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METHODS

Upper Bounded Minimal Solution of the Max-Min Fuzzy Relation Inequality System

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ABSTRACT Resolution of the minimal solutions plays an important role in the research on fuzzy relation equations or inequalities system. Most of the existing works focused on the general minimal solutions or some specific minimal solutions that optimize particular objective functions. In a recently published work, the restricted minimal solution of fuzzy relation inequalities with addition-min composition was studied. Motivated by such an idea, we investigate the so-called upper bounded minimal solution of fuzzy relation inequalities with max-min composition in this work. The upper bounded minimal solution is defined as the minimal solution that is less than or equal to a given vector. Here, the given vector can be viewed as the upper bound. The major content in this work consists of two components: the existence and the resolution of the upper bounded minimal solution. First, we provide some necessary and sufficient conditions to determine whether the upper bounded minimal solution exists with respect to a given vector. Second, when it exists, we further develop two algorithms to search for the upper bounded minimal solution in a step-by-step approach. The validity of our proposed Algorithms I and II is formally proved in theory. The computational complexities of Algorithms I and II are $\mathbf{O}(mn)$ and $\mathbf{O}(mn^2)$, respectively. Moreover, our proposed algorithms are illustrated by some numerical examples.

INDEX TERMS Fuzzy relation inequality, fuzzy relation equation, max-min composition, minimal solution, upper bounded.

I. INTRODUCTION

A. MAX-MIN FUZZY RELATION EQUATIONS

A crisp relation can be represented by a Boolean matrix, in which the elements are either 0 or 1. As an extension of a crisp relation, a fuzzy relation is usually represented by a fuzzy matrix, with the entries belonging to the unit interval [0, 1]. The most commonly used operations in fuzzy algebra are the logical operators *max* ($\vee$) and *min* ($\wedge$). These two operations were widely applied in fuzzy comprehensive evaluation methods [2], [3], [4], replacing the classical *addition* ($+$) and *multiplication* ($\times$). The fuzzy relation equation was indeed the inverse problem of the fuzzy comprehensive evaluation method. The resolution of fuzzy relation equations was first investigated by Sanchez [1], with application in medical diagnosis. In general, the mathematical formula of fuzzy relation equations with max-min composition is

$$\begin{cases}(a_{i1} \wedge x_1) \vee (a_{i2} \wedge x_2) \vee \cdots \vee (a_{in} \wedge x_n) = b_i,\\ \forall i = \{1, 2, \cdots, m\},\end{cases} \tag{1}$$

where $a_{ij}, x_j, b_i \in [0, 1]$, $i = 1, 2, \cdots, m$, and $j = 1, 2, \cdots, n$. One of the most important research issues with regard to fuzzy relation equations is finding the entire solution set. There are several methods for completely solving system (1) [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. If a system of max-min fuzzy relation equations is consistent (solvable), then its solution set is usually composed of one maximum solution and finitely many minimal solutions. The solution

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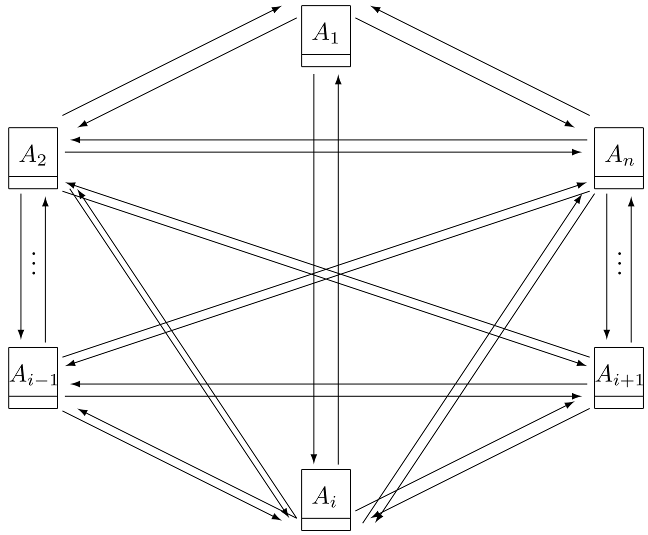

FIGURE 1. P2P (peer-to-peer) network system with *n* users.

set of system (1) can be written as

$$\bigcup_{\check{x} \in \check{X}(A,b)} [\check{x}, \hat{x}], \tag{2}$$

where $\check{X}(A, b)$ represents the set of all minimal solutions of system (1), and $\hat{x}$ is the unique maximum solution. In fact, the maximum solution $\hat{x}$ can be obtained by direct calculation. Following the expression in (2), it is clear that the key process for solving system (1) is to compute all its minimal solutions [15], [16], [17], [18], [19], [20]. However, although the number of minimal solutions should be finite, it might increase exponentially with the growth of the problem size. Solving all the minimal solutions of system (3) is highly related to the set covering problem [25], [26], which is a typical NP-hard problem. As a consequence, it is difficult to compute all the minimal solutions of a system of fuzzy relation equations.

B. MAX-MIN FUZZY RELATION INEQUALITIES AND THEIR APPLICATION IN A P2P (PEER-TO-PEER) EDUCATIONAL INFORMATION RESOURCE SHARING SYSTEM

Here, we would like to point out that the resolution approach and structure of the solution set of a fuzzy relation equation system are similar to those of a fuzzy relation inequality system [24], [41], [42]. Wang *et al.* [24] first introduced the conservative path approach for obtaining the minimal solution set of a system of fuzzy relation inequalities with max-min composition. It was verified that each conservative path corresponded to a unique minimal solution. In other words, there existed a one-to-one correspondence between the set of all conservative paths and the minimal solution set. Based on the obtained minimal solution set, the optimal solutions for minimizing a latticized linear objective function subject to a system of fuzzy relation inequalities could be further derived [24]. The fuzzy relation inequalities with max-min composition were also studied in [21], [22], and [23]. In [22] and [23], the conservative path was modified to be the FRI (fuzzy relation inequality) path. These two concepts are essentially identical. They are powerful tools for computing all the minimal solutions.

The max-min fuzzy relation inequalities were recently applied to the P2P educational information resource sharing system [38], [39], [40] (see Fig. 1).

Assume that the educational information resources are stored in some terminals in a P2P network system. All the terminals are denoted by $A_1, A_2, \cdots, A_n$. Each terminal is connected to any other terminal and free to download its required educational information resources. There exists a line with bandwidth between each pair of terminals. The bandwidth between the terminals A_i and A_j is assumed to be a_{ij}. That is, when the ith terminal A_i downloads its required resources from the jth terminal A_j, the actual quality level is

$$a_{ij} \wedge x_j,$$

where x_j (measure: Mbps) denotes the quality level on which A_j shares (sends out) its local resources. $a_{ij} \wedge x_j$ represents the receiving quality level at A_i from A_j. In general, A_i will select the terminal with the highest receiving quality level to download the resources. For example, if

$$(a_{i1} \wedge x_1) \vee (a_{i2} \wedge x_2) \vee \cdots \vee (a_{in} \wedge x_n) = a_{i1} \wedge x_1,$$

then A_i will select A_1 to download its required resources. Additionally, we assume that the download traffic requirement of A_i is no less than b_i and no more than d_i. Then, such a requirement can be represented by

$$b_i \leq (a_{i1} \wedge x_1) \vee (a_{i2} \wedge x_2) \vee \cdots \vee (a_{in} \wedge x_n) \leq d_i.$$

Without loss of generality, we assume that some of the terminals, denoted by $\{A_1, A_2, \cdots, A_m\}$, have a download traffic requirement. As a consequence, all the requirements of $\{A_1, A_2, \cdots, A_m\}$ can be represented by the following max-min fuzzy relation inequalities (after normalization):

$$\begin{cases} b_1 \leq (a_{11} \wedge x_1) \vee (a_{12} \wedge x_2) \vee \cdots \vee (a_{1n} \wedge x_n) \leq d_1, \\ b_2 \leq (a_{21} \wedge x_1) \vee (a_{22} \wedge x_2) \vee \cdots \vee (a_{2n} \wedge x_n) \leq d_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_m \leq (a_{m1} \wedge x_1) \vee (a_{m2} \wedge x_2) \vee \cdots \vee (a_{mn} \wedge x_n) \leq d_m, \end{cases} \tag{3}$$

where $a_{ij}, x_j \in [0, 1]$, $0 < b_i \leq d_i \leq 1$, $i \in I$, $j \in J$, and

$$I = \{1, 2, \cdots, m\}, \quad J = \{1, 2, \cdots, n\}.$$

A solution of system (3) is indeed a feasible flow control scheme for the terminals in the P2P network system. To decrease network congestion, a minimal solution is usually required. Considering the fixed priority grade of all terminals in a P2P network system, Ma *et al.* [40] studied the lexicographic minimum solution to the corresponding max-min FRIs. The authors proposed a detailed round-robin algorithm for computing the lexicographic minimum solution. The computational complexity is polynomial. On the other hand, considering the stability of a given feasible flow control scheme, Chen *et al.* [39] defined and investigated the interval solution. The middle point of the widest interval solution has the largest fluctuation range. Thus, it is the most stable solution for the max-min FRIs. In addition, the inconsistent max-min fuzzy relation equation system was further considered in [38]. For such an inconsistent system, the target is to find the approximate solution(s). By introducing an auxiliary parameter system, the authors designed an effective algorithm to search for an approximate solution to the inconsistent max-min system [38].

C. MOTIVATION AND CONTRIBUTIONS OF THIS WORK

In this work, we aim to define and investigate another kind of specific minimal solution, namely, the *upper bounded minimal solution* (see Definition 3 in Section 3). The concept of the upper bounded minimal solution is motivated by the recently published work [43]. In an FRI system with addition-min composition, Li *et al.* [43] formally proved that there exists a minimal solution x' such that x' is less than or equal to a given solution x''. Moreover, they developed an efficient algorithm to find such a minimal solution. Motivated by the idea presented in [43], we attempt to study the upper bounded minimal solution in an FRI system with max-min composition, i.e., system (3).

In fact, an upper bounded minimal solution is actually a minimal solution with an upper bound. In this work, we always assume that

$$\bar{x} \in [0, 1]^n$$

is a given vector. We aim to investigate the upper bounded minimal solution x^* with the upper bound $\bar{x}$. Here, x^* is a minimal solution of system (3), satisfying $x^* \leq \bar{x}$.

The contributions of this work can be summarized by the following two points.

(i) The necessary and sufficient conditions are provided for the existence of the upper bounded minimal solution with respect to a given vector.

(ii) An effective approach is developed for solving the upper bounded minimal solution when it exists.

D. ORGANIZATION OF THIS WORK

The rest of this work is organized as follows. Section 2 presents some necessary preliminaries. In Section 3, we discuss the consistency checking of system (3). In Section 4, we develop two detailed algorithms, which contribute to the resolution of the upper bounded minimal solution. Section 5 presents the conclusion.

II. PRELIMINARIES

In this section, we present some basic concepts and results pertaining to system (3). We will introduce three aspects for system (3): (i) its maximum solution; (ii) the system consistency check; and (iii) the structure of its solution set.

For arbitrary $x, y \in [0, 1]^n$, $x \leq y$ $(x = y)$ denotes $x_j \leq y_j$ $(x_j = y_j)$, $\forall j \in J$. Hence, the matrix form of system (3) is

$$b^T \leq A \circ x^T \leq d^T, \tag{4}$$

where $A = (a_{ij})_{m\times n}$, $x = (x_j)_{1\times n}$, $b = (b_i)_{1\times m}$, and $d = (d_i)_{1\times m}$. Moreover, the solution set of the system can be represented by

$$X(A, b, d) = \{x \in [0, 1]^n | b^T \leq A \circ x^T \leq d^T\}, \tag{5}$$

Definition 1: System (3) is called consistent (inconsistent) if $X(A, b, d) \neq \emptyset$ $(X(A, b, d) = \emptyset)$.

Definition 2: In system (3), a solution $\hat{x}$ is called the maximum solution if $\hat{x} \geq x$ for any $x \in X(A, b, d)$. A solution $\check{x}$ is called a minimal solution if there exists some $x \in X(A, b, d)$ such that $x \leq \check{x}$; then, we have $x = \check{x}$.

Let $x \in X(A, b, d)$ be an arbitrary solution of system (3). Then, it is obvious that

$$(a_{i1} \wedge x_1) \vee (a_{i2} \wedge x_2) \vee \cdots \vee (a_{in} \wedge x_n) \leq d_i, \quad \forall i \in I, \tag{6}$$

That is,

$$a_{ij} \wedge x_j \leq d_i, \quad \forall i \in I, j \in J. \tag{7}$$

In fact, inequality (7) implies that

$$x_j \leq a_{ij} @ d_i, \quad \forall i \in I, j \in J, \tag{8}$$

where

$$a_{ij} @ d_i = \begin{cases} 1, & \text{if } a_{ij} \leq d_i, \\ d_i, & \text{if } a_{ij} > d_i. \end{cases} \tag{9}$$

Furthermore, inequality (8) is equivalent to

$$x_j \leq \bigwedge_{i\in I} a_{ij} @ d_i, \quad \forall j \in J. \tag{10}$$

Denote $\hat{x} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)$, where

$$\hat{x}_j = \bigwedge_{i\in I} a_{ij} @ d_i, \quad \forall j \in J. \tag{11}$$

Then, by (10) and (11), we obtain the following Proposition 1.

Proposition 1: If x is a solution of system (3), then it holds that $x \leq \hat{x}$.

According to Proposition 1, the vector $\hat{x}$ can be viewed as the potential maximum solution of system (3). In fact, it can be used to check the consistency of system (3) as follows.

Theorem 1: System (3) is consistent if and only if $\hat{x} \in X(A, b, d)$.

Following Proposition 1 and Theorem 1, system (3) is consistent if and only if $\hat{x}$ is its (unique) maximum solution.

Proposition 2: If x is a solution of system (3), then it holds that $[x, \hat{x}] \subseteq X(A, b, d)$.

Proof: It follows from Proposition 1 that $x \leq \hat{x}$. Hence, the notation $[x, \hat{x}]$ is meaningful. Taking an arbitrary $y \in [x, \hat{x}]$ and considering Theorem 1, we have $x, \hat{x} \in X(A, b, d)$. Therefore, it holds that

$$b_i \leq \bigvee_{j\in J} a_{ij} \wedge x_j \leq d_i, \quad \forall i \in I,$$

$$b_i \leq \bigvee_{j\in J} a_{ij} \wedge \hat{x}_j \leq d_i, \quad \forall i \in I. \tag{12}$$

Since $x \le y \le \hat{x}$, we have

$$b_i \le \bigvee_{j\in J} a_{ij} \wedge x_j \le \bigvee_{j\in J} a_{ij} \wedge y_j \le \bigvee_{j\in J} a_{ij} \wedge \hat{x}_j \le d_i, \quad \forall i \in I. \tag{13}$$

This indicates that y is a solution of system (3), i.e., $y \in X(A, b, d)$. The proof is complete. □

The complete solution set of system (3) (when it is consistent) can be characterized by the following Theorem 2.

Theorem 2: If system (3) is consistent, then its solution set is

$$X(A, b, d) = \bigcup_{\check{x}\in\check{X}(A,b,d)} \{x|\check{x} \le x \le \hat{x}\}.$$

Here, $\check{X}(A, b, d)$ represents the set of all minimal solutions, while $\hat{x}$ is the unique maximum solution of system (3).

III. EXISTENCE OF THE UPPER BOUNDED MINIMAL SOLUTION

In this section, we define the concept of the upper bounded minimal solution of system (3). Moreover, some approaches for checking the existence of the upper bounded minimal solution are provided.

Definition 3 (Upper Bounded Minimal Solution): In system (3), a minimal solution x^* is said to be an upper bounded minimal solution (with respect to $\bar{x}$) if it holds that $x^* \le \bar{x}$ for the given vector $\bar{x}$.

Adding the inequality $x \le \bar{x}$ to system (3), we construct the following system:

$$\begin{cases} b_1 \le (a_{11} \wedge x_1) \vee (a_{12} \wedge x_2) \vee \cdots \vee (a_{1n} \wedge x_n) \le d_1, \\ b_2 \le (a_{21} \wedge x_1) \vee (a_{22} \wedge x_2) \vee \cdots \vee (a_{2n} \wedge x_n) \le d_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_m \le (a_{m1} \wedge x_1) \vee (a_{m2} \wedge x_2) \vee \cdots \vee (a_{mn} \wedge x_n) \le d_m, \\ x \le \bar{x}. \end{cases} \tag{14}$$

According to Definition 3, the following Corollary 1 below is self-evident.

Corollary 1: x^* is an upper bounded minimal solution of system (3) if and only if it is a minimal solution of system (14).

Theorem 3: System (3) has an upper bounded minimal solution with respect to $\bar{x}$ if and only if system (14) is consistent, i.e., there exists at least one solution of system (14).

Proof: ($\Rightarrow$) It is obvious according to Corollary 1.

($\Leftarrow$) Let y be a solution of system (14). Then, it holds that $y \in X(A, b, d)$ and $y \le \bar{x}$. Following Theorem 2, there exists $\check{y} \in \check{X}(A, b, d)$ such that $\check{y} \le y \le \hat{x}$. Therefore, $\check{y}$ is a minimal solution of system (3), satisfying $\check{y} \le y \le \bar{x}$. It follows from Definition 3 that $\check{y}$ is an upper bounded minimal solution of system (3) with respect to $\bar{x}$. □

Theorem 4: System (3) has an upper bounded minimal solution with respect to $\bar{x}$ if and only if $x^0 = \bar{x} \wedge \hat{x}$ is a solution of system (3).

Proof: ($\Rightarrow$) According to Theorem 3, system (14) is consistent. Suppose x is a solution of (14). Then, it holds that $x \in X(A, b, d)$ and $x \le \bar{x}$. However, it follows from Proposition 1 that $x \le \hat{x}$. Therefore, we obtain

$$x \le \bar{x} \wedge \hat{x} \le \hat{x}.$$

According to Proposition 2, it holds that $x^0 = \bar{x} \wedge \hat{x} \in X(A, b, d)$.

($\Leftarrow$) Note that $x^0 = \bar{x} \wedge \hat{x} \le \bar{x}$. It is clear that x^0 is a solution of system (3) if and only if x^0 is a solution of system (14). Then, the rest of the proof is due to Theorem 3. □

Lemma 1: For any $i \in I, j \in J$, it holds that $a_{ij} \wedge \hat{x}_j \le d_i$.

Proof: If $a_{ij} \le d_i$, then it holds that $a_{ij} \wedge \hat{x}_j \le a_{ij} \le d_i$. Otherwise, if $a_{ij} > d_i$, then it follows from (9) that

$$\hat{x}_j = \bigwedge_{k\in I} a_{kj} @ d_k \le a_{ij} @ d_i = d_i. \tag{15}$$

Hence, $a_{ij} \wedge \hat{x}_j \le a_{ij} \wedge d_i \le d_i$. □

Theorem 5: System (3) has an upper bounded minimal solution with respect to $\bar{x}$ if and only if for any $i \in I$, there exists $j \in J$ such that $a_{ij} \wedge \bar{x}_j \wedge \hat{x}_j \ge b_i$.

Proof: ($\Rightarrow$) Suppose system (3) has an upper bounded minimal solution. It follows from Theorem 4 that $x^0 = \bar{x} \wedge \hat{x}$ is a solution of system (3). By system (3), we have

$$\bigvee_{j\in J} a_{ij} \wedge (\bar{x}_j \wedge \hat{x}_j) \ge b_i. \tag{16}$$

This indicates that for any $i \in I$, there exists $j' \in J$ such that

$$a_{ij'} \wedge \bar{x}_{j'} \wedge \hat{x}_{j'} = a_{ij'} \wedge (\bar{x}_{j'} \wedge \hat{x}_{j'}) \ge b_i. \tag{17}$$

($\Leftarrow$) If for any $i \in I$, there exists $j \in J$ such that $a_{ij} \wedge \bar{x}_j \wedge \hat{x}_j \ge b_i$, then we have

$$\bigvee_{j\in J} a_{ij} \wedge x_j^0 \ge b_i, \quad \forall i \in I. \tag{18}$$

However, since $x_j^0 = \bar{x}_j \wedge \hat{x}_j \le \hat{x}_j, \forall j \in J$, it follows from Lemma 1 that

$$a_{ij} \wedge x_j^0 \le a_{ij} \wedge \hat{x}_j \le d_i, \quad \forall i \in I, j \in J, \tag{19}$$

That is,

$$\bigvee_{j\in J} a_{ij} \wedge x_j^0 \le d_i, \quad \forall i \in I. \tag{20}$$

Inequalities (18) and (20) contribute to $x^0 = \bar{x} \wedge \hat{x} \in X(A, b, d)$. As a consequence, it follows from Theorem 4 that system (3) has an upper bounded minimal solution. □

IV. RESOLUTION OF THE UPPER BOUNDED MINIMAL SOLUTION

In this section, we aim to provide an effective resolution approach for the upper bounded minimal solution of system (3).

To obtain an upper bounded minimal solution with respect to the given vector $\bar{x}$, we construct n subproblems as follows, based on the vector $x^0 = \bar{x} \wedge \hat{x}$.

The first subproblem P_1 is formulated as

$$\begin{aligned}&P_1\\&\min\ x_1\\&\text{s.t. } (x_1, x_2, \cdots, x_n) \in X(A, b, d),\\&\quad x_2 = x_2^0, x_3 = x_3^0, \cdots, x_n = x_n^0.\end{aligned} \tag{21}$$

Suppose the optimal solution of the subproblem P_1 is x_1^*. Then, we further construct the second subproblem P_2 as

$$\begin{aligned}&P_2\\&\min\ x_2\\&\text{s.t. } (x_1, x_2, \cdots, x_n) \in X(A, b, d),\\&\quad x_1 = x_1^*, x_3 = x_3^0, x_4 = x_4^0, \cdots, x_n = x_n^0.\end{aligned} \tag{22}$$

Thus, the j'th subproblem is

$$\begin{aligned}&P_{j'}\\&\min\ x_{j'}\\&\text{s.t. } (x_1, x_2, \cdots, x_n) \in X(A, b, d),\\&\quad x_1 = x_1^*, \cdots, x_{j'-1} = x_{j'-1}^*,\\&\quad x_{j'+1} = x_{j'+1}^0, \cdots, x_n = x_n^0,\end{aligned} \tag{23}$$

for $j' = 3, \cdots, n$. The last subproblem turns out to be

$$\begin{aligned}&P_n\\&\min\ x_n\\&\text{s.t. } (x_1, x_2, \cdots, x_n) \in X(A, b, d),\\&\quad x_1 = x_1^*, x_2 = x_2^*, \cdots, x_{n-1} = x_{n-1}^*.\end{aligned} \tag{24}$$

If all the above presented subproblems are solvable with the optimal solutions $x_1^*, x_2^*, \cdots, x_n^*$, then there exists an upper bounded minimal solution with respect to $\bar{x}$. Moreover, one is able to generate an upper bounded minimal solution by these optimal solutions. Next, we present the related results.

A. RESOLUTION ALGORITHM FOR THE SUBPROBLEMS

Taking arbitrary $k \in J$, we consider the following problem:

$$\begin{aligned}&\min\ sx_k\\&\text{s.t. } (x_1, x_2, \cdots, x_n) \in X(A, b, d),\\&\quad x_j = c_j, \quad j \in J,\ j \neq k,\end{aligned} \tag{25}$$

where c_j is a given constant for any $j \in J - \{k\}$.

It is clear that each subproblem in $\{P_1, P_2, \cdots, P_n\}$ can be viewed as the above problem (25).

Theorem 6: Problem (25) is solvable (i.e., has an optimal solution), if and only if

$$(c_1, \cdots, c_{k-1}, \hat{x}_k, c_{k+1}, \cdots, c_n) \in X(A, b, d).$$

Proof: ($\Rightarrow$) If Problem (25) has an optimal solution, denoted by x_k^*, then it is also a feasible solution. Following the constraints of (25), we have

$$(c_1, \cdots, c_{k-1}, x_k^*, c_{k+1}, \cdots, c_n) \in X(A, b, d). \tag{26}$$

Since $\hat{x}$ is the maximum solution of system (3), it follows that

$$(c_1, \cdots, c_{k-1}, x_k^*, c_{k+1}, \cdots, c_n) \leq \hat{x}. \tag{27}$$

This implies that

$$c_j \leq \hat{x}_j, \quad \forall j \in J, j \neq k, \tag{28}$$

and

$$x_k^* \leq \hat{x}_k. \tag{29}$$

Therefore, we further obtain

$$\begin{aligned}&(c_1, \cdots, c_{k-1}, x_k^*, c_{k+1}, \cdots, c_n)\\&\quad \leq (c_1, \cdots, c_{k-1}, \hat{x}_k, c_{k+1}, \cdots, c_n)\\&\quad \leq \hat{x}.\end{aligned} \tag{30}$$

Considering (26), it follows from Proposition 2 that $(c_1, \cdots, c_{k-1}, \hat{x}_k, c_{k+1}, \cdots, c_n) \in X(A, b, d)$.

($\Leftarrow$) Suppose

$$(c_1, \cdots, c_{k-1}, \hat{x}_k, c_{k+1}, \cdots, c_n) \in X(A, b, d).$$

According to Theorem 2,

$$X(A, b, d) = \bigcup_{\check{x} \in \check{X}(A,b,d)} [\check{x}, \hat{x}].$$

$\check{X}(A, b, d)$ is the set of all minimal solutions of system (3). Hence, there exists $\check{x}' \in \check{X}(A, b, d)$, such that

$$\check{x}' \leq (c_1, \cdots, c_{k-1}, \hat{x}_k, c_{k+1}, \cdots, c_n) \leq \hat{x}. \tag{31}$$

Let

$$\begin{aligned}\check{X}^k(A, b, d) = \{&\check{x} \in \check{X}(A, b, d) | \check{x}\\&\leq (c_1, \cdots, c_{k-1}, \hat{x}_k, c_{k+1}, \cdots, c_n)\}.\end{aligned} \tag{32}$$

Then, it holds that $\check{x}' \in \check{X}^k(A, b, d) \neq \emptyset$. In addition, since system (3) has at most finitely many minimal solutions, we have

$$1 \leq |\check{X}^k(A, b, d)| < \infty, \tag{33}$$

i.e., $\check{X}^k(A, b, d)$ is a nonempty finite set. Let

$$\check{x}_k^{min} = \min\{\check{x}_k | (\check{x}_1, \check{x}_2, \cdots, \check{x}_n) \in \check{X}^k(A, b, d)\}. \tag{34}$$

Then, it is easy to verify that $[\check{x}_k^{min}, \hat{x}_k]$ is the feasible domain of Problem (25). As a consequence, Problem (25) has an optimal solution as $\check{x}_k^{min}$. □

Corollary 2: Problem (25) is solvable if and only if $c_j \leq \hat{x}_j$ for any $j \in J, j \neq k$, and $\hat{x}_k$ is a feasible solution of Problem (25).

Corollary 3: Problem (25) is solvable if and only if it has at least a feasible solution.

For arbitrary $(x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n) \in [0, 1]^n$, we denote

$$(x_1, x_2, \cdots, x_n) \circ \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \cdots \vee (x_n \wedge y_n). \tag{35}$$

Let

$$I^k = \{i \in I | (a_{i1}, a_{i2}, \cdots, a_{in}) \circ \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \\ 0 \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix} < b_i\}. \quad (36)$$

Lemma 2: Suppose Problem (25) is solvable. If $I^k \neq \emptyset$, then for any $i \in I^k$, it holds that $a_{ik} \geq b_i$.

Proof: Following Theorem 6,

$$(c_1, \cdots, c_{k-1}, \hat{x}_k, c_{k+1}, \cdots, c_n) \in X(A, b, d)$$

is a solution of system (3). Hence,

$$(a_{i1}, a_{i2}, \cdots, a_{in}) \circ \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \\ \hat{x}_k \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix} \geq b_i, \quad \forall i \in I. \quad (37)$$

Therefore, we have

$$(a_{i1}, a_{i2}, \cdots, a_{in}) \circ \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \\ \hat{x}_k \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix} \geq b_i, \quad \forall i \in I^k, \quad (38)$$

That is,

$$\left(\bigvee_{j \neq k} (a_{ij} \wedge c_j)\right) \vee (a_{ik} \wedge \hat{x}_k) \geq b_i, \quad \forall i \in I^k. \quad (39)$$

However, (36) shows that

$$(a_{i1}, a_{i2}, \cdots, a_{in}) \circ \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \\ 0 \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix} < b_i, \quad \forall i \in I^k, \quad (40)$$

That is,

$$\left(\bigvee_{j \neq k} (a_{ij} \wedge c_j)\right) \vee (a_{ik} \wedge 0) < b_i, \quad \forall i \in I^k. \quad (41)$$

Inequalities (39) and (41) contribute to

$$a_{ik} \wedge \hat{x}_k \geq b_i, \quad \forall i \in I^k. \quad (42)$$

Hence, it holds that $a_{ik} \geq b_i, \forall i \in I^k$. □

Theorem 7: Suppose Problem (25) is solvable. Then,

$$x_k^* = \begin{cases} 0, & \text{if } I^k = \emptyset, \\ \bigvee_{i \in I^k} b_i, & \text{if } I^k \neq \emptyset, \end{cases} \quad (43)$$

is the optimal solution (also the objective value) of Problem (25).

Proof: (i) Feasibility.

Case 1. If $I^k = \emptyset$, then $x_k^* = 0$. According to (36), we have

$$(a_{i1}, a_{i2}, \cdots, a_{in}) \circ \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \\ x_k^* \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix} = (a_{i1}, a_{i2}, \cdots, a_{in}) \circ \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \\ 0 \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix} \geq b_i, \quad \forall i \in I. \quad (44)$$

Hence, $(c_1, \cdots, c_{k-1}, x_k^*, c_{k+1}, \cdots, c_n) \in X(A, b, d)$ is a solution of system (3). This indicates that x_k^* is a feasible solution of Problem (25).

Case 2. If $I^k \neq \emptyset$, then $x_k^* = \bigvee_{i' \in I^k} b_{i'} \geq 0$. For arbitrary $i \notin I^k$, it follows from (36) that

$$(a_{i1}, a_{i2}, \cdots, a_{in}) \circ \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \\ x_k^* \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix} \geq (a_{i1}, a_{i2}, \cdots, a_{in}) \circ \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \\ 0 \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix} \geq b_i, \quad \forall i \notin I^k. \quad (45)$$

On the other hand, for an arbitrary $i \in I^k$, it is obvious that

$$\bigvee_{i' \in I^k} b_{i'} \geq b_i. \quad (46)$$

Hence,

$$(a_{i1}, a_{i2}, \cdots, a_{in}) \circ \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \\ x_k^* \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix}$$

$$= (a_{i1}, a_{i2}, \cdots, a_{in}) \circ \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \\ \bigvee_{i' \in I^k} b_{i'} \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix}$$

$$\geq a_{ik} \wedge \left(\bigvee_{i' \in I^k} b_{i'} \right)$$

$$\geq a_{ik} \wedge b_i, \quad \forall i \in I^k. \tag{47}$$

According to Lemma 2,

$$a_{ik} \geq b_i, \quad \forall i \in I^k. \tag{48}$$

Therefore, we have

$$(a_{i1}, a_{i2}, \cdots, a_{in}) \circ \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \\ x_k^* \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix} \geq a_{ik} \wedge b_i = b_i, \quad \forall i \in I^k. \tag{49}$$

Inequalities (45) and (49) contribute to

$$(a_{i1}, a_{i2}, \cdots, a_{in}) \circ \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \\ x_k^* \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix} \geq a_{ik} \wedge b_i = b_i, \quad \forall i \in I. \tag{50}$$

Hence, $(c_1, \cdots, c_{k-1}, x_k^*, c_{k+1}, \cdots, c_n) \in X(A, b, d)$ and x_k^* is a feasible solution of Problem (25).

(ii) Optimality.

Let y_k be an arbitrary feasible solution of Problem (25). Then,

$$(c_1, \cdots, c_{k-1}, y_k, c_{k+1}, \cdots, c_n) \in X(A, b, d). \tag{51}$$

According to system (3), we have

$$(a_{i1}, a_{i2}, \cdots, a_{in}) \circ \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \\ y_k \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix} \geq b_i, \quad \forall i \in I. \tag{52}$$

This indicates

$$(a_{i1}, a_{i2}, \cdots, a_{in}) \circ \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \\ y_k \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix} \geq b_i, \quad \forall i \in I^k, \tag{53}$$

That is,

$$\left(\bigvee_{j \neq k} (a_{ij} \wedge c_j) \right) \vee (a_{ik} \wedge y_k) \geq b_i, \quad \forall i \in I^k. \tag{54}$$

However, it follows from (36) that

$$(a_{i1}, a_{i2}, \cdots, a_{in}) \circ \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \\ 0 \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix} < b_i, \quad \forall i \in I^k, \tag{55}$$

That is,

$$\bigvee_{j \neq k} (a_{ij} \wedge c_j) < b_i, \quad \forall i \in I^k. \tag{56}$$

Inequalities (54) and (56) contribute to

$$a_{ik} \wedge y_k \geq b_i, \quad \forall i \in I^k. \tag{57}$$

Hence,

$$y_k \geq a_{ik} \wedge y_k \geq b_i, \quad \forall i \in I^k. \tag{58}$$

Thus, $y_k \geq \bigvee_{i \in I^k} b_i = x_k^*$. □

Summarizing the above presented results, we obtain the following Algorithm I regarding the resolution of Problem (25)).

Algorithm I (For Solving Problem (25))

Step 1. According to (9) and (11), compute the potential maximum solution $\hat{x}$ of system (3).

Step 2. Check the consistency of system (3) by Theorem 1. If system (3) is consistent, then continue to Step 3. Otherwise, system (3) is inconsistent, i.e., $X(A, b, d) = \emptyset$. Problem (25) does not have an optimal solution; therefore, stop.

Step 3. Determine whether Problem (25) is solvable by Theorem 6. If Problem (25) is solvable, then continue to Step 4. Otherwise, Problem (25) does not have an optimal solution; therefore, stop.

Step 4. Compute the index set I^k by (36).

Step 5. Compute the value of x_k^* by (43). Then, x_k^* is the unique optimal solution of Problem (25). The optimal objective function value is also x_k^*.

Example 1: Consider the max-min fuzzy relation inequality system

$$b^T \le A \circ x^T \le d^T, \tag{59}$$

where

$$A = \begin{bmatrix} 0.6 & 0.7 & 0.4 & 0.8 & 0.6 & 0.7 \\ 0.8 & 0.7 & 0.4 & 0.2 & 0.4 & 0.8 \\ 0.7 & 0.6 & 0.7 & 0.4 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.5 & 0.6 & 0.5 & 0.6 \\ 0.6 & 0.5 & 0.7 & 0.4 & 0.8 & 0.3 \\ 0.5 & 0.4 & 0.4 & 0.8 & 0.9 & 0.5 \end{bmatrix},$$

$x = (x_1, x_2, \cdots, x_6)$, $b = (0.6, 0.5, 0.55, 0.55, 0.6, 0.6)$, $d = (0.7, 0.7, 0.6, 0.8, 0.9, 0.8)$. Suppose the solution set of system (59) is $X(A, b, d)$. Then, we attempt to find the optimal solution of the following Problem (60)

$$\begin{aligned} &\min x_6 \\ &\text{s.t. } (x_1, x_2, \cdots, x_6) \in X(A, b, d), \\ &\quad x_1 = 0.4, x_2 = 0.3, x_3 = 0.6, x_4 = 0.5, x_5 = 0.6. \end{aligned} \tag{60}$$

Solution:

Step 1. According to (9) and (11), we are able to compute the potential maximum solution of system (59) as $\hat{x} = (0.6, 1, 0.6, 0.7, 0.8, 0.7)$.

Step 2. After calculation, we obtain

$$A \circ \hat{x}^T = \begin{bmatrix} 0.6 & 0.7 & 0.4 & 0.8 & 0.6 & 0.7 \\ 0.8 & 0.7 & 0.4 & 0.2 & 0.4 & 0.8 \\ 0.7 & 0.6 & 0.7 & 0.4 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.5 & 0.6 & 0.5 & 0.6 \\ 0.6 & 0.5 & 0.7 & 0.4 & 0.8 & 0.3 \\ 0.5 & 0.4 & 0.4 & 0.8 & 0.9 & 0.5 \end{bmatrix} \circ \begin{bmatrix} 0.6 \\ 1 \\ 0.6 \\ 0.7 \\ 0.8 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.7 \\ 0.6 \\ 0.6 \\ 0.8 \\ 0.8 \end{bmatrix}.$$

It is clear that

$$b^T = \begin{bmatrix} 0.6 \\ 0.5 \\ 0.55 \\ 0.55 \\ 0.6 \\ 0.6 \end{bmatrix} \le \begin{bmatrix} 0.7 \\ 0.7 \\ 0.6 \\ 0.6 \\ 0.8 \\ 0.8 \end{bmatrix} \le \begin{bmatrix} 0.7 \\ 0.7 \\ 0.6 \\ 0.8 \\ 0.9 \\ 0.8 \end{bmatrix} = d^T.$$

Hence, following Theorem 1, system (59) is consistent, and we continue to Step 3.

Step 3. After calculation, we obtain

$$A \circ (c_1, c_2, c_3, c_4, c_5, \hat{x}_6)^T = \begin{bmatrix} 0.6 & 0.7 & 0.4 & 0.8 & 0.6 & 0.7 \\ 0.8 & 0.7 & 0.4 & 0.2 & 0.4 & 0.8 \\ 0.7 & 0.6 & 0.7 & 0.4 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.5 & 0.6 & 0.5 & 0.6 \\ 0.6 & 0.5 & 0.7 & 0.4 & 0.8 & 0.3 \\ 0.5 & 0.4 & 0.4 & 0.8 & 0.9 & 0.5 \end{bmatrix} \circ \begin{bmatrix} 0.4 \\ 0.3 \\ 0.6 \\ 0.5 \\ 0.6 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.7 \\ 0.6 \\ 0.6 \\ 0.6 \\ 0.6 \end{bmatrix}.$$

It is clear that

$$b^T = \begin{bmatrix} 0.6 \\ 0.5 \\ 0.55 \\ 0.55 \\ 0.6 \\ 0.6 \end{bmatrix} \le \begin{bmatrix} 0.7 \\ 0.7 \\ 0.6 \\ 0.6 \\ 0.6 \\ 0.6 \end{bmatrix} \le \begin{bmatrix} 0.7 \\ 0.7 \\ 0.6 \\ 0.8 \\ 0.9 \\ 0.8 \end{bmatrix} = d^T.$$

Hence, following Theorem 6, Problem (60) is solvable, and we continue to Step 4.

Step 4. Since

$$A \circ (c_1, c_2, c_3, c_4, c_5, 0)^T = \begin{bmatrix} 0.6 & 0.7 & 0.4 & 0.8 & 0.6 & 0.7 \\ 0.8 & 0.7 & 0.4 & 0.2 & 0.4 & 0.8 \\ 0.7 & 0.6 & 0.7 & 0.4 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.5 & 0.6 & 0.5 & 0.6 \\ 0.6 & 0.5 & 0.7 & 0.4 & 0.8 & 0.3 \\ 0.5 & 0.4 & 0.4 & 0.8 & 0.9 & 0.5 \end{bmatrix} \circ \begin{bmatrix} 0.4 \\ 0.3 \\ 0.6 \\ 0.5 \\ 0.6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \\ 0.6 \\ 0.5 \\ 0.6 \\ 0.6 \end{bmatrix},$$

we have $I^6 = \{2, 4\}$ by (36).

Step 5. Since $I^6 = \{2, 4\} \neq \emptyset$, we have

$$x_6^* = \bigvee_{i \in I^6} b_i = b_2 \vee b_4 = 0.5 \vee 0.55 = 0.55. \tag{61}$$

Therefore, the optimal solution of Problem (60) is $x_6^* = 0.55$. □

B. RESOLUTION ALGORITHM OF THE UPPER BOUNDED MINIMAL SOLUTION

Proposition 3: If $x^0 \in X(A, b, d)$, then for any $j' \in \{1, 2, \cdots, n\}$, the subproblem $P_{j'}$ is solvable, i.e., has an optimal solution.

Proof: Observing the constraints in Problem P_1, it follows from $x^0 = (x_1^0, x_2^0, \cdots, x_n^0) \in X(A, b, d)$ that x_1^0 is a feasible solution of Problem P_1. According to Corollary 3, Problem P_1 is solvable.

Note that x_1^* is the optimal solution of Problem P_1. It is also a feasible solution. Thus, it holds that $(x_1^*, x_2^0, \cdots, x_n^0) \in X(A, b, d)$. This indicates that x_2^0 is a feasible solution of Problem P_2. Again, by Corollary 3, Problem P_2 is solvable.

In the same way, it is easy to verify that all the other subproblems $P_3, \cdots, P_n$ are solvable. □

For an arbitrary $j' \in \{1, 2, \cdots, n\}$, we construct the following problem corresponding to Problem $P_{j'}$.

$$\begin{aligned} &P_{j'}^{\le} \\ &\min x_{j'} \\ &\text{s.t. } (x_1, x_2, \cdots, x_n) \in X(A, b, d), \\ &\quad x_1 = x_1^*, \cdots, x_{j'-1} = x_{j'-1}^*, \\ &\quad x_{j'+1} \le x_{j'+1}^0, \cdots, x_n \le x_n^0. \end{aligned} \tag{62}$$

In Problem $P_{j'}^{\le}$, the decision variables can be viewed as $x_{j'}, x_{j'+1}, \cdots, x_n$.

Lemma 3: Problems $P_{j'}$ and $P_{j'}^{\le}$ have the same optimal objective value.

Proof: The optimal objective value of Problem $P_{j'}$ is $x_{j'}^*$. Denote the feasible domains of Problems $P_{j'}$ and $P_{j'}^{\leq}$ by D and $D^{\leq}$. Comparing the constraints in these two subproblems, it is easily found that $D \subseteq D^{\leq}$. Hence, if the optimal objective value of Problem $P_{j'}^{\leq}$ is $x_{j'}^{\leq}$, then it holds that $x_{j'}^{\leq} \leq x_{j'}^*$. Next, we must verify that $x_{j'}^{\leq} = x_{j'}^*$.

Assume (by contradiction) that $x_{j'}^{\leq} \neq x_{j'}^*$. Then, it turns out to be

$$x_{j'}^{\leq} < x_{j'}^*. \tag{63}$$

Since $x_{j'}^{\leq}$ is the optimal objective value of Problem $P_{j'}^{\leq}$, there exists a corresponding optimal solution, denoted by $(x_{j'}^{\leq}, x_{j'+1}^{\leq}, \cdots, x_n^{\leq})$, satisfying $x_{j'+1}^{\leq} \leq x_{j'+1}^0, \cdots, x_n^{\leq} \leq x_n^0$ and

$$(x_1^*, \cdots, x_{j'-1}^*, x_{j'}^{\leq}, x_{j'+1}^{\leq}, \cdots, x_n^{\leq}) \in X(A, b, d). \tag{64}$$

Note that $\hat{x}$ is the maximum solution in $X(A, b, d)$. It is clear that

$$x_{j'+1}^{\leq} \leq \hat{x}_{j'+1}, \cdots, x_n^{\leq} \leq \hat{x}_n. \tag{65}$$

It follows from Proposition 2 that

$$(x_1^*, \cdots, x_{j'-1}^*, x_{j'}^{\leq}, \hat{x}_{j'+1}, \cdots, \hat{x}_n) \in X(A, b, d). \tag{66}$$

This indicates that $x_{j'}^{\leq}$ is a feasible solution of Problem $P_{j'}$. As a consequence, it holds that

$$x_{j'}^{\leq} \geq x_{j'}^*, \tag{67}$$

due to the optimality of $x_{j'}^*$. Inequalities (63) and (67) lead to a contradiction. □

Theorem 8: If $x^0 \in X(A, b, d)$ and $x_{j'}^*$ is the optimal solution of Problem $P_{j'}$, for each $j' \in \{1, 2, \cdots, n\}$, then $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is an upper bounded minimal solution of system (3) with respect to $\bar{x}$.

Proof: (i) $x^* \leq \bar{x}$.

Since $x^0 = (x_1^0, x_2^0, \cdots, x_n^0) \in X(A, b, d)$, observing the constraints in Problem P_1, we have x_1^0 is a feasible solution of Problem P_1. Therefore, we obtain $x_1^* \leq x_1^0$.

Since x_1^* is the optimal solution of Problem P_1, it is clear that $(x_1^*, x_2^0, \cdots, x_n^0) \in X(A, b, d)$. This indicates that x_2^0 is a feasible solution of Problem P_1. Therefore, we obtain $x_2^* \leq x_2^0$.

Similarly, we obtain $x_j^* \leq x_j^0$, $\forall j \in J$. Consequently, it holds that $x^* \leq x^0 = \bar{x} \wedge \hat{x} \leq \bar{x}$.

(ii) x^* is a minimal solution of system (3).

Since x_n^* is the optimal solution of Problem P_n, it is also a feasible solution. Hence, $x^* = (x_1^*, x_2^*, \cdots, x_n^*) \in X(A, b, d)$ is a solution of system (3).

Suppose $y = (y_1, y_2, \cdots, y_n) \in X(A, b, d)$ is an arbitrary solution of system (3) such that $y \leq x^*$, i.e.,

$$y_j \leq x_j^*, \quad \forall j \in J. \tag{68}$$

To check the minimality of x^*, we must verify that $y = x^*$.

Assume (by contradiction) that $y \neq x^*$. Considering $y \leq x^*$, there exists $j' \in \{1, 2, \cdots, n\}$ such that

$$y_1 = x_1^*, \cdots, y_{j'-1} = x_{j'-1}^*, \tag{69}$$

and

$$y_{j'} < x_{j'}^*. \tag{70}$$

Inequality (68) indicates

$$y_{j'+1} \leq x_{j'+1}^*, \cdots, y_n \leq x_n^*. \tag{71}$$

On the other hand, it has been proven in (i) that $x^* \leq x^0$. Hence,

$$y_{j'+1} \leq x_{j'+1}^0, \cdots, y_n \leq x_n^0. \tag{72}$$

Considering $y = (y_1, y_2, \cdots, y_n) \in X(A, b, d)$ and (69) and (72), $(y_{j'}, y_{j'+1}, \cdots, y_n)$ is a feasible solution of Problem $P_{j'}^{\leq}$ with objective value $y_{j'}$. Suppose the optimal objective value of Problem $P_{j'}^{\leq}$ is $x_{j'}^{\leq}$. Then it holds that $y_{j'} \geq x_{j'}^{\leq}$. However, it follows from Lemma 3 that $x_{j'}^{\leq} = x_{j'}^*$. Therefore, we obtain $y_{j'} \geq x_{j'}^{\leq}$.

$$y_{j'} \geq x_{j'}^*. \tag{73}$$

Inequalities (70) and (73) lead to a contradiction. □

Algorithm II (For Solving the Upper Bounded Minimal Solution of System (3))

Step 1. According to (9) and (11), compute the potential maximum solution $\hat{x}$ of system (3).

Step 2. Check the consistency of system (3) by Theorem 1. If system (3) is consistent, then continue to Step 3. Otherwise, system (3) is inconsistent, i.e., $X(A, b, d) = \emptyset$, it does not have any upper bounded minimal solution; therefore, stop.

Step 3. Compute the vector $x^0 = \bar{x} \wedge \hat{x}$.

Step 4. Determine whether x^0 is a solution of system (3). If $x^0 \in X(A, b, d)$, then by Theorem 4, system (3) has an upper bounded minimal solution with respect to $\bar{x}$, and we continue to Step 5. Otherwise, if $x^0 \notin X(A, b, d)$, then by Theorem 4, system (3) has no upper bounded minimal solution with respect to $\bar{x}$; therefore, stop.

Step 5. Let $k := 1$.

Step 6. Solving the subproblem P_k by Algorithm I, assume that the obtained optimal solution of Problem P_k is x_k^*.

Step 7. If $k = n$, then go to Step 8. Otherwise, if $k < n$, then let $k := k + 1$ and return to Step 6.

Step 8. Generate the vector $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ by the optimal solutions of the subproblems $P_1, P_2, \cdots, P_n$. Then, by Theorem 8, x^* is an upper bounded minimal solution of system (3) with respect to $\bar{x}$.

Example 2: We still consider the max-min fuzzy relation inequality system presented in Example (1), i.e., system (59). For the given vector $\bar{x} = (0.8, 0.5, 0.7, 0.55, 0.6, 0.4)$, find an upper bounded minimal solution with respect to $\bar{x}$.

Solution:

Steps 1 & 2. It has been checked in Example (1) that $\hat{x} = (0.6, 1, 0.6, 0.7, 0.8, 0.7)$ and system (59) is consistent. Thus, we continue to Step 3.

Step 3. Computing the vector x^0, we have

$$\begin{aligned} x^0 &= \bar{x} \wedge \hat{x} \\ &= (0.8, 0.5, 0.7, 0.55, 0.6, 0.4) \\ &\quad \wedge (0.6, 1, 0.6, 0.7, 0.8, 0.7) \\ &= (0.6, 0.5, 0.6, 0.55, 0.6, 0.4). \end{aligned} \tag{74}$$

Step 4. After calculation, we obtain

$$A \circ x^{0T} = \begin{bmatrix} 0.6 & 0.7 & 0.4 & 0.8 & 0.6 & 0.7 \\ 0.8 & 0.7 & 0.4 & 0.2 & 0.4 & 0.8 \\ 0.7 & 0.6 & 0.7 & 0.4 & 0.5 & 0.5 \\ 0.4 & 0.3 & 0.5 & 0.6 & 0.5 & 0.6 \\ 0.6 & 0.5 & 0.7 & 0.4 & 0.8 & 0.3 \\ 0.5 & 0.4 & 0.4 & 0.8 & 0.9 & 0.5 \end{bmatrix} \circ \begin{bmatrix} 0.6 \\ 0.5 \\ 0.6 \\ 0.55 \\ 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.6 \\ 0.6 \\ 0.55 \\ 0.6 \\ 0.6 \end{bmatrix}.$$

It is clear that

$$b^T = \begin{bmatrix} 0.6 \\ 0.5 \\ 0.55 \\ 0.55 \\ 0.6 \\ 0.6 \end{bmatrix} \leq \begin{bmatrix} 0.6 \\ 0.6 \\ 0.6 \\ 0.55 \\ 0.6 \\ 0.6 \end{bmatrix} \leq \begin{bmatrix} 0.7 \\ 0.7 \\ 0.6 \\ 0.8 \\ 0.9 \\ 0.8 \end{bmatrix} = d^T.$$

This indicates that $x^0 = (0.6, 0.5, 0.6, 0.55, 0.6, 0.4)$ is a solution of system (59). It follows from Theorem 4 that system (59) has an upper bounded minimal solution with respect to $\bar{x}$.

Steps 5-7. For $k = 1$, the subproblem P_k is

$$\begin{aligned} &P_1 \\ &\min x_1 \\ &\text{s.t. } (x_1, x_2, \cdots, x_6) \in X(A, b, d), \\ &\quad x_2 = 0.5, x_3 = 0.6, x_4 = 0.55, x_5 = 0.6, x_6 = 0.4, \end{aligned} \tag{75}$$

where $X(A, b, d)$ represents the solution set of system (59). Applying Algorithm I to solve Problem P_1, we have $x_1^* = 0$.

For $k = 2$, the subproblem P_k is

$$\begin{aligned} &P_2 \\ &\min x_2 \\ &\text{s.t. } (x_1, x_2, \cdots, x_6) \in X(A, b, d), \\ &\quad x_1 = 0, x_3 = 0.6, x_4 = 0.55, x_5 = 0.6, x_6 = 0.4. \end{aligned} \tag{76}$$

Applying Algorithm I to solve Problem P_2, we have $x_2^* = 0.5$.

For $k = 3$, the subproblem P_k is

$$\begin{aligned} &P_3 \\ &\min x_3 \\ &\text{s.t. } (x_1, x_2, \cdots, x_6) \in X(A, b, d), \\ &\quad x_1 = 0, x_2 = 0.5, x_4 = 0.55, x_5 = 0.6, x_6 = 0.4. \end{aligned} \tag{77}$$

Applying Algorithm I to solve Problem P_3, we have $x_3^* = 0.55$.

For $k = 4$, the subproblem P_k is

$$\begin{aligned} &P_4 \\ &\min x_4 \\ &\text{s.t. } (x_1, x_2, \cdots, x_6) \in X(A, b, d), \\ &\quad x_1 = 0, x_2 = 0.5, x_3 = 0.55, x_5 = 0.6, x_6 = 0.4. \end{aligned} \tag{78}$$

Applying Algorithm I to solve Problem P_4, we have $x_4^* = 0.55$.

For $k = 5$, the subproblem P_k is

$$\begin{aligned} &P_5 \\ &\min x_5 \\ &\text{s.t. } (x_1, x_2, \cdots, x_6) \in X(A, b, d), \\ &\quad x_1 = 0, x_2 = 0.5, x_3 = 0.55, x_4 = 0.55, x_6 = 0.4. \end{aligned} \tag{79}$$

Applying Algorithm I to solve Problem P_5, we have $x_5^* = 0.6$.

For $k = 6$, the subproblem P_k is

$$\begin{aligned} &P_6 \\ &\min x_6 \\ &\text{s.t. } (x_1, x_2, \cdots, x_6) \in X(A, b, d), \\ &\quad x_1 = 0, x_2 = 0.5, x_3 = 0.55, x_4 = 0.55, x_5 = 0.6. \end{aligned} \tag{80}$$

Applying Algorithm I to solve Problem P_6, we have $x_6^* = 0$.

Step 8. It follows from Theorem 8 that $x^* = (0, 0.5, 0.55, 0.55, 0.6, 0)$ is an upper bounded minimal solution of system (59) with respect to $\bar{x} = (0.8, 0.5, 0.7, 0.55, 0.6, 0.4)$. □

Example 3: In this example, a P2P network system consisting of 6 terminals $\{T_1, T_2, \cdots, T_6\}$ is considered (see Fig. 2). In such a system, each pair of terminals is connected with a directed line. Suppose the bandwidth is represented by a_{ij}, $i = 1, 2, \cdots, 6$, and $j = 1, 2, \cdots, 6$. However, the requirement of the highest download traffic to the ith terminal, i.e., T_i, is assumed to be no less than b_i and no more than d_i. The quality level on which the jth terminal shares (sends out) its local resources is denoted by x_j. The measurement unit of a_{ij}, b_i, d_i or x_j is Mbps (million bits per second). The following Tables 1 and 2 store the values of all the parameters a_{ij}, b_i and d_i.

All the requirements of the terminals can be characterized by the following inequalities system:

$$b^T \leq A \circ x^T \leq d^T, \tag{81}$$

where $A = (a_{ij})_{6\times6}$, $b = (b_1, b_2, \cdots, b_6)$, and $d = (d_1, d_2, \cdots, d_6)$. Moreover, if we normalize all the parameters and variables in system (81), dividing by 50 (Mbps), then system (81) turns out to be a system of max-min fuzzy relation inequalities. After normalization, suppose the obtained fuzzy relation system is

$$b'^T \leq A' \circ x'^T \leq d'^T. \tag{82}$$

Here, $A' = (a'_{ij})$, $b' = (b'_i)$, $d' = (d'_i)$, $x' = (x'_j)$,

$$a'_{ij} = \frac{a_{ij}}{50}, \quad b'_i = \frac{b_i}{50}, \; d'_i = \frac{d_i}{50}, \; x'_j = \frac{x_j}{50}. \tag{83}$$

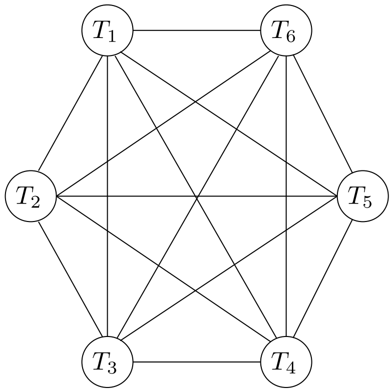

FIGURE 2. A P2P network system including 6 terminals.

TABLE 1. Value of a_{ij} (Measurement unit: Mbps).

i \ j	1	2	3	4	5	6
1	0	35	40	40	40	45
2	45	0	35	43	39	48
3	42	44	0	35	33	47
4	45	30	34	0	40	32
5	43	44	48	36	0	40
6	40	45	43	49	35	0

TABLE 2. Values of b_i and d_i (Measurement unit: Mbps).

i	1	2	3	4	5	6
b_i	40	38	42	35	40	42
d_i	45	44	48	45	46	47

We aim to find a minimal solution x^* with the upper bound $\bar{x} = (45, 42, 40, 35, 34, 39)$.

Solution:

It is trivial to find the maximum solution of system (82). After calculation, its maximum solution is

$$\hat{x}' = (0.88, 1, 0.92, 0.94, 1, 0.88).$$

As a consequence, the maximum solution of system (81) is

$$\hat{x} = (44, 50, 46, 47, 50, 44).$$

Therefore, we have

$$x^0 = \bar{x} \wedge \hat{x} = (44, 42, 40, 35, 34, 39).$$

Following our proposed Algorithms I and II, an eligible solution is

$$x^* = (35, 42, 40, 0, 0, 39).$$

That is, the quality levels of $T_1, T_2, \cdots, T_6$ are

$$35\text{Mbps},\ 42\text{Mbps},\ 40\text{Mbps},\ 0\text{Mbps},\ 0\text{Mbps},\ 39\text{Mbps},$$

respectively. □

V. DISCUSSION AND RESULT

A. COMPARISON WITH THE EXISTING WORKS

To embody the novelty and the technical contributions of this work, we compare our studied problem to the related ones in some existing works.

- Distinguishing from the objective
  In the existing works [27], [28], [29], [30], [31], although the constraint (fuzzy relation system with max-min composition) is the same as that in our work, the optimization objective is much different from that in this work. The objective (of the optimization problems) in [27], [28], [29], [30], and [31] is a typical linear function, i.e.,

$$\min c_1x_1 + c_2x_2 + \cdots + c_nx_n.$$

  By minimizing such an objective function under the fuzzy relation constraints, one is able to find a minimal solution. However, the objective of the problem in this work is to find a minimal solution that is no more than a given solution $\bar{x}$, i.e.,

$$x \leq \bar{x}.$$

  In addition, the constraint in [34], [35], [36], and [37] is the same as that in this work. However, the objective functions of the problems in these works turn out to be linear.
- Distinguishing from the constraint
  The objectives of the problems in [43] and this work are identical, i.e., to find a minimal solution of a fuzzy relation system such that it is no more than a given solution. However, their constraint systems are different. The constraint system in [43] is composed of addition-min operations, whereas the constraint system in this work is composed of max-min operations. As shown in [15], [16], and [45], the number and the resolution approach of the minimal solutions to the max-min system are much different from those to the addition-min system. As a consequence, the resolution method in [43] is no longer effective for our studied upper bounded minimal solution to system (3).

B. ADVANTAGES OF OUR PROPOSED ALGORITHMS

- **Computational complexity of Algorithm I**

Step 1 in Algorithm I costs $2mn - n$ operations for computing the potential maximum solution $\hat{x}$. To examine the consistency of system (3), Step 2 costs $m(2n + 1)$ operations. Similarly, it costs $m(2n + 1)$ operations for determining the solvability of Problem (25). Computing the index set I^k, it costs $2mn$ operations in Step 4. Finally, Step 5 costs m operations for obtaining the optimal solution x_k^*. As a result, all steps in Algorithm I cost

$$2mn - n + m(2n + 1) + m(2n + 1) + 2mn + m = 8mn + 3m - n$$

operations in total. Hence, the computational complexity of Algorithm I is $\mathbf{O}(mn)$. Algorithm I has a polynomial computational complexity.

- **Computational complexity of Algorithm II**

Similar to those in Algorithm I, Steps 1 and 2 in Algorithm II cost $2mn - n$ and $m(2n + 1)$ operations, respectively. Computing the vector x^0 in Step 3, it costs n operations.

Furthermore, determining whether the vector x^0 is a solution of system (3), it costs $m(2n+1)$ operations in Step 4. Steps 5-8 have n loops of operations. They cost $(2mn + m + 1) \times n$ operations in total. As a result, all steps in Algorithm II cost

$$2mn - n + m(2n + 1) \times 2 + (2mn + m + 1) \times n = 2mn^2 + 7mn + 2m$$

operations in total. Hence, the computational complexity of Algorithm II is $\mathbf{O}(mn^2)$. Algorithm II has a polynomial computational complexity.

- Comparing our proposed algorithms to the existing ones

In this paper, we proposed Algorithms I and II for obtaining the upper bounded minimal solution. The problem for solving the upper bounded minimal solution is separated into n subproblems. Each subproblem can be solved by Algorithm I, with a polynomial computational complexity. Combining all the optimal solutions of these subproblems, the upper bounded minimal solution can be generated following Algorithm II. Next, we further compare our proposed algorithms to the existing ones.

(i) In the existing works, some scholars have studied optimization problems with a linear objective function and fuzzy relation equation constraints [27], [28], [29], [30], [31], [32], [33] or fuzzy relation inequality constraints [22], [23]. The original problem was separated into two subproblems. Furthermore, one of the subproblems was equivalently converted to a 0-1 integer programming problem. As a result, the resolution of the subproblems is NP-hard. The computational complexity is not polynomial. However, as demonstrated above, solving the subproblems presented in this work has polynomial computational complexity.

(ii) The approach to separate the main problem into some subproblems was also adopted in the works [36], [37], [49], [50]. The number of subproblems coincides with the number of minimal solutions of a system of fuzzy relation equations (or inequalities). However, it is well known that the number of minimal solutions exponentially increases with the size of the fuzzy relation equations. This indicates that the number of subproblems is nonpolynomial. As a consequence, the computational complexity of the resolution algorithms presented in [36], [37], [49], and [50] is also nonpolynomial.

(iii) A genetic algorithm was introduced for dealing with nonlinear or multiobjective programming problems subject to fuzzy relation equations or inequality systems [34], [51], [52], [53], [54]. However, applying the genetic algorithm, one was only able to find an approximate optimal solution but not an exact optimal solution. The convergence was not formally proven in the works [34], [51], [52], [53], [54]. The error of the approximate optimal solution and the convergence are the significant defects of the genetic algorithm. However, Algorithms I and II presented in our work enable us to find the exact upper bounded minimal solution.

(iv) Similar to the resolution approach employed in our work, the original optimization problem in [55], [56], and [57] was also divided into n subproblems. Moreover, an efficient algorithm with polynomial computational complexity was developed for solving each subproblem. Finally, an optimal solution of the original optimization problem can be constructed by the optimal solutions of all these subproblems. As mentioned above, the computational complexity of the resolution algorithm for the subproblems is polynomial. As a consequence, from the perspective of computational complexity, the resolution algorithm in [55], [56], and [57] has the same order efficiency as Algorithm II proposed in this work. However, the fuzzy relation system employed in [55], [56], and [57] is composed of addition-min operations. However, the fuzzy relation system studied in our work is composed of max-min operations. Due to the different compositions, the resolution algorithm in [55], [56], and [57] is inapplicable to our studied problem.

C. GENERALIZATION OF THE UPPER BOUNDED MINIMAL SOLUTION TO SYSTEM (3)

It is clear that system (3) consists of a group of fuzzy relation inequalities with max-min composition. The corresponding upper bounded minimal solution is defined and studied. In system (3), the parameters are the common type-1 fuzzy numbers. As an extension, one could further investigate the upper bounded minimal solution to system (3) by representing the parameters as types of fuzzy numbers, such as interval values, triangular fuzzy numbers, trapezoidal fuzzy numbers and polygonal fuzzy numbers [46], [47], [48].

In addition, another generalization direction is to extend the upper bounded minimal solution to the fuzzy relation system with other kinds of compositions, such as max-product or max-Łukasiewicz.

D. CHARACTERISTICS OF THE UPPER BOUNDED MINIMAL SOLUTION TO SYSTEM (3)

To enable easy use of the upper bounded minimal solution to system (3), we further depict its characteristics in this subsection.

Following Definition 3, an upper bounded minimal solution to system (3), denoted by x^*, possesses the following three characteristics:

(i) *Feasible*

x^* should be a solution of system (3), i.e., $x^* \in X(A, b, d)$.

(ii) *Upper bounded*

x^* has an upper bound $\bar{x}$, i.e., $x^* \leq \bar{x}$ holds for the given solution $\bar{x}$.

(iii) *Minimal*

x^* is minimal, i.e., if there exists $y^* \in X(A, b, d)$ such that $y^* \leq x^*$, then we have $y^* = x^*$.

VI. CONCLUSION

In the fuzzy relation inequality system with addition-min composition, the minimal solution, which is no more than a given solution, was studied by Li and Wang [43]. Obviously, it is a restricted minimal solution, different from the general minimal solution. An effective resolution approach was proposed by the authors for obtaining such a minimal solution of the addition-min system. In this paper, motivated by the idea presented in [43], we extended the restricted minimal solution to the fuzzy relation inequality system with max-min composition, i.e. system (3).

In the max-min system (3), the upper bounded minimal solution with respect to $\bar{x}$ was defined as the minimal solution that is no more than $\bar{x}$. As a consequence, an upper bounded minimal solution was indeed a minimal solution with a given upper bound (denoted by $\bar{x}$ in this work).

To generate an upper bounded minimal solution of system (3) with the given vector $\bar{x}$, we constructed n subproblems based on system (3) and the vector $\bar{x}$. Algorithm I was developed to search for the optimal solutions of the subproblems. Furthermore, Algorithm II was proposed to find an upper bounded minimal solution of system (3). Our proposed resolution algorithms were illustrated by numerical examples.

As noted in Section V, the computational complexities of Algorithms I and II are $\mathbf{O}(mn)$ and $\mathbf{O}(mn^2)$, respectively. Algorithm I is the foundation of Algorithm II. Following Algorithm II, one is able to find the upper bounded minimal solution exactly. Obviously, Algorithm II has polynomial computational complexity. It is an efficient algorithm. It can be applied to obtain the upper bounded minimal solution with a large problem size.

In our future work, we will focus on other kinds of minimal solutions to the fuzzy relation system with max-min or addition-min composition. In addition, we will explore the application of fuzzy relation inequalities in the supply chain system [58], [59], [60] and the energy system [61], [62], [63], [64], [65], [66]. Moreover, we may try to find the specific solution by some machine learning techniques [67].

REFERENCES

[1] E. Sanchez, "Resolution of composite fuzzy relation equations," *Inf. Control*, vol. 30, pp. 38–48, Jan. 1976.

[2] S. Feng and L. D. Xu, "Decision support for fuzzy comprehensive evaluation of urban development," *Fuzzy Sets Syst.*, vol. 105, no. 1, pp. 1–12, Jul. 1999.

[3] S.-H. Cao, P.-P. Ming, and X. Zhao, "Fuzzy comprehensive evaluation of human thermal comfort in simulating natural wind environment," *Building Environ.*, vol. 188, Jan. 2021, Art. no. 107447.

[4] H. Guo, C. Zheng, X. Yang, H. Lin, and X. Yang, "Fuzzy comprehensive evaluation for the laboratory performance in the university under multi-judgments situation," *J. Liaocheng Univ.*, vol. 33, no. 1, pp. 10–16, 2020.

[5] A. Di Nola, S. Sessa, W. Pedrycz, and E. Sanchez, *Fuzzy Relation Equations and Their Applications to Knowledge Engineering*. Dordrecht, The Netherlands: Kluwer, 1989.

[6] P. Li and S.-C. Fang, "On the resolution and optimization of a system of fuzzy relational equations with sup-T composition," *Fuzzy Optim. Decis. Making*, vol. 7, no. 2, pp. 169–214, Jun. 2008.

[7] X. Li and D. Ruan, "Novel neural algorithms based on fuzzy δ rules for solving fuzzy relation equations: Part I," *Fuzzy Sets Syst.*, vol. 90, pp. 11–23, Jan. 1997.

[8] B. De Baets, "Analytical solution methods for fuzzy relational equations," in *Fundamentals of Fuzzy Sets*, D. Dubois and H. Prade, Eds. Boston, MA, USA: Kluwer, 2000, pp. 291–340.

[9] M. Allame and B. Vatankhahan, "Iteration algorithm for solving $Ax = b$ in max–min algebra," *Appl. Math. Comput.*, vol. 175, no. 1, pp. 269–276, Apr. 2006.

[10] S. Abbasbandy, E. Babolian, and M. Allame, "Numerical solution of fuzzy max–min systems," *Appl. Math. Comput.*, vol. 174, no. 2, pp. 1321–1328, Mar. 2006.

[11] B.-S. Shieh, "New resolution of finite fuzzy relation equations with max–min composition," *Int. J. Uncertainty, Fuzziness Knowl.-Based Syst.*, vol. 16, no. 1, pp. 19–33, Feb. 2008.

[12] D. Cheng, J.-E. Feng, and H. Lv, "Solving fuzzy relational equations via semitensor product," *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 2, pp. 390–396, Apr. 2012.

[13] A. A. Molai and E. Khorram, "An algorithm for solving fuzzy relation equations with max-T composition operator," *Inf. Sci.*, vol. 178, no. 5, pp. 1293–1308, Mar. 2008.

[14] H. Nobuhara, W. Pedrycz, and K. Hirota, "Fast solving method of fuzzy relational equation and its application to lossy image compression/reconstruction," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 3, pp. 325–334, Jun. 2000.

[15] Z. Chai, "Solving the minimal solutions of max–min fuzzy relation equation by graph method and branch method," in *Proc. 7th Int. Conf. Fuzzy Syst. Knowl. Discovery*, Aug. 2010, pp. 319–324.

[16] C.-T. Yeh, "On the minimal solutions of max–min fuzzy relational equations," *Fuzzy Sets Syst.*, vol. 159, no. 1, pp. 23–39, Jan. 2008.

[17] E. Bartl and P. Prochazka, "Do we need minimal solutions of fuzzy relational equations in advance?" *IEEE Trans. Fuzzy Syst.*, vol. 25, no. 5, pp. 1356–1363, Oct. 2017.

[18] B.-S. Shieh, "Deriving minimal solutions for fuzzy relation equations with max-product composition," *Inf. Sci.*, vol. 178, no. 19, pp. 3766–3774, Oct. 2008.

[19] L. Chen and P. P. Wang, "Fuzzy relation equations (II): The branch-point-solutions and the categorized minimal solutions," *Soft Comput.*, vol. 11, no. 1, pp. 33–40, Jan. 2007.

[20] F. Sun, X.-B. Qu, and X.-P. Wang, "Remarks on minimal solutions of a system of fuzzy relation equations over complete infinitely distributive lattices," *Soft Comput.*, vol. 20, no. 2, pp. 423–428, Feb. 2016.

[21] S. Wang and H. Li, "Column stacking approach to resolution of systems of fuzzy relational inequalities," *J. Franklin Inst.*, vol. 356, no. 6, pp. 3314–3332, Apr. 2019.

[22] F.-F. Guo and Z.-Q. Xia, "An algorithm for solving optimization problems with one linear objective function and finitely many constraints of fuzzy relation inequalities," *Fuzzy Optim. Decis. Making*, vol. 5, no. 1, pp. 33–47, Jan. 2006.

[23] F.-F. Guo, L.-P. Pang, D. Meng, and Z.-Q. Xia, "An algorithm for solving optimization problems with fuzzy relational inequality constraints," *Inf. Sci.*, vol. 252, pp. 20–31, Dec. 2013.

[24] P. Z. Wang, D. Z. Zhang, E. Sanchez, and E. S. Lee, "Latticized linear programming and fuzzy relation inequalities," *J. Math. Anal. Appl.*, vol. 159, no. 1, pp. 72–87, Jul. 1991.

[25] L. Chen and P. P. Wang, "Fuzzy relation equations (I): The general and specialized solving algorithms," *Soft Comput. Fusion Found., Methodologies Appl.*, vol. 6, no. 6, pp. 428–435, Sep. 2002.

[26] J.-L. Lin, Y.-K. Wu, and S.-M. Guu, "On fuzzy relational equations and the covering problem," *Inf. Sci.*, vol. 181, no. 14, pp. 2951–2963, Jul. 2011.

[27] S.-C. Fang and G. Li, "Solving fuzzy relation equations with a linear objective function," *Fuzzy Sets Syst.*, vol. 103, no. 1, pp. 107–113, Apr. 1999.

[28] Y.-K. Wu and S.-M. Guu, "Minimizing a linear function under a fuzzy max–min relational equation constraint," *Fuzzy Sets Syst.*, vol. 150, no. 1, pp. 147–162, Feb. 2005.

[29] H.-C. Lee and S.-M. Guu, "On the optimal three-tier multimedia streaming services," *Fuzzy Optim. Decis. Making*, vol. 2, pp. 31–39, Mar. 2003.

[30] L. Jian-Xin, "A new algorithm for minimizing a linear objective function with fuzzy relation equation constraints," *Fuzzy Sets Syst.*, vol. 159, no. 17, pp. 2278–2298, Sep. 2008.

[31] C.-W. Chang and B.-S. Shieh, "Linear optimization problem constrained by fuzzy max–min relation equations," *Inf. Sci.*, vol. 234, pp. 71–79, Jun. 2013.

[32] Z. Mashayekhi and E. Khorram, "On optimizing a linear objective function subjected to fuzzy relation inequalities," *Fuzzy Optim. Decis. Making*, vol. 8, no. 1, pp. 103–114, Mar. 2009.

[33] A. Thapar, D. Pandey, and S. K. Gaur, "Optimization of linear objective function with max-t fuzzy relation equations," *Appl. Soft Comput.*, vol. 9, no. 3, pp. 1097–1101, Jun. 2009.

[34] J. Lu and S.-C. Fang, "Solving nonlinear optimization problems with fuzzy relation equation constraints," *Fuzzy Sets Syst.*, vol. 119, no. 1, pp. 1–20, Apr. 2001.

[35] X.-G. Zhou, X.-P. Yang, and B.-Y. Cao, "Posynomial geometric programming problem subject to max–min fuzzy relation equations," *Inf. Sci.*, vol. 328, pp. 15–25, Jan. 2016.

[36] X.-P. Yang, "Linear programming method for solving semi-latticized fuzzy relation geometric programming with max–min composition," *Int. J. Uncertainty, Fuzziness Knowl.-Based Syst.*, vol. 23, no. 5, pp. 781–804, Oct. 2015.

[37] X.-P. Yang, X.-G. Zhou, B.-Y. Cao, and Y.-H. Hong, "Variable substitution method for solving single-variable term fuzzy relation geometric programming problem and its application," *Int. J. Uncertainty, Fuzziness Knowl.-Based Syst.*, vol. 27, no. 4, pp. 537–557, Aug. 2019.

[38] G. Xiao, T. Zhu, Y. Chen, and X. Yang, "Linear searching method for solving approximate solution to system of max–min fuzzy relation equations with application in the instructional information resources allocation," *IEEE Access*, vol. 7, pp. 65019–65028, 2019.

[39] Y. Chen, X. Liu, and L. Zhang, "Interval solution to fuzzy relation inequality with application in P2P educational information resource sharing systems," *IEEE Access*, vol. 9, pp. 96166–96175, 2021.

[40] Y. Ma, X. Yang, and B.-Y. Cao, "Fuzzy-relation-based lexicographic minimum solution to the P2P network system," *IEEE Access*, vol. 8, pp. 195447–195458, 2020.

[41] Z. Matusiewicz and J. Drewniak, "Increasing continuous operations in fuzzy max–∗ equations and inequalities," *Fuzzy Sets Syst.*, vol. 232, pp. 120–133, Dec. 2013.

[42] X.-P. Yang, X.-G. Zhou, and B.-Y. Cao, "Latticized linear programming subject to max-product fuzzy relation inequalities with application in wireless communication," *Inf. Sci.*, vols. 358–359, pp. 44–55, Sep. 2016.

[43] M. Li and X.-P. Wang, "Remarks on minimal solutions of fuzzy relation inequalities with addition-min composition," *Fuzzy Sets Syst.*, vol. 410, pp. 19–26, May 2021.

[44] Y.-B. Zhong, G. Xiao, and X.-P. Yang, "Fuzzy relation lexicographic programming for modelling P2P file sharing system," *Soft Comput.*, vol. 23, no. 11, pp. 3605–3614, Jun. 2019.

[45] X.-P. Yang, H.-T. Lin, X.-G. Zhou, and B.-Y. Cao, "Addition-min fuzzy relation inequalities with application in BitTorrent-like peer-to-peer file sharing system," *Fuzzy Sets Syst.*, vol. 343, pp. 126–140, Jul. 2018.

[46] C. Suo, Y. Li, and Z. Li, "On *n*-polygonal interval-valued fuzzy sets," *Fuzzy Sets Syst.*, vol. 417, pp. 46–70, Aug. 2021.

[47] P. Gao and Y. Gao, "Quadrilateral interval type-2 fuzzy regression analysis for data outlier detection," *Math. Problems Eng.*, vol. 2019, pp. 1–9, Aug. 2019.

[48] X. Li, F. Li, X. Zhang, C. Yang, and W. Gui, "Exponential stability analysis for delayed semi-Markovian recurrent neural networks: A homogeneous polynomial approach," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 12, pp. 6374–6384, Dec. 2018.

[49] A. A. Molai, "The quadratic programming problem with fuzzy relation inequality constraints," *Comput. Ind. Eng.*, vol. 62, no. 1, pp. 256–263, Feb. 2012.

[50] A. Abbasi Molai, "A new algorithm for resolution of the quadratic programming problem with fuzzy relation inequality constraints," *Comput. Ind. Eng.*, vol. 72, pp. 306–314, Jun. 2014.

[51] E. Khorram and R. Hassanzadeh, "Solving nonlinear optimization problems subjected to fuzzy relation equation constraints with max–average composition using a modified genetic algorithm," *Comput. Ind. Eng.*, vol. 55, no. 1, pp. 1–14, Aug. 2008.

[52] R. Hassanzadeh, E. Khorram, I. Mahdavi, and N. Mahdavi-Amiri, "A genetic algorithm for optimization problems with fuzzy relation constraints using max-product composition," *Appl. Soft Comput.*, vol. 11, no. 1, pp. 551–560, Jan. 2011.

[53] A. Ghodousian and A. Babalhavaeji, "An efficient genetic algorithm for solving nonlinear optimization problems defined with fuzzy relational equations and max-Lukasiewicz composition," *Appl. Soft Comput.*, vol. 69, pp. 475–492, Aug. 2018.

[54] E. Khorram, R. Ezzati, and Z. Valizadeh, "Solving nonlinear multi-objective optimization problems with fuzzy relation inequality constraints regarding Archimedean triangular norm compositions," *Fuzzy Optim. Decis. Making*, vol. 11, no. 3, pp. 299–335, Sep. 2012.

[55] X.-P. Yang, X.-G. Zhou, and B.-Y. Cao, "Multi-level linear programming subject to addition-min fuzzy relation inequalities with application in peer-to-peer file sharing system," *J. Intell. Fuzzy Syst.*, vol. 28, no. 6, pp. 2679–2689, Aug. 2015.

[56] X.-P. Yang, X.-G. Zhou, and B.-Y. Cao, "Min–max programming problem subject to addition-min fuzzy relation inequalities," *IEEE Trans. Fuzzy Syst.*, vol. 24, no. 1, pp. 111–119, Feb. 2016.

[57] Y.-L. Chiu, S.-M. Guu, J. Yu, and Y.-K. Wu, "A single-variable method for solving min–max programming problem with addition-min fuzzy relational inequalities," *Fuzzy Optim. Decis. Making*, vol. 18, no. 4, pp. 433–449, Dec. 2019.

[58] X. Zhou, X. Zhong, H. Lin, Z. Qin, and X. Yang, "Lexicographic maximum solution of min-product fuzzy relation inequalities for modeling the optimal pricing with fixed priority grade in supply chain," *IEEE Access*, vol. 6, pp. 71306–71316, 2018.

[59] X. Yang, "Solutions and strong solutions of min-product fuzzy relation inequalities with application in supply chain," *Fuzzy Sets Syst.*, vol. 384, pp. 54–74, Apr. 2020.

[60] P. Akbary, M. Ghiasi, M. R. R. Pourkheranjani, H. Alipour, and N. Ghadimi, "Extracting appropriate nodal marginal prices for all types of committed reserve," *Comput. Econ.*, vol. 53, no. 1, pp. 1–26, Jan. 2019.

[61] Z. Yang, M. Ghadamyari, H. Khorramdel, S. M. S. Alizadeh, S. Pirouzi, M. Milani, F. Banihashemi, and N. Ghadimi, "Robust multi-objective optimal design of islanded hybrid system with renewable and diesel sources/stationary and mobile energy storage systems," *Renew. Sustain. Energy Rev.*, vol. 148, Sep. 2021, Art. no. 111295.

[62] J. Liu, C. Chen, Z. Liu, K. Jermsittiparsert, and N. Ghadimi, "An IGDT-based risk-involved optimal bidding strategy for hydrogen storage-based intelligent parking lot of electric vehicles," *J. Energy Storage*, vol. 27, Feb. 2020, Art. no. 101057.

[63] M. Mir, M. Shafieezadeh, M. A. Heidari, and N. Ghadimi, "Application of hybrid forecast engine based intelligent algorithm and feature selection for wind signal prediction," *Evolving Syst.*, vol. 11, no. 4, pp. 559–573, Dec. 2020.

[64] M. Mehrpooya, N. Ghadimi, M. Marefati, and S. A. Ghorbanian, "Numerical investigation of a new combined energy system includes parabolic dish solar collector, stirling engine and thermoelectric device," *Int. J. Energy Res.*, vol. 45, no. 11, pp. 16436–16455, Sep. 2021.

[65] F. Mirzapour, M. Lakzaei, G. Varamini, M. Teimourian, and N. Ghadimi, "A new prediction model of battery and wind-solar output in hybrid power system," *J. Ambient Intell. Humanized Comput.*, vol. 10, no. 1, pp. 77–87, Jan. 2019.

[66] W. Cai, R. Mohammaditab, G. Fathi, K. Wakil, A. G. Ebadi, and N. Ghadimi, "Optimal bidding and offering strategies of compressed air energy storage: A hybrid robust-stochastic approach," *Renew. Energy*, vol. 143, pp. 1–8, Dec. 2019.

[67] A. S. D. Murthy, P. S. Murthy, V. Rajesh, S. H. Ahammad, and B. O. L. Jagan, "Execution of natural random forest machine learning techniques on multi spectral image compression," *Int. J. Pharmaceutical Res.*, vol. 11, no. 4, pp. 1241–1255, 2019.

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