{\iota=1}^{\flat} w_\iota = 1$. Especially, if $w = \left(\frac{1}{\flat}, \frac{1}{\flat}, \ldots, \frac{1}{\flat}\right)^T$, then the Cq-ROFAAWA operator reduces to Cq-ROFAAA operator of dimension $\flat$, which is described as follows:

$$Cq-ROFAAA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) = \frac{1}{\flat}\oplus_{\iota=1}^{\flat}\left(\mathbb{Z}_\iota\right). \quad (10)$$

Theorem 2: Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi\eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi\eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, then the result obtained by utilizing Cq-ROFAAWA operator is still a Cq-ROFN, and (11), as shown at the bottom of page 9.

Proof: We can prove Theorem 2 with the help of the mathematical induction method in the following way:

For $\flat = 2$, we have $Cq-ROFAAWA\,(\mathbb{Z}_1, \mathbb{Z}_2)$, as shown at the bottom of page 9.

Hence, the result is true for $\flat = 2$.

Suppose that Eq. (11) is true for $\flat = k$, then we have $Cq-ROFAAWA\,(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_k)$, as shown at the bottom of page 9.

Now for $\flat = k+1$, we have $Cq-ROFAAWA\big(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_k, \mathbb{Z}_{k+1}\big)$, as shown at the bottom of page 10.

Thus, Eq. (11) is legitimate for $\flat = k+1$, and hence, by the principle of mathematical induction, the result given in Eq. (11) is true for all positive integer $\flat$. ■

Theorem 3 (Idempotency): Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi\eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi\eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, if $\mathbb{Z}_\iota = \mathbb{Z}\ \forall\ \iota$, then

$$Cq-ROFAAWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) = \mathbb{Z}. \quad (12)$$

$$S_A^{\zeta}\left(\hbar_1, \hbar_2\right) = \begin{cases} S_D\left(\hbar_1, \hbar_2\right), & \text{if } \zeta = 0 \\ \max\left\{\hbar_1, \hbar_2\right\}, & \text{if } \zeta = \infty\ \forall\ \hbar_1, \hbar_2 \in [0,1] \\ 1 - \exp^{-\left((-\ln(1-\hbar_1))^{\zeta} + (-\ln(1-\hbar_2))^{\zeta}\right)^{1/\zeta}}, & \text{otherwise} \end{cases} \quad (2)$$

---

1)

$$\mathbb{Z}_1 \oplus \mathbb{Z}_2 = \left(\left(\mu_1^q + \mu_2^q - \mu_1^q\mu_2^q\right)^{1/q} \mathbf{e}^{i2\pi\left(\eth_{\mu_1}^q + \eth_{\mu_2}^q - \eth_{\mu_1}^q\eth_{\mu_2}^q\right)^{1/q}}, \nu_1\nu_2 \mathbf{e}^{i2\pi\left(\eth_{\nu_1}\eth_{\nu_2}\right)}\right);$$

2)

$$\mathbb{Z}_1 \otimes \mathbb{Z}_2 = \left(\mu_1\mu_2 \mathbf{e}^{i2\pi\left(\eth_{\mu_1}\eth_{\mu_2}\right)}, \left(\nu_1^q + \nu_2^q - \nu_1^q\nu_2^q\right)^{1/q} \mathbf{e}^{i2\pi\left(\eth_{\nu_1}^q + \eth_{\nu_2}^q - \eth_{\nu_1}^q\eth_{\nu_2}^q\right)^{1/q}}\right);$$

3)

$$\mathbb{Z}_1^{\eta} = \left(\mu_1^{\eta} \mathbf{e}^{i2\pi\left(\eth_{\mu_1}^{\eta}\right)}, \left(1 - \left(1 - \nu_1^q\right)^{\eta}\right)^{1/q} \mathbf{e}^{i2\pi\left(1 - \left(1 - \eth_{\nu_1}^q\right)^{\eta}\right)^{1/q}}\right);$$

4)

$$\eta\mathbb{Z}_1 = \left(\left(1 - \left(1 - \mu_1^q\right)^{\eta}\right)^{1/q} \mathbf{e}^{i2\pi\left(1 - \left(1 - \eth_{\mu_1}^q\right)^{\eta}\right)^{1/q}}, \nu_1^{\eta} \mathbf{e}^{i2\pi\left(\eth_{\nu_1}^{\eta}\right)}\right);$$

5)

$$\mathbb{Z}_1^c = \left(\nu_1 \mathbf{e}^{i2\pi\eth_{\nu_1}}, \mu_{Q_1} \mathbf{e}^{i2\pi\eth_{\mu_1}}\right)$$

Proof: Since $\mathbb{Z}_\iota = \mathbb{Z}\ \forall\ \iota$, and $\sum_{\iota=1}^{\flat} w_\iota = 1$ so by Theorem 2, we have $Cq - ROFAAWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right)$, as shown at the bottom of page 10.

■

Theorem 4 (Monotonicity): Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ and $\dot{\mathbb{Z}}_\iota = \left(\dot{\mu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\mu}_\iota}}, \dot{\nu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\nu}_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be two families of Cq-ROFNs, such that $\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}} \geq \dot{\mu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\mu}_\iota}}$ and $\nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}} \leq \dot{\nu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\nu}_\iota}}\ \forall\ \iota$, then

$$\begin{aligned} &Cq - ROFAAWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &\quad \geq Cq - ROFAAWA\left(\dot{\mathbb{Z}}_1, \dot{\mathbb{Z}}_2, \ldots, \dot{\mathbb{Z}}_\flat\right). \end{aligned} \tag{13}$$

Proof: Since $\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}} \geq \dot{\mu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\mu}_\iota}}$ and $\nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}} \leq \dot{\nu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\nu}_\iota}}\ \forall\ \iota$. Based on these, we have the subsequent inequalities the equation can be derived, as shown at the bottom of page 11, which implies that the equation can be derived, as shown at the bottom of page 11.

Hence $Cq - ROFAAWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \geq Cq - ROFAAWA\left(\dot{\mathbb{Z}}_1, \dot{\mathbb{Z}}_2, \ldots, \dot{\mathbb{Z}}_\flat\right)$. ■

Theorem 5 (Boundedness): Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, and let $\mathbb{Z}^- = \min\left\{\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right\}$ and $\mathbb{Z}^+ = \max\left\{\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right\}$, then

$$\mathbb{Z}^- \leq Cq - ROFAAWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \leq \mathbb{Z}^+. \tag{14}$$

Proof: As given that

$$\mathbb{Z}^- = \min\left\{\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right\} = \left(\mu^- \mathbf{e}^{2i\pi \eth_{\mu^-}}, \nu^- \mathbf{e}^{2i\pi \eth_{\nu^-}}\right)$$

and

$$\begin{aligned} &\mathbb{Z}^+ \\ &\quad = \max\left\{\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right\} = \left(\mu^+ \mathbf{e}^{2i\pi \eth_{\mu^+}}, \nu^+ \mathbf{e}^{2i\pi \eth_{\nu^+}}\right), \end{aligned}$$

where

$$\begin{aligned} &\mu^- \mathbf{e}^{2i\pi \eth_{\mu^-}} \\ &\quad = \min\left\{\mu_1 \mathbf{e}^{2i\pi \eth_{\mu 1}}, \mu_2 \mathbf{e}^{2i\pi \eth_{\mu 2}}, \ldots, \mu_\flat \mathbf{e}^{2i\pi \eth_{\mu_\flat}}\right\}, \end{aligned}$$

1)

$$\mathbb{Z}_1 \oplus \mathbb{Z}_2 = \left(\begin{array}{l} \sqrt[q]{1 - \mathbf{e}^{-\left(\left(-\ln\left(1-\mu_1^q\right)\right)^\zeta + \left(-\ln\left(1-\mu_2^q\right)\right)^\zeta\right)^{1/\zeta}}}\ \mathbf{e}^{2i\pi \sqrt[q]{1-\mathbf{e}^{-\left(\left(-\ln\left(1-\eth_{\mu 1}^q\right)\right)^\zeta + \left(-\ln\left(1-\eth_{\mu 2}^q\right)\right)^\zeta\right)^{1/\zeta}}}}, \\ \sqrt[q]{\mathbf{e}^{-\left(\left(-\ln \nu_1^q\right)^\zeta + \left(-\ln \nu_2^q\right)^\zeta\right)^{1/\zeta}}}\ \mathbf{e}^{2i\pi \sqrt[q]{\mathbf{e}^{-\left(\left(-\ln \eth_{\nu 1}^q\right)^\zeta + \left(-\ln \eth_{\nu 2}^q\right)^\zeta\right)^{1/\zeta}}}} \end{array}\right);$$

2)

$$\mathbb{Z}_1 \otimes \mathbb{Z}_2 = \left(\begin{array}{l} \sqrt[q]{\mathbf{e}^{-\left(\left(-\ln \mu_1^q\right)^\zeta + \left(-\ln \mu_2^q\right)^\zeta\right)^{1/\zeta}}}\ \mathbf{e}^{2i\pi \sqrt[q]{\mathbf{e}^{-\left(\left(-\ln \eth_{\mu 1}^q\right)^\zeta + \left(-\ln \eth_{\mu 2}^q\right)^\zeta\right)^{1/\zeta}}}}, \\ \sqrt[q]{1 - \mathbf{e}^{-\left(\left(-\ln\left(1-\nu_1^q\right)\right)^\zeta + \left(-\ln\left(1-\nu_2^q\right)\right)^\zeta\right)^{1/\zeta}}}\ \mathbf{e}^{2i\pi \sqrt[q]{1-\mathbf{e}^{-\left(\left(-\ln\left(1-\eth_{\nu 1}^q\right)\right)^\zeta + \left(-\ln\left(1-\eth_{\nu 2}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \end{array}\right);$$

3)

$$\mathbb{Z}_1^\lambda = \left(\begin{array}{l} \sqrt[q]{\mathbf{e}^{-\left(\lambda\left(-\ln \mu_1^q\right)^\zeta\right)^{1/\zeta}}}\ \mathbf{e}^{2i\pi \sqrt[q]{\mathbf{e}^{-\left(\lambda\left(-\ln \eth_{\mu 1}^q\right)^\zeta\right)^{1/\zeta}}}}, \\ \sqrt[q]{1 - \mathbf{e}^{-\left(\lambda\left(-\ln\left(1-\nu_1^q\right)\right)^\zeta\right)^{1/\zeta}}}\ \mathbf{e}^{2i\pi \sqrt[q]{1-\mathbf{e}^{-\left(\lambda\left(-\ln\left(1-\eth_{\nu 1}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \end{array}\right);$$

4)

$$\lambda \mathbb{Z}_1 = \left(\begin{array}{l} \sqrt[q]{1 - \mathbf{e}^{-\left(\lambda\left(-\ln\left(1-\mu_1^q\right)\right)^\zeta\right)^{1/\zeta}}}\ \mathbf{e}^{2i\pi \sqrt[q]{1-\mathbf{e}^{-\left(\lambda\left(-\ln\left(1-\eth_{\mu 1}^q\right)\right)^\zeta\right)^{1/\zeta}}}}, \\ \sqrt[q]{\mathbf{e}^{-\left(\lambda\left(-\ln \nu_1^q\right)^\zeta\right)^{1/\zeta}}}\ \mathbf{e}^{2i\pi \sqrt[q]{\mathbf{e}^{-\left(\lambda\left(-\ln \eth_{\nu 1}^q\right)^\zeta\right)^{1/\zeta}}}} \end{array}\right)$$

1)

$$\begin{aligned}\mathbb{Z}_1 \oplus \mathbb{Z}_2 &= \left(\sqrt[q]{1-\mathbf{e}^{-\left((-\ln(1-\mu_1^q))^{\zeta}+(-\ln(1-\mu_2^q))^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left((-\ln(1-\eth^q_{\mu 1}))^{\zeta}+(-\ln(1-\eth^q_{\mu 2}))^{\zeta}\right)^{1/\zeta}}}},\right.\\
&\left.\sqrt[q]{\mathbf{e}^{-\left((-\ln \nu_1^q)^{\zeta}+(-\ln \nu_2^q)^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left((-\ln \eth^q_{\nu 1})^{\zeta}+(-\ln \eth^q_{\nu 2})^{\zeta}\right)^{1/\zeta}}}}\right)\\
&= \left(\sqrt[q]{1-\mathbf{e}^{-\left((-\ln(1-\mu_2^q))^{\zeta}+(-\ln(1-\mu_1^q))^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left((-\ln(1-\eth^q_{\mu 2}))^{\zeta}+(-\ln(1-\eth^q_{\mu 1}))^{\zeta}\right)^{1/\zeta}}}},\right.\\
&\left.\sqrt[q]{\mathbf{e}^{-\left((-\ln \nu_2^q)^{\zeta}+(-\ln \nu_1^q)^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left((-\ln \eth^q_{\nu 1})^{\zeta}+(-\ln \eth^q_{\nu 1})^{\zeta}\right)^{1/\zeta}}}}\right) = \mathbb{Z}_2 \oplus \mathbb{Z}_1.\end{aligned}$$

2)

$$\begin{aligned}&\left(\sqrt[q]{\mathbf{e}^{-\left((-\ln \mu_1^q)^{\zeta}+(-\ln \mu_2^q)^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left((-\ln \eth^q_{\mu 1})^{\zeta}+(-\ln \eth^q_{\mu 2})^{\zeta}\right)^{1/\zeta}}}},\right.\\
&\left.\sqrt[q]{1-\mathbf{e}^{-\left((-\ln(1-\nu_1^q))^{\zeta}+(-\ln(1-\nu_2^q))^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left((-\ln(1-\eth^q_{\nu 1}))^{\zeta}+(-\ln(1-\eth^q_{\nu 2}))^{\zeta}\right)^{1/\zeta}}}}\right)\\
&= \left(\sqrt[q]{\mathbf{e}^{-\left((-\ln \mu_2^q)^{\zeta}+(-\ln \mu_1^q)^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left((-\ln \eth^q_{\mu 2})^{\zeta}+(-\ln \eth^q_{\mu 1})^{\zeta}\right)^{1/\zeta}}}},\right.\\
&\left.\sqrt[q]{1-\mathbf{e}^{-\left((-\ln(1-\nu_2^q))^{\zeta}+(-\ln(1-\nu_1^q))^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left((-\ln(1-\eth^q_{\nu 2}))^{\zeta}+(-\ln(1-\eth^q_{\nu 1}))^{\zeta}\right)^{1/\zeta}}}}\right) = \mathbb{Z}_2 \otimes \mathbb{Z}_1.\end{aligned}$$

3)

$$\begin{aligned}\lambda\left(\mathbb{Z}_1 \oplus \mathbb{Z}_2\right) &= \left(\sqrt[q]{1-\mathbf{e}^{-\left(\lambda(-\ln(1-\mu_1^q))^{\zeta}+(-\ln(1-\mu_2^q))^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\lambda(-\ln(1-\eth^q_{\mu 1}))^{\zeta}+(-\ln(1-\eth^q_{\mu 2}))^{\zeta}\right)^{1/\zeta}}}},\right.\\
&\left.\sqrt[q]{\mathbf{e}^{-\left(\lambda(-\ln \nu_1^q)^{\zeta}+(-\ln \nu_2^q)^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\lambda(-\ln \eth^q_{\nu 1})^{\zeta}+(-\ln \eth^q_{\nu 2})^{\zeta}\right)^{1/\zeta}}}}\right)\\
&= \left(\sqrt[q]{1-\mathbf{e}^{-\left(\lambda(-\ln(1-\mu_1^q))^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\lambda(-\ln(1-\eth^q_{\mu 1}))^{\zeta}\right)^{1/\zeta}}}},\ \sqrt[q]{\mathbf{e}^{-\left(\lambda(-\ln \nu_1^q)^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\lambda(-\ln \eth^q_{\nu 1})^{\zeta}\right)^{1/\zeta}}}}\right)\\
&\oplus \left(\sqrt[q]{1-\mathbf{e}^{-\left(\lambda(-\ln(1-\mu_2^q))^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\lambda(-\ln(1-\eth^q_{\mu 2}))^{\zeta}\right)^{1/\zeta}}}},\ \sqrt[q]{\mathbf{e}^{-\left(\lambda(-\ln \nu_2^q)^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\lambda(-\ln \eth^q_{\nu 2})^{\zeta}\right)^{1/\zeta}}}}\right)\\
&= \lambda\mathbb{Z}_1 \oplus \lambda\mathbb{Z}_2.\end{aligned}$$

4)

$$\begin{aligned}\lambda_1\mathbb{Z} \oplus \lambda_2\mathbb{Z} &= \left(\sqrt[q]{1-\mathbf{e}^{-\left(\lambda_1(-\ln(1-\mu^q))^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\lambda_1(-\ln(1-\eth^q_{\mu}))^{\zeta}\right)^{1/\zeta}}}},\right.\\
&\left.\sqrt[q]{\mathbf{e}^{-\left(\lambda_1(-\ln \nu^q)^{\zeta}\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\lambda_1(-\ln \eth^q_{\nu})^{\zeta}\right)^{1/\zeta}}}}\right)\end{aligned}$$

$$\oplus\left(\sqrt[q]{1-\mathbf{e}^{-\left(\lambda_2(-\ln(1-\mu^q))^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\lambda_2\left(-\ln\left(1-\eth_\mu^q\right)\right)^\zeta\right)^{1/\zeta}}}},\ \sqrt[q]{\mathbf{e}^{-\left(\lambda_2(-\ln\nu^q)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\lambda_2\left(-\ln\eth_\nu^q\right)^\zeta\right)^{1/\zeta}}}}\right)$$

$$=\left(\sqrt[q]{1-\mathbf{e}^{-\left((\lambda_1+\lambda_2)(-\ln(1-\mu^q))^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left((\lambda_1+\lambda_2)\left(-\ln\left(1-\eth_\mu^q\right)\right)^\zeta\right)^{1/\zeta}}}},\right.$$
$$\left.\sqrt[q]{\mathbf{e}^{-\left((\lambda_1+\lambda_2)(-\ln\nu^q)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left((\lambda_1+\lambda_2)\left(-\ln\eth_\nu^q\right)^\zeta\right)^{1/\zeta}}}}\right)=(\lambda_1+\lambda_2)\,\mathbb{Z}.$$

5)

$$(\mathbb{Z}_1\otimes\mathbb{Z}_2)^\lambda=\left(\sqrt[q]{\mathbf{e}^{-\left((-\ln\mu_1^q)^\zeta+(-\ln\mu_2^q)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left((-\ln\eth_{\mu 1}^q)^\zeta+(-\ln\eth_{\mu 2}^q)^\zeta\right)^{1/\zeta}}}},\right.$$
$$\left.\sqrt[q]{1-\mathbf{e}^{-\left((-\ln(1-\nu_1^q))^\zeta+(-\ln(1-\nu_2^q))^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left((-\ln(1-\eth_{\nu 1}^q))^\zeta+(-\ln(1-\eth_{\nu 2}^q))^\zeta\right)^{1/\zeta}}}}\right)^\lambda$$

$$=\left(\sqrt[q]{\mathbf{e}^{-\left(\lambda\left((-\ln\mu_1^q)^\zeta+(-\ln\mu_2^q)^\zeta\right)\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\lambda\left((-\ln\eth_{\mu 1}^q)^\zeta+(-\ln\eth_{\mu 2}^q)^\zeta\right)\right)^{1/\zeta}}}},\right.$$
$$\left.\sqrt[q]{1-\mathbf{e}^{-\left(\lambda\left((-\ln(1-\nu_1^q))^\zeta+(-\ln(1-\nu_2^q))^\zeta\right)\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\lambda\left((-\ln(1-\eth_{\nu 1}^q))^\zeta+(-\ln(1-\eth_{\nu 2}^q))^\zeta\right)\right)^{1/\zeta}}}}\right)$$

$$=\left(\sqrt[q]{\mathbf{e}^{-\left(\lambda\left((-\ln\mu_1^q)^\zeta\right)\right)^{1/\zeta}}},\sqrt[q]{1-\mathbf{e}^{-\left(\lambda\left((-\ln(1-\nu_1^q))^\zeta\right)\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\lambda\left((-\ln(1-\eth_{\nu 1}^q))^\zeta\right)\right)^{1/\zeta}}}}\right)$$
$$\otimes\left(\sqrt[q]{\mathbf{e}^{-\left(\lambda\left((-\ln\mu_2^q)^\zeta\right)\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\lambda\left((-\ln\eth_{\mu 2}^q)^\zeta\right)\right)^{1/\zeta}}}},\right.$$
$$\left.\sqrt[q]{1-\mathbf{e}^{-\left(\lambda\left((-\ln(1-\nu_2^q))^\zeta\right)\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\lambda\left((-\ln(1-\eth_{\nu 2}^q))^\zeta\right)\right)^{1/\zeta}}}}\right)=\mathbb{Z}_1^\lambda\otimes\mathbb{Z}_2^\lambda.$$

6)

$$\mathbb{Z}^{\lambda_1}\otimes\mathbb{Z}^{\lambda_2}=\left(\sqrt[q]{\mathbf{e}^{-\left(\lambda_1(-\ln\mu^q)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\lambda_1\left(-\ln\eth_\mu^q\right)^\zeta\right)^{1/\zeta}}}},\right.$$
$$\left.\sqrt[q]{1-\mathbf{e}^{-\left(\lambda_1(-\ln(1-\nu^q))^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\lambda_1\left(-\ln\left(1-\eth_\nu^q\right)\right)^\zeta\right)^{1/\zeta}}}}\right)$$
$$\otimes\left(\sqrt[q]{\mathbf{e}^{-\left(\lambda_2(-\ln\mu^q)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\lambda_2\left(-\ln\eth_\mu^q\right)^\zeta\right)^{1/\zeta}}}},\right.$$
$$\left.\sqrt[q]{1-\mathbf{e}^{-\left(\lambda_2(-\ln(1-\nu^q))^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\lambda_2\left(-\ln\left(1-\eth_\nu^q\right)\right)^\zeta\right)^{1/\zeta}}}}\right)$$
$$=\left(\sqrt[q]{\mathbf{e}^{-\left((\lambda_1+\lambda_2)(-\ln\mu^q)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left((\lambda_1+\lambda_2)\left(-\ln\eth_\mu^q\right)^\zeta\right)^{1/\zeta}}}},\right.$$
$$\left.\sqrt[q]{1-\mathbf{e}^{-\left((\lambda_1+\lambda_2)(-\ln(1-\nu^q))^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left((\lambda_1+\lambda_2)\left(-\ln\left(1-\eth_\nu^q\right)\right)^\zeta\right)^{1/\zeta}}}}\right)=\mathbb{Z}^{(\lambda_1+\lambda_2)}.$$

$$\nu^{-}\mathbf{e}^{2i\pi\eth_{\nu^{-}}} = \max\left\{\nu_1\mathbf{e}^{2i\pi\eth_{\nu 1}}, \nu_2\mathbf{e}^{2i\pi\eth_{\nu 2}}, \ldots, \nu_\flat\mathbf{e}^{2i\pi\eth_{\mu_\flat}}\right\},$$

$$\mu^{+}\mathbf{e}^{2i\pi\eth_{\mu^{+}}} = \max\left\{\mu_1\mathbf{e}^{2i\pi\eth_{\mu 1}}, \mu_2\mathbf{e}^{2i\pi\eth_{\mu 2}}, \ldots, \mu_\flat\mathbf{e}^{2i\pi\eth_{\mu_\flat}}\right\},$$

$$Cq-ROFAAWA\left(\mathbb{Z}_1,\mathbb{Z}_2,\ldots,\mathbb{Z}_\flat\right) = \left(\begin{array}{l}\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\mu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\eth_{\mu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}}, \\ \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\nu_\iota^q\right)^\zeta\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\eth_{\nu_\iota}^q\right)^\zeta\right)^{1/\zeta}}}}\end{array}\right) \tag{11}$$

$$\begin{aligned}Cq-ROFAAWA\left(\mathbb{Z}_1,\mathbb{Z}_2\right) &= w_1\mathbb{Z}_1\oplus w_2\mathbb{Z}_2\\ &= \left(\begin{array}{l}\sqrt[q]{1-\mathbf{e}^{-\left(w_1\left(-\ln\left(1-\mu_1^q\right)\right)^\zeta\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(w_1\left(-\ln\left(1-\eth_{\mu 1}^q\right)\right)^\zeta\right)^{1/\zeta}}}}, \\ \sqrt[q]{\mathbf{e}^{-\left(w_1\left(-\ln\nu_1^q\right)^\zeta\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(w_1\left(-\ln\eth_{\nu 1}^q\right)^\zeta\right)^{1/\zeta}}}}\end{array}\right)\\ &\oplus \left(\begin{array}{l}\sqrt[q]{1-\mathbf{e}^{-\left(w_2\left(-\ln\left(1-\mu_2^q\right)\right)^\zeta\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(w_2\left(-\ln\left(1-\eth_{\mu 2}^q\right)\right)^\zeta\right)^{1/\zeta}}}}, \\ \sqrt[q]{\mathbf{e}^{-\left(w_2\left(-\ln\nu_2^q\right)^\zeta\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(w_2\left(-\ln\eth_{\nu 2}^q\right)^\zeta\right)^{1/\zeta}}}}\end{array}\right)\\ &= \left(\begin{array}{l}\sqrt[q]{1-\mathbf{e}^{-\left(w_1\left(-\ln\left(1-\mu_1^q\right)\right)^\zeta+w_2\left(-\ln\left(1-\mu_2^q\right)\right)^\zeta\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(w_1\left(-\ln\left(1-\eth_{\mu 1}^q\right)\right)^\zeta+w_2\left(-\ln\left(1-\eth_{\mu 2}^q\right)\right)^\zeta\right)^{1/\zeta}}}}, \\ \sqrt[q]{\mathbf{e}^{-\left(w_1\left(-\ln\nu_1^q\right)^\zeta+w_2\left(-\ln\nu_2^q\right)^\zeta\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(w_1\left(-\ln\eth_{\nu 1}^q\right)^\zeta+w_2\left(-\ln\eth_{\nu 2}^q\right)^\zeta\right)^{1/\zeta}}}}\end{array}\right)\\ &= \left(\begin{array}{l}\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{2} w_\iota\left(-\ln\left(1-\mu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{2} w_\iota\left(-\ln\left(1-\eth_{\mu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}}, \\ \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{2} w_\iota\left(-\ln\nu_\iota^q\right)^\zeta\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{2} w_\iota\left(-\ln\eth_{\nu_\iota}^q\right)^\zeta\right)^{1/\zeta}}}}\end{array}\right)\end{aligned}$$

$$\begin{aligned}Cq-ROFAAWA\left(\mathbb{Z}_1,\mathbb{Z}_2,\ldots,\mathbb{Z}_k\right) &= \oplus_{\iota=1}^{k}\left(w_\iota\mathbb{Z}_\iota\right)\\ &= \left(\begin{array}{l}\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln\left(1-\mu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln\left(1-\eth_{\mu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}}, \\ \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln\nu_\iota^q\right)^\zeta\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln\eth_{\nu_\iota}^q\right)^\zeta\right)^{1/\zeta}}}}\end{array}\right)\end{aligned}$$

and

$$\nu^{+}\mathbf{e}^{2i\pi\eth_{\nu^{+}}}=\min\left\{\nu_{1}\mathbf{e}^{2i\pi\eth_{\nu 1}},\nu_{2}\mathbf{e}^{2i\pi\eth_{\nu 2}},\ldots,\nu_{\flat}\mathbf{e}^{2i\pi\eth_{\nu\flat}}\right\}.$$

As a result, there are ongoing inequities:

Thereby, $\mathbb{Z}^{-}\leq Cq-ROFAAWA\left(\mathbb{Z}_{1},\mathbb{Z}_{2},\ldots,\mathbb{Z}_{\flat}\right)\leq\mathbb{Z}^{+}$. ■

$$\begin{aligned}
&Cq-ROFAAWA\left(\mathbb{Z}_{1},\mathbb{Z}_{2},\ldots,\mathbb{Z}_{k},\mathbb{Z}_{k+1}\right)\\
&=\oplus_{\iota=1}^{k}\left(w_{\iota}\mathbb{Z}_{\iota}\right)\oplus\left(w_{k+1}\mathbb{Z}_{k+1}\right)\\
&=\left(\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{k}w_{\iota}\left(-\ln\left(1-\mu_{\iota}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{k}w_{\iota}\left(-\ln\left(1-\eth_{\mu_{\iota}}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}},\right.\\
&\qquad\left.\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{k}w_{\iota}\left(-\ln\nu_{\iota}^{q}\right)^{\zeta}\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{k}w_{\iota}\left(-\ln\eth_{\nu_{\iota}}^{q}\right)^{\zeta}\right)^{1/\zeta}}}}\right)\\
&\quad\oplus\left(\sqrt[q]{1-\mathbf{e}^{-\left(w_{k+1}\left(-\ln\left(1-\mu_{k+1}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(w_{k+1}\left(-\ln\left(1-\eth_{\mu k+1}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}},\sqrt[q]{\mathbf{e}^{-\left(w_{k+1}\left(-\ln\nu_{\mathbb{Z}_{k+1}}^{q}\right)^{\zeta}\right)^{1/\zeta}}}\right)\\
&=\left(\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{k+1}w_{\iota}\left(-\ln\left(1-\mu_{\iota}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{k+1}w_{\iota}\left(-\ln\left(1-\eth_{\mu_{\iota}}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}},\right.\\
&\qquad\left.\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{k+1}w_{\iota}\left(-\ln\nu_{\iota}^{q}\right)^{\zeta}\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{k+1}w_{\iota}\left(-\ln\eth_{\nu_{\iota}}^{q}\right)^{\zeta}\right)^{1/\zeta}}}}\right)
\end{aligned}$$

$$\begin{aligned}
&Cq-ROFAAWA\left(\mathbb{Z}_{1},\mathbb{Z}_{2},\ldots,\mathbb{Z}_{\flat}\right)\\
&=\left(\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}w_{\iota}\left(-\ln\left(1-\mu^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}w_{\iota}\left(-\ln\left(1-\eth_{\mu}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}},\right.\\
&\qquad\left.\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}w_{\iota}\left(-\ln\nu^{q}\right)^{\zeta}\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}w_{\iota}\left(-\ln\eth_{\nu}^{q}\right)^{\zeta}\right)^{1/\zeta}}}}\right)\\
&=\left(\sqrt[q]{1-\mathbf{e}^{-\left(\left(-\ln\left(1-\mu^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\left(-\ln\left(1-\eth_{\mu}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}},\sqrt[q]{\mathbf{e}^{-\left(\left(-\ln\nu^{q}\right)^{\zeta}\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\left(-\ln\eth_{\nu}^{q}\right)^{\zeta}\right)^{1/\zeta}}}}\right)\\
&=\left(\mu\mathbf{e}^{2i\pi\eth_{\mu}},\nu\mathbf{e}^{2i\pi\eth_{\nu}}\right)=\mathbb{Z}
\end{aligned}$$

$$\begin{aligned}
&\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}w_{\iota}\left(-\ln\left(1-\mu_{\iota}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}w_{\iota}\left(-\ln\left(1-\eth_{\mu_{\iota}}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}}\\
&\quad\geq\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}w_{\iota}\left(-\ln\left(1-\dot{\mu}_{\iota}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}w_{\iota}\left(-\ln\left(1-\eth_{\mu_{\iota}}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}}
\end{aligned}$$

and

$$\begin{aligned}
&\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}w_{\iota}\left(-\ln\left(\nu_{\iota}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}w_{\iota}\left(-\ln\left(\eth_{\nu_{\iota}}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}}\\
&\quad\leq\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}w_{\iota}\left(-\ln\left(\dot{\nu}_{\iota}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}w_{\iota}\left(-\ln\left(\eth_{\nu_{\iota}}^{q}\right)\right)^{\zeta}\right)^{1/\zeta}}}}
\end{aligned}$$

Theorem 6 (Symmetry): Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi\eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi\eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs. Then, if $\breve{\mathbb{Z}}_\iota = \left(\breve{\mu}_\iota \mathbf{e}^{2i\pi\eth_{\breve{\mu}_\iota}}, \breve{\nu}_\iota \mathbf{e}^{2i\pi\eth_{\breve{\nu}_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be any permutation of $\mathbb{Z}_\iota$, then we have

$$\begin{aligned} &Cq-ROFAAWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &\quad = Cq-ROFAAWA\left(\breve{\mathbb{Z}}_1, \breve{\mathbb{Z}}_2, \ldots, \breve{\mathbb{Z}}_\flat\right). \end{aligned} \tag{15}$$

Proof: The proof is obvious and thus omitted. ■

Next, we introduce complex q-rung orthopair fuzzy Aczel–Alsina ordered weighted averaging (Cq-ROFAAOWA) operator.

Definition 15: Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi\eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi\eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, then the Cq-ROFAAOWA operator is:

$$Cq-ROFAAOWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) = \oplus_{\iota=1}^{\flat}\left(\varpi_\iota \mathbb{Z}_{\delta(\iota)}\right), \tag{16}$$

where $\varpi = \left(\varpi_1, \varpi_2, \ldots, \varpi_\flat\right)^T$ is the position weights of $\mathbb{Z}_\iota$ $(\iota = 1, 2, \ldots, \flat)$ such that $\varpi_\iota > 0$ and $\sum_{\iota=1}^{\flat} \varpi_\iota = 1$. $(\delta(1), \delta(2), \ldots, \delta(\flat))$ is a permutation of $(1, 2, \ldots, \flat)$ such that $\mathbb{Z}_{\delta(\iota-1)} \geq \mathbb{Z}_{\delta(\iota)}$ for $\iota = 1, 2, \ldots, \flat$.

Theorem 7: Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi\eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi\eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, then the result obtained by utilizing Cq-ROFAAOWA operator is still a Cq-ROFN, and (17), as shown at the bottom of the next page.

Proof: We skip the proof of this theorem since it is analogous to that of Theorem 2. ■

The following features may be efficiently shown by using the Cq-ROFAAOWA operator.

Theorem 8 (Idempotency): Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi\eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi\eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, if $\mathbb{Z}_\iota = \mathbb{Z}\ \forall\ \iota$, then

$$Cq-ROFAAOWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) = \mathbb{Z}. \tag{18}$$

Theorem 9 (Monotonicity): Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi\eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi\eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ and $\dot{\mathbb{Z}}_\iota = \left(\dot{\mu}_\iota \mathbf{e}^{2i\pi\eth_{\dot{\mu}_\iota}}, \dot{\nu}_\iota \mathbf{e}^{2i\pi\eth_{\dot{\nu}_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be two families of Cq-ROFNs, such that $\mu_\iota \mathbf{e}^{2i\pi\eth_{\mu_\iota}} \geq \dot{\mu}_\iota \mathbf{e}^{2i\pi\eth_{\dot{\mu}_\iota}}$ and $\nu_\iota \mathbf{e}^{2i\pi\eth_{\nu_\iota}} \leq \dot{\nu}_\iota \mathbf{e}^{2i\pi\eth_{\dot{\nu}_\iota}}$ $\forall\ \iota$, then

$$\begin{aligned} &Cq-ROFAAOWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &\quad \geq Cq-ROFAAOWA\left(\dot{\mathbb{Z}}_1, \dot{\mathbb{Z}}_2, \ldots, \dot{\mathbb{Z}}_\flat\right). \end{aligned} \tag{19}$$

$$\begin{aligned} &\left(\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\mu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\eth_{\mu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}},\right. \\ &\qquad \left.\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln \nu_\iota^q\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln \eth_{\nu_\iota}^q\right)^\zeta\right)^{1/\zeta}}}}\right) \\ &\geq \left(\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\dot{\mu}_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\eth_{\dot{\mu}_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}},\right. \\ &\qquad \left.\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln \dot{\nu}_\iota^q\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln \eth_{\dot{\nu}_\iota}^q\right)^\zeta\right)^{1/\zeta}}}}\right) \end{aligned}$$

$$\begin{aligned} &\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\mu^{-q}\right)\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\eth_{\mu-}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \\ &\quad \leq \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\mu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\eth_{\mu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \\ &\quad \leq \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\mu^{+q}\right)\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\eth_{\mu+}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \\ &\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\nu^{+q}\right)\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\eth_{\nu+}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \\ &\quad \leq \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\nu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\eth_{\nu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \\ &\quad \leq \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\nu^{-q}\right)\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\eth_{\nu-}^q\right)\right)^\zeta\right)^{1/\zeta}}}}. \end{aligned}$$

Theorem 10 (Boundedness): Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, and let $\mathbb{Z}^- = \min\left\{\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right\}$ and $\mathbb{Z}^+ = \max\left\{\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right\}$, then

$$\mathbb{Z}^- \leq Cq - ROFAAOWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \leq \mathbb{Z}^+. \quad (20)$$

Theorem 11 (Symmetry): Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs. Then, if $\breve{\mathbb{Z}}_\iota = \left(\breve{\mu}_\iota \mathbf{e}^{2i\pi \eth_{\breve{\mu}_\iota}}, \breve{\nu}_\iota \mathbf{e}^{2i\pi \eth_{\breve{\nu}_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be any permutation of $\mathbb{Z}_\iota$, then we have

$$\begin{aligned} &Cq - ROFAAWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &\quad = Cq - ROFAAWA\left(\breve{\mathbb{Z}}_1, \breve{\mathbb{Z}}_2, \ldots, \breve{\mathbb{Z}}_\flat\right). \end{aligned} \quad (21)$$

The q-ROFAAWA operator weights just the Cq-ROFNs, as defined by Definition 14, while the q-ROFAAOWA operator weights only the ordered locations of the Cq-ROFNs, as defined by Definition 15. As a result, weights represent various aspects of the q-ROFAAWA and q-ROFAAOWA operators. Nonetheless, one of the operators, as well as the other operators, consider just one of them. To address this shortcoming, we study the q-rung orthopair fuzzy Aczel–Alsina hybrid averaging (q-ROFAAHA) operator, which weights all of the provided Cq-ROFN and their appropriate ordered position.

Definition 16: Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, then the Cq-ROFAAHA operator is:

$$Cq - ROFAAHA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) = \oplus_{\iota=1}^{\flat}\left(\varpi_\iota \hat{\mathbb{Z}}_{\delta(\iota)}\right), \quad (22)$$

where $\varpi = \left(\varpi_1, \varpi_2, \ldots, \varpi_\flat\right)^T$ is the weight vector associated with Cq-ROFAAHA $\mathbb{Z}_\iota\,(\iota = 1, 2, \ldots, \flat)$ such that $\varpi_\iota > 0$ and $\sum_{\iota=1}^{\flat} \varpi_\iota = 1$, $w = \left(w_1, w_2, \ldots, w_\flat\right)^T$ is the weight vector of $\mathbb{Z}_\iota\,(\iota = 1, 2, \ldots, \flat)$ such that $w_\iota > 0$ and $\sum_{\iota=1}^{\flat} w_\iota = 1$. $\hat{\mathbb{Z}}_{\delta(\iota)}$ is the ιth largest of the weighted Cq-ROFNs $\hat{\mathbb{Z}}_\iota \left(\hat{\mathbb{Z}}_\iota = (\flat w_\iota)\,\mathbb{Z}_\iota\right)$, $(\iota = 1, 2, \ldots, \flat)$ and $\flat$ is the balancing coefficient.

Theorem 12: Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)\left(\iota = 1, 2, \ldots, \flat\right)$ be a family of Cq-ROFNs, then the result obtained by utilizing q-ROFAAHA operator is still a Cq-ROFN, and (23), as shown at the bottom of the next page.

Proof: We skip the proof of this theorem since it is analogous to that of Theorem 2. ■

As analogous to those of Cq-ROFAAWA operator and Cq-ROFAAOWA, the Cq-ROFAAHA operator also follows the idempotency, monotonicity, boundedness and symmetry properties. Besides the aforesaid characteristics, the q-ROFAAHA operator has the following special cases.

Corollary 1: Cq-ROFAAWA operator is a special case of the Cq-ROFAAHA operator.

Proof: Let $\varpi = \left(\frac{1}{\flat}, \frac{1}{\flat}, \ldots, \frac{1}{\flat}\right)^T$, then

$$\begin{aligned} &Cq - ROFAAHA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &\quad = \varpi_1 \hat{\mathbb{Z}}_{\delta(1)} \oplus \varpi_2 \hat{\mathbb{Z}}_{\delta(2)} \oplus \ldots \oplus \varpi_\flat \hat{\mathbb{Z}}_{\delta(\flat)} \\ &\quad = \frac{1}{\flat}\left(\hat{\mathbb{Z}}_{\delta(1)} \oplus \hat{\mathbb{Z}}_{\delta(2)} \oplus \ldots \oplus \hat{\mathbb{Z}}_{\delta(\flat)}\right) \\ &\quad = w_1 \mathbb{Z}_1 \oplus w_2 \mathbb{Z}_2 \oplus \ldots \oplus w_\flat \mathbb{Z}_\flat \\ &\quad = Cq - ROFAAWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right). \end{aligned}$$

■

Corollary 2: Cq-ROFAAOWA operator is a special case of the Cq-ROFAAHA operator.

Proof: Let $w = \left(\frac{1}{\flat}, \frac{1}{\flat}, \ldots, \frac{1}{\flat}\right)^T$, then

$$\begin{aligned} &Cq - ROFAAHA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &\quad = \varpi_1 \hat{\mathbb{Z}}_{\delta(1)} \oplus \varpi_2 \hat{\mathbb{Z}}_{\delta(2)} \oplus \ldots \oplus \varpi_\flat \hat{\mathbb{Z}}_{\delta(\flat)} \\ &\quad = \varpi_1 \mathbb{Z}_{\delta(1)} \oplus \varpi_2 \mathbb{Z}_{\delta(2)} \oplus \ldots \oplus \varpi_\flat \mathbb{Z}_{\delta(\flat)} \\ &\quad = Cq - ROFAAOWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right). \end{aligned}$$

■

B. COMPLEX Q-RUNG ORTHOPAIR FUZZY ASZEL-ALSINA GEOMETRIC AGGREGATION OPERATORS

Based on the designed operations, in this section, we put forward some novel geometric aggregation operators including complex q-rung orthopair fuzzy Aszel-Alsina geometric (Cq-ROFAAG) operator, complex q-rung orthopair fuzzy Aszel-Alsina weighted geometric (Cq-ROFAAWG) operator, complex q-rung orthopair fuzzy Aczel–Alsina ordered weighted averaging (Cq-ROFAAOWA) operator, complex q-rung orthopair fuzzy Aczel–Alsina ordered weighted geometric (Cq-ROFAAOWG) operator, and complex q-rung orthopair fuzzy Aczel–Alsina hybrid geometric

$$\begin{aligned} &Cq - ROFAAOWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &= \left(\begin{array}{l} \sqrt[q]{1 - \mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} \varpi_\iota \left(-\ln\left(1-\mu_{\delta(\iota)}^q\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} \varpi_\iota \left(-\ln\left(1-\eth_{\mu\,\delta(\iota)}^q\right)\right)^\zeta\right)^{1/\zeta}}}}, \\ \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} \varpi_\iota \left(-\ln \nu_{\delta(\iota)}^q\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} \varpi_\iota \left(-\ln \eth_{\nu\,\delta(\iota)}^q\right)^\zeta\right)^{1/\zeta}}}} \end{array}\right) \end{aligned} \quad (17)$$

(Cq-ROFAAHG) operator. In addition, we investigate some special cases and properties of these operators.

Definition 17: Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, then the Cq-ROFAAWG operator is:

$$Cq-ROFAAWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) = \otimes_{\iota=1}^{\flat} \left(\mathbb{Z}_\iota\right)^{w_\iota}, \quad (24)$$

where $w = \left(w_1, w_2, \ldots, w_\flat\right)^T$ is the weight vector of $\mathbb{Z}_\iota$ $(\iota = 1, 2, \ldots, \flat)$ such that $w_\iota > 0$ and $\sum_{\iota=1}^{\flat} w_\iota = 1$. Especially, if $w = \left(\frac{1}{\flat}, \frac{1}{\flat}, \ldots, \frac{1}{\flat}\right)^T$, then the Cq-ROFAAWG operator reduces to Cq-ROFAAG operator of dimension $\flat$, which is described as follows:

$$Cq-ROFAAG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) = \otimes_{\iota=1}^{\flat} \left(\mathbb{Z}_\iota\right)^{\frac{1}{\flat}}. \quad (25)$$

Theorem 13: Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, then the result obtained by utilizing Cq-ROFAAWG operator is still a Cq-ROFN, and (26), as shown at the bottom of the next page.

Proof: We can prove Theorem 13 with the help of the mathematical induction method in the following way:

For $\flat = 2$, we have the equation can be derived, as shown at the bottom of next page.

Hence, the result is true for $\flat = 2$.

Suppose that Eq. (26) is true for $\flat = k$, then we have the equation can be derived, as shown at the bottom of next page.

Now for $\flat = k + 1$, we have the equation can be derived, as shown at the bottom of page 15.

Thus, Eq. (26) is legitimate for $\flat = k+1$ and hence, by the principle of mathematical induction, result given in Eq. (26) is true for all positive integer $\flat$. ■

Theorem 14 (Idempotency): Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, if $\mathbb{Z}_\iota = \mathbb{Z}$ $\forall$ ι, then

$$Cq-ROFAAWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) = \mathbb{Z}. \quad (27)$$

Proof: Since $\mathbb{Z}_\iota = \mathbb{Z}$ $\forall$ ι, and $\sum_{\iota=1}^{\flat} w_\iota = 1$ so by Theorem 13, we have the equation can be derived, as shown at the bottom of page 15. ■

Theorem 15 (Monotonicity): Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ and $\dot{\mathbb{Z}}_\iota = \left(\dot{\mu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\mu}_\iota}}, \dot{\nu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\nu}_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be two families of Cq-ROFNs, such that $\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}} \geq \dot{\mu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\mu}_\iota}}$ and $\nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}} \leq \dot{\nu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\nu}_\iota}}$ $\forall$ ι, then

$$\begin{aligned} &Cq-ROFAAWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &\quad \geq Cq-ROFAAWG\left(\dot{\mathbb{Z}}_1, \dot{\mathbb{Z}}_2, \ldots, \dot{\mathbb{Z}}_\flat\right). \end{aligned} \quad (28)$$

Proof: Since $\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}} \geq \dot{\mu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\mu}_\iota}}$ and $\nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}} \leq \dot{\nu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\nu}_\iota}}$ $\forall$ ι. Based on these, we have the subsequent inequalities the equation can be derived, as shown at the bottom of page 15, which implies that the equation can be derived, as shown at the bottom of page 16.

Hence

$$\begin{aligned} &Cq-ROFAAWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &\quad \geq Cq-ROFAAWG\left(\dot{\mathbb{Z}}_1, \dot{\mathbb{Z}}_2, \ldots, \dot{\mathbb{Z}}_\flat\right). \end{aligned}$$

■

Theorem 16 (Boundedness): Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, and let $\mathbb{Z}^- = \min\left\{\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right\}$ and $\mathbb{Z}^+ = \max\left\{\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right\}$, then

$$\mathbb{Z}^- \leq Cq-ROFAAWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \leq \mathbb{Z}^+. \quad (29)$$

Proof: As given that

$$\mathbb{Z}^- = \min\left\{\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right\} = \left(\mu^- \mathbf{e}^{2i\pi \eth_{\mu^-}}, \nu^- \mathbf{e}^{2i\pi \eth_{\nu^-}}\right)$$

and

$$\begin{aligned} &\mathbb{Z}^+ \\ &\quad = \max\left\{\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right\} = \left(\mu^+ \mathbf{e}^{2i\pi \eth_{\mu^+}}, \nu^+ \mathbf{e}^{2i\pi \eth_{\nu^+}}\right), \end{aligned}$$

where

$$\begin{aligned} &\mu^- \mathbf{e}^{2i\pi \eth_{\mu^-}} \\ &\quad = \min\left\{\mu_1 \mathbf{e}^{2i\pi \eth_{\mu_1}}, \mu_2 \mathbf{e}^{2i\pi \eth_{\mu_2}}, \ldots, \mu_\flat \mathbf{e}^{2i\pi \eth_{\mu_\flat}}\right\}, \\ &\nu^- \mathbf{e}^{2i\pi \eth_{\nu^-}} \\ &\quad = \max\left\{\nu_1 \mathbf{e}^{2i\pi \eth_{\nu_1}}, \nu_2 \mathbf{e}^{2i\pi \eth_{\nu_2}}, \ldots, \nu_\flat \mathbf{e}^{2i\pi \eth_{\nu_\flat}}\right\}, \\ &\mu^+ \mathbf{e}^{2i\pi \eth_{\mu^+}} \\ &\quad = \max\left\{\mu_1 \mathbf{e}^{2i\pi \eth_{\mu_1}}, \mu_2 \mathbf{e}^{2i\pi \eth_{\mu_2}}, \ldots, \mu_\flat \mathbf{e}^{2i\pi \eth_{\mu_\flat}}\right\}, \end{aligned}$$

and

$$\nu^+ \mathbf{e}^{2i\pi \eth_{\nu^+}} = \min\left\{\nu_1 \mathbf{e}^{2i\pi \eth_{\nu_1}}, \nu_2 \mathbf{e}^{2i\pi \eth_{\nu_2}}, \ldots, \nu_\flat \mathbf{e}^{2i\pi \eth_{\nu_\flat}}\right\}.$$

$$\begin{aligned} &Cq-ROFAAHA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &= \left(\sqrt[q]{1 - \mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} \varpi_\iota \left(-\ln\left(1-\hat{\mu}_{\delta(\iota)}^q\right)\right)^\zeta\right)^{1/\zeta}}} \mathbf{e}^{2i\pi \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} \varpi_\iota \left(-\ln\left(1-\eth_{\hat{\mu}\delta(\iota)}^q\right)\right)^\zeta\right)^{1/\zeta}}}}, \right. \\ &\left. \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} \varpi_\iota \left(-\ln \hat{\nu}_{\delta(\iota)}^q\right)^\zeta\right)^{1/\zeta}}} \mathbf{e}^{2i\pi \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} \varpi_\iota \left(-\ln \eth_{\hat{\nu}\delta(\iota)}^q\right)^\zeta\right)^{1/\zeta}}}} \right) \end{aligned} \quad (23)$$

As a result, there are ongoing inequities the equation can be derived, as shown at the bottom of page 16.

Thereby, $\mathbb{Z}^{-} \leq Cq-ROFAAWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \leq \mathbb{Z}^{+}$. ■

$$Cq-ROFAAWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) = \left(\begin{array}{l}\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln \mu_\iota^q\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln \eth_{\mu_\iota}^q\right)^\zeta\right)^{1/\zeta}}}},\\ \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\nu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\eth_{\nu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}}\end{array}\right) \tag{26}$$

$$\begin{aligned} Cq-ROFAAWG\left(\mathbb{Z}_1, \mathbb{Z}_2\right) &= \mathbb{Z}_1^{w_1} \otimes \mathbb{Z}_2^{w_2} \\ &= \left(\begin{array}{l}\sqrt[q]{\mathbf{e}^{-\left(w_1\left(-\ln \mu_1^q\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(w_1\left(-\ln \eth_{\mu 1}^q\right)^\zeta\right)^{1/\zeta}}}},\\ \sqrt[q]{1-\mathbf{e}^{-\left(w_1\left(-\ln\left(1-\nu_1^q\right)\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(w_1\left(-\ln\left(1-\eth_{\nu 1}^q\right)\right)^\zeta\right)^{1/\zeta}}}}\end{array}\right) \\ &\otimes \left(\begin{array}{l}\sqrt[q]{\mathbf{e}^{-\left(w_2\left(-\ln \mu_2^q\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(w_2\left(-\ln \eth_{\mu 2}^q\right)^\zeta\right)^{1/\zeta}}}},\\ \sqrt[q]{1-\mathbf{e}^{-\left(w_2\left(-\ln\left(1-\nu_2^q\right)\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(w_2\left(-\ln\left(1-\eth_{\nu 2}^q\right)\right)^\zeta\right)^{1/\zeta}}}}\end{array}\right) \\ &= \left(\begin{array}{l}\sqrt[q]{\mathbf{e}^{-\left(w_1\left(-\ln \mu_1^q\right)^\zeta+w_2\left(-\ln \nu_2^q\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(w_1\left(-\ln \eth_{\mu 1}^q\right)^\zeta+w_2\left(-\ln \nu_2^q\right)^\zeta\right)^{1/\zeta}}}},\\ \sqrt[q]{1-\mathbf{e}^{-\left(w_1\left(-\ln\left(1-\nu_1^q\right)\right)^\zeta+w_2\left(-\ln\left(1-\mu_2^q\right)\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(w_1\left(-\ln\left(1-\eth_{\nu 1}^q\right)\right)^\zeta+w_2\left(-\ln\left(1-\mu_2^q\right)\right)^\zeta\right)^{1/\zeta}}}}\end{array}\right) \\ &= \left(\begin{array}{l}\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{2} w_\iota\left(-\ln \mu_\iota^q\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{2} w_\iota\left(-\ln \eth_{\mu_\iota}^q\right)^\zeta\right)^{1/\zeta}}}},\\ \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{2} w_\iota\left(-\ln\left(1-\nu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{2} w_\iota\left(-\ln\left(1-\eth_{\nu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}}\end{array}\right) \end{aligned}$$

$$\begin{aligned} Cq-ROFAAWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_k\right) &= \otimes_{\iota=1}^{k}\left(\mathbb{Z}_\iota\right)^{w_\iota} \\ &= \left(\begin{array}{l}\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln \mu_\iota^q\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln \eth_{\mu_\iota}^q\right)^\zeta\right)^{1/\zeta}}}},\\ \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln\left(1-\nu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\,\mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln\left(1-\eth_{\nu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}}\end{array}\right) \end{aligned}$$

Theorem 17: (Symmetry) Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi\eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi\eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs. Then, if $\breve{\mathbb{Z}}_\iota = (\breve{\mu}_\iota, \breve{\nu}_\iota)$ $(\iota = 1, 2, \ldots, \flat)$ be any permutation of $\mathbb{Z}_\iota$, then we have

$$\begin{aligned} &Cq-ROFAAWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &\quad = Cq-ROFAAWG\left(\breve{\mathbb{Z}}_1, \breve{\mathbb{Z}}_2, \ldots, \breve{\mathbb{Z}}_\flat\right). \end{aligned} \tag{30}$$

$$\begin{aligned} &Cq-ROFAAWA\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_k, \mathbb{Z}_{k+1}\right) \\ &= \oplus_{\iota=1}^{k}\left(w_\iota \mathbb{Z}_\iota\right) \oplus \left(w_{k+1}\mathbb{Z}_{k+1}\right) \\ &= \left(\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln\left(1-\mu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln\left(1-\eth_{\mu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}}, \right.\\ &\quad \left. \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln \nu_\iota^q\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln \eth_{\nu_\iota}^q\right)^\zeta\right)^{1/\zeta}}}} \right) \\ &\quad \oplus \left(\sqrt[q]{1-\mathbf{e}^{-\left(w_{k+1}\left(-\ln\left(1-\mu_{k+1}^q\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(w_{k+1}\left(-\ln\left(1-\eth_{\mu k+1}^q\right)\right)^\zeta\right)^{1/\zeta}}}}, \sqrt[q]{\mathbf{e}^{-\left(w_{k+1}\left(-\ln \nu_{\mathbb{Z}_{k+1}}^q\right)^\zeta\right)^{1/\zeta}}} \right) \\ &= \left(\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{k+1} w_\iota\left(-\ln\left(1-\mu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{k+1} w_\iota\left(-\ln\left(1-\eth_{\mu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}}, \right.\\ &\quad \left. \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{k+1} w_\iota\left(-\ln \nu_\iota^q\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{k+1} w_\iota\left(-\ln \eth_{\nu_\iota}^q\right)^\zeta\right)^{1/\zeta}}}} \right) \end{aligned}$$

$$\begin{aligned} &Cq-ROFAAWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &= \left(\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln \mu^q\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln \eth_\mu^q\right)^\zeta\right)^{1/\zeta}}}}, \right.\\ &\quad \left. \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln(1-\nu^q)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\eth_\nu^q\right)\right)^\zeta\right)^{1/\zeta}}}} \right) = \left(\sqrt[q]{\mathbf{e}^{-\left(\left(-\ln \mu^q\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\left(-\ln \eth_\mu^q\right)^\zeta\right)^{1/\zeta}}}}, \right.\\ &\quad \left. \sqrt[q]{1-\mathbf{e}^{-\left(\left(-\ln(1-\nu^q)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\left(-\ln\left(1-\eth_\nu^q\right)\right)^\zeta\right)^{1/\zeta}}}} \right) = \left(\mu \mathbf{e}^{2i\pi\eth_\mu}, \nu \mathbf{e}^{2i\pi\eth_\nu}\right) = \mathbb{Z}. \end{aligned}$$

$$\begin{aligned} &\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\mu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\eth_{\mu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \\ &\quad \geq \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\dot{\mu}_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\eth_{\dot{\mu}_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \end{aligned}$$

and

$$\begin{aligned} &\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\nu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\eth_{\nu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \\ &\quad \leq \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\dot{\nu}_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\eth_{\dot{\nu}_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \end{aligned}$$

Proof: The proof is obvious and thus omitted. ∎

Next, we introduce complex q-rung orthopair fuzzy Aczel–Alsina ordered weighted geometric (Cq-ROFAAOWG) operator.

Definition 18: Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $\left(\iota = 1, 2, \ldots, \flat\right)$ be a family of Cq-ROFNs, then the Cq-ROFAAOWG operator is:

$$Cq-ROFAAOWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) = \otimes_{\iota=1}^{\flat} \left(\mathbb{Z}_{\delta(\iota)}\right)^{\varpi_\iota}, \tag{31}$$

where $\varpi = \left(\varpi_1, \varpi_2, \ldots, \varpi_\flat\right)^T$ is the position weights of $\mathbb{Z}_\iota$ $(\iota = 1, 2, \ldots, \flat)$ such that $\varpi_\iota > 0$ and $\sum_{\iota=1}^{\flat} \varpi_\iota = 1$. $(\delta(1), \delta(2), \ldots, \delta(\flat))$ is a permutation of $(1, 2, \ldots, \flat)$ such that $\mathbb{Z}_{\delta(\iota-1)} \geq \mathbb{Z}_{\delta(\iota)}$ for $\iota = 1, 2, \ldots, \flat$.

Theorem 18: Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $\left(\iota = 1, 2, \ldots, \flat\right)$ be a family of Cq-ROFNs, then the result obtained by utilizing Cq-ROFAAOWG operator is still a Cq-ROFN, and (32), as shown at the bottom of the next page.

Proof: We skip the proof of this theorem since it is analogous to that of Theorem 13. ∎

The following features may be efficiently shown by using the q-ROFAAOWG operator.

Theorem 19 (Idempotency): Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, if $\mathbb{Z}_\iota = \mathbb{Z}\ \forall\ \iota$, then

$$Cq-ROFAAOWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) = \mathbb{Z}. \tag{33}$$

Theorem 20 (Monotonicity): Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ and $\dot{\mathbb{Z}}_\iota = \left(\dot{\mu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\mu}_\iota}}, \dot{\nu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\nu}_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be two families of Cq-ROFNs, such that $\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}} \geq \dot{\mu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\mu}_\iota}}$ and $\nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}} \leq \dot{\nu}_\iota \mathbf{e}^{2i\pi \eth_{\dot{\nu}_\iota}}\ \forall\ \iota$, then

$$\begin{aligned} &Cq-ROFAAOWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &\quad \geq Cq-ROFAAOWG\left(\dot{\mathbb{Z}}_1, \dot{\mathbb{Z}}_2, \ldots, \dot{\mathbb{Z}}_\flat\right). \end{aligned} \tag{34}$$

Theorem 21 (Boundedness): Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, and let $\mathbb{Z}^- = \min\left\{\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right\}$ and $\mathbb{Z}^+ = \max\left\{\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right\}$, then

$$\mathbb{Z}^- \leq Cq-ROFAAOWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \leq \mathbb{Z}^+. \tag{35}$$

$$\begin{aligned} &\left(\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln \mu_\iota^q\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln \eth_{\mu_\iota}^q\right)^\zeta\right)^{1/\zeta}}}}, \right. \\ &\qquad \left. \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\nu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\eth_{\nu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \right) \\ &\geq \left(\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln \dot{\mu}_\iota^q\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln \eth_{\dot{\mu}_\iota}^q\right)^\zeta\right)^{1/\zeta}}}}, \right. \\ &\qquad \left. \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\dot{\nu}_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\eth_{\dot{\nu}_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \right) \end{aligned}$$

$$\begin{aligned} &\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\mu^{-q}\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\eth_{\mu^-}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \\ &\leq \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\mu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\eth_{\mu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \\ &\leq \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\mu^{+q}\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(1-\eth_{\mu^+}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \\ &\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\nu^{+q}\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\eth_{\nu^+}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \\ &\leq \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\nu_\iota^q\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\eth_{\nu_\iota}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \\ &\leq \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\nu^{-q}\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} w_\iota\left(-\ln\left(\eth_{\nu^-}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \end{aligned}$$

Theorem 22 (Symmetry): Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs. Then, if $\breve{\mathbb{Z}}_\iota = \left(\breve{\mu}_\iota \mathbf{e}^{2i\pi \eth_{\breve{\mu}_\iota}}, \breve{\nu}_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be any permutation of $\mathbb{Z}_\iota$, then we have

$$\begin{aligned} &Cq-ROFAAWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &\quad = Cq-ROFAAWG\left(\breve{\mathbb{Z}}_1, \breve{\mathbb{Z}}_2, \ldots, \breve{\mathbb{Z}}_\flat\right). \end{aligned} \tag{36}$$

The Cq-ROFAAWG operator weights just the Cq-ROFNs, as defined by Definition 17, while the Cq-ROFAAOWG operator weights only the ordered locations of the Cq-ROFNs, as defined by Definition 18. As a result, weights represent various aspects of the Cq-ROFAAWG and Cq-ROFAAOWG operators. Nonetheless, one of the operators and the other operators consider just one of them. To address this shortcoming, we study the complex q-rung orthopair fuzzy Aczel–Alsina hybrid geometric (Cq-ROFAAHG) operator, which weights all of the provided Cq-ROFN and their appropriate ordered position.

Definition 19: Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, then the Cq-ROFAAHG operator is:

$$Cq-ROFAAHG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) = \otimes_{\iota=1}^{\flat}\left(\hat{\mathbb{Z}}_{\delta(\iota)}\right)^{\varpi_\iota}, \tag{37}$$

where $\varpi = \left(\varpi_1, \varpi_2, \ldots, \varpi_\flat\right)^T$ is the weight vector associated with q-ROFAAHG $\mathbb{Z}_\iota$ $(\iota = 1, 2, \ldots, \flat)$ such that $\varpi_\iota > 0$ and $\sum_{\iota=1}^{\flat} \varpi_\iota = 1$, $w = \left(w_1, w_2, \ldots, w_\flat\right)^T$ is the weight vector of $\mathbb{Z}_\iota$ $(\iota = 1, 2, \ldots, \flat)$ such that $w_\iota > 0$ and $\sum_{\iota=1}^{\flat} w_\iota = 1$. $\hat{\mathbb{Z}}_{\delta(\iota)}$ is the ιth largest of the weighted Cq-ROFNs $\hat{\mathbb{Z}}_\iota$ $\left(\hat{\mathbb{Z}}_\iota = \mathbb{Z}_\iota^{(\flat w_\iota)}\right)$, $(\iota = 1, 2, \ldots, \flat)$ and $\flat$ is the balancing coefficient.

Theorem 23: Let $\mathbb{Z}_\iota = \left(\mu_\iota \mathbf{e}^{2i\pi \eth_{\mu_\iota}}, \nu_\iota \mathbf{e}^{2i\pi \eth_{\nu_\iota}}\right)$ $(\iota = 1, 2, \ldots, \flat)$ be a family of Cq-ROFNs, then the result obtained by utilizing Cq-ROFAAHG operator is still a Cq-ROFN, and (38), as shown at the bottom of the next page.

Proof: We skip the proof of this theorem since it is analogous to that of Theorem 13. ■

As analogous to those of Cq-ROFAAWG operator and Cq-ROFAAOWG, the Cq-ROFAAHG operator also follows the idempotency, monotonicity, boundedness and symmetry properties. Besides the aforesaid characteristics, the Cq-ROFAAHG operator has the following special cases.

Corollary 3: Cq-ROFAAWG operator is a special case of the Cq-ROFAAHG operator.

Proof: Let $\varpi = \left(\frac{1}{\flat}, \frac{1}{\flat}, \ldots, \frac{1}{\flat}\right)^T$, then

$$\begin{aligned} &Cq-ROFAAHG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &\quad = \hat{\mathbb{Z}}_{\delta(1)}^{\varpi_1} \otimes \hat{\mathbb{Z}}_{\delta(2)}^{\varpi_2} \otimes \ldots \otimes \hat{\mathbb{Z}}_{\delta(\flat)}^{\varpi_\flat} \\ &\quad = \left(\hat{\mathbb{Z}}_{\delta(1)} \otimes \hat{\mathbb{Z}}_{\delta(2)} \otimes \ldots \otimes \hat{\mathbb{Z}}_{\delta(\flat)}\right)^{\frac{1}{\flat}} \\ &\quad = \mathbb{Z}_1^{w_1} \otimes \mathbb{Z}_2^{w_2} \otimes \ldots \otimes \mathbb{Z}_\flat^{w_\flat} \\ &\quad = Cq-ROFAAWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right). \end{aligned}$$

■

Corollary 4: Cq-ROFAAOWG operator is a special case of the Cq-ROFAAHG operator.

Proof: Let $w = \left(\frac{1}{\flat}, \frac{1}{\flat}, \ldots, \frac{1}{\flat}\right)^T$, then

$$\begin{aligned} &Cq-ROFAAHG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) \\ &\quad = \hat{\mathbb{Z}}_{\delta(1)}^{\varpi_1} \otimes \hat{\mathbb{Z}}_{\delta(2)}^{\varpi_2} \otimes \ldots \otimes \hat{\mathbb{Z}}_{\delta(\flat)}^{\varpi_\flat} \\ &\quad = \mathbb{Z}_{\delta(1)}^{\varpi_1} \otimes \mathbb{Z}_{\delta(2)}^{\varpi_2} \otimes \ldots \otimes \mathbb{Z}_{\delta(\flat)}^{\varpi_\flat} \\ &\quad = Cq-ROFAAOWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right). \end{aligned}$$

■

V. NOVEL COMPLEX Q-RUNG ORTHOPAIR FUZZY ENTROPY MEASURE

This section is dedicated to construct a new entropy of C-qROFS to overcome the shortcomings of the existing complex q-rung orthopair fuzzy entropies. First, we let the universe be $X = \{\hbar_1, \hbar_2, \ldots, \hbar_n\}$, any C-qROFS on X is $\mathbb{Z} = \left\{\left(\hbar, \mu(\hbar)\mathbf{e}^{i2\pi(\eth_\mu)}, \nu(\hbar)\mathbf{e}^{i2\pi(\eth_\nu)}\right) | \hbar \in X\right\}$. Next, a new complex q-rung orthopair fuzzy is created by combining membership and non-membership differences with degree of hesitation $\pi(\hbar)$ in order to more accurately characterise the uncertainty of information. Additionally, a comparison is made to underline the shortcomings of the present ones.

In what follows, we first review the existing complex q-rung orthopair fuzzy entropy measure given by Mahmood and Ali [49] and at the end we draw a comparison through solid examples.

Definition 20: [49] For the Cq-ROFS $\mathbb{Z} = \left\{\left(\hbar, \mu(\hbar)\mathbf{e}^{i2\pi(\eth_\mu)}, \nu(\hbar)\mathbf{e}^{i2\pi(\eth_\nu)}\right) | \hbar \in X\right\}$, the complex q-rung orthopair fuzzy entropy of $\mathbb{Z}$ is given as takes after (39) and (40), as shown at the bottom of the next page.

$$\begin{aligned} Cq-ROFAAOWG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) = \Bigg(&\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} \varpi_\iota\left(-\ln \mu_{\delta(\iota)}^q\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} \varpi_\iota\left(-\ln \eth_{\mu\,\delta(\iota)}^q\right)^\zeta\right)^{1/\zeta}}}}, \\ &\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} \varpi_\iota\left(-\ln\left(1-\nu_{\delta(\iota)}^q\right)\right)^\zeta\right)^{1/\zeta}}}\, \mathbf{e}^{2i\pi \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat} \varpi_\iota\left(-\ln\left(1-\eth_{\nu\,\delta(\iota)}^q\right)\right)^\zeta\right)^{1/\zeta}}}} \Bigg) \end{aligned} \tag{32}$$

In Eqs. (39) and (40), difference of membership and non-membership degrees are considered when calculating entropy measures. However, in many complicated practical problems, DEs are uncertain due to the limitations of their cognition and the complicatedness of objective things. Thereby, the role of hesitation in complex q-rung orthopair fuzzy entropy should not be depreciated. In addition, some counterintuitive situations still exist for the entropy measure mentioned above; that is, when the difference between the membership degree and non-membership degree is the same, it cannot effectively distinguish the fuzziness of two Cq-ROFSs. Some examples are included as follows:

Example 1: Let the universal set X be a single point set, and $\mathbb{Z}_1 = \left\{\left(\hbar, 0.2\mathbf{e}^{i2\pi(0.3)}, 0.5\mathbf{e}^{i2\pi(0.6)}\right)\right\}$ and $\mathbb{Z}_2 = \left\{\left(\hbar, 0.3\mathbf{e}^{i2\pi(0.2)}, 0.6\mathbf{e}^{i2\pi(0.5)}\right)\right\}$ be two Cq-ROFSs on X. Utilizing Eqs. (39) and (40), we get

$$E_1(\mathbb{Z}_1) = E_1(\mathbb{Z}_2) = E_2(\mathbb{Z}_1) = E_2(\mathbb{Z}_2) = 0.3751.$$

Clearly, $\mathbb{Z}_1 \neq \mathbb{Z}_2$, but the computed result shows that the entropy of Cq-ROFSs $\mathbb{Z}_1$ and $\mathbb{Z}_2$ is equal, which is contrary to the actual situation.

Example 2: Let the universal set X be a single point set, and $\mathbb{Z}_1 = \left\{\left(\hbar, 0.4\mathbf{e}^{i2\pi(0.5)}, 0.6\mathbf{e}^{i2\pi(0.2)}\right)\right\}$ and $\mathbb{Z}_2 = \left\{\left(\hbar, 0.55\mathbf{e}^{i2\pi(0.35)}, 0.25\mathbf{e}^{i2\pi(0.55)}\right)\right\}$ be two Cq-ROFSs on X. Utilizing Eqs. (39) and (40), we get

$$E_1(\mathbb{Z}_1) = E_1(\mathbb{Z}_2) = E_2(\mathbb{Z}_1) = E_2(\mathbb{Z}_2) = 0.4131.$$

Again, we get the same entropy of two different Cq-ROFSs. Thus, E_1 and E_2 are counterintuitive.

The new complex q-rung orthopair fuzzy entropy can be characterized in the following way:

Definition 21: For the Cq-ROFS $\mathbb{Z} = \left\{\left(\hbar, \mu(\hbar)\mathbf{e}^{i2\pi(\eth_\mu)}, \nu(\hbar)\mathbf{e}^{i2\pi(\eth_\nu)}\right) | \hbar \in X\right\}$, the complex q-rung orthopair fuzzy entropy of $\mathbb{Z}$ is given as takes after (41), as shown at the bottom of the next page.

In the subsequent lines, we are going to verify several characteristics associated with the suggested complex q-rung orthopair fuzzy entropy measure.

Theorem 24: The mapping $E : \mathbb{Z} \longrightarrow [0, 1]$ is stated as complex q-rung orthopair fuzzy entropy if it meets the following properties:

1) $E(\mathbb{Z}) = 0$ if and only if $\mathbb{Z}$ is a crisp set;
2) $E(\mathbb{Z}) = 1$ if and only if $\mu(\hbar) = \nu(\hbar)$ and $\eth_\mu(\hbar) = \eth_\nu(\hbar)\ \forall\ \hbar \in X$;
3) $E(\mathbb{Z}_1) \leq E(\mathbb{Z}_2)$ if $\mu_1(\hbar) + \nu_1(\hbar) \geq \mu_2(\hbar) + \nu_2(\hbar)$, $\eth_{\mu_1}(\hbar) + \eth_{\nu_1}(\hbar) \geq \eth_{\mu_2}(\hbar) + \eth_{\mu_2}(\hbar)$, and $|\mu_1(\hbar) - \nu_1(\hbar)| \geq |\mu_2(\hbar) - \nu_2(\hbar)|$, $\left|\eth_{\mu_1}(\hbar) - \eth_{\nu_1}(\hbar)\right| \geq \left|\eth_{\mu_2}(\hbar) - \eth_{\mu_2}(\hbar)\right|$, $\forall\ \hbar \in X$;
4) $E(\mathbb{Z}) = E(\mathbb{Z}^c)$.

Proof:

1) If $E(\mathbb{Z}) = 0$, then $2 = |\mu^q(\hbar) - \nu^q(\hbar)| + \left|\eth^q_\mu(\hbar) - \eth^q_\nu(\hbar)\right| - \pi(\hbar)$. We have $\mu(\hbar) = 0$, $\nu(\hbar) = 1$, $\eth_\mu(\hbar) = 0$, $\eth_\nu(\hbar) = 1$, $\pi(\hbar) = 0$ or $\mu(\hbar) = 1$, $\nu(\hbar) = 0$, $\eth_\mu(\hbar) = 1$, $\eth_\nu(\hbar) = 0$, $\pi(\hbar) = 1$ or $\mu(\hbar) = 0$, $\nu(\hbar) = 1$, $\eth_\mu(\hbar) = 1$, $\eth_\nu(\hbar) = 0$, $\pi(\hbar) = 0$ or $\mu(\hbar) = 1$, $\nu(\hbar) = 0$, $\eth_\mu(\hbar) = 0$, $\eth_\nu(\hbar) = 1$, $\pi(\hbar) = 0$. Hence, in each case $\mathbb{Z}$ is a crisp set. Conversely if $\mathbb{Z}$ is a crisp set, then clearly $E(\mathbb{Z}) = 0$.
2) If $E(\mathbb{Z}) = 0$, then $n = \sum_{\hbar\in X} \frac{2-|\mu^q(\hbar)-\nu^q(\hbar)|-\left|\eth^q_\mu(\hbar)-\eth^q_\nu(\hbar)\right|+\pi(\hbar)}{2+|\mu^q(\hbar)-\nu^q(\hbar)|+\left|\eth^q_\mu(\hbar)-\eth^q_\nu(\hbar)\right|+\pi(\hbar)}$, this implies $\frac{2-|\mu^q(\hbar)-\nu^q(\hbar)|-\left|\eth^q_\mu(\hbar)-\eth^q_\nu(\hbar)\right|+\pi(\hbar)}{2+|\mu^q(\hbar)-\nu^q(\hbar)|+\left|\eth^q_\mu(\hbar)-\eth^q_\nu(\hbar)\right|+\pi(\hbar)} = 1\ \forall\ \hbar \in X$. This further implies that $-|\mu^q(\hbar) - \nu^q(\hbar)| - \left|\eth^q_\mu(\hbar) - \eth^q_\nu(\hbar)\right| = |\mu^q(\hbar) - \nu^q(\hbar)| + \left|\eth^q_\mu(\hbar) - \eth^q_\nu(\hbar)\right|$. Thus, we have $\mu(\hbar) = \nu(\hbar)$ and $\eth_\mu(\hbar) = \eth_\nu(\hbar)\ \forall\ \hbar \in X$.

$$Cq-ROFAAHG\left(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_\flat\right) = \left(\begin{array}{l} \sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}\varpi_\iota\left(-\ln\hat{\mu}^q_{\delta(\iota)}\right)^\zeta\right)^{1/\zeta}}}\ \mathbf{e}^{2i\pi\sqrt[q]{\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}\varpi_\iota\left(-\ln\eth^q_{\hat{\mu}\delta(\iota)}\right)^\zeta\right)^{1/\zeta}}}}, \\ \sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}\varpi_\iota\left(-\ln\left(1-\hat{\nu}^q_{\delta(\iota)}\right)\right)^\zeta\right)^{1/\zeta}}}\ \mathbf{e}^{2i\pi\sqrt[q]{1-\mathbf{e}^{-\left(\sum_{\iota=1}^{\flat}\varpi_\iota\left(-\ln\left(1-\eth^q_{\hat{\nu}\delta(\iota)}\right)\right)^\zeta\right)^{1/\zeta}}}} \end{array}\right) \tag{38}$$

$$E_1(\mathbb{Z}) = \frac{1}{n}\sum_{\hbar\in X}\left\{\left\{\begin{array}{l} \sin\left(\frac{\pi\times\left(2+\mu^q(\hbar)-\nu^q(\hbar)+\eth^q_\mu(\hbar)-\eth^q_\nu(\hbar)\right)}{8}\right)+ \\ \sin\left(\frac{\pi\times\left(2-\mu^q(\hbar)+\nu^q(\hbar)-\eth^q_\mu(\hbar)+\eth^q_\nu(\hbar)\right)}{8}\right)-1 \end{array}\right\}\times\frac{1}{2^{\frac{1}{q}}-1}\right\}, \tag{39}$$

$$E_2(\mathbb{Z}) = \frac{1}{n}\sum_{\hbar\in X}\left\{\left\{\begin{array}{l} \cos\left(\frac{\pi\times\left(2+\mu^q(\hbar)-\nu^q(\hbar)+\eth^q_\mu(\hbar)-\eth^q_\nu(\hbar)\right)}{8}\right)+ \\ \cos\left(\frac{\pi\times\left(2-\mu^q(\hbar)+\nu^q(\hbar)-\eth^q_\mu(\hbar)+\eth^q_\nu(\hbar)\right)}{8}\right)-1 \end{array}\right\}\times\frac{1}{2^{\frac{1}{q}}-1}\right\} \tag{40}$$

Conversely, if we take $\mu(\hbar) = \nu(\hbar)$ and $\eth_{\mu}(\hbar) = \eth_{\nu}(\hbar)\ \forall\ \hbar \in X$, then clearly $E(\mathbb{Z}) = 1$.

3) if $\mu_1(\hbar) + \nu_1(\hbar) \geq \mu_2(\hbar) + \nu_2(\hbar)$, $\eth_{\mu 1}(\hbar) + \eth_{\nu 1}(\hbar) \geq \eth_{\mu 2}(\hbar) + \eth_{\mu 2}(\hbar)$, then $\pi_1(\hbar) \leq \pi_2(\hbar)$; and $|\mu_1(\hbar) - \nu_1(\hbar)| \geq |\mu_2(\hbar) - \nu_2(\hbar)|$, $\left|\eth_{\mu 1}(\hbar) - \eth_{\nu 1}(\hbar)\right| \geq \left|\eth_{\mu 2}(\hbar) - \eth_{\mu 2}(\hbar)\right|$. Therefore, we get $E(\mathbb{Z}_1) \leq E(\mathbb{Z}_2)$.

4) $E(\mathbb{Z}) = \frac{1}{n}\sum_{\hbar\in X} \frac{2-|\mu^q(\hbar)-\nu^q(\hbar)|-\left|\eth^q_{\mu}(\hbar)-\eth^q_{\nu}(\hbar)\right|+\pi(\hbar)}{2+|\mu^q(\hbar)-\nu^q(\hbar)|+\left|\eth^q_{\mu}(\hbar)-\eth^q_{\nu}(\hbar)\right|+\pi(\hbar)} = \frac{1}{n}\sum_{\hbar\in X} \frac{2-|\nu^q(\hbar)-\mu^q(\hbar)|-\left|\eth^q_{\nu}(\hbar)-\eth^q_{\mu}(\hbar)\right|+\pi(\hbar)}{2+|\nu^q(\hbar)-\mu^q(\hbar)|+\left|\eth^q_{\nu}(\hbar)-\eth^q_{\mu}(\hbar)\right|+\pi(\hbar)} = E(\mathbb{Z}^c)$.

■

Following Example 1 above, the entropy measures for the Cq-ROFSs $\mathbb{Z}_1$ and $\mathbb{Z}_2$ are worked out by Eq. (41), as follows:

$$E(\mathbb{Z}_1) = 0.5385, \quad E(\mathbb{Z}_2) = 0.5819.$$

Obviously, the proposed E overcomes the counterintuitive situation of E_1 and E_2.

Analogously, following Example 2, we get

$$E(\mathbb{Z}_1) = 0.6000, \quad E(\mathbb{Z}_2) = 0.5952.$$

Thus, the counterintuitive situation of E_1 and E_2 is address.

Through Examples 1 and 2, it is clear that the entropy measure presented in this study takes into account not only the difference between the membership degree and non-membership degree, but also the hesitation of DEs. It represents the fuzzy degree of the fuzzy set in terms of uncertainty and unknown aspects in a more complete and objective manner. In addition, taking into account the influence of the hesitation degree on complex q-rung orthopair fuzzy entropy, Eq. (41) can be utilised to differentiate the scenario when the deviation of the membership degree and non-membership degree are identical. Meanwhile, the use of Eq. (41) can effectively avoid the emergence of counterintuitive phenomena, which further elaborates the effectiveness of the novel entropy measure.

VI. MCDM APPROACH

In this part, we use our proposed Cq-ROFAAHA and Cq-ROFAAHG operators to provide an MCDM method for dealing with MCDM problems in q-rung orthopair fuzzy situations.

Let $o = \{o_1, o_2, \ldots, o_m\}$ be a discrete set of alternatives and $\kappa = \{\kappa_1, \kappa_2, \ldots, \kappa_\flat\}$ be the corresponding set of criteria with weight vector $w = \{w_1, w_2, \ldots, w_\flat\}$ where $w_\iota \in [0, 1]$ such that $\sum_{\iota=1}^{\flat} w_\iota = 1$. A team of experts is assembled to evaluate each alternative $o_i\ (i = 1, 2, \ldots m)$ in relation to the relevant criteria $\kappa_\iota\ (\iota = 1, 2, \ldots, \flat)$. The experts provide the evaluation information in the form of Cq-ROFNs marked by $\mathbb{Z}_{i\iota} = \left(\mu_{i\iota}\mathbf{e}^{2i\pi\eth_{\mu_{i\iota}}}, \nu_{i\iota}\mathbf{e}^{2i\pi\eth_{\nu_{i\iota}}}\right)$ where according to experts $\mu_{i\iota}\mathbf{e}^{2i\pi\eth_{\mu_{i\iota}}}$ denotes membership, and $\nu_{i\iota}\mathbf{e}^{2i\pi\eth_{\nu_{i\iota}}}$ denotes non-membership grades to which alternative o_i meets that the criteria κ_ι having the constraint that $0 \leq \mu^q_{i\iota} + \nu^q_{i\iota} \leq 1$ for $q \geq 1$.

A. ENTROPY MODEL

Due to limited knowledge, time constraints, and the complexity of problems, it is difficult to provide weight information in advance. To address such a problem efficiently, we calculate the weights of criteria based on the proposed entropy measure as follows:

$$w_\flat = \frac{1 - E_\flat}{\sum_{\iota=1}^{\flat} 1 - E_\flat}. \tag{42}$$

Here $E_\flat \in [0, 1]$, $\flat = 1, 2, \ldots, n$ is defined as

$$E_\flat = \frac{1}{m}\sum_{i=1}^{m} \frac{2 - \left|\mu^q_{i\flat} - \nu^q_{i\flat}\right| - \left|\eth^q_{\mu i\flat} - \eth^q_{\nu i\flat}\right| + \pi_{i\flat}}{2 + \left|\mu^q_{i\flat} - \nu^q_{i\flat}\right| + \left|\eth^q_{\mu i\flat} - \eth^q_{\nu i\flat}\right| + \pi_{i\flat}}. \tag{43}$$

Setting the value of $q = 1, 2$ in the above equation, it reduced to CPyFS (for $q = 2$) and CIFS (for $q = 1$). Further, if we take the imaginary part zero, then in that case Eq. (42) reduced to PyFS (for $q = 2$) and IFS (for $q = 1$).

B. ALGORITHM

In the subsequent steps, we outline the suggested model's decision process.

Step 1 From the preceding analysis gather the expert's evaluation information provided for each alternative to their corresponding criteria and then build a decision matrix as follows the equation can be derived, as shown at the bottom of next page.

Step 2 Build the normalized decision matrix $N = \left(\widetilde{\mathbb{Z}}_{i\iota}\right)_{m\times\flat}$ by use of the following transformation

$$\widetilde{\mathbb{Z}}_{i\iota} = \begin{cases} \mathbb{Z}_{i\iota}, & \kappa_\iota \text{ is benefit criteria,} \\ \mathbb{Z}^c_{i\iota}, & \kappa_\iota \text{ is cost criteria.} \end{cases} \tag{44}$$

where $\mathbb{Z}^c_{i\iota} = \left(\nu_{i\iota}\mathbf{e}^{2i\pi\eth_{\nu_{i\iota}}}, \mu_{i\iota}\mathbf{e}^{2i\pi\eth_{\mu_{i\iota}}}\right)$ is the complement of $\mathbb{Z}_{i\iota}$.

Step 3 In the light of Eq. (43), determine the criteria weight vector.

Step 4 Use the newly designed Cq-ROFAAHA or Cq-ROFAAHG operator to obtain the overall aggregated result from matrix N row-wise for each alternative o_i.

Step 5 Employ Eq. (4) to determine the score value of each aggregated result derived in Step 2.

$$E(\mathbb{Z}) = \frac{1}{n}\sum_{\hbar\in X} \frac{2 - |\mu^q(\hbar) - \nu^q(\hbar)| - \left|\eth^q_{\mu}(\hbar) - \eth^q_{\nu}(\hbar)\right| + \pi(\hbar)}{2 + |\mu^q(\hbar) - \nu^q(\hbar)| + \left|\eth^q_{\mu}(\hbar) - \eth^q_{\nu}(\hbar)\right| + \pi(\hbar)} \tag{41}$$

Step 6 Rank the alternatives $o_i\,(i = 1, 2, \ldots, m)$ in descending order according to their score values and get the optimal one.

VII. AN ILLUSTRATIVE EXAMPLE

In this part, in order to demonstrate the implementation of the proposed MCDM method, we solve a problem (adopted from Ref. [22]) to examine the sector that had the most impact on the Pakistan Stock Exchange (PSX) during calendar year 2021.

A. BACKGROUND DESCRIPTION

The PSX was established on 11 January 2016 after the merger of the Karachi Stock Exchange, Lahore Stock Exchange and Islamabad Stock Exchange. As of January 2022, over 375 firms with a total market value of PKR 7,756 billion (USD \$52 billion) are listed on the PSX.

There are about 220,000 individual investors in addition to 1,886 international institutional investors and 883 local institutional investors on the exchanges. There are also over 400 brokerage firms and 21 asset management firms that are members of the PSX. The Karachi Stock Exchange, one of the PSX's component stock exchanges, was ranked among the world's top performing frontier stock markets: between 2009 and 2015, it had an average annual return of 26%. In December 2016, PSX sold \$40 million worth of strategic shares to a Chinese consortium. On May 27, 2021, record-breaking trade volumes of 2.20 billion shares were achieved. It surpassed the previous record volume of 1.56 billion shares, which was achieved on May 26, 2021. There are numerous sectors which affect PSX including Sec_1= automobile sector, Sec_2=construction, Sec_3=banking and Financial services, and Sec_4=pharmaceutical sector. we are interested in finding out the most important sector, out of these sectors.

Taking into account the above four sectors $Sec_i; (i = 1, 2, \ldots, 4)$. Our goal is to order these four sectors into decreasing order from the most important to the least important that affected PSX during 2021. Further, we consider the four major factors $\kappa_\iota; (\iota = 1, 2, 3, 4)$ which influenced the role of these sectors $Sec_i; (i = 1, 2, \ldots, 4)$ in PSX and these are κ_1= crude oil price movement, κ_2 = budget 2021, κ_3=performance of the debt market, and κ_4= reduction in repo rate by the State bank of Pakistan (SBP). Experts use Cq-ROFNs to examine sectors since it is time-periodic problem. The influence of a given sector on PSX fluctuates during the course of the year. Some sectors affect stock market for few months only and not for the whole year. Since complex q-rung orthopair fuzzy environment is the ideal environment for handling time-periodic problems, thereby experts employ Cq-ROFNs to describe their evaluation values. In this problem, the amplitude terms represent the influence degree of sectors under the mentioned criteria on PSX, whilst the phase term represents the duration of this influence.

B. THE DECISION-MAKING PROCESS

The steps for addressing the aforesaid MCDM problem are detailed as:

Step 1: The assessment values given by economic experts are listed in Table 1.

In Table 1, we have the assessment value of Sec_1 under criteria κ_1 is given as $\left(0.7\mathbf{e}^{i2\pi(\frac{5}{12})}, 0.1\mathbf{e}^{i2\pi(\frac{2}{12})}\right)$. The membership part 0.7 reveals that the experts believe that there is 70 percent influence of Sec_1 on PSX during the year 2021 under criteria κ_1 and the duration of this influence is of 5 months out of 12 months. Analogously, for non-membership part 0.1, the experts agree that with a degree of 10 percent there is no influence of Sec_1 on PSX during the year 2021 under criteria κ_1 and the time span with no influence is of 2 months out of 12 months. Analogously, the remaining data can be decoded.

Step 2: In the considered problem the criteria κ_2 and κ_3 are benefit type, κ_1 and κ_4 are cost type. Based on Eq. (44) and Table 1, the original decision matrix $M = (\mathbb{Z}_{i\iota})_{4\times 4}$ can be updated to the following normalized decision matrix $M = (\mathbb{Z}_{i\iota})_{4\times 4}$, which is documented in Table 2.

Step 3: According to proposed entropy model, the criteria weight vector is acquired as takes after:

$$w_1 = 0.3471, w_2 = 0.2316, w_3 = 0.2308, w_4 = 0.1905.$$
$$\flat w_1 = 1.388, \flat w_2 = 0.9264, \flat w_3 = 0.9232, \flat w_4 = 0.7620.$$

Step 4: Using the complex q-rung orthoair fuzzy information detailed in Table 1, the values of $\hat{\mathbb{Z}}_{i\iota} = (\flat w_\iota)\,\mathbb{Z}_{i\iota}$ are computed as shown follows:

$$\hat{\mathbb{Z}}_{11} = \left(0.1179\mathbf{e}^{i2\pi(0.1960)}, 0.6095\mathbf{e}^{i2\pi(0.2967)}\right),$$
$$\hat{\mathbb{Z}}_{12} = \left(0.5820\mathbf{e}^{i2\pi(0.3216)}, 0.4278\mathbf{e}^{i2\pi(0.5262)}\right),$$

$$\begin{array}{c} \\ o_1 \\ \vdots \\ o_i \\ \vdots \\ o_m \end{array}
\begin{array}{c}
\begin{array}{ccccc} \kappa_1 & \cdots & \kappa_\iota & \cdots & \kappa_\flat \end{array} \\
\begin{pmatrix}
\left(\mu_{11}\mathbf{e}^{2i\pi\eth_{\mu 11}}, \nu_{11}\mathbf{e}^{2i\pi\eth_{\nu 11}}\right) & \cdots & \left(\mu_{1\iota}\mathbf{e}^{2i\pi\eth_{\mu 1\iota}}, \nu_{1\iota}\mathbf{e}^{2i\pi\eth_{\nu 1\iota}}\right) & \cdots & \left(\mu_{1\flat}\mathbf{e}^{2i\pi\eth_{\mu 1\flat}}, \nu_{1\flat}\mathbf{e}^{2i\pi\eth_{\mu 1\flat}}\right) \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\left(\mu_{i1}\mathbf{e}^{2i\pi\eth_{\mu i1}}, \nu_{i1}\mathbf{e}^{2i\pi\eth_{\nu i1}}\right) & \cdots & \left(\mu_{i\iota}\mathbf{e}^{2i\pi\eth_{\mu i\iota}}, \nu_{i\iota}\mathbf{e}^{2i\pi\eth_{\nu i\iota}}\right) & \cdots & \left(\mu_{i\flat}\mathbf{e}^{2i\pi\eth_{\mu i\flat}}, \nu_{i\flat}\mathbf{e}^{2i\pi\eth_{\nu i\flat}}\right) \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\left(\mu_{m1}\mathbf{e}^{2i\pi\eth_{\mu m1}}, \nu_{m1}\mathbf{e}^{2i\pi\eth_{\nu m1}}\right) & \cdots & \left(\mu_{m\iota}\mathbf{e}^{2i\pi\eth_{\mu m\iota}}, \nu_{m\iota}\mathbf{e}^{2i\pi\eth_{\nu m\iota}}\right) & \cdots & \left(\mu_{m\flat}\mathbf{e}^{2i\pi\eth_{\mu m\flat}}, \nu_{m\flat}\mathbf{e}^{2i\pi\eth_{\nu m\flat}}\right)
\end{pmatrix}
\end{array}$$

TABLE 1. Complex q-rung orthopair fuzzy decision matrix ***C***.

	κ_1	κ_2	κ_3	κ_4
Sec_1	$\left(0.7\mathbf{e}^{i2\pi(\frac{5}{12})}, 0.1\mathbf{e}^{i2\pi(\frac{2}{12})}\right)$	$\left(0.6\mathbf{e}^{i2\pi(\frac{4}{12})}, 0.4\mathbf{e}^{i2\pi(\frac{6}{12})}\right)$	$\left(0.4\mathbf{e}^{i2\pi(\frac{5}{12})}, 0.8\mathbf{e}^{i2\pi(\frac{7}{12})}\right)$	$\left(0.5\mathbf{e}^{i2\pi(\frac{5}{12})}, 0.3\mathbf{e}^{i2\pi(\frac{1}{12})}\right)$
Sec_2	$\left(0.7\mathbf{e}^{i2\pi(\frac{7}{12})}, 0.2\mathbf{e}^{i2\pi(\frac{1}{12})}\right)$	$\left(0.8\mathbf{e}^{i2\pi(\frac{8}{12})}, 0.2\mathbf{e}^{i2\pi(\frac{1}{12})}\right)$	$\left(0.9\mathbf{e}^{i2\pi(\frac{3}{12})}, 0.3\mathbf{e}^{i2\pi(\frac{5}{12})}\right)$	$\left(0.3\mathbf{e}^{i2\pi(\frac{4}{12})}, 0.3\mathbf{e}^{i2\pi(\frac{2}{12})}\right)$
Sec_3	$\left(0.6\mathbf{e}^{i2\pi(\frac{10}{12})}, 0.3\mathbf{e}^{i2\pi(\frac{4}{12})}\right)$	$\left(0.7\mathbf{e}^{i2\pi(\frac{6}{12})}, 0.3\mathbf{e}^{i2\pi(\frac{4}{12})}\right)$	$\left(0.6\mathbf{e}^{i2\pi(\frac{7}{12})}, 0.5\mathbf{e}^{i2\pi(\frac{8}{12})}\right)$	$\left(0.7\mathbf{e}^{i2\pi(\frac{9}{12})}, 0.4\mathbf{e}^{i2\pi(\frac{3}{12})}\right)$
Sec_4	$\left(0.9\mathbf{e}^{i2\pi(\frac{8}{12})}, 0.4\mathbf{e}^{i2\pi(\frac{3}{12})}\right)$	$\left(0.5\mathbf{e}^{i2\pi(\frac{4}{12})}, 0.3\mathbf{e}^{i2\pi(\frac{5}{12})}\right)$	$\left(0.4\mathbf{e}^{i2\pi(\frac{5}{12})}, 0.6\mathbf{e}^{i2\pi(\frac{6}{12})}\right)$	$\left(0.7\mathbf{e}^{i2\pi(\frac{3}{12})}, 0.2\mathbf{e}^{i2\pi(\frac{4}{12})}\right)$

TABLE 2. Normalized complex q-rung orthopair fuzzy decision matrix ***N***.

	κ_1	κ_2	κ_3	κ_4
Sec_1	$\left(0.1\mathbf{e}^{i2\pi(\frac{2}{12})}, 0.7\mathbf{e}^{i2\pi(\frac{5}{12})}\right)$	$\left(0.6\mathbf{e}^{i2\pi(\frac{4}{12})}, 0.4\mathbf{e}^{i2\pi(\frac{6}{12})}\right)$	$\left(0.4\mathbf{e}^{i2\pi(\frac{5}{12})}, 0.8\mathbf{e}^{i2\pi(\frac{7}{12})}\right)$	$\left(0.3\mathbf{e}^{i2\pi(\frac{1}{12})}, 0.5\mathbf{e}^{i2\pi(\frac{5}{12})}\right)$
Sec_2	$\left(0.2\mathbf{e}^{i2\pi(\frac{1}{12})}, 0.7\mathbf{e}^{i2\pi(\frac{7}{12})}\right)$	$\left(0.8\mathbf{e}^{i2\pi(\frac{8}{12})}, 0.2\mathbf{e}^{i2\pi(\frac{1}{12})}\right)$	$\left(0.9\mathbf{e}^{i2\pi(\frac{3}{12})}, 0.3\mathbf{e}^{i2\pi(\frac{5}{12})}\right)$	$\left(0.3\mathbf{e}^{i2\pi(\frac{2}{12})}, 0.3\mathbf{e}^{i2\pi(\frac{4}{12})}\right)$
Sec_3	$\left(0.3\mathbf{e}^{i2\pi(\frac{4}{12})}, 0.6\mathbf{e}^{i2\pi(\frac{10}{12})}\right)$	$\left(0.7\mathbf{e}^{i2\pi(\frac{6}{12})}, 0.3\mathbf{e}^{i2\pi(\frac{4}{12})}\right)$	$\left(0.6\mathbf{e}^{i2\pi(\frac{7}{12})}, 0.5\mathbf{e}^{i2\pi(\frac{8}{12})}\right)$	$\left(0.4\mathbf{e}^{i2\pi(\frac{3}{12})}, 0.7\mathbf{e}^{i2\pi(\frac{9}{12})}\right)$
Sec_4	$\left(0.4\mathbf{e}^{i2\pi(\frac{3}{12})}, 0.9\mathbf{e}^{i2\pi(\frac{8}{12})}\right)$	$\left(0.5\mathbf{e}^{i2\pi(\frac{4}{12})}, 0.3\mathbf{e}^{i2\pi(\frac{5}{12})}\right)$	$\left(0.4\mathbf{e}^{i2\pi(\frac{5}{12})}, 0.6\mathbf{e}^{i2\pi(\frac{6}{12})}\right)$	$\left(0.2\mathbf{e}^{i2\pi(\frac{4}{12})}, 0.7\mathbf{e}^{i2\pi(\frac{3}{12})}\right)$

$$
\begin{aligned}
\hat{\mathbb{Z}}_{13} &= \left(0.3856\mathbf{e}^{i2\pi(0.4019)}, 0.8138\mathbf{e}^{i2\pi(0.6079)}\right), \\
\hat{\mathbb{Z}}_{14} &= \left(0.2632\mathbf{e}^{i2\pi(0.07280)}, 0.5897\mathbf{e}^{i2\pi(0.5132)}\right), \\
\hat{\mathbb{Z}}_{21} &= \left(0.2347\mathbf{e}^{i2\pi(0.09798)}, 0.6095\mathbf{e}^{i2\pi(0.4733)}\right), \\
\hat{\mathbb{Z}}_{22} &= \left(0.7823\mathbf{e}^{i2\pi(0.6479)}, 0.2251\mathbf{e}^{i2\pi(0.1000)}\right), \\
\hat{\mathbb{Z}}_{23} &= \left(0.8855\mathbf{e}^{i2\pi(0.2404)}, 0.3291\mathbf{e}^{i2\pi(0.4455)}\right), \\
\hat{\mathbb{Z}}_{24} &= \left(0.2632\mathbf{e}^{i2\pi(0.1459)}, 0.3995\mathbf{e}^{i2\pi(0.4328)}\right), \\
\hat{\mathbb{Z}}_{31} &= \left(0.3503\mathbf{e}^{i2\pi(0.3883)}, 0.4919\mathbf{e}^{i2\pi(0.7764)}\right), \\
\hat{\mathbb{Z}}_{32} &= \left(0.6812\mathbf{e}^{i2\pi(0.4836)}, 0.3277\mathbf{e}^{i2\pi(0.3612)}\right), \\
\hat{\mathbb{Z}}_{33} &= \left(0.5811\mathbf{e}^{i2\pi(0.5647)}, 0.5273\mathbf{e}^{i2\pi(0.6877)}\right), \\
\hat{\mathbb{Z}}_{34} &= \left(0.3527\mathbf{e}^{i2\pi(0.2191)}, 0.7620\mathbf{e}^{i2\pi(0.8031)}\right), \\
\hat{\mathbb{Z}}_{41} &= \left(0.4637\mathbf{e}^{i2\pi(0.2927)}, 0.8639\mathbf{e}^{i2\pi(0.5695)}\right), \\
\hat{\mathbb{Z}}_{42} &= \left(0.4836\mathbf{e}^{i2\pi(0.3216)}, 0.3277\mathbf{e}^{i2\pi(0.4444)}\right), \\
\hat{\mathbb{Z}}_{43} &= \left(0.3856\mathbf{e}^{i2\pi(0.4019)}, 0.6239\mathbf{e}^{i2\pi(0.5275)}\right), \\
\hat{\mathbb{Z}}_{44} &= \left(0.1749\mathbf{e}^{i2\pi(0.2929)}, 0.7620\mathbf{e}^{i2\pi(0.3477)}\right).
\end{aligned}
$$

Based on the score function Eq. (7), we have

$$
\begin{aligned}
\hat{\mathbb{Z}}_{\delta(11)} &= \hat{\mathbb{Z}}_{12} \\
&= \left(0.5820\mathbf{e}^{i2\pi(0.3216)}, 0.4278\mathbf{e}^{i2\pi(0.5262)}\right), \\
\hat{\mathbb{Z}}_{\delta(12)} &= \hat{\mathbb{Z}}_{11} \\
&= \left(0.1179\mathbf{e}^{i2\pi(0.1960)}, 0.6095\mathbf{e}^{i2\pi(0.2967)}\right), \\
\hat{\mathbb{Z}}_{\delta(13)} &= \hat{\mathbb{Z}}_{14} \\
&= \left(0.2632\mathbf{e}^{i2\pi(0.07280)}, 0.5897\mathbf{e}^{i2\pi(0.5132)}\right), \\
\hat{\mathbb{Z}}_{\delta(14)} &= \hat{\mathbb{Z}}_{13} \\
&= \left(0.3856\mathbf{e}^{i2\pi(0.4019)}, 0.8138\mathbf{e}^{i2\pi(0.6079)}\right), \\
\hat{\mathbb{Z}}_{\delta(21)} &= \hat{\mathbb{Z}}_{22} \\
&= \left(0.7823\mathbf{e}^{i2\pi(0.6479)}, 0.2251\mathbf{e}^{i2\pi(0.1000)}\right), \\
\hat{\mathbb{Z}}_{\delta(22)} &= \hat{\mathbb{Z}}_{23} \\
&= \left(0.8855\mathbf{e}^{i2\pi(0.2404)}, 0.3291\mathbf{e}^{i2\pi(0.4455)}\right), \\
\hat{\mathbb{Z}}_{\delta(23)} &= \hat{\mathbb{Z}}_{24} \\
&= \left(0.2632\mathbf{e}^{i2\pi(0.1459)}, 0.3995\mathbf{e}^{i2\pi(0.4328)}\right), \\
\hat{\mathbb{Z}}_{\delta(24)} &= \hat{\mathbb{Z}}_{21} \\
&= \left(0.2347\mathbf{e}^{i2\pi(0.09798)}, 0.6095\mathbf{e}^{i2\pi(0.4733)}\right), \\
\hat{\mathbb{Z}}_{\delta(31)} &= \hat{\mathbb{Z}}_{32} \\
&= \left(0.6812\mathbf{e}^{i2\pi(0.4836)}, 0.3277\mathbf{e}^{i2\pi(0.3612)}\right), \\
\hat{\mathbb{Z}}_{\delta(32)} &= \hat{\mathbb{Z}}_{33} \\
&= \left(0.5811\mathbf{e}^{i2\pi(0.5647)}, 0.5273\mathbf{e}^{i2\pi(0.6877)}\right), \\
\hat{\mathbb{Z}}_{\delta(33)} &= \hat{\mathbb{Z}}_{31} \\
&= \left(0.3503\mathbf{e}^{i2\pi(0.3883)}, 0.4919\mathbf{e}^{i2\pi(0.7764)}\right), \\
\hat{\mathbb{Z}}_{\delta(34)} &= \hat{\mathbb{Z}}_{34} \\
&= \left(0.3527\mathbf{e}^{i2\pi(0.2191)}, 0.7620\mathbf{e}^{i2\pi(0.8031)}\right), \\
\hat{\mathbb{Z}}_{\delta(41)} &= \hat{\mathbb{Z}}_{42} \\
&= \left(0.4836\mathbf{e}^{i2\pi(0.3216)}, 0.3277\mathbf{e}^{i2\pi(0.4444)}\right), \\
\hat{\mathbb{Z}}_{\delta(42)} &= \hat{\mathbb{Z}}_{43} \\
&= \left(0.3856\mathbf{e}^{i2\pi(0.4019)}, 0.6239\mathbf{e}^{i2\pi(0.5275)}\right), \\
\hat{\mathbb{Z}}_{\delta(43)} &= \hat{\mathbb{Z}}_{44} \\
&= \left(0.1749\mathbf{e}^{i2\pi(0.2929)}, 0.7620\mathbf{e}^{i2\pi(0.3477)}\right), \\
\hat{\mathbb{Z}}_{\delta(44)} &= \hat{\mathbb{Z}}_{41} \\
&= \left(0.4637\mathbf{e}^{i2\pi(0.2927)}, 0.8639\mathbf{e}^{i2\pi(0.5695)}\right).
\end{aligned}
$$

Now utilizing Cq-ROFAAHA operator i.e., Eq. (22) (q=2 and $\zeta = 1$), having associated weight vector $\varpi =$

{0.2, 0.2, 0.3, 0.3} to work out the overall value of each alternative o_i, shown as follows:

$$\mathbb{Z}_1 = \left(0.4005\mathbf{e}^{i2\pi(0.2786)}, 0.5770\mathbf{e}^{i2\pi(0.4539)}\right),$$
$$\mathbb{Z}_2 = \left(0.6434\mathbf{e}^{i2\pi(0.3494)}, 0.3891\mathbf{e}^{i2\pi(0.3335)}\right),$$
$$\mathbb{Z}_3 = \left(0.5520\mathbf{e}^{i2\pi(0.4600)}, 0.4853\mathbf{e}^{i2\pi(0.5993)}\right),$$
$$\mathbb{Z}_4 = \left(0.4088\mathbf{e}^{i2\pi(0.3381)}, 0.5712\mathbf{e}^{i2\pi(0.4679)}\right).$$

Step 5: In the light of Eq. (7), figure out the score value of each alternative Sec_i, derived as follows:

$$S(Sec_1) = 0.4248, S(Sec_2) = 0.5684,$$
$$S(Sec_3) = 0.4804, S(Sec_4) = 0.4341.$$

Step 5: The ranking of alternatives is $Sec_2 \succ Sec_3 \succ Sec_4 \succ Sec_1$. Hence, Sec_2 is the most important sector.

Now, we are leveraging the Cq-ROFAAHG operator to emulate the decision-making process.

According to the Cq-ROFAAHG operator, the main steps are as follows:

Step 1-2: These are identical to above Steps 1-2.

Step 3: Using the complex q-rung orthoair fuzzy information detailed in Table 2, the values of $\hat{\mathbb{Z}}_{\iota\iota} = \mathbb{Z}_{\iota\iota}^{(\flat w_\iota)}$ are computed as shown follows:

$$\hat{\mathbb{Z}}_{11} = \left(0.04093\mathbf{e}^{i2\pi(0.08312)}, 0.7792\mathbf{e}^{i2\pi(0.4823)}\right),$$
$$\hat{\mathbb{Z}}_{12} = \left(0.6229\mathbf{e}^{i2\pi(0.3612)}, 0.3863\mathbf{e}^{i2\pi(0.4836)}\right),$$
$$\hat{\mathbb{Z}}_{13} = \left(0.4292\mathbf{e}^{i2\pi(0.4455)}, 0.7815\mathbf{e}^{i2\pi(0.5647)}\right),$$
$$\hat{\mathbb{Z}}_{14} = \left(0.3995\mathbf{e}^{i2\pi(0.1505)}, 0.4436\mathbf{e}^{i2\pi(0.3677)}\right),$$
$$\hat{\mathbb{Z}}_{21} = \left(0.1071\mathbf{e}^{i2\pi(0.03178)}, 0.7792\mathbf{e}^{i2\pi(0.6623)}\right),$$
$$\hat{\mathbb{Z}}_{22} = \left(0.8132\mathbf{e}^{i2\pi(0.6869)}, 0.1926\mathbf{e}^{i2\pi(0.0800)}\right),$$
$$\hat{\mathbb{Z}}_{23} = \left(0.9073\mathbf{e}^{i2\pi(0.2780)}, 0.2888\mathbf{e}^{i2\pi(0.4019)}\right),$$
$$\hat{\mathbb{Z}}_{24} = \left(0.3995\mathbf{e}^{i2\pi(0.2552)}, 0.2632\mathbf{e}^{i2\pi(0.2929)}\right),$$
$$\hat{\mathbb{Z}}_{31} = \left(0.1881\mathbf{e}^{i2\pi(0.2175)}, 0.6796\mathbf{e}^{i2\pi(0.8983)}\right),$$
$$\hat{\mathbb{Z}}_{32} = \left(0.7186\mathbf{e}^{i2\pi(0.5262)}, 0.2893\mathbf{e}^{i2\pi(0.3216)}\right),$$
$$\hat{\mathbb{Z}}_{33} = \left(0.6239\mathbf{e}^{i2\pi(0.6079)}, 0.4830\mathbf{e}^{i2\pi(0.6471)}\right),$$
$$\hat{\mathbb{Z}}_{34} = \left(0.4973\mathbf{e}^{i2\pi(0.3477)}, 0.6336\mathbf{e}^{i2\pi(0.6837)}\right),$$
$$\hat{\mathbb{Z}}_{41} = \left(0.2803\mathbf{e}^{i2\pi(0.1460)}, 0.9488\mathbf{e}^{i2\pi(0.7468)}\right),$$
$$\hat{\mathbb{Z}}_{42} = \left(0.5262\mathbf{e}^{i2\pi(0.3612)}, 0.2893\mathbf{e}^{i2\pi(0.4025)}\right),$$
$$\hat{\mathbb{Z}}_{43} = \left(0.4292\mathbf{e}^{i2\pi(0.4455)}, 0.5811\mathbf{e}^{i2\pi(0.4830)}\right),$$
$$\hat{\mathbb{Z}}_{44} = \left(0.2933\mathbf{e}^{i2\pi(0.4328)}, 0.6336\mathbf{e}^{i2\pi(0.2191)}\right).$$

Based on the score function Eq. (7), we have

$$\hat{\mathbb{Z}}_{\delta(11)} = \hat{\mathbb{Z}}_{12} = \left(0.6229\mathbf{e}^{i2\pi(0.3612)}, 0.3863\mathbf{e}^{i2\pi(0.4836)}\right),$$
$$\hat{\mathbb{Z}}_{\delta(12)} = \hat{\mathbb{Z}}_{14} = \left(0.3995\mathbf{e}^{i2\pi(0.1505)}, 0.4436\mathbf{e}^{i2\pi(0.3677)}\right),$$
$$\hat{\mathbb{Z}}_{\delta(13)} = \hat{\mathbb{Z}}_{13} = \left(0.4292\mathbf{e}^{i2\pi(0.4455)}, 0.7815\mathbf{e}^{i2\pi(0.5647)}\right),$$
$$\hat{\mathbb{Z}}_{\delta(14)} = \hat{\mathbb{Z}}_{11} = \left(0.04093\mathbf{e}^{i2\pi(0.08312)}, 0.7792\mathbf{e}^{i2\pi(0.4823)}\right),$$
$$\hat{\mathbb{Z}}_{\delta(21)} = \hat{\mathbb{Z}}_{22} = \left(0.8132\mathbf{e}^{i2\pi(0.6869)}, 0.1926\mathbf{e}^{i2\pi(0.0800)}\right),$$
$$\hat{\mathbb{Z}}_{\delta(22)} = \hat{\mathbb{Z}}_{23} = \left(0.9073\mathbf{e}^{i2\pi(0.2780)}, 0.2888\mathbf{e}^{i2\pi(0.4019)}\right),$$
$$\hat{\mathbb{Z}}_{\delta(23)} = \hat{\mathbb{Z}}_{24} = \left(0.3995\mathbf{e}^{i2\pi(0.2552)}, 0.2632\mathbf{e}^{i2\pi(0.2929)}\right),$$
$$\hat{\mathbb{Z}}_{\delta(24)} = \hat{\mathbb{Z}}_{21} = \left(0.1071\mathbf{e}^{i2\pi(0.03178)}, 0.7792\mathbf{e}^{i2\pi(0.6623)}\right),$$
$$\hat{\mathbb{Z}}_{\delta(31)} = \hat{\mathbb{Z}}_{32} = \left(0.7186\mathbf{e}^{i2\pi(0.5262)}, 0.2893\mathbf{e}^{i2\pi(0.3216)}\right),$$
$$\hat{\mathbb{Z}}_{\delta(32)} = \hat{\mathbb{Z}}_{33} = \left(0.6239\mathbf{e}^{i2\pi(0.6079)}, 0.4830\mathbf{e}^{i2\pi(0.6471)}\right),$$
$$\hat{\mathbb{Z}}_{\delta(33)} = \hat{\mathbb{Z}}_{34} = \left(0.4973\mathbf{e}^{i2\pi(0.3477)}, 0.6336\mathbf{e}^{i2\pi(0.6837)}\right),$$
$$\hat{\mathbb{Z}}_{\delta(34)} = \hat{\mathbb{Z}}_{31} = \left(0.1881\mathbf{e}^{i2\pi(0.2175)}, 0.6796\mathbf{e}^{i2\pi(0.8983)}\right),$$
$$\hat{\mathbb{Z}}_{\delta(41)} = \hat{\mathbb{Z}}_{42} = \left(0.5262\mathbf{e}^{i2\pi(0.3612)}, 0.2893\mathbf{e}^{i2\pi(0.4025)}\right),$$
$$\hat{\mathbb{Z}}_{\delta(42)} = \hat{\mathbb{Z}}_{44} = \left(0.2933\mathbf{e}^{i2\pi(0.4328)}, 0.6336\mathbf{e}^{i2\pi(0.2191)}\right),$$
$$\hat{\mathbb{Z}}_{\delta(43)} = \hat{\mathbb{Z}}_{43} = \left(0.4292\mathbf{e}^{i2\pi(0.4455)}, 0.5811\mathbf{e}^{i2\pi(0.4830)}\right),$$
$$\hat{\mathbb{Z}}_{\delta(44)} = \hat{\mathbb{Z}}_{41} = \left(0.2803\mathbf{e}^{i2\pi(0.1460)}, 0.9488\mathbf{e}^{i2\pi(0.7468)}\right).$$

Now utilizing Cq-ROFAAHG operator i.e., Eq. (37) (suppose q=2 and $\zeta = 1$), having associated weight vector $\varpi = \{0.3, 0.2, 0.25, 0.25\}$ to work out the overall value of each alternative o_i, shown as follows:

$$\mathbb{Z}_1 = \left(0.2938\mathbf{e}^{i2\pi(0.2160)}, 0.6223\mathbf{e}^{i2\pi(0.4730)}\right),$$
$$\mathbb{Z}_2 = \left(0.4524\mathbf{e}^{i2\pi(0.2105)}, 0.4638\mathbf{e}^{i2\pi(0.4224)}\right),$$

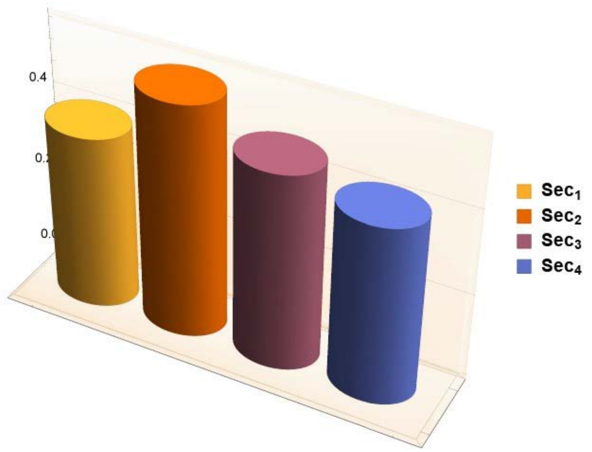

FIGURE 1. Ranking of sectors by Cq-ROFAAHA operator.

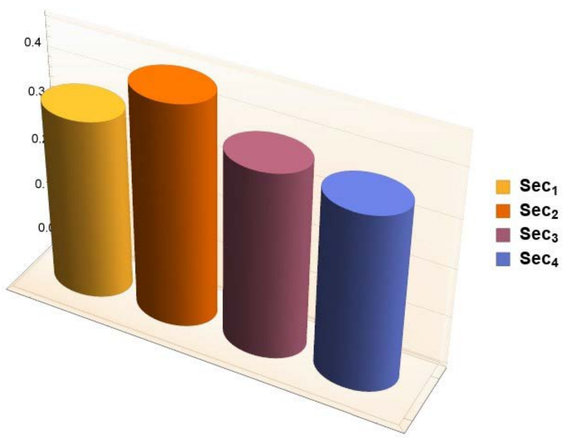

FIGURE 2. Ranking of sectors by Cq-ROFAAHG operator.

$$\mathbb{Z}_3 = \left(0.4717\mathbf{e}^{i2\pi(0.4066)}, 0.5593\mathbf{e}^{i2\pi(0.7104)}\right),$$
$$\mathbb{Z}_4 = \left(0.3738\mathbf{e}^{i2\pi(0.3317)}, 0.7176\mathbf{e}^{i2\pi(0.4973)}\right).$$

Step 4: According to Eq. (7), compute the score value of each alternative o_i, derived as follows:

$S(Sec_1) = 0.3805$, $S(Sec_2) = 0.4686$, $S(Sec_3) = 0.3926$, $S(Sec_4) = 0.3719$.

Step 5: The ranking of alternatives is $Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$. Hence, Sec_2 is the most important sector. It is the same as Cq-ROFAAHA operator. The results obtained by Cq-ROFAAHA operator and Cq-ROFAAHG operator are graphically visualized in Fig. 1 and 2, respectively.

C. INFLUENCE STUDY

To demonstrate the effect of different magnitudes of the parameter ζ, we use different parametric values of ζ inside our given technique to rank the alternatives. Tables 3 and 4 demonstrate the ranking implications of the sectors $Sec_i(i = 1, 2, \ldots m)$ based on the Cq-ROFAAHA and Cq-ROFAAHG operators, as seen in Figs. 3 and 4. It is evident that as the magnitude of ζ for the Cq-ROFAAHA operator grows, so do the score values of the alternatives; nevertheless, when the magnitude of ζ for the Cq-ROFAAHG operator increases, the score values of the alternatives decreases. However, in both circumstances, the associated ranking stays constant, suggesting that the suggested technique always has the isotonicity quality, enabling DEs to choose the optimal value depending on their preferences. Based on these numbers and analysis, it is concluded that a DE may choose the appropriate value of ζ based on its decision-making behaviour. For example, if the DE is the most hopeful about the choice, he or she might choose the Cq-ROFAAHG operator with smaller ζ values. However, if he/she uses the Cq-ROFAAHA operator to aggregate the process, he/she may use greater values of the parameters ζ. Furthermore, if a DE employs the Cq-ROFAAHA operator throughout the aggregation process to get the most pessimistic choice, he or she might choose lower values of ζ. The influence of ζ values on decision making makes our proposed strategy more flexible since DEs may change the parameters based on their preferences and practical scenarios. Furthermore, as shown in Figs. 3 and 4, even though the values of ζ vary throughout the presentation, the outputs of the choices seem to be the same, confirming the consistency of the recommended operators.

VIII. COMPARATIVE ANALYSIS

To study the efficacy of the schemed aggregation operators, this section compares several preexisting aggregation operators such as complex q-rung orthopair fuzzy weighted averaging (Cq-ROFWA) [56], complex q-rung orthopair fuzzy weighted geometric (Cq-ROFWG) [56], complex pythagorean fuzzy einstein ordered weighted averaging (CPFEOWA) [57], complex pythagorean fuzzy einstein ordered weighted geometric (CPFEOWG) [57], complex pythagorean fuzzy einstein hybrid averaging (CPFEHA) [57], complex pythagorean fuzzy einstein hybrid geometric (CPFEHG) [57], intuitionistic fuzzy Aczel–alsina hybrid averaging (IFAAHA) [44], intuitionistic fuzzy Aczel–alsina hybrid geometric (IFAAHG) [44], q-rung orthopair fuzzy weighted averaging (q-ROFWA) [54], q-rung orthopair fuzzy weighted geometric (q-ROFWG) [54], q-rung orthopair fuzzy frank ordered weighted averaging (q-ROFFOWA) [39], q-rung orthopair fuzzy frank ordered weighted geometric (q-ROFFOWG) [39], q-rung orthopair fuzzy frank hybrid averaging (q-ROFFHA) [39], q-rung orthopair fuzzy frank hybrid geometric (q-ROFFHG) [39], and the proposed aggregation operators Cq-ROFAAHA and Cq-ROFAAHG in the same working environment. We discovered some interesting findings, which are detailed as follows:

i). According to the Table 5, the most significant sector derived from all the operators under consideration is the same, with the exception of the IFAAHA [44], and IFAAHG [44] operators, which are invalid since the IFS requirement is not met in the studied data. Except for [44] operators, which can

TABLE 3. Ranking results by Cq-ROFAAHA with various ζ.

ζ	$S(Sec_1)$	$S(Sec_2)$	$S(Sec_3)$	$S(Sec_4)$	Ranking
$\zeta=1$	0.4248	0.5684	0.4804	0.4341	$Sec_2 \succ Sec_3 \succ Sec_4 \succ Sec_1$
$\zeta=3$	0.4798	0.6928	0.5303	0.4885	$Sec_2 \succ Sec_3 \succ Sec_4 \succ Sec_1$
$\zeta=5$	0.4979	0.7331	0.5908	0.5136	$Sec_2 \succ Sec_3 \succ Sec_4 \succ Sec_1$
$\zeta=7$	0.5098	0.7519	0.6082	0.5277	$Sec_2 \succ Sec_3 \succ Sec_4 \succ Sec_1$
$\zeta=9$	0.5277	0.7656	0.6187	0.5389	$Sec_2 \succ Sec_3 \succ Sec_4 \succ Sec_1$
$\zeta=50$	0.5442	0.7950	0.6504	0.5623	$Sec_2 \succ Sec_3 \succ Sec_4 \succ Sec_1$
$\zeta=75$	0.5461	0.7973	0.6528	0.5642	$Sec_2 \succ Sec_3 \succ Sec_4 \succ Sec_1$
$\zeta=100$	0.5471	0.7985	0.6543	0.5651	$Sec_2 \succ Sec_3 \succ Sec_4 \succ Sec_1$
$\zeta=150$	0.5481	0.7996	0.6550	0.5660	$Sec_2 \succ Sec_3 \succ Sec_4 \succ Sec_1$

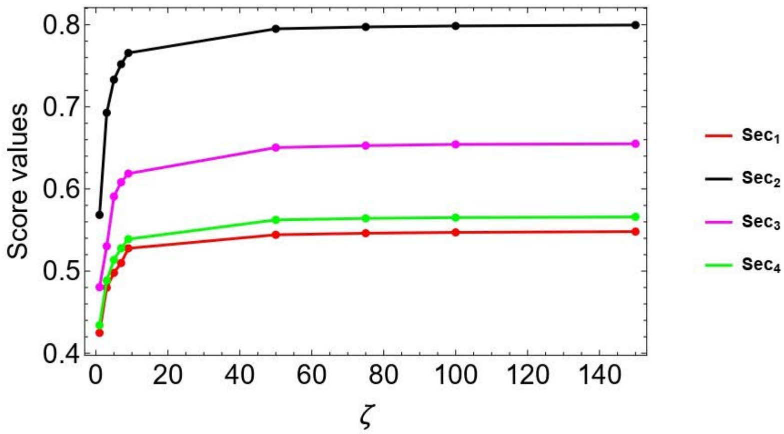

FIGURE 3. Score values of the alternatives for various values ζ by Cq-ROFAAHA operator.

TABLE 4. Ranking results by Cq-ROFAAHG with various ζ.

ζ	$S(Sec_1)$	$S(Sec_2)$	$S(Sec_3)$	$S(Sec_4)$	Ranking
$\zeta=1$	0.3805	0.4686	0.3926	0.3719	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$
$\zeta=3$	0.3669	0.3805	0.3363	0.3022	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$
$\zeta=5$	0.3125	0.3529	0.3094	0.2720	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$
$\zeta=7$	0.2998	0.3397	0.2935	0.2564	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$
$\zeta=9$	0.2917	0.3321	0.2834	0.2468	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$
$\zeta=50$	0.2650	0.3091	0.2496	0.2181	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$
$\zeta=75$	0.2631	0.3075	0.2471	0.2161	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$
$\zeta=100$	0.2621	0.3064	0.2458	0.2151	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$
$\zeta=150$	0.2611	0.3059	0.2449	0.2142	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$

be treated as a special version of the developed operators, none of the existing operators listed in Table 5, consider Aczel Alsina operations, which enable the decision aggregation procedure more flexible.

ii). From Table 5 and Fig. 5, it is evident that the ranking results derived by the Cq-ROFWA and Cq-ROFWG [56] totally match with the proposed aggregation operator Cq-ROFAAHA results. But both the operators weights just the Cq-ROFNs and neglect the ordered positions of the Cq-ROFNs, whereas the proposed operators encounter both the Cq-ROFNs as well as their ordered positions weights. Furthermore, the Liu *et al.* [56] aggregation-based technique is based on a score function witch has many weak aspects. For better comparison, we have calculated the score values using the formulation provided by [55]. However, using the score function of [56] may provide ineffective results.

iii). The operators given by Janani *et al.* [57] are based on complex pythagorean data. For the considered problem these operators work and the results obtained from these operators under investigation are almost same, except CPFEHA and CPFEHG operators which is due to the structural difference of these operators. But, the range of CPyFS data is

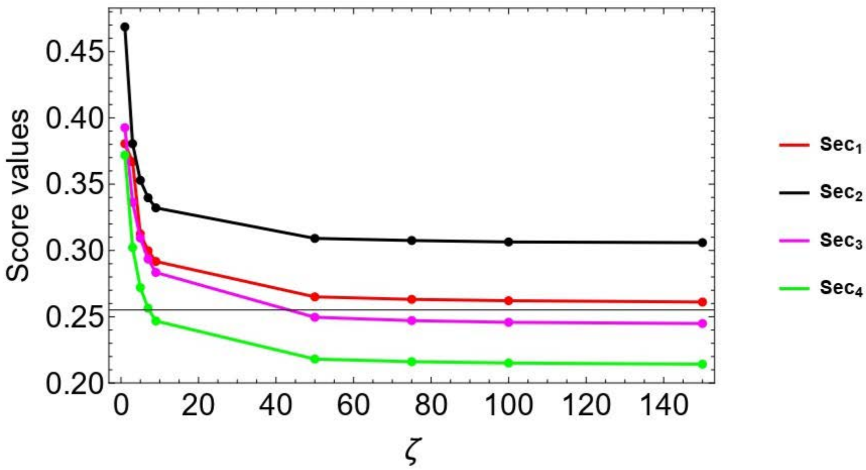

FIGURE 4. Score values of the alternatives for various values ζ by Cq-ROFAAHG operator.

TABLE 5. Ranking results based on different aggregation operators.

Operator	$S(o_1)$	$S(o_2)$	$S(o_3)$	$S(o_4)$	Ranking
Cq-ROFWA [56]	0.4142	0.5990	0.4558	0.4220	$Sec_2 \succ Sec_3 \succ Sec_4 \succ Sec_1$
Cq-ROFWG [56]	0.3616	0.4448	0.3828	0.3487	$Sec_2 \succ Sec_3 \succ Sec_4 \succ Sec_1$
CPFEOWA [57]	0.4338	0.6137	0.4949	0.4499	$Sec_2 \succ Sec_3 \succ Sec_4 \succ Sec_1$
CPFEOWG [57]	0.3816	0.4807	0.4213	0.3863	$Sec_2 \succ Sec_3 \succ Sec_4 \succ Sec_1$
CPFEHA [57]	0.4248	0.6066	0.4297	0.4341	$Sec_2 \succ Sec_4 \succ Sec_3 \succ Sec_1$
CPFEHG [57]	0.3805	0.4639	0.3926	0.3719	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$
IFAAHA [44]	$\cdots$	$\cdots$	$\cdots$	$\cdots$	Failed
IFAAHG [44]	$\cdots$	$\cdots$	$\cdots$	$\cdots$	Failed
q-ROFWA [54]	−0.2016	0.3447	0.02279	−0.2064	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$
q-ROFWG [54]	−0.3654	−0.06875	−0.1055	−0.4275	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$
q-ROFFOWA [39]	−0.2202	0.3111	−0.01570	−0.2364	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$
q-ROFFOWG [39]	−0.3787	−0.06477	−0.1399	−0.4390	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$
q-ROFFHA [39]	0.1013	0.2629	−0.02599	−0.02760	$Sec_2 \succ Sec_1 \succ Sec_3 \succ Sec_4$
q-ROFFHG [39]	−0.4217	−0.1452	−0.1615	−0.4882	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$
Proposed Cq-ROFAAHA	0.4248	0.5684	0.4804	0.4341	$Sec_2 \succ Sec_3 \succ Sec_4 \succ Sec_1$
Proposed Cq-ROFAAHG	0.3805	0.4686	0.3926	0.3719	$Sec_2 \succ Sec_3 \succ Sec_1 \succ Sec_4$

still limited and constrained by rule $0 \leq \mu^2 + \nu^2 \leq 1$ and $0 \leq (\eth_\mu)^2 + (\eth_\nu)^2 \leq 1$. However, if a person gives information in which the sum of the squares of the real parts (and imaginary parts) of both grades exceeds the unit interval. For instance, if we take $0.9\mathbf{e}^{i2\pi(0.8)}$ for membership grade and $0.8\mathbf{e}^{i2\pi(0.9)}$ for non-membership grade, then by utilizing the constraints of the CPyFS, $0.9^2 + 0.8^2 = 1.45 > 1$ and $0.8^2 + 0.9^2 = 1.45 > 1$ indicate that the CPyFS have failed.

iv). From Table 5, it is evident that the results derived on the basis of q-ROFWA and q-ROFWG operators [54] are little different from the suggested operators results. This difference is mainly due to the ignorance of amplitude terms. This cause a serious loss of information and thus these operators are not suitable for handling two dimensional problems. Further, in terms of operational rules, Liu *et al.*'s technique [54] only addresses the computation of aggregation operators under algebraic operation rules and accordingly have no any additional parameter, which limits its capacity to deal with challenges. On the other hand, the suggested approach in this study is based on Aczel Alsina operations. As a result, our technique is more adaptable and successful in choosing operational rules according to real problems.

v). For good measures, likewise the presented aggregation operators, Sheikh and Mandal [39] operators also permits the DEs to choose their preferences with respect to the different values of the parameter but these operators deal with one dimensional information at a time, which often results data loss. This shortfall implies that the proposed work is superior than [39].

To demonstrate the suggested aggregating operator's superiority over other considered existing operators, the following noteworthy properties have been bullated:

1). *Generality:* The aggregation operators utilized in the proposed method are the generalization of certain prevailing aggregation operators. For instance, the operators given in [44], [54], and [56] are special cases of the suggested Cq-ROFAAHA and Cq-ROFAAHG operators, when we assign specific values to the parameters ζ and q. So the approach described in this article is more comprehensive.

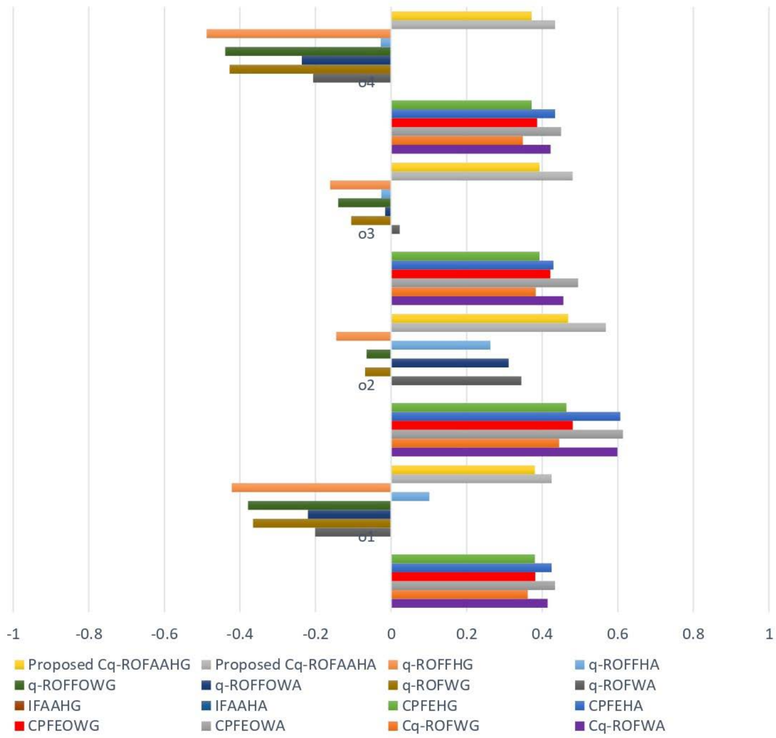

FIGURE 5. Comparative analysis with existing aggregation operators.

2). *Parameter ζ:* The designed aggregation operators include a parameter ζ that allows DEs to alter the aggregate value based on real-world decision demands, and they capture numerous existing complex q-rung orthopair aggregation operators. Along these lines, the benefit is that the proposed operators have a better level of consensus and flexibility.

3). *Property of Isotonicity:* The operators presented in this article adhere the property of isotonicity. The Cq-ROFAAHA (Cq-ROFAAHG) operator values grow (reduce) monotonically with the increase of parameter ζ, allowing DEs to select the right value based on their risk preferences. If the DEs are risk preference, they may take the parameter value as low as fairly practicable; if the DEs are risk aversion, they can take the parameter value as high as reasonably achievable in the case of the Cq-ROFAAHA operator, and vice versa for the Cq-ROFAAHG operator. Thus, the DEs can use the appropriate parameter value based on their risk tolerance and real demands.

4). *Criteria Weights:* The existing approaches [39], [44], [54], [56], [57] are incapable of dealing with MCDM situations with unknown weight information. Whilst the presented approach is effective in such situations due to the proposed entropy measure. Though Mahmood and Ali developed an entropy model in 2021 [49] to address the circumstance of unknown weight information in a complex q-rung orthopair fuzzy environment, but there are several shortcomings in their supplied model (please see the discussion in Section V).

IX. CONCLUSION

The Cq-ROFS is an effective track to represent ambiguous data than the CIFSs and CPyFSs. Its distinctive feature is that the sum of the qth power of the amplitude term (similar to the phase term) of the complex-valued membership grades and the qth power of the amplitude term (similar to the phase term) of the complex-valued non-membership grades is equal to or less than 1. In this article, some new complex q-rung orthopair fuzzy Aczel Alsina operations and complex q-rung orthopair fuzzy Aczel Alsina aggregation operators, such as Cq-ROFAAWA operator, Cq-ROFAAOWA operator, Cq-ROFAAHA operator, Cq-ROFAAWG operator, Cq-ROFAAOWG and Cq-ROFAAHG operator have been framed for aggregating Cq-ROFNs. Meanwhile, some desirable characteristics of the propound operators that include idempotency, monotonicity, boundedness and commutativity have been described and validated. Afterwards, the existing complex q-rung orthopair fuzzy entropy measures were criticized and a novel entropy model for criteria weight determination based on the proposed entropy measure was put forward. Following that, a decision framework based on the proposed operators was built to solve the MCDM problems with fully unknown weight information, and it was applied to the selection of most important sector that effects the Pakistan Stock Exchange. We further analyzed the behaviour of the presented operators by altering the values of the parameter ζ. Finally, a detailed comparative analysis was conducted with preexisting approaches to manifest the credibility of the concepts set out in the decision-making procedure. In future, our work will be extended to other aggregation operators, namely, HM [58], partitioned BM [59], power BM [60], MSM [33], Hamy mean [61] and so on, with Cq-ROFNs to generate different efficient aggregation operators which may, subsequently, be utilized for resolving MCDM problems.

Availability of Data and Materials: All data generated or analysed during this study are included in this published article.

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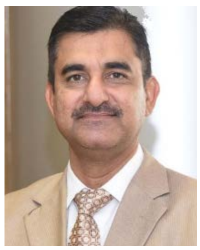

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