IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 8 July 2022, accepted 26 July 2022, date of publication 5 August 2022, date of current version 16 August 2022.

Digital Object Identifier 10.1109/ACCESS.2022.3196664

COMMENT

Comments on "On Scale Parameter Monitoring of the Rayleigh Distributed Data Using a New Design"

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ABSTRACT We provide comments on a research paper proposing a control chart to monitor the scale parameter of a Rayleigh distribution. Our primary focus is on improving its average run length profile. We determine limits such that, on average, one can detect any change in this scale parameter more quickly than to trigger a false alarm.

INDEX TERMS Average run length (ARL), ARL-unbiased charts, statistical process monitoring, Weibull distribution.

I. INTRODUCTION

The authors in [1] considered the monitoring of the scale factor σ ($\sigma > 0$) of a quality characteristic T with a Rayleigh distribution, i.e., with a cumulative distribution function equal to

$$F_T(t) = 1 - e^{-\left[t/(\sqrt{2}\sigma)\right]^2}, \quad t > 0.$$

It was assumed that the output process observations are independent between and within samples of size n. Moreover, they suggested the maximum likelihood estimator (MLE) of the *scale factor* σ, $V_{SQR} = \sqrt{\sum_{i=1}^{n} T_i^2/(2n)}$, as the control statistic of a Phase II chart they proposed to detect sustained shifts in σ from a known target value σ_0 to $\theta\,\sigma_0$, where θ is a known positive constant. The observed Rayleigh random variables T_i are independent and identically distributed.

The control limits of this chart were given by

$$LCL_{SQR} = \sigma_0 \cdot \left[A(n) - l\sqrt{1 - A^2(n)}\right] \quad (1)$$

$$UCL_{SQR} = \sigma_0 \cdot \left[A(n) + l\sqrt{1 - A^2(n)}\right], \quad (2)$$

where: $A(n) = E_{\sigma_0}(V_{SQR}/\sigma_0) = \Gamma(n + 1/2)/(\sqrt{n}\,\Gamma(n))$; l is chosen in such a way that the probability of a false alarm is equal to a pre-specified value α and can be found in [1, Table 1] for different values of the sample size ($n \in \{2, 3, \ldots, 10\}$) and false alarm rate ($\alpha \in \{0.002, 0.01, 0.025\}$).

The associate editor coordinating the review of this manuscript and approving it for publication was Jenny Mahoney.

Since $Z = 2nV_{SQR}^2/(\theta\sigma_0)^2 \sim \chi^2_{(2n)}$ is a pivotal quantity for σ, we can obtain the following probability that the chart triggers a signal, given that $\sigma = \theta\,\sigma_0$:

$$\xi_{SQR}(\theta) = 1 - F_{\chi^2_{(2n)}}\left(\frac{2n\,UCL_{SQR}^2}{\theta^2\sigma_0^2}\right) + F_{\chi^2_{(2n)}}\left(\frac{2n\,LCL_{SQR}^2}{\theta^2\sigma_0^2}\right),$$

for $\theta > 0$. The corresponding average run length (ARL), that is, the expected number of samples we collect until a signal is triggered by what we shall call the V_{SQR} chart, is given by $\mathrm{ARL}_{SQR}(\theta) = 1/\xi_{SQR}(\theta)$.

Quality control practitioners favour charts with suitably large ARL values when the process is in-control, thus leading to infrequent false alarms, and small ARL values when the process is out-of-control, yielding valid signals quickly.

II. ARL PERFORMANCE CONCERNS

Now, we offer some comments on [1, Fig. 2] with four plots of the power function $\xi_{SQR}(\theta)$, for $\alpha = 0.025$, $n \in \{1, 2, 3, 4\}$, and $\theta \in [1, 6]$.

We believe that these power curves should have also been plotted for $\theta < 1$, in order to give the quality practitioner an idea of the behaviour of the proposed chart in the presence of decreases in σ.

Hence, in Figures 1 and 2, the reader can find plots of $\xi_{SQR}(\theta)$, for $\alpha = 0.025$, $n \in \{1, 2, 3, 4\}$, and θ in a wider range and also in a range around the in-control situation, [0.75, 1.6].

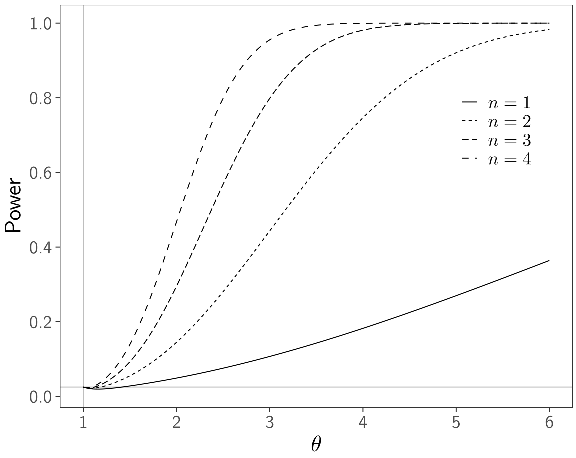

FIGURE 1. **Power function of the V_{SQR} chart, for $\alpha = 0.025$ and $n \in \{1, 2, 3, 4\}$.**

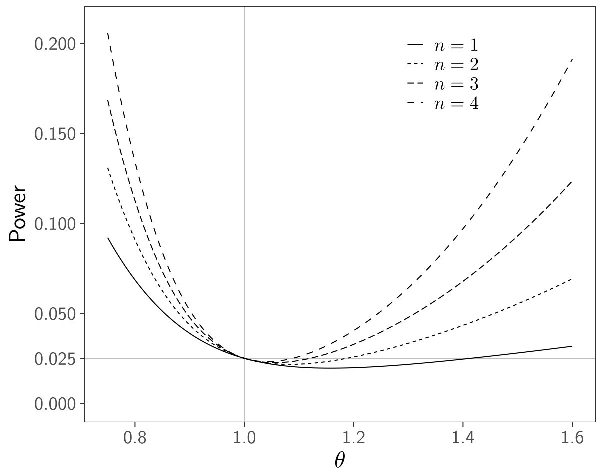

FIGURE 2. **Power function of the V_{SQR} chart in the vicinity of $\theta = 1$, for $\alpha = 0.025$ and $n \in \{1, 2, 3, 4\}$.**

We can see that the probability that the chart proposed by [1] detects a considerable range of increases in σ is smaller than the probability that the chart triggers a false alarm.

Moreover, [1, Fig. 3] illustrated the ARL profiles of the V_{SQR} chart, for $n \in \{1, 2, 3, 4\}$ and in-control ARL equal to $\text{ARL}_0 = 1/\alpha = 40$. The curves were truncated at ARL = 30 and values of θ in the interval (0, 1] were omitted.

For comparison, we plotted the ARL profiles for $\theta \in [0.5, 6]$ in Figure 3. All the plots in Figure 3 show that the out-of-control ARL is larger than the in-control ARL values for some small increases in σ. For example, when $n = 3$, the ARL associated with a 5.6 % increase in σ is equal to $\text{ARL}_{SQR}(1.056) \simeq 44.23$ and 10.58 % larger than the in-control ARL. This behaviour of the ARL function of the V_{SQR} chart will be less severe if we consider larger sample sizes. However, in practice, sample sizes are often between 1 and 5. Sample sizes could be much higher, but we recommend careful consideration if data are aggregated over time, as discussed by [2].

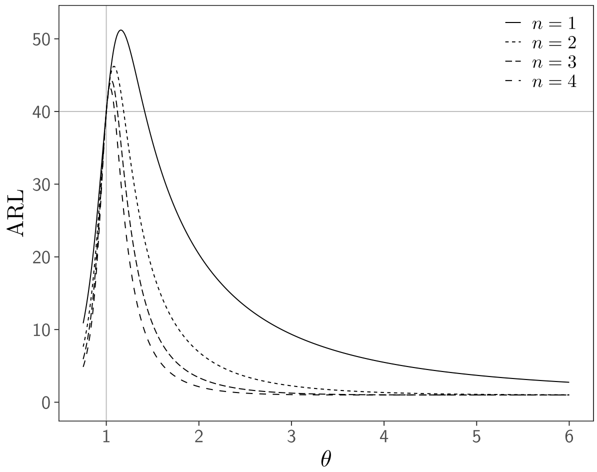

FIGURE 3. **ARL profiles of the V_{SQR} chart, for $\text{ARL}_0 = 1/\alpha = 40$ and $n \in \{1, 2, 3, 4\}$.**

III. SOME ARL PERFORMANCE COMPARISONS

The control chart we have discussed so far was suggested as an alternative to the one proposed by [3], with the control statistic $V_R = V_{SQR}^2$, an unbiased estimator of σ^2, and control limits

$$LCL_R = \sigma_0^2 - L \cdot \sigma_0^2/\sqrt{n}$$
$$UCL_R = \sigma_0^2 + L \cdot \sigma_0^2/\sqrt{n},$$

where L is also chosen so that the probability of a false alarm equals α.

Provided that $\sigma = \theta\,\sigma_0$, the V_R chart for σ^2 triggers a signal with probability

$$\xi_R(\theta) = 1 - F_{\chi^2_{(2n)}}\left(\frac{2n\,UCL_R^2}{\theta^2\sigma_0^2}\right) + F_{\chi^2_{(2n)}}\left(\frac{2n\,LCL_R^2}{\theta^2\sigma_0^2}\right)$$

and $\text{ARL}_R(\theta) = 1/\xi_R(\theta)$, for $\theta > 0$.

We now reassess the performance comparison between the V_{SQR} and V_R charts in [1]. The authors' Figure 7 provides plots of the power functions of these two charts, for $\alpha = 0.025$, $n \in \{2, 4, 8, 10\}$, and $\theta \in [1, 6]$; decreases in σ were not considered.

In our opinion, the power curves in [1, Fig. 7] should have been complemented by the ARL profiles of the V_{SQR} and V_R charts found in our Figure 4 with $\theta \in [0.5, 2]$.

This figure allows us to conclude that the Phase II chart proposed by [3] offers less protection than the V_{SQR} chart against decreases in the scale factor and takes, on average, longer to detect small decreases in σ than to emit a false alarm. Furthermore, when it comes to detecting increases in σ, the V_R chart seems to provide more protection against small increases in the process variance ($2\sigma^2(1 - \pi/4)$) than the V_{SQR} chart.

The monitoring of σ can also be done by adopting an alternative chart with *probability limits*.

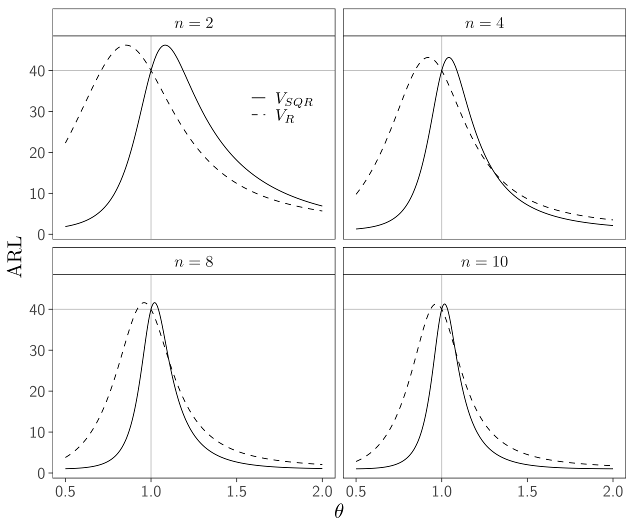

FIGURE 4. ARL profiles of the V_{SQR} and V_R charts, for $\text{ARL}_0 = 1/\alpha = 40$ and $n \in \{2, 4, 8, 10\}$.

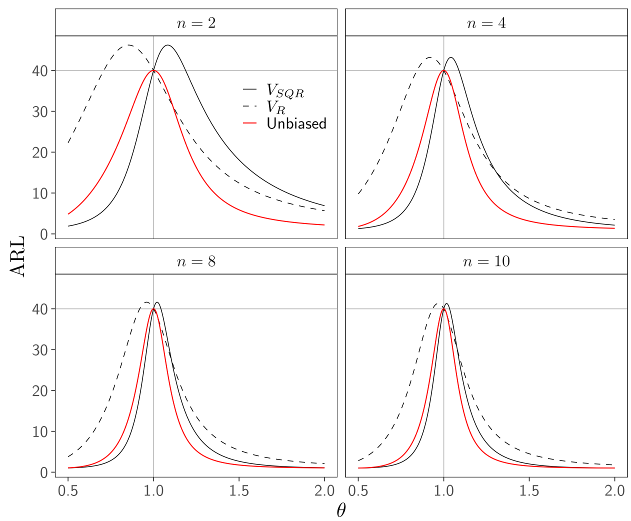

FIGURE 5. ARL profiles of the V_{SQR} and V_R charts and the ARL-unbiased chart for σ, with $\text{ARL}_0 = 1/\alpha = 40$ and $n \in \{2, 4, 8, 10\}$.

IV. AN ALTERNATIVE: THE ARL-UNBIASED CHART FOR σ

Ideally, the ARL function should achieve a maximum when the process is in-control, leading to what [4] termed an *ARL-unbiased chart*. An ARL-biased chart will have out-of-control ARL values larger than the in-control ARL for some range of small shifts in the parameter being monitored.

To achieve an ARL-unbiased chart for σ with in-control ARL equal to $\text{ARL}_0 = 1/\alpha$, where ARL_0 is a pre-specified suitably large value of the in-control ARL, we can capitalize on [5] or on two associated papers, [6] and [7]. These references propose ARL-unbiased charts (with fixed and variable sampling intervals) for the scale parameter of a process with Weibull output. The Rayleigh distribution is a particular case of the Weibull distribution with shape and scale parameters equal to 2 and $\sqrt{2}\sigma$, respectively. Consequently, we can use the same control statistic V_{SQR} and adopt the control limits

$$LCL^\star = \sigma_0 \cdot \sqrt{a^\star/n} \tag{3}$$
$$UCL^\star = \sigma_0 \cdot \sqrt{b^\star/n}, \tag{4}$$

where the quantiles of the $\chi^2_{(2n)}$ distribution, $a^\star \equiv a^\star(\alpha, n)$ and $b^\star \equiv b^\star(\alpha, n)$, satisfy, according to [5], [6], and [7], the following system of equations:

$$F_{\chi^2_{(2n)}}(b^\star) - F_{\chi^2_{(2n)}}(a^\star) = 1 - \alpha; \tag{5}$$

$$f_{\chi^2_{(2(n+1))}}(b^\star) - f_{\chi^2_{(2(n+1))}}(a^\star) = 0. \tag{6}$$

The probability that this chart triggers a signal, given that $\sigma = \theta\, \sigma_0$, is given by

$$\xi^\star(\theta) = 1 - \left[F_{\chi^2_{2n}}(b^\star/\theta^2) - F_{\chi^2_{2n}}(a^\star/\theta^2)\right],$$

for $\theta > 0$; the associated ARL function is equal to $\text{ARL}^\star(\theta) = 1/\xi^\star(\theta)$.

The use of (5) and (6) guarantees that $\text{ARL}^\star(1) = 1/\alpha$ and $d\text{ARL}^\star(\theta)/d\theta\,|_{\theta=1} = 0$, respectively. The resulting chart has the desired in-control ARL, $\text{ARL}_0 = 1/\alpha$, and the ARL curve attains a maximum in the in-control situation, as illustrated by the plots in Figures 5 and 6.

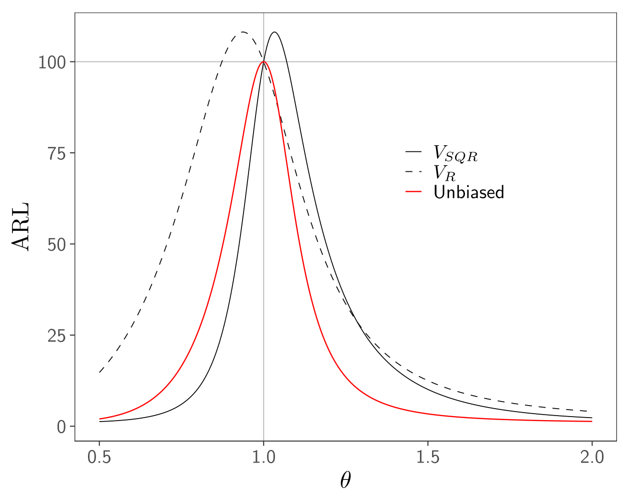

FIGURE 6. ARL profiles of the V_{SQR} and V_R charts and the ARL-unbiased chart for σ, with $\text{ARL}_0 = 1/\alpha = 100$ and $n = 5$.

When $\text{ARL}_0 = 1/\alpha = 40$ and $n \in \{2, 4, 8, 10\}$, we have $a^\star = 0.422171, 1.954684, 6.355643, 8.889265$ and $b^\star = 14.593993, 20.917543, 32.432957, 37.881789$, respectively. For the in-control ARL value $\text{ARL}_0 = 1/\alpha = 100$, and the commonly used sample size $n = 5$, we have $a^\star = 2.34441$ and $b^\star = 26.65311$.

Both figures also illustrate the fact that the V_{SQR} chart is, on average, quicker than its ARL-unbiased counterpart in detecting decreases in σ^2, namely because these two charts have the same in-control ARL and the derivative of the $\text{ARL}_{SQR}(\theta)$ is positive at $\theta = 1$, whereas the one of $\text{ARL}^\star(\theta)$ is zero. Basically, for the same reason, the ARL-unbiased chart performs better than the V_{SQR} in the presence of any increases in the scale factor.

In addition, Figures 5 and 6 suggest that the V_R chart takes longer, on average, than the ARL-unbiased chart to detect any decreases and most increases in σ.

We provide the source code in R language [8] used to numerically obtain the quantiles $a^\star$ and $b^\star$ after the reference list. Alternatively, we could have adapted the code found in [9] and used the R function `uniroot`. Complete scripts to reproduce all results presented in [1], and this article are available at "Code Ocean". Alternatively, these programs will be made available to those who are interested and request them from the corresponding author.

V. CONCLUSION

Equations (1) and (2) are very similar in form to the control limits of the traditional S-chart for the standard deviation of a normally distributed quality characteristic,

$$\sigma_0 \cdot (c_4 \pm 3\sqrt{1-c_4^2});$$

instead of $c_4 = \sqrt{2/(n-1)} \cdot \Gamma(n/2)/\Gamma[(n-1)/2]$ we are dealing with the constant $A(n)$.

We should also mention that V_{SQR} is a biased estimator of σ while V_R is an unbiased estimator of σ^2. In any case, the control statistics are skewed distributions. Moreover, pairs of control limits defining a symmetric interval around an expected value were considered. Consequently, the ARL profiles of the V_{SQR} and V_R charts are not unbiased.

Furthermore, the (equal tails) probability limits in equation (21) for the V_{SQR} chart proposed by [1] can be obtained from the ones in formula (11) for the V_R chart introduced by [3] through square root transformations. Both charts are ARL-biased.

Paraphrasing [10], the ARL-unbiased chart we discussed "balances" control limits so that increases in the scale factor of the Rayleigh distribution are detected about as well as decreases.

Cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) charts, see [11] and [12], would be more efficient in detecting small process shifts.

SOURCE CODE FOR QUANTILES

```
f <- function(n, L0) {
  al     <- 1 / L0
  ainit <- qchisq(al/2,   df = 2*n)
  binit <- qchisq(1-al/2, df = 2*n)

  loss <- function(X) {
    a <- X[1]
    b <- X[2]

   return(sum(c(
     1 - al  - (pchisq(b, df = 2*n) - pchisq(a, df = 2*n)),
     dchisq(a, df = 2*(n+1)) - dchisq(b, df = 2*(n+1))
   )^2))
  }
 nlm(loss, c(ainit, binit), gradtol = 1e-14)$estimate
}
```

Listing 1. **R source code to determine the values a^* and b^*.**

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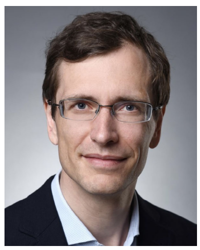

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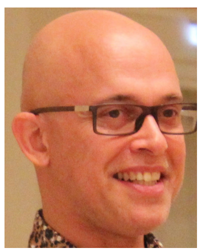

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