IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received 9 July 2022, accepted 26 July 2022, date of publication 1 August 2022, date of current version 8 November 2022.

Digital Object Identifier 10.1109/ACCESS.2022.3195885

RESEARCH ARTICLE

Pythagorean Fuzzy Linguistic Power Generalized Maclaurin Symmetric Mean Operators and Their Application in Multiple Attribute Group Decision-Making

JUNHUI CHEN[1,2] AND RUNTONG ZHANG[1], (Senior Member, IEEE)
[1]School of Economics and Management, Beijing Jiaotong University, Beijing 100044, China
[2]School of E-Commerce and Logistics Management, Henan University of Economics and Law, Zhengzhou, Henan 450000, China

Corresponding author: Runtong Zhang (rtzhang@bjtu.edu.cn)

This work was supported by the National Social Science Foundation of China under Grant 18ZDA086.

ABSTRACT As an extension of Pythagorean fuzzy sets and linguistic term sets, Pythagorean fuzzy linguistic sets (PFSs) are powerful to describe decision-making information quantificational and qualitatively, which have received much scholars' attention. The purpose of this paper is to propose a new multiple attribute group decision-making (MAGDM) approach with Pythagorean fuzzy linguistic (PFL) information. To this end, we firstly analyze the drawbacks of existing operations of PFL numbers and propose new operational rules based on linguistic scale function. The power average (PA) operator is famous for its capacity of reducing the negative influence of unreasonable evaluation values provided by prejudiced decision makers on the decision results. The generalized Maclaurin symmetric mean (GMSM) can not only capture the interrelationship among multiple inputs but also manipulate the effect of related properties by adjusting the parameters. When considering aggregation operators of PFL numbers, we combine PA with GMSM and propose the PFL power generalized Maclaurin symmetric and the PFL power generalized weighted Maclaurin symmetric operators. We also study important properties and special cases of these operators. We continue to investigate MAGDM problems with PFL decision information and propose a novel method to determine the optimal alternative. Finally, we conduct numerical examples to demonstrate the effectiveness of our proposed method. We also attempt to illustrate the advantages and superiorities of the proposed method via comparative analysis.

INDEX TERMS Pythagorean fuzzy linguistic sets, linguistic scale function, power average operator, generalized Maclaurin symmetric mean, multiple attribute group decision making.

I. INTRODUCTION

The Pythagorean fuzzy sets (PFSs) originated by Prof. Yager [1] are an efficient to portray decision makers' (DMs) fuzzy and complicated judgements in realistic multi-attribute group decision-making (MAGDM) process. The prominent characteristic of PFS is $\mu^2 + v^2 \leq 1$, where μ and v represent the membership grade (MG) and non-membership grade (NMG) respectively. Due to this feature, PFSs have been extensively employed in expressing fuzzy decision information and PFSs based MAGDM has been a hot research topic in modern decision-making science. Generally, recent researches on PFSs in decision-making can be roughly divided three categories. The first category is utility values-based PF-MAGDM method. For handling different decision-making situations, scholars proposed different AOs to integrate PF numbers (PFNs) to compute the overall preference information of alternatives. For example, to capture the interrelationship among attributes the PF Bonferroni mean [2], [3] and Maclaurin symmetric mean [4], [5] operators were proposed. To improve the reliability of the final decision results by reducing the bad influence of DMs' extreme evaluation values, Wei and Lu [6] proposed a series of PF power average operators. To effectively deal with the

The associate editor coordinating the review of this manuscript and approving it for publication was Giovanni Pau.

heterogeneous interrelationship among PFNs, Liang *et al.* [7] proposed a set of PF partitioned Bonferroni mean operators. To make the decision results more reliable and consider the interrelationship among any number of attributes, Li *et al.* [8] put forward the PF power Muirhead mean operator. For more AOs of PFNs, we suggest to refer [9]–[12]. In addition, to enrich the PF operation rules theories some scholars also investigated operations of PF numbers (PFNs) under different t-norms and t-corms (TNTC), such as the Einstein TNTC [13], [14], Hamacher TNTC [15], Frank TNTC [16], Dombi TNTC [17], and Archimedean TNTC [18]. So other representative utility value based MAGDM methods are PF-TOPSIS [19], PF-VIKOR [20], PF-MOORA [21] and so forth. The second type is based on outranking method. Scholars extended the traditional outranking methods to PFSs and proposed the PF-ELECTRE [22] and PF-PROMETHEE [23]. The third category is based on information measures. Scholars investigated the distance and similarity measures [24]–[27], entropy [28], and correlation coefficient of PFSs [29], and studied their applications in PF-MAGDM problems.

Besides, some scholars have focused on extensions of traditional PFSs to improve their efficiency in depicting fuzzy information, among which the Pythagorean fuzzy linguistic set (PFLS) [30] is the one the representative. The PFLS is a combination of PFSs with linguistic term set so that it can portray both DMs' quantitative and qualitative evaluation information. Afterwards, in Ref. [30]–[32], the authors proposed the PFL weighted average (PFLWA), Muirhead mean, and power MSM (PMSM) operators and employed them in MAGDM. Additionally, the authors discussed their advantages and superiorities through numerical examples. However, the decision-making methods proposed in [30]–[32] still have limitations. First, the operations of PFL numbers (PFLNs) proposed in [30]–[32] are not so reasonable, as they are simply calculated by using the subscript of the linguistic terms (LTs). For example, let $S = \{s_\theta \,|\theta = 0, 1, \ldots, 6\}$ be a predefined LTS and $\alpha_1 = \langle s_3, (0.4, 0.5)\rangle$, and $\alpha_2 = \langle s_4, (0.6, 0.7)\rangle$ be two PFLN, then according to the PFL operations proposed by Peng *et al.* [30] we have $\alpha_1 \oplus \alpha_2 = \langle s_7, (0.68, 0.38)\rangle$. Obviously, the linguistic part of the result exceeds the upper bound of the given LTS, which is irrational, counterintuitive and meaningless. Second, the MAGDM method introduced by Liu and Qin has the ability of considering the interrelationship among attributes, but it is powerless to effectively handle decision makers' extreme evaluation values. Third, although the method proposed by Liu *et al.* [32] based on the PMSM operator can reduce the negative influence of DMs' unreasonable evaluation values and capture the interrelationship among multiple attributes meanwhile, it fails to reflect the individual importance of aggregated arguments. Hence, existing MAGDM methods based on PFLSs still have flaws when dealing with realistic decision-making problems.

Considering that existing MAGDM methods under PFLSs still have some limitations, it is necessary to propose novel PFLSs based MAGDM method, which is the main motivation of this study. To this end, we conduct our research from following aspects. First, to overcome drawback of existing PFL operations, we introduce the linguistic scale function (LSF) into PFLSs and propose novel operational rues of PFLNs. The new operations not only have good closure but also can flexibly adapt to the semantic changes of DMs. As a matter of fact, LSFs has been widely applied in intuitionistic uncertain linguistic sets [33], interval-valued intuitionistic uncertain linguistic sets [34] and picture fuzzy linguistic sets [35]. Hence, it is necessary and worth extending LSFs into PFL sets. Second, to overcome the drawbacks of existing the aggregation operators when fusing Pythagorean fuzzy linguistic information, some novel aggregation operators are developed. Based on the new operational rules, the power average (PA) [36] operator is extended to PFL environment, and the Pythagorean fuzzy linguistic power average (PFLPA) operator and the Pythagorean fuzzy linguistic power weighted average (PFLPWA) operator are introduced. These two operators can be used to aggregate individual decision matrix to calculate the comprehensive decision matrix. In addition, the generalized Maclaurin symmetric mean (GMSM) operator is a powerful aggregation function proposed by Wang *et al.* [37], which not only reduces the bad effect of decision makers' extreme values and capture the interrelationship among attributes, but also reflects individual importance of aggregated values. Due to these advantages, the GMSM is has been widely used to solve MAGDM problems under q-rung orthopair fuzzy sets [38], Pythagorean fuzzy sets [4], intuitionistic fuzzy soft set [39], and probabilistic linguistic terms set [40]. Motivated by these researches, in this study we combine PA with GMSMS under Pythagorean fuzzy linguistic context and propose the Pythagorean fuzzy linguistic power generalized Maclaurin symmetric mean (PFLPGMSM) operator and the Pythagorean fuzzy linguistic power weighted generalized Maclaurin symmetric mean (PFLPWGMSM) operator. These two operators can be applied in calculating the final overall evaluation values of alternatives. Obviously, these two operators can overcome the drawbacks of the operators proposed in [31], [32]. Final, we propose a new MAGDM method with PFL information. In the proposed method, the PFLPWA operator is employed to compute the collective decision matrix and the PFLPWGMSM is employed to calculate the comprehensive evaluation value of each alternative. Hence, the final decision results are more reasonable and reliable. The main contribution of this study is to propose a novel MAGDM method under PFLSs, which can overcome shortcomings of some existing decision-making method. More specifically, contributions of this paper contain the following four aspects.

1) Novel operational rules for PFLNs based on LSF are proposed. The proposed novel operations not only are closed but also can flexibly adapt to the semantic

changes of DMs, making them more powerful and reasonable than existing operations.

2) Based on the new operations, the PFLPA and PFLPWA operators are proposed, which have the ability of effectively dealing with DMs' unduly high or low evaluation values can be applied in computing the overall evaluation matrix.
3) Some novel aggregation operators, i.e., PFLPGMSM and PFLPWGMSM operators are developed, which integrate PA and GMSM under PFL sets. These two operators absorb the advantages of both PA and GMSM, making them more powerful than some existing operators.
4) A new MAGDM method is presented based on the new operational rules as well as aggregation operators. Moreover, our proposed method is applied to solve realistic MAGDM problems to illustrate its effectiveness.

To better illustrate the main findings of this study, we present the rest of our paper as follows. Section 2 reviews basic concepts and proposes novel operations of PFLNs based on LSF. Section 3 introduces some new AOs of PFLNs and discusses their properties. Section 4 introduces a new MAGDM method and gives their detailed steps. Section 5 illustrates the performance of the new method and analyzes its advantages. Conclusions are presented in Section 6.

II. PRELIMINARIES

In the present section, we will briefly review some fundamental notions which will be used in the following sections.

A. PYTHAGOREAN LINGUISTIC SETS AND THEIR NOVEL OPERATIONS

Definition 1 [30]: Let X be an ordinary set and $s_{\theta(x)} \in \overline{S}$, then a Pythagorean fuzzy linguistic set A defined on X is expressed as

$$A = \left\{\left\langle x, s_{\theta(x)}, (\mu_A(x), \nu_A(x))\right\rangle | x \in X\right\} \tag{1}$$

where $s_{\theta(x)}$ is a linguistic term in $\overline{S}$, $\mu_A(x) : X \rightarrow [0, 1], \nu_A(x) : X \rightarrow [0, 1]$, denoting the MD and NMD of $x \in X$ belong to the linguistic term $s_{\theta(x)}$, satisfying $\mu_A(x)^2 + \nu_A(x)^2 \leq 1$. Then is hesitancy degree is expressed as $\pi_A(x) = \left(1 - \mu_A(x)^2 - \nu_A(x)^2\right)^{1/2}$. The ordered pair $\left\langle s_{\theta(x)}, (\mu_A(x), \nu_A(x))\right\rangle$ is called a PFLN for convenience, which can be denoted as $\alpha = \langle s_\theta, (\mu, \nu)\rangle$ for simplicity.

Existing operational rules of PFLNs are shown as follows.

Definition 2 [30]: Let $\alpha_1 = \left\langle s_{\theta_1}, (\mu_1, \nu_1)\right\rangle, \alpha_2 = \left\langle s_{\theta_2}, (\mu_2, \nu_2)\right\rangle$ and $\alpha = \langle s_\theta, (\mu, \nu)\rangle$ be any three PFLNs and λ be positive real number, then

(1) $\alpha_1 \oplus \alpha_2 = \left\langle s_{\theta_1+\theta_2}, \left(\left(\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2\right)^{1/2}, \nu_1\nu_2\right)\right\rangle$;

(2) $\alpha_1 \otimes \alpha_2 = \left\langle s_{\theta_1\theta_2}, \left(\mu_1\mu_2, \left(\nu_1^2 + \nu_2^2 - \nu_1^2\nu_2^2\right)^{1/2}\right)\right\rangle$;

(3) $\lambda\alpha = \left\langle s_{\lambda\theta}, \left(\left(1 - \left(1 - \mu^2\right)^\lambda\right)^{1/2}, \nu^\lambda\right)\right\rangle$;

(4) $\alpha^\lambda = \left\langle s_{\lambda\theta}, \left(\mu^\lambda, \left(1 - \left(1 - \nu^2\right)^\lambda\right)^{1/2}\right)\right\rangle$.

However, the above operations of PFLNs have some drawbacks. In the following we attempt to propose novel operational rules of PFLNs based on LSF. In order to so this, we first review the concept of LSF.

Definition 3 [41]: Let $S = \{s_i | i = 0, 1, \ldots, 2t\}$ be a linguistic term se, $s_i \in S$ be a linguistic term and $\tau_i \in [0, 1]$ be a real number. A linguistic scale function (LSF) f is a mapping from is mapping from s_i to τ_i $(i = 1, 2, \ldots, 2t)$ such that

$$f : s_i \rightarrow \tau_i \, (i = 1, 2, \ldots, 2t) \tag{2}$$

where $0 \leq \tau_0 < \tau_1 < \ldots < \tau_{2t}$. Hence, f is a strictly monotonically increasing function with regard to linguistic subscript i. Generally, there are three types of LSFs and we give a brief review in the following.

(1) The most widely used LSF is expressed as

$$f_1(s_i) = \theta_i = \frac{i}{2t} \, (i = 1, 2, \ldots, 2t) \tag{3}$$

which is a simple average calculation of the subscripts of linguistic terms.

(2) The second type of LSF is expressed s follows

$$f_2(s_i) = \theta_i = \begin{cases} \dfrac{a^t - a^{t-i}}{2a^t - 2} & (i = 0, 1, 2, \ldots, t) \\ \dfrac{a^t + a^{i-t} - 2}{2a^t - 2} & (i = t+1, t+2, \ldots, 2t) \end{cases} \tag{4}$$

(3) The third type of LSF is expressed as

$$f_3(s_i) = \theta_i = \begin{cases} \dfrac{t^\alpha - (t-i)^\alpha}{2t^\alpha} & (i = 0, 1, 2, \ldots, t) \\ \dfrac{t^\beta + (i-t)^\beta}{2t^\beta} & (i = t+1, t+2, \ldots, 2t) \end{cases} \tag{5}$$

Based on the LSF, we introduce new operational rules for PFLNs.

Definition 4: Let $\alpha_1 = \left\langle s_{\theta_1}, (\mu_1, \nu_1)\right\rangle, \alpha_2 = \left\langle s_{\theta_2}, (\mu_2, \nu_2)\right\rangle$ and $\alpha = \langle s_\theta, (\mu, \nu)\rangle$ be any three PFLNs and λ be positive real number, then

(1) $\alpha_1 \oplus \alpha_2 = \left\langle \begin{matrix} f^{*-1}\left(f^*(\theta_1) + f^*(\theta_2) - f^*(\theta_1)f^*(\theta_2)\right), \\ \left(\left(\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2\right)^{1/2}, \nu_1\nu_2\right) \end{matrix} \right\rangle$;

(2) $\alpha_1 \otimes \alpha_2 = \left\langle f^{*-1}\left(f^*(\theta_1) \times f^*(\theta_2)\right), \left(\mu_1\mu_2, \left(\nu_1^2 + \nu_2^2 - \nu_1^2\nu_2^2\right)^{1/2}\right)\right\rangle$;

(3) $\lambda\alpha = \left\langle f^{*-1}\left(1 - \left(1 - f^*(\theta)\right)^\lambda\right), \left(\left(1 - \left(1 - \mu^2\right)^\lambda\right)^{1/2}, \nu^\lambda\right)\right\rangle$;

(4) $\alpha^\lambda = \left\langle f^{*-1}\left(\left(f^*(\theta)\right)^\lambda\right), \left(\mu^\lambda, \left(1 - \left(1 - \nu^2\right)^\lambda\right)^{1/2}\right)\right\rangle$.

Based on the newly developed operational rules, it is easy to prove the following theorem.

Theorem 1: Let α, α_1 and α_2 be any three PFLNs and $\lambda, \lambda_1, \lambda_2 > 0$, then

(1) $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$;

(2) $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$;

(3) $\lambda(\alpha_1 \oplus \alpha_2) = \lambda\alpha_1 \oplus \lambda\alpha_2$;
(4) $\lambda_1\alpha \oplus \lambda_2\alpha = (\lambda_1 + \lambda_2)\alpha$;
(5) $\alpha^{\lambda_1} \otimes \alpha^{\lambda_2} = \alpha^{\lambda_1+\lambda_2}$;
(6) $\alpha_1^{\lambda} \otimes \alpha_2^{\lambda} = (\alpha_1 \otimes \alpha_2)^{\lambda}$.

Based on the LSF, we propose new score function (SF) and accuracy function (AF) of PFLNs.

Definition 5: Let $\alpha = \langle s_\theta, (\mu, \nu)\rangle$ be a PFLN, then the SF of α is expressed as

$$S(\alpha) = \frac{1}{2}\left(1 + \mu^2 - \nu^2\right) \times f^*(\theta) \tag{6}$$

and the AF is given as

$$H(\alpha) = \left(\mu^2 + \nu^2\right) \times f^*(\theta) \tag{7}$$

Based on the SF and AF of PFLNs, in the following we propose a comparison method of PFLNs.

Definition 6: Let $\alpha_1 = \left\langle s_{\theta_1}, (\mu_1, \nu_1)\right\rangle$ and $\alpha_2 = \left\langle s_{\theta_2}, (\mu_2, \nu_2)\right\rangle$ be any two PFLNs, $S(\alpha_1)$ and $S(\alpha_2)$ denote the SF of α and α_1, $H(\alpha_1)$ and $H(\alpha_2)$ denote the AF of α and α_1, then

(1) If $S(\alpha_1) > S(\alpha_2)$, then $\alpha_1 > \alpha_2$;
(2) If $S(\alpha_1) = S(\alpha_2)$, then
if $H(\alpha_1) > H(\alpha_2)$, then $\alpha_1 > \alpha_2$;
if $H(\alpha_1) = H(\alpha_2)$, then $\alpha_1 = \alpha_2$;

Based on LSF, the distance between any two PFLNs are defined as follows.

Definition 7: Let $\alpha_1 = \left\langle s_{\theta_1}, (\mu_1, \nu_1)\right\rangle$ and $\alpha_2 = \left\langle s_{\theta_2}, (\mu_2, \nu_2)\right\rangle$ be any two PFLNs, then the Hamming distance between α_1 and α_2 is defined as

$$d(\alpha_1, \alpha_2) = \frac{1}{2}\left|\left(1 + \mu_1^2 - \nu_1^2\right) \times f^*(\theta_1) - \left(1 + \mu_2^2 - \nu_2^2\right) \times f^*(\theta_2)\right| \tag{8}$$

B. POWER AVERAGE OPERATOR AND GENERALIZED MACLAURIN SYMMETRIC MEAN

Yager [36] initiated the concept of power average (PA) operator, which is presented as follows.

Definition 8 [36]: Let $a_i\,(i = 1, 2, \ldots, n)$ be a collection of non-negative crisp numbers, then the PA operator is defined as

$$PA(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^{n}(1 + T(a_i))\,a_i}{\sum_{i=1}^{n}(1 + T(a_i))} \tag{9}$$

where $T(a_i) = \sum_{j=1, i\neq j}^{n} Sup\left(a_i, a_j\right)$, $Sup\left(a_i, a_j\right)$ denotes the support for a_i from a_j, satisfying the conditions

(1) $0 \le Sup\left(a_i, a_j\right) \le 1$
(2) $Sup\left(a_i, a_j\right) = Sup\left(a_j, a_i\right)$;
(3) $Sup(a, b) \le Sup(c, d)$, if $|a, b| \ge |c, d|$.

The Definition of PGMSM operator is provided as follows.

Definition 9 [37]: Let $a_i\,(i = 1, 2, \ldots, n)$ be a set of non-negative real numbers, $k \in [1, n]$ be an integer, and $p_1, p_2, \ldots, p_k \ge 1$. If

$$GMSM^{(k, p_1, p_2, \ldots, p_k)}(a_1, a_2, \ldots, a_n) = \left(\frac{\sum_{1\le i_1 < i_2 < \ldots < i_k \le n} \prod_{j=1}^{k} a_{i_j}^{p_j}}{C_n^k}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}} \tag{10}$$

then $GMSM^{(k, p_1, p_2, \ldots, p_k)}$ is called the generalized Maclaurin symmetric mean (GMSMS) operator, where $(i_1, i_2, \ldots, i_k)$ traversals all the k-tuple combination of $(1, 2, \ldots, n)$ and C_n^k is the binominal coefficient.

III. SOME NEW AGGREGATION OPERATORS OF PYTHAGOREAN FUZZY LINGUISTIC NUMBERS AND THEIR PROPERTIES

Based on the new operational rules of PFLNs, we propose some AOs to fuse Pythagorean fuzzy linguistic information.

A. THE PYTHAGOREAN FUZZY LINGUISTIC POWER AVERAGE OPERATOR

Definition 10: Let $\alpha_i = \left\langle s_{\theta_i}, (\mu_i, \nu_i)\right\rangle$ be a collection of PFLNs. The Pythagorean fuzzy linguistic power average (PFLPA) operator is expressed as

$$PFLPA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{\bigoplus_{i=1}^{n}(1 + T(\alpha_i))\,\alpha_i}{\sum_{i=1}^{n}(1 + T(\alpha_i))} \tag{11}$$

where $T(\alpha_i) = \sum_{j=1, i\neq j}^{n} Sup\left(\alpha_i, \alpha_j\right)$, $Sup\left(\alpha_i, \alpha_j\right)$ denotes the support for α_i from α_j, satisfying the conditions

(1) $0 \le Sup\left(\alpha_i, \alpha_j\right) \le 1$;
(2) $Sup\left(\alpha_i, \alpha_j\right) = Sup\left(\alpha_j, \alpha_i\right)$;
(3) $Sup(\alpha, \beta) \le Sup(\chi, \delta)$, if $d(\alpha, \beta) \ge d(\chi, \delta)$, and $d(\alpha, \beta)$ is the Hamming distance between α and β.

If we let

$$\omega_i = \frac{(1 + T(\alpha_i))\,\alpha_i}{\sum_{i=1}^{n}(1 + T(\alpha_i))} \tag{12}$$

then Eq. (11) can be simplified as

$$PFLPA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \bigoplus_{i=1}^{n} \omega_i\alpha_i \tag{13}$$

where $0 \le \omega_i \le 1$ and $\sum_{i=1}^{n} \omega_i = 1$.

According to Definition4, we can obtain the following theorem.

Theorem 2: Let $\alpha_i = \left\langle s_{\theta_i}, (\mu_i, \nu_i)\right\rangle (i = 1, 2, \ldots, n)$ be a collection of PFLNs, then the aggregated value by the PFLPA operator is still a PFLN and

$$PFLPA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left\langle f^{*-1}\left(1 - \prod_{i=1}^{n}\left(1 - f^*(\theta_i)\right)^{\omega_i}\right),\right.$$

$$\left(\left(1-\prod_{i=1}^{n}\left(1-\mu_i^2\right)^{\omega_i}\right)^{1/2},\prod_{i=1}^{n}v^{\omega_i}\right)\Bigg\rangle \tag{14}$$

Theorem 2 is trivial and we omit its proof. In addition, the proposed PFLPA also has the following properties.

Property 1 (Idempotency): Let α_i $(i = 1, 2, \ldots, n)$ be a set of PFLNs, if $\alpha_i = \alpha = \langle s_\theta, (\mu, v)\rangle$ for any i, then

$$PFLPA\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) = \alpha_i \tag{15}$$

Property 2 (Boundedness): Let $\alpha_i = \left\langle s_{\theta_i}, (\mu_i, v_i)\right\rangle$ $(i = 1, 2, \ldots, n)$ be a set of PFLNs, if

$$\alpha^+ = \left\langle \max_{i=1}^{n}\left(s_{\theta_i}\right), \left(\max_{i=1}^{n}\mu_i, \min_{i=1}^{n}v_i\right)\right\rangle$$

and

$$\alpha^- = \left\langle \min_{i=1}^{n}\left(s_{\theta_i}\right), \left(\min_{i=1}^{n}\mu_i, \max_{i=1}^{n}v_i\right)\right\rangle,$$

then

$$\alpha^- \le PFLPA\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) \le \alpha^+ \tag{16}$$

The proofs of Property 1 and 2 are trivial and we omit them.

B. THE PYTHAGOREAN FUZZY LINGUISTIC POWER WEIGHTED AVERAGE OPERATOR

Definition 11: Let $\alpha_i = \left\langle s_{\theta_i}, (\mu_i, v_i)\right\rangle$ be a collection of PFLNs, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of α_i $(i = 1, 2, \ldots, n)$, such that $0 \le w_i \le 1$ and $\sum_{i=1}^{n} w_i = 1$. The Pythagorean fuzzy linguistic power weighted average (PFLPWA) operator is expressed as

$$PFLPWA\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) = \frac{\bigoplus_{i=1}^{n} w_i\left(1 + T\left(\alpha_i\right)\right)\alpha_i}{\sum_{i=1}^{n} w_i\left(1 + T\left(\alpha_i\right)\right)} \tag{17}$$

where $T\left(\alpha_i\right) = \sum_{j=1, i\neq j}^{n} Sup\left(\alpha_i, \alpha_j\right)$, $Sup\left(\alpha_i, \alpha_j\right)$ denotes the support for α_i from α_j, satisfying the properties in Definition 9 (37). Similarity, if we assume

$$\eta_i = \frac{w_i\left(1 + T\left(\alpha_i\right)\right)\alpha_i}{\sum_{i=1}^{n} w_i\left(1 + T\left(\alpha_i\right)\right)} \tag{18}$$

then Eq. (12) can be written as

$$PFLPWA\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) = \bigoplus_{i=1}^{n} \eta_i\alpha_i \tag{19}$$

where $0 \le \eta_i \le 1$ and $\sum_{i=1}^{n} \eta_i = 1$.

The aggregated value by the PFLPWA operator can be obtained on the basis of Definition4.

Theorem 3: Let $\alpha_i = \left\langle s_{\theta_i}, (\mu_i, v_i)\right\rangle$ be a collection of PFLNs, then the aggregated value by the PFLPWA operator is still a PFLN and

$$PFLPWA\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right)$$

$$= \left\langle f^{*-1}\left(1-\prod_{i=1}^{n}\left(1-f^*\left(\theta_i\right)\right)^{\eta_i}\right),\right.$$

$$\left.\left(\left(1-\prod_{i=1}^{n}\left(1-\mu_i^2\right)^{\eta_i}\right)^{1/2},\prod_{i=1}^{n}v^{\eta_i}\right)\right\rangle \tag{20}$$

It is easy to prove that PFLPWA operator has the properties of idempotency and boundedness.

C. THE PYTHAGOREAN FUZZY LINGUISTIC POWER GENERALIZED MACLAURIN SYMMETRIC MEAN OPERATOR

Definition 12: Let be a collection of PFLNs, be an integer, and The Pythagorean fuzzy linguistic power generalized Maclaurin symmetric mean (PFLPGMSM) operator is defined as

$$PFLPGMSM^{(k,p_1,p_2,\ldots,p_k)}\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right)$$
$$= \left(\frac{1}{C_n^k}\bigoplus_{1\le i_1<i_2<\ldots<i_k\le n}\bigotimes_{j=1}^{k}\left(n\frac{\left(1+T\left(\alpha_{i_j}\right)\right)\alpha_{i_j}}{\sum_{t=1}^{n}\left(1+T\left(\alpha_t\right)\right)}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}} \tag{21}$$

where $(i_1, i_2, \ldots, i_k)$ traversals all the k-tuple combination of $(1, 2, \ldots, n)$ and C_n^k is the binominal coefficient. In addition, $T\left(\alpha_i\right) = \sum_{j=1, i\neq j}^{n} Sup\left(\alpha_i, \alpha_j\right)$, $Sup\left(\alpha_i, \alpha_j\right)$ denotes the support for α_i from α_j, satisfying the conditions presented in Definition 11. To simplify Eq. (21), we assume

$$\frac{\left(1+T\left(\alpha_{i_j}\right)\right)}{\sum_{t=1}^{n}\left(1+T\left(\alpha_t\right)\right)} = \varsigma_{i_j} \tag{22}$$

then Eq. (21) can be written as

$$PFLPGMSM^{(k,p_1,p_2,\ldots,p_k)}\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right)$$
$$= \left(\frac{1}{C_n^k}\bigoplus_{1\le i_1<i_2<\ldots<i_k\le n}\bigotimes_{j=1}^{k}\left(n\varsigma_{i_j}\alpha_{i_j}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}} \tag{23}$$

where $0 \le \varsigma_i \le 1$ and $\sum_{i=1}^{n} \varsigma_i = 1$.

Theorem 4: Let $\alpha_i = \left\langle s_{\theta_i}, (\mu_i, v_i)\right\rangle$ be a collection of PFLNs, $k \in [1, n]$ be an integer, and $p_1, p_2, \ldots, p_k \ge 1$. Then the aggregated value by the PFLPGMSM operator is still a PFLN and (24), as shown at the bottom of the next page,

Proof: According to the operations, we can obtain

$$n\varsigma_{i_j}\alpha_{i_j} = \left\langle f^{*-1}\left(1-\left(1-f^*\left(\theta_{i_j}\right)\right)^{n\varsigma_{i_j}}\right),\right.$$
$$\left.\left(\left(1-\left(1-\mu_{i_j}^2\right)^{n\varsigma_{i_j}}\right)^{1/2}, v_{i_j}^{n\varsigma_{i_j}}\right)\right\rangle$$

and

$$\left(n\varsigma_{i_j}\alpha_{i_j}\right)^{p_j} = \left\langle \begin{array}{l} f^{*-1}\left(\left(1-\left(1-f^*\left(\theta_{i_j}\right)\right)^{n\varsigma_{i_j}}\right)^{p_j}\right), \\ \left(\left(\left(1-\left(1-\mu_{i_j}^2\right)^{n\varsigma_{i_j}}\right)^{1/2p_j}, \right.\right. \\ \left.\left(1-\left(1-v_{i_j}^{2n\varsigma_{i_j}}\right)^{p_j}\right)^{1/2}\right) \end{array} \right\rangle$$

Further,

$$\overset{k}{\underset{j=1}{\otimes}}\left(n\varsigma_{i_j}\alpha_{i_j}\right)^{p_j} = \left\langle \begin{array}{l} f^{*-1}\left(\prod_{j=1}^{k}\left(1-\left(1-f^*\left(\theta_{i_j}\right)\right)^{n\varsigma_{i_j}}\right)^{p_j}\right), \\ \left(\left(\prod_{j=1}^{k}\left(1-\left(1-\mu_{i_j}^2\right)^{n\varsigma_{i_j}}\right)^{1/2p_j}, \right.\right. \\ \left.\left(1-\prod_{j=1}^{k}\left(1-v_{i_j}^{2n\varsigma_{i_j}}\right)^{p_j}\right)^{1/2}\right) \end{array} \right\rangle$$

and $\underset{1\le i_1<i_2<\ldots<i_k\le n}{\oplus}\overset{k}{\underset{j=1}{\otimes}}\left(n\varsigma_{i_j}\alpha_{i_j}\right)^{p_j}$, as shown at the bottom of the next page.

Therefore, $\frac{1}{C_n^k}\underset{1\le i_1<i_2<\ldots<i_k\le n}{\oplus}\overset{k}{\underset{j=1}{\otimes}}\left(n\varsigma_{i_j}\alpha_{i_j}\right)^{p_j}$, as shown at the bottom of the next page.

Finally, $\left(\frac{1}{C_n^k}\underset{1\le i_1<i_2<\ldots<i_k\le n}{\oplus}\overset{k}{\underset{j=1}{\otimes}}\left(n\varsigma_{i_j}\alpha_{i_j}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}}$, as shown at the bottom of the next page.

Property 3 (Idempotency): Let α_i $(i=1,2,\ldots,n)$ be a set of PFLNs, if $\alpha_i=\alpha=\langle s_\theta,(\mu,v)\rangle$ for any i, then

$$PFLPGMSM^{(k,p_1,p_2,\ldots,p_k)}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)=\alpha \tag{25}$$

Proof: When $\alpha_i=\alpha=\langle s_\theta,(\mu,v)\rangle$, then according to Theorem 1, we can obtain $PFLPGMSM^{(k,p_1,p_2,\ldots,p_k)}$ $(\alpha_1,\alpha_2,\ldots,\alpha_n)$, as shown at the bottom of page 8.

Property 4 (Boundedness): Let $\alpha_i=\left\langle s_{\theta_i},(\mu_i,v_i)\right\rangle$ $(i=1,2,\ldots,n)$ be a set of PFLNs, if

$$\alpha^+=\left\langle \max_{i=1}^{n}\left(s_{\theta_i}\right),\left(\max_{i=1}^{n}\left(\mu_i\right),\min_{i=1}^{n}\left(v_i\right)\right)\right\rangle$$

and

$$\alpha^-=\left\langle \min_{i=1}^{n}\left(s_{\theta_i}\right),\left(\min_{i=1}^{n}\left(\mu_i\right),\max_{i=1}^{n}\left(v_i\right)\right)\right\rangle$$

then

$$\alpha^-\le PFLPGMSM^{(k,p_1,p_2,\ldots,p_k)}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)\le\alpha^+ \tag{26}$$

Proof: As the LSF f is a strictly monotonically increasing function, then it is easy to prove (at the bottom of page 8).

Then according to Definition5, we have $\alpha^-\le PFLPGMSM^{(k,p_1,p_2,\ldots,p_k)}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)$. Similarly, we can prove $PFLPGMSM^{(k,p_1,p_2,\ldots,p_k)}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)\le\alpha^+$ and so that the proof of Property 4 is completed.

In the followings, we discuss special cases of the PFLPGMSM operator with respect to the parameters k and $p_1,p_2,\ldots,p_k$.

Special Case 1: When $k=1$, then the PFLPGMSM operator reduces to

$$PFLPGMSM^{(1,p_1)}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)=\left(\frac{1}{n}\overset{n}{\underset{j=1}{\oplus}}\left(n\varsigma_j\alpha_j\right)^{p_1}\right)^{\frac{1}{p_1}}$$

$$=\left\langle \begin{array}{l} f^{*-1}\left(\left(1-\left(\prod_{j=1}^{n}\left(1-\left(1-\left(1-f^*\left(\theta_j\right)\right)^{n\varsigma_j}\right)^{p_1}\right)\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}}\right), \\ \left(\left(1-\left(\prod_{j=1}^{n}\left(1-\left(1-\left(1-\mu_j^2\right)^{n\varsigma_j}\right)^{p_1}\right)\right)^{\frac{1}{n}}\right)^{\frac{1}{2p_1}}, \right. \\ \left.\left(1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-v_j^{2n\varsigma_j}\right)^{p_1}\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}}\right)^{\frac{1}{2}}\right) \end{array}\right\rangle \tag{27}$$

In this case, if $Sup\left(\alpha_i,\alpha_j\right)=t\,(t>0)$, then the PFLPGMSM operator reduces to the generalized Pythagorean fuzzy linguistic average (GPFLA) operator, i.e.

$$PFLPGMSM^{(1,p_1)}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)$$
$$=\left(\frac{1}{n}\overset{n}{\underset{j=1}{\oplus}}\left(n\varsigma_j\alpha_j\right)^{p_1}\right)^{1/p_1}=\left(\frac{1}{n}\overset{n}{\underset{j=1}{\oplus}}\alpha_j^{p_1}\right)^{1/p_1}$$
$$=\left\langle f^{*-1}\left(\left(1-\left(\prod_{j=1}^{n}\left(1-\left(f^*\left(\theta_j\right)\right)^{p_1}\right)\right)^{1/n}\right)^{1/p_1}\right),\right.$$

$$\left(\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\mu_{i_j}^2\right)^{n\varsigma_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{2(p_1+p_2+\ldots+p_k)}},\right.$$
$$\left.\left(1-\left(1-\left(1-\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-v_{i_j}^{2n\varsigma_{i_j}}\right)^{p_j}\right)\right)^{1/C_n^k}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}}\right)^{1/2}\right)\right\rangle \tag{24}$$

$$\left(\begin{array}{l}\left(1-\left(\prod_{j=1}^{n}\left(1-\mu_{j}^{2p_1}\right)\right)^{1/n}\right)^{1/2p_1}, \\ \left(1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-v_{j}^{2}\right)^{p_1}\right)^{1/n}\right)^{1/p_1}\right)^{1/2}\end{array}\right)\Bigg\rangle \tag{28}$$

Special Case 2: When $k = 2$, then the PFLPGMSM operator reduces to (29), as shown at the bottom of page 9, which is the Pythagorean fuzzy linguistic power Bonferroni mean (PFLPBM) operator with the parameters p_1 and p_2.

In this case, if $Sup\left(\alpha_i, \alpha_j\right) = t\,(t > 0)$, then the PFLPGMSM operator reduces to the Pythagorean fuzzy linguistic Bonferroni mean (PFLBM) operator, i.e. (30), as shown at the bottom of page 9.

$$\begin{aligned}&\bigoplus_{1\le i_1<i_2<\ldots<i_k\le n}\bigotimes_{j=1}^{k}\left(n\varsigma_{i_j}\alpha_{i_j}\right)^{p_j}\\ &=\left\langle f^{*-1}\left(1-\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-f^{*}\left(\theta_{i_j}\right)\right)^{n\varsigma_{i_j}}\right)^{p_j}\right)\right),\right.\\ &\left.\left(\begin{array}{l}\left(1-\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\mu_{i_j}^{2}\right)^{n\varsigma_{i_j}}\right)^{p_j}\right)\right)^{1/2},\\ \prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-v_{i_j}^{2n\varsigma_{i_j}}\right)^{p_j}\right)^{1/2}\end{array}\right)\right\rangle.\end{aligned}$$

$$\begin{aligned}&\frac{1}{C_n^k}\bigoplus_{1\le i_1<i_2<\ldots<i_k\le n}\bigotimes_{j=1}^{k}\left(n\varsigma_{i_j}\alpha_{i_j}\right)^{p_j}\\ &=\left\langle f^{*-1}\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-f^{*}\left(\theta_{i_j}\right)\right)^{n\varsigma_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^k}}\right),\right.\\ &\left.\left(\begin{array}{l}\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\mu_{i_j}^{2}\right)^{n\varsigma_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^k}}\right)^{1/2},\\ \prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-v_{i_j}^{2n\varsigma_{i_j}}\right)^{p_j}\right)^{1/2C_n^k}\end{array}\right)\right\rangle.\end{aligned}$$

$$\begin{aligned}&\left(\frac{1}{C_n^k}\bigoplus_{1\le i_1<i_2<\ldots<i_k\le n}\bigotimes_{j=1}^{k}\left(n\varsigma_{i_j}\alpha_{i_j}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}}\\ &=\left\langle f^{*-1}\left(\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-f^{*}\left(\theta_{i_j}\right)\right)^{n\varsigma_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}}\right),\right.\\ &\left(\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\mu_{i_j}^{2}\right)^{n\varsigma_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{2(p_1+p_2+\ldots+p_k)}},\right.\\ &\left.\left.\left(1-\left(1-\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-v_{i_j}^{2n\varsigma_{i_j}}\right)^{p_j}\right)^{1/C_n^k}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}}\right)^{1/2}\right)\right\rangle\end{aligned}$$

Special Case 3: When $k = 3$, then the PFLPGMSM operator reduces to (31), as shown at the bottom of page 10, which is the Pythagorean fuzzy linguistic power generalized Bonferroni mean (PFLPGBM) operator with the parameters p_1, p_2, and p_3.

In this case, if $Sup(\alpha_i, \alpha_j) = t\,(t > 0)$, then the PFLPGMSM operator reduces to the generalized Pythagorean fuzzy linguistic Bonferroni mean (GPFLBM) operator, i.e. (32), as shown at the bottom of page 10.

Special Case 4: When $p_1 = p_2 = \ldots = p_k = 1$, then the PFLPGMSM operator reduces to (33), as shown at the bottom of page 11, which is the Pythagorean fuzzy linguistic power Maclaurin symmetric mean (PFLPMSM) operator with the parameter k.

$$PFLPGMSM^{(k,p_1,p_2,\ldots,p_k)}(\alpha_1,\alpha_2,\ldots,\alpha_n)$$
$$= \left\langle f^{*-1}\left(\left(\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-f^*(\theta)\right)^{n\frac{1}{n}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}}\right)\right),\right.$$
$$\left(\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\mu^2\right)^{n\frac{1}{n}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{2(p_1+p_2+\ldots+p_k)}},\right.$$
$$\left.\left.\left(1-\left(1-\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\nu^{2n\frac{1}{n}}\right)^{p_j}\right)^{1/C_n^k}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}}\right)^{1/2}\right)\right\rangle$$
$$= \langle s_\theta, (\mu, \nu)\rangle$$

$$f^{*-1}\left(\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-f^*(\theta_{i_j})\right)^{n\varsigma_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}}\right)$$
$$\ge f^{*-1}\left(\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-f^*\left(\min_{i=1}^{n}(\theta_i)\right)\right)^{n\varsigma_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}}\right),$$
$$\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\mu_{i_j}^2\right)^{n\varsigma_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{2(p_1+p_2+\ldots+p_k)}}$$
$$\ge \left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\min_{i=1}^{n}(\mu_i)\right)^2\right)^{n\varsigma_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{2(p_1+p_2+\ldots+p_k)}},$$

and

$$\left(1-\left(1-\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\nu_{i_j}^{2n\varsigma_{i_j}}\right)^{p_j}\right)^{1/C_n^k}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}}\right)^{1/2}$$
$$\le \left(1-\left(1-\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(\max_{i=1}^{n}(\nu_i)\right)^{2n\varsigma_{i_j}}\right)^{p_j}\right)^{1/C_n^k}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}}\right)^{1/2}$$

In this case, if $Sup(\alpha_i, \alpha_j) = t\,(t > 0)$, then the PFLPGMSM operator reduces to the Pythagorean fuzzy linguistic Maclaurin symmetric mean (PFLMSM) operator, i.e. (34), as shown at the bottom of page 11.

Special Case 5: When $p_1 = p_2 = \ldots = p_k = 1/n$, then the PFLPGMSM operator reduces to

$$\begin{aligned}
&PFLPGMSM^{(k,1/n,1/n,\ldots,1/n)}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\
&= \bigotimes_{j=1}^{n} \left(n\varsigma_j \alpha_j\right)^{1/n}\\
&= \left\langle f^{*-1}\left(\prod_{j=1}^{n}\left(1-\left(1-f^{*}(\theta_j)\right)^{n\varsigma_j}\right)^{1/n}\right),\right.\\
&\quad \left(\prod_{j=1}^{n}\left(1-\left(1-\mu_j^2\right)^{n\varsigma_j}\right)^{1/2n},\right.\\
&\quad \left.\left.\left(1-\prod_{j=1}^{n}\left(1-v_j^{2n\varsigma_j}\right)^{1/n}\right)^{1/2}\right)\right\rangle
\end{aligned} \tag{35}$$

In this case, if $Sup(\alpha_i, \alpha_j) = t\,(t > 0)$, then the PFLPGMSM operator reduces to the Pythagorean fuzzy linguistic geometric (PFLG) operator, i.e.

$$\begin{aligned}
&PFLPGMSM^{(k,1/n,1/n,\ldots,1/n)}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\
&= \bigotimes_{j=1}^{n} \alpha_j^{1/n}\\
&= \left\langle f^{*-1}\left(\prod_{j=1}^{n}\left(f^{*}(\theta_j)\right)^{1/n}\right),\right.
\end{aligned}$$

$$\begin{aligned}
&PFLPGMSM^{(2,p_1,p_2)}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\
&= \left(\frac{1}{n(n-1)} \bigoplus_{1\le i<j\le n}\left(\left(n\varsigma_i\alpha_i\right)^{p_1} \otimes \left(n\varsigma_j\alpha_j\right)^{p_2}\right)\right)^{\frac{1}{p_1+p_2}}\\
&\times \left\langle f^{*-1}\left(\left(1-\left(\prod_{1\le i<j\le n}\left(1-\left(1-\left(1-f^{*}(\theta_i)\right)^{n\varsigma_i}\right)^{p_1}\left(1-\left(1-f^{*}(\theta_j)\right)^{n\varsigma_j}\right)^{p_2}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p_1+p_2}}\right),\right.\\
&\left(\left(\left(1-\left(\prod_{1\le i<j\le n}\left(1-\left(1-\left(1-\mu_i^2\right)^{n\varsigma_i}\right)^{p_1}\left(1-\left(1-\mu_j^2\right)^{n\varsigma_j}\right)^{p_2}\right)\right)^{\frac{1}{n(n-1)}}\right)^{1/2}\right)^{\frac{1}{p_1+p_2}},\right.\\
&\left.\left.\left(1-\left(1-\left(\prod_{1\le i<j\le n}\left(1-\left(1-v_i^{2n\varsigma_i}\right)^{p_1}\left(1-v_j^{2n\varsigma_j}\right)^{p_2}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p_1+p_2}}\right)^{1/2}\right)\right\rangle
\end{aligned} \tag{29}$$

$$\begin{aligned}
&PFLPGMSM^{(2,p_1,p_2)}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\
&= \left(\frac{1}{n(n-1)} \bigoplus_{1\le i<j\le n}\left(\alpha_i^{p_1} \otimes \alpha_j^{p_2}\right)\right)^{\frac{1}{p_1+p_2}}\\
&\left\langle f^{*-1}\left(\left(1-\left(\prod_{1\le i<j\le n}\left(1-\left(f^{*}(\theta_i)\right)^{p_1}\left(f^{*}(\theta_j)\right)^{p_2}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p_1+p_2}}\right),\right.\\
&\left(\left(\left(1-\left(\prod_{1\le i<j\le n}\left(1-\mu_i^{2p_1}\mu_j^{2p_2}\right)\right)^{\frac{1}{n(n-1)}}\right)^{1/2}\right)^{\frac{1}{p_1+p_2}},\right.\\
&\left.\left.\left(1-\left(1-\left(\prod_{1\le i<j\le n}\left(1-\left(1-v_i^2\right)^{p_1}\left(1-v_j^2\right)^{p_2}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p_1+p_2}}\right)^{1/2}\right)\right\rangle
\end{aligned} \tag{30}$$

$$\left.\left(\prod_{j=1}^{n}\mu_j^{1/n},\ \left(1-\prod_{j=1}^{n}\left(1-v_j^2\right)^{1/n}\right)^{1/2}\right)\right\rangle \tag{36}$$

D. THE PYTHAGOREAN FUZZY LINGUISTIC POWER WEIGHTED GENERALIZED MACLAURIN SYMMETRIC MEAN OPERATOR

Definition 13: Let $\alpha_i = \langle s_{\theta_i}, (\mu_i, v_i)\rangle$ be a collection of PFLNs, $k \in [1, n]$ be an integer, and $p_1, p_2, \ldots, p_k \geq 1$. Let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\alpha_j\ (j = 1, 2, \ldots, n)$, such that $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} w_j = 1$. The Pythagorean fuzzy linguistic power weighted generalized Maclaurin symmetric mean (PFLPWGMSM) operator is defined as

$$PFLPWGMSM^{(k,p_1,p_2,\ldots,p_k)}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{1}{C_n^k}\bigoplus_{1\leq i_1<i_2<\ldots<i_k\leq n}\bigotimes_{j=1}^{k}\left(n\frac{w_{i_j}\left(1+T\left(\alpha_{i_j}\right)\right)\alpha_{i_j}}{\sum_{t=1}^{n} w_t\left(1+T\left(\alpha_t\right)\right)}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}} \tag{37}$$

where $(i_1, i_2, \ldots, i_k)$ traversals all the k-tuple combination of $(1, 2, \ldots, n)$ and C_n^k is the binominal coefficient. In addition, $T(\alpha_i) = \sum_{j=1, i\neq j}^{n} Sup\left(\alpha_i, \alpha_j\right)$, $Sup\left(\alpha_i, \alpha_j\right)$ denotes the support for α_i from α_j, satisfying the conditions presented in

$$\begin{aligned}
&PFLPGMSM^{(3,p_1,p_2,p_3)}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\
&= \left(\frac{1}{n(n-1)(n-2)}\bigoplus_{i,j,s=1.i\neq j\neq s}^{n}\left((n\varsigma_i\alpha_i)^{p_1}\otimes\left(n\varsigma_j\alpha_j\right)^{p_2}\otimes(n\varsigma_s\alpha_s)^{p_3}\right)\right)^{\frac{1}{p_1+p_2+p_3}}\\
&\left\langle f^{*-1}\left(\left(1-\left(\prod_{,j,s=1.i\neq j\neq s}\left(1-\left(1-\left(1-f^*(\theta_i)\right)^{n\varsigma_i}\right)^{p_1}\left(1-\left(1-f^*(\theta_j)\right)^{n\varsigma_j}\right)^{p_2}\left(1-\left(1-f^*(\theta_s)\right)^{n\varsigma_s}\right)^{p_3}\right)\right)^{\frac{1}{n(n-1)(n-2)}}\right)^{\frac{1}{p_1+p_2+p_3}}\right),\right.\\
&\left(\left(\left(1-\left(\prod_{i,j,s=1.i\neq j\neq s}\left(1-\left(1-\left(1-\mu_i^2\right)^{n\varsigma_i}\right)^{p_1}\left(1-\left(1-\mu_j^2\right)^{n\varsigma_j}\right)^{p_2}\left(1-\left(1-\mu_s^2\right)^{n\varsigma_s}\right)^{p_3}\right)\right)^{\frac{1}{n(n-1)(n-2)}}\right)^{1/2}\right)^{\frac{1}{p_1+p_2+p_3}},\\
&\left.\left(1-\left(1-\left(\prod_{i,j,s=1.i\neq j\neq s}\left(1-\left(1-v_i^{2n\varsigma_i}\right)^{p_1}\left(1-v_j^{2n\varsigma_j}\right)^{p_2}\left(1-v_s^{2n\varsigma_s}\right)^{p_3}\right)\right)^{\frac{1}{n(n-1)(n-2)}}\right)^{\frac{1}{p_1+p_2+p_3}}\right)^{1/2}\right\rangle
\end{aligned} \tag{31}$$

$$\begin{aligned}
&PFLPGMSM^{(3,p_1,p_2,p_3)}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\
&= \left(\frac{1}{n(n-1)(n-2)}\bigoplus_{i,j,s=1.i\neq j\neq s}^{n}\left(\alpha_i^{p_1}\otimes\alpha_j^{p_2}\otimes\alpha_s^{p_3}\right)^{p_3}\right)^{\frac{1}{p_1+p_2+p_3}}\\
&= \left\langle f^{*-1}\left(\left(1-\left(\prod_{i,j,s=1.i\neq j\neq s}\left(1-\left(f^*(\theta_i)\right)^{p_1}\left(f^*(\theta_j)\right)^{p_2}\left(f^*(\theta_s)\right)^{p_3}\right)\right)^{\frac{1}{n(n-1)(n-2)}}\right)^{\frac{1}{p_1+p_2+p_3}}\right),\right.\\
&\left(\left(\left(1-\left(\prod_{i,j,s=1.i\neq j\neq s}\left(1-\mu_i^{2p_1}\mu_j^{2p_2}\mu_s^{2p_3}\right)\right)^{\frac{1}{n(n-1)(n-2)}}\right)^{1/2}\right)^{\frac{1}{p_1+p_2+p_3}},\\
&\left.\left(1-\left(1-\left(\prod_{i,j,s=1.i\neq j\neq s}\left(1-\left(1-v_i^2\right)^{p_1}\left(1-v_j^2\right)^{p_2}\left(1-v_s^2\right)^{p_3}\right)\right)^{\frac{1}{n(n-1)(n-2)}}\right)^{\frac{1}{p_1+p_2+p_3}}\right)^{1/2}\right\rangle
\end{aligned} \tag{32}$$

Definition 11. To simplify Eq. (37), we assume

$$\delta_j = \frac{w_j\left(1+T\left(\alpha_j\right)\right)}{\sum_{t=1}^{n} w_t\left(1+T\left(\alpha_t\right)\right)} \tag{38}$$

then Eq. (37) can be written as

$$PFLPWGMSM^{(k,p_1,p_2,\ldots,p_k)}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left(\frac{1}{C_n^k}\bigoplus_{1\le i_1<i_2<\ldots<i_k\le n}\bigotimes_{j=1}^{k}\left(n\delta_{i_j}\alpha_{i_j}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}} \tag{39}$$

where $0\le\delta_i\le 1$ and $\sum_{i=1}^{n}\delta_i = 1$.

Theorem 6: Let $\alpha_i = \langle s_{\theta_i},(\mu_i,v_i)\rangle$ be a collection of PFLNs, $k\in[1,n]$ be an integer, and $p_1,p_2,\ldots,p_k\ge 1$. Then the aggregated value by the PFLPWGMSM operator is still a PFLN and (40), as shown at the bottom of the next page.

Similarly, the proposed PFLPWGMSM operator has the properties of boundedness.

IV. A NOVEL METHOD TO MULTIPLE ATTRIBUTE GROUP DECISION-MAKING WITH PYTHAGOREAN FUZZY LINGUISTIC NUMBERS

This section introduces the main steps of solving multiple attribute group decision-making (MAGDM) problems in which DMs' judgements over alternatives are expressed by Pythagorean fuzzy linguistic numbers (PFLNs). Suppose there are m feasible alternative which are to be evaluated by DMs under n attributes. For the convenience of description, let the alternative set be $A=\{A_1,A_2,\ldots,A_m\}$ and attribute set be $G=\{G_1,G_2,\ldots,G_n\}$. The weight vector of attributes is $w=(w_1,w_2,\ldots,w_n)^T$, such that $0\le w_j\le 1$ and $\sum_{j=1}^{n} w_j = 1$. To make a reliable decision, a set of DMs are invited to evaluate the performance of each alternative. Let $D=\{D_1,D_2,\ldots,D_t\}$ be the DM set, whose importance vector is $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_t)^T$, satisfying $0\le\lambda_l\le 1$ and $\sum_{l=1}^{t}\lambda_l=1$. The DM D_l $(l=1,2,\ldots,t)$ employs a PFLN $\alpha_{ij}^l = \left\langle s_{\theta_{ij}^l},\left(\mu_{ij}^l,v_{ij}^l\right)\right\rangle$ to denote his/her judgement of the performance of alternative A_i $(i=1,2,\ldots,m)$ under the attribute G_j $(j=1,2,\ldots,n)$. Finally, l Pythagorean fuzzy

$$\begin{aligned} &PFLPGMSM^{(k,1,1,\ldots,1)}(\alpha_1,\alpha_2,\ldots,\alpha_n) \\ &= \left(\frac{1}{C_n^k}\bigoplus_{1\le i_1<i_2<\ldots<i_k\le n}\bigotimes_{j=1}^{k}\left(n\varsigma_{i_j}\alpha_{i_j}\right)\right)^{1/k} \\ &= \left\langle f^{*-1}\left(\left(\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-f^*\left(\theta_{i_j}\right)\right)^{n\varsigma_{i_j}}\right)\right)\right)^{1/C_n^k}\right)^{1/k}\right)\right. \right. \\ &\left.\left(\begin{array}{l}\left(\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\mu_{i_j}^2\right)^{n\varsigma_j}\right)\right)\right)^{1/C_n^k}\right)^{1/2k}, \\ \left(1-\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-v_{i_j}^{2n\varsigma_{i_j}}\right)\right)\right)^{1/C_n^k}\right)^{1/k}\right)^{1/2}\end{array}\right)\right\rangle \end{aligned} \tag{33}$$

$$\begin{aligned} &PFLPGMSM^{(k,1,1,\ldots,1)}(\alpha_1,\alpha_2,\ldots,\alpha_n) \\ &= \left(\frac{1}{C_n^k}\bigoplus_{1\le i_1<i_2<\ldots<i_k\le n}\bigotimes_{j=1}^{k}\alpha_{i_j}\right)^{1/k} \\ &= \left\langle f^{*-1}\left(\left(\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(f\left(\theta_{i_j}\right)\right)\right)\right)^{1/C_n^k}\right)^{1/k}\right)\right., \\ &\left.\left(\begin{array}{l}\left(\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\mu_{i_j}^2\right)\right)^{1/C_n^k}\right)^{1/2k}, \\ \left(1-\left(1-\left(\prod_{1\le i_1<i_2<\ldots<i_k}\left(1-\prod_{j=1}^{k}\left(1-v_{i_j}^2\right)\right)\right)^{1/C_n^k}\right)^{1/k}\right)^{1/2}\end{array}\right)\right\rangle \end{aligned} \tag{34}$$

linguistic decision matrices are obtained and in the following we present a method to determine the optimal based on the proposed aggregation operators.

Step 1: Normalize the original Pythagorean fuzzy linguistic decision matrices. Due to the existence of benefit and cost types of attributes, to make alternatives more comparable, it is necessary to normalize the original decision matrices according to the following formula

$$\alpha_{ij}^{l} = \begin{cases} \left\langle s_{\theta_{ij}^{l}}, \left(\mu_{ij}^{l}, v_{ij}^{l}\right)\right\rangle & G_j \in T_1 \\ \left\langle s_{\theta_{ij}^{l}}, \left(v_{ij}^{l}, \mu_{ij}^{l}\right)\right\rangle & G_j \in T_2 \end{cases} \tag{41}$$

where T_1 and T_2 denote the benefit and cost types of attributes, respectively. Go to the next step.

Step 2: Compute the support between α_{ij}^{h} and α_{ij}^{g} with respective to DMs,

$$Sup\left(\alpha_{ij}^{h}, \alpha_{ij}^{g}\right) = 1 - d\left(\alpha_{ij}^{h}, \alpha_{ij}^{g}\right) \; (h, g = 1, 2, \ldots, n;\, h \neq g) \tag{42}$$

where $d\left(\alpha_{ij}^{h}, \alpha_{ij}^{g}\right)$ is the Hamming distance between α_{ij}^{h} and α_{ij}^{g}. Go to the next step.

Step 3: Compute the weighted overall supports $T\left(\alpha_{ij}^{h}\right)$ of the PFLN by

$$T\left(\alpha_{ij}^{h}\right) = \sum_{h=1; h\neq g}^{t} Sup\left(\alpha_{ij}^{h}, \alpha_{ij}^{g}\right) \tag{43}$$

Step 4: Based on $T\left(\alpha_{ij}^{h}\right)$ and the weight vector of DMs, compute the power weight associated with PFLN α_{ij}^{h} given by DM D_h. Go to the next step.

$$\omega^{h} = \frac{\lambda_h\left(1 + T\left(\alpha_{ij}^{h}\right)\right)}{\sum_{h=1}^{t} \lambda_h\left(1 + T\left(\alpha_{ij}^{h}\right)\right)} \tag{44}$$

Step 5: Aggregate individual decision matrix by the PFLPWA operator to compute the comprehensive decision matrix, i.e.

$$\alpha_{ij} = PFLPWA\left(\alpha_{ij}^{1}, \alpha_{ij}^{2}, \ldots, \alpha_{ij}^{t}\right) \tag{45}$$

Then, a collective decision matrix is derived. Go to the next step.

Step 6: Calculate the support between α_{ij} and α_{is}

$$Sup\left(\alpha_{ij}, \alpha_{is}\right) = 1 - d\left(\alpha_{ij}, \alpha_{is}\right) \quad (i = 1, 2, \ldots, m; j, s = 1, 2, \ldots, n; j \neq s) \tag{46}$$

where $d\left(\alpha_{ij}, \alpha_{is}\right)$ is the Hamming distance between α_{ij} and α_{is}. Go to the next step.

Step 7: Compute the weighted overall supports $T\left(\alpha_{ij}\right)$ according to the following formula and go the next step.

$$T\left(\alpha_{ij}\right) = \sum_{j=1; j\neq s}^{n} Sup\left(\alpha_{ij}, \alpha_{is}\right) \tag{47}$$

Step 8: Compute the power weight associated with the PFLN α_{ij} by the following formula and go to the next step.

$$\delta_{ij} = \frac{w_j\left(1 + T\left(\alpha_{ij}\right)\right)}{\sum_{j=1}^{n} w_j\left(1 + T\left(\alpha_{ij}\right)\right)} \tag{48}$$

Step 9: For each alternative, utilize the

$$\alpha_i = PFLPWGMSM^{(k, p_1, p_2, \ldots, p_k)}\left(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}\right) \tag{49}$$

and for each alternative $A_i\ (i = 1, 2, \ldots, m)$, a corresponding overall evaluation value $\alpha_i\ (i = 1, 2, \ldots, m)$ is determined. Go to the next step.

Step 10: Compute the scores of $\alpha_i\ (i = 1, 2, \ldots, m)$ according to Eq. (8) and then go to the next step.

Step 11: Rank the corresponding alternatives and select the optimal one.

$$\begin{aligned} &PFLPWGMSM^{(k, p_1, p_2, \ldots, p_k)}\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) \\ &= \left\langle f^{*-1}\left(\left(\left(1 - \left(\prod_{1\leq i_1 < i_2 < \ldots < i_k \leq n}\left(1 - \prod_{j=1}^{k}\left(1 - \left(1 - f^{*}\left(\theta_{i_j}\right)\right)^{n\delta_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}}\right)\right), \right. \\ &\left(\left(1 - \left(\prod_{1\leq i_1 < i_2 < \ldots < i_k \leq n}\left(1 - \prod_{j=1}^{k}\left(1 - \left(1 - \mu_{i_j}^{2}\right)^{n\delta_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{2(p_1+p_2+\ldots+p_k)}}\right), \\ &\left.\left(1 - \left(1 - \prod_{1\leq i_1 < i_2 < \ldots < i_k \leq n}\left(1 - \prod_{j=1}^{k}\left(1 - v_{i_j}^{2n\delta_{i_j}}\right)^{p_j}\right)^{1/C_n^k}\right)^{\frac{1}{p_1+p_2+\ldots+p_k}}\right)^{1/2}\right\rangle \end{aligned} \tag{40}$$

V. NUMERICAL EXAMPLE

Example 1: Let's consider a low-carbon tourism destination selection (LCTDS) problem. With the increasing popularity of environmental protection concepts, low-carbon tourism has gradually gained widespread attention. The so-called low-carbon tourism is a kind of tourism that reduces carbon. In other words, in tourism activities, tourists try to reduce carbon dioxide emissions as much as possible. Low-carbon tourism is green travel based on low energy consumption and low pollution, advocating to minimize carbon footprint and carbon dioxide emissions during travel, which is also a deep-level expression of environmental protection tourism. In low-carbon tourism, one of the most important problems is LCTDS, i.e., choosing a suitable destination which is low-carbon. When evaluating the performance of low carbon of tourism destinations, decision makers usually have to consider multiple aspects. Generally, when considering a LCTDS problem, the following four attributes should be taken into consideration, i.e., traffic conditions (G_1), attractions of the low carbon tourism destination (G_2); tourist consumption satisfaction (G_3), and environmental quality (G_4). Let's consider a realistic LCTDS problems. Suppose there are four destinations (alternatives), which can be denoted as A_1, A_2, A_3, and A_4. The weight vector of attributes is $w = (0.32, 0.26, 0.18, 0.24)^T$. In order to comprehensively evaluate the performance of the four destinations, three decision experts are invited to express their opinions over the four alternatives. Let $S = \{s_0 =$ "*extremely poor*", $s_1 =$ "*very poor*", $s_2 =$ " *poor*", $s_3 =$ "*slightly poor*", $s_4 =$ "*fair*", $s_5 =$ "*slightly good*", $s_6 =$ "good", $s_7 =$ "very good", $s_8 =$ "*extremely good*"$\}$ be a linguistic term set. Decision experts use PFLNs over S to express their evaluation information over alternatives and their evaluation matrices are listed in Tables 1-3. The weight vector of the three decision experts is $\lambda = (0.4, 0.32, 0.28)^T$. In the following sections, our proposed novel MAGDM method is employed to select the best low-carbon destination.

TABLE 1. The intuitionistic linguistic decision matrix R1 of example 1 provided by decision maker E1.

	C_1	C_2	C_3	C_4
A_1	$\langle s_5,(0.2,0.7)\rangle$	$\langle s_2,(0.4,0.6)\rangle$	$\langle s_5,(0.5,0.5)\rangle$	$\langle s_3,(0.2,0.6)\rangle$
A_2	$\langle s_4,(0.4,0.6)\rangle$	$\langle s_5,(0.4,0.5)\rangle$	$\langle s_3,(0.1,0.8)\rangle$	$\langle s_4,(0.5,0.5)\rangle$
A_3	$\langle s_3,(0.2,0.7)\rangle$	$\langle s_4,(0.2,0.7)\rangle$	$\langle s_4,(0.3,0.7)\rangle$	$\langle s_5,(0.2,0.7)\rangle$
A_4	$\langle s_6,(0.5,0.4)\rangle$	$\langle s_2,(0.2,0.8)\rangle$	$\langle s_3,(0.2,0.6)\rangle$	$\langle s_3,(0.3,0.6)\rangle$

A. THE PROCEDURE OF CHOOSING THE OPTIMAL ALTERNATIVE

Step 1: As all the attributes are benefit type, the original decision matrix does need to be normalized.

Step 2: Compute the support between two PFLNs α_{ij}^h and α_{ij}^g, where $i, j = 1, 2, 3, 4$, $h, g = 1, 2, 3$ and $h \neq g$ according to Eq. (42). For the facility of expression, we S_g^h to represent the support of α_{ij}^k from α_{ij}^d and we obtain the following results.

TABLE 2. The intuitionistic linguistic decision matrix R_2 of Example 1 provided by decision maker e_2.

	C_1	C_2	C_3	C_4
A_1	$\langle s_4,(0.1,0.7)\rangle$	$\langle s_3,(0.2,0.7)\rangle$	$\langle s_3,(0.2,0.8)\rangle$	$\langle s_6,(0.4,0.5)\rangle$
A_2	$\langle s_5,(0.4,0.5)\rangle$	$\langle s_3,(0.3,0.6)\rangle$	$\langle s_4,(0.2,0.6)\rangle$	$\langle s_3,(0.2,0.7)\rangle$
A_3	$\langle s_4,(0.2,0.6)\rangle$	$\langle s_4,(0.2,0.7)\rangle$	$\langle s_2,(0.4,0.6)\rangle$	$\langle s_3,(0.3,0.7)\rangle$
A_4	$\langle s_5,(0.3,0.6)\rangle$	$\langle s_4,(0.4,0.5)\rangle$	$\langle s_2,(0.3,0.6)\rangle$	$\langle s_4,(0.2,0.6)\rangle$

TABLE 3. The intuitionistic linguistic decision matrix R3 of Example 1 provided by decision maker e_3.

	C_1	C_2	C_3	C_4
A_1	$\langle s_5,(0.2,0.6)\rangle$	$\langle s_3,(0.3,0.7)\rangle$	$\langle s_4,(0.4,0.5)\rangle$	$\langle s_4,(0.2,0.7)\rangle$
A_2	$\langle s_4,(0.3,0.7)\rangle$	$\langle s_5,(0.3,0.6)\rangle$	$\langle s_2,(0.1,0.8)\rangle$	$\langle s_3,(0.4,0.6)\rangle$
A_3	$\langle s_4,(0.2,0.7)\rangle$	$\langle s_5,(0.3,0.6)\rangle$	$\langle s_1,(0.1,0.8)\rangle$	$\langle s_4,(0.2,0.7)\rangle$
A_4	$\langle s_3,(0.2,0.7)\rangle$	$\langle s_3,(0.1,0.7)\rangle$	$\langle s_4,(0.3,0.6)\rangle$	$\langle s_5,(0.4,0.5)\rangle$

$$S_2^1 = S_1^2 = \begin{bmatrix} 0.9442 & 0.9958 & 0.6833 & 0.7150 \\ 0.8875 & 0.8033 & 0.8658 & 0.8042 \\ 0.9108 & 1.0000 & 0.9333 & 0.9208 \\ 0.7592 & 0.7633 & 0.9517 & 0.9558 \end{bmatrix}$$

$$S_3^1 = S_3^1 = \begin{bmatrix} 0.9458 & 0.9833 & 0.8867 & 0.9867 \\ 0.9333 & 0.9250 & 0.9692 & 0.8667 \\ 0.9542 & 0.8792 & 0.8308 & 0.9542 \\ 0.5925 & 0.9367 & 0.9267 & 0.8033 \end{bmatrix}$$

$$S_3^2 = S_2^3 = \begin{bmatrix} 0.8900 & 0.9875 & 0.7967 & 0.7283 \\ 0.8208 & 0.8783 & 0.9692 & 0.9375 \\ 0.9567 & 0.8792 & 0.8975 & 0.9667 \\ 0.8333 & 0.8267 & 0.8783 & 0.8475 \end{bmatrix}$$

Step 3: Compute the weighted overall supports $T\left(\alpha_{ij}^h\right)$ associated with the PFLN α_{ij}^h by Eq. (43). We use the symbol T^h $(h = 1, 2, 3)$ to represent $T\left(\alpha_{ij}^h\right)$ and we have

$$T^1 = \begin{bmatrix} 1.8900 & 1.9792 & 1.5700 & 1.7017 \\ 1.8208 & 1.7283 & 1.8350 & 1.6708 \\ 1.8650 & 1.8792 & 1.7642 & 1.8750 \\ 1.3517 & 1.7000 & 1.8783 & 1.7592 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 1.8342 & 1.9833 & 1.4800 & 1.4433 \\ 1.7083 & 1.6817 & 1.7008 & 1.7417 \\ 1.8675 & 1.8792 & 1.8303 & 1.8875 \\ 1.5925 & 1.5900 & 1.8300 & 1.8033 \end{bmatrix}$$

$$T^3 = \begin{bmatrix} 1.8358 & 1.9708 & 1.6833 & 1.7150 \\ 1.7542 & 1.8033 & 1.8042 & 1.8042 \\ 1.9108 & 1.7583 & 1.7283 & 1.9208 \\ 1.4258 & 1.7633 & 1.8050 & 1.6508 \end{bmatrix}$$

TABLE 4. The comprehensive Pythagorean fuzzy linguistic decision matrix.

	C_1	C_2	C_3	C_4
A_1	$\langle s_{4.7539},(0.1749,0.6706)\rangle$	$\langle s_{2.6340},(0.3216,0.6581)\rangle$	$\langle s_{4.2818},(0.4043,0.5780)\rangle$	$\langle s_{6},(0.2783,0.5942)\rangle$
A_2	$\langle s_{4.3904},(0.3756,0.5916)\rangle$	$\langle s_{4.5883},(0.3444,0.5579)\rangle$	$\langle s_{3.1313},(0.1393,0.7316)\rangle$	$\langle s_{3.4400},(0.4015,0.5871)\rangle$
A_3	$\langle s_{3.6496},(0.2000,0.6664)\rangle$	$\langle s_{4.3430},(0.2320,0.6713)\rangle$	$\langle s_{2.7729},(0.3044,0.6906)\rangle$	$\langle s_{4.2718},(0.2372,7000)\rangle$
A_4	$\langle s_{6},(0.3775,0.5359)\rangle$	$\langle s_{3.0282},(0.2657,0.6657)\rangle$	$\langle s_{3.0605},(0.2647,0.6000)\rangle$	$\langle s_{4.0482},(0.3062,0.5711)\rangle$

Step 4: Based on DMs' weight vector and the weighted overall supports $T\left(\alpha_{ij}^{h}\right)$, compute the power weight associated with the PFLN α_{ij}^{h} by Eq. (44), and we obtain

$$\omega^1 = \begin{bmatrix} 0.4046 & 0.4001 & 0.3995 & 0.4120 \\ 0.4079 & 0.3991 & 0.4074 & 0.3912 \\ 0.3981 & 0.4048 & 0.3984 & 0.3977 \\ 0.3840 & 0.4026 & 0.4051 & 0.4024 \end{bmatrix}$$

$$\omega^2 = \begin{bmatrix} 0.3174 & 0.3206 & 0.3084 & 0.2981 \\ 0.3133 & 0.3138 & 0.3105 & 0.3213 \\ 0.3188 & 0.3238 & 0.3264 & 0.3195 \\ 0.3387 & 0.3090 & 0.3183 & 0.3270 \end{bmatrix}$$

$$\omega^3 = \begin{bmatrix} 0.2779 & 0.2793 & 0.2920 & 0.2899 \\ 0.2788 & 0.2871 & 0.2821 & 0.2875 \\ 0.2831 & 0.2714 & 0.2752 & 0.2828 \\ 0.2773 & 0.2884 & 0.2763 & 0.2706 \end{bmatrix}$$

Step 5: Compute the overall decision matrix by the PFLPWA operator and the results are shown in Table 4.

Step 6: Calculate the support between $Sup\left(\alpha_{ij}, \alpha_{is}\right)$ by Eq. (46). Similarly, we employ Sup_s^j to denote $Sup\left(\alpha_{ij}, \alpha_{is}\right)$, and we have

$$Sup_2^1 = Sup_1^2 = (0.9170, 0.9807, 0.9629, 0.7307)$$
$$Sup_3^1 = Sup_1^3 = (0.9342, 0.8369, 0.9611, 0.7534)$$
$$Sup_4^1 = Sup_1^4 = (0.8679, 0.9447, 0.9796, 0.8313)$$
$$Sup_3^2 = Sup_2^3 = (0.8512, 0.8176, 0.9240, 0.9772)$$
$$Sup_4^2 = Sup_2^4 = (0.7850, 0.9254, 0.9833, 0.8994)$$
$$Sup_4^3 = Sup_3^4 = (0.9337, 0.8922, 0.9407, 0.9221)$$

Step 7: Compute the weighted overall supports $T\left(\alpha_{ij}\right)$ by Eq. (47) and we can get

$$T = \begin{bmatrix} 2.7191 & 2.5532 & 2.7191 & 2.5866 \\ 2.7623 & 2.7237 & 2.5468 & 2.7623 \\ 2.9036 & 2.8702 & 2.8258 & 2.9036 \\ 2.3154 & 2.6073 & 2.6528 & 2.6528 \end{bmatrix}$$

Step 8: Compute the power weight δ_{ij} associated with PFLN α_{ij} and we obtain

$$\delta = \begin{bmatrix} 0.3266 & 0.2535 & 0.1837 & 0.2362 \\ 0.3242 & 0.2607 & 0.1719 & 0.2432 \\ 0.3219 & 0.2593 & 0.1774 & 0.2414 \\ 0.3003 & 0.2655 & 0.1861 & 0.2481 \end{bmatrix}$$

TABLE 5. Decision results with different *K* in the pflpwgmsm operator.

k	Scores $S(\alpha_i)=(i=1,2,3,4)$	Ranking results
$k=1$	$S(\alpha_1)=0.3437$ $S(\alpha_2)=0.2571$ $S(\alpha_3)=0.1929$ $S(\alpha_4)=0.3757$	A_4 f A_1 f A_2 f A_3
k=2	$S(\alpha_1)=0.2584$ $S(\alpha_2)=0.2344$ $S(\alpha_3)=0.1838$ $S(\alpha_4)=0.2415$	A_1 f A_4 f A_2 f A_3
$k=3$	$S(\alpha_1)=0.2426$ $S(\alpha_2)=0.2203$ $S(\alpha_3)=0.1782$ $S(\alpha_4)=0.2296$	A_1 f A_4 f A_2 f A_3
$k=4$	$S(\alpha_1)=0.2349$ $S(\alpha_2)=0.2056$ $S(\alpha_3)=0.1733$ $S(\alpha_4)=0.2229$	A_1 f A_4 f A_2 f A_3

Step 9: Compute the collective evaluation values of alternatives by the PFLPWGMSM operator. Without loss of generality, we assume $k = 2$ and $p_1 = p_2 = 1$, then we can obtain

$$\alpha_1 = \langle s_{4.5399}, (0.2885, 0.6326)\rangle$$
$$\alpha_2 = \langle s_{3.8708}, (0.3356, 0.6213)\rangle$$
$$\alpha_3 = \langle s_{3.7644}, (0.2386, 0.6864)\rangle$$
$$\alpha_4 = \langle s_{3.9446}, (0.3066, 0.5996)\rangle$$

Step 10: Compute the scores of the alternatives and we have

$$S(\alpha_1) = 0.2584 \quad S(\alpha_2) = 0.2344$$
$$S(\alpha_3) = 0.1838 \quad S(\alpha_4) = 0.2415$$

Step 11: Rank alternatives according to their scores and we have $A_1 \succ A_4 \succ A_2 \succ A_3$, and A_4 is the optimal alternative.

B. ANALYSIS OF THE INFLUENCE OF PARAMETERS ON THE DECISION RESULTS

1) THE IMPACT OF K ON THE RESULTS

The parameter k in the PFLPWGMSM operator has important impact on the decision results and in this subsection we investigate its influence. To this end, we assign different value to k in Step 9 and present the scores and ranking results in Table 5. Without loss of generality, we assume $p_1 = \ldots = p_k = 1$ and the LST is taken as $f(\theta) = \frac{\theta}{2t}$ $(t = 3)$.

As we can see from Table 5, different scores and ranking orders are obtained with different parameter values k

TABLE 6. The score values and ranking orders of Example 1 with different parameters ($k = 2$).

p_1	p_2	Scores $S(\alpha_i) = (i = 1,2,3,4)$	Ranking results
1	0	$S(\alpha_1)=0.2559$ $S(\alpha_2)=0.3132$ $S(\alpha_3)=0.2167$ $S(\alpha_4)=0.4036$	A_4 f A_2 f A_1 f A_3
0	1	$S(\alpha_1)=0.3448$ $S(\alpha_2)=0.1937$ $S(\alpha_3)=0.1678$ $S(\alpha_4)=0.1992$	A_1 f A_4 f A_2 f A_3
1	2	$S(\alpha_1)=0.2742$ $S(\alpha_2)=0.2273$ $S(\alpha_3)=0.1801$ $S(\alpha_4)=0.2249$	A_1 f A_2 f A_4 f A_3
1	3	$S(\alpha_1)=0.2870$ $S(\alpha_2)=0.2293$ $S(\alpha_3)=0.1807$ $S(\alpha_4)=0.2204$	A_1 f A_2 f A_4 f A_3
1	4	$S(\alpha_1)=0.2971$ $S(\alpha_2)=0.2337$ $S(\alpha_3)=0.1826$ $S(\alpha_4)=0.2196$	A_1 f A_2 f A_4 f A_3
2	1	$S(\alpha_1)=0.2572$ $S(\alpha_2)=0.2613$ $S(\alpha_3)=0.1947$ $S(\alpha_4)=0.2758$	A_4 f A_2 f A_1 f A_3
3	1	$S(\alpha_1)=0.2616$ $S(\alpha_2)=0.2793$ $S(\alpha_3)=0.2024$ $S(\alpha_4)=0.3026$	A_4 f A_2 f A_1 f A_3
4	1	$S(\alpha_1)=0.2667$ $S(\alpha_2)=0.2923$ $S(\alpha_3)=0.2083$ $S(\alpha_4)=0.3236$	A_4 f A_2 f A_1 f A_3
0.5	0.5	$S(\alpha_1)=0.2548$ $S(\alpha_2)=0.2267$ $S(\alpha_3)=0.1817$ $S(\alpha_4)=0.2377$	A_1 f A_4 f A_2 f A_3
1	1	$S(\alpha_1)=0.2584$ $S(\alpha_2)=0.2344$ $S(\alpha_3)=0.1838$ $S(\alpha_4)=0.2415$	A_1 f A_4 f A_2 f A_3
2	2	$S(\alpha_1)=0.2660$ $S(\alpha_2)=0.2480$ $S(\alpha_3)=0.1884$ $S(\alpha_4)=0.2496$	A_1 f A_4 f A_2 f A_3
3	3	$S(\alpha_1)=0.2732$ $S(\alpha_2)=0.2593$ $S(\alpha_3)=0.1931$ $S(\alpha_4)=0.2577$	A_1 f A_2 f A_4 f A_3
4	4	$S(\alpha_1)=0.2799$ $S(\alpha_2)=0.2691$ $S(\alpha_3)=0.1974$ $S(\alpha_4)=0.2653$	A_1 f A_2 f A_4 f A_3

in the PFLPWGMSM operator. This is because the value k determines the number of dependent attributes among which the interrelationship among them is taken into consideration. When $k = 1$, our method does not consider the interrelationship between attributes. When $k = 2$ and 3, the method captures the interrelationship between any two or three attributes. When $k = 4$, then the interrelationship among all the four attributes. As there is usually interrelationship among attributes, our proposed method is flexible to deal with practical MAGDM problems. Moreover, we find out that the increase of the value k leads to the decrease of the score values of each alternative. Hence, the value of k can be regarded as DMs' attitude towards the performance of alternatives. If DMs are optimistic to the alternatives, then they should choose a smaller value of k. If DMs are pessimistic to the alternatives, then they can select a larger value of k. If DMs are neutral, then they can set $k = [n/2]$, where [] is the round function and n is the number of attributes.

2) THE EFFECT OF THE PARAMETER VECTOR ON THE RESULTS

In this section, we investigate the influence of the parameter vector $(p_1, p_2, \ldots, p_k)$ on the final decision results. The score values of alternatives and the final ranking orders with different parameters are presented in Tables 6 and 7.As seen from Tables 6 and 7, different score values and ranking orders are derived with different values of the parameters. In Table 6, we find out that let p_1 be fixed and then the increase of the value of parameter p_2 leads to the decrease of the score values of each alternative. Similarly, if p_2 is fixed then the score values of alternatives will also decrease if the value of p_1 increase. However, the ranking orders are always the same. In addition, when both the values of p_1 and p_2 increase, the score values of alternatives also decrease. Hence, we can determine appropriate values according to actual needs and basically we should choose the value of p_1 and p_2, such that $0 \leq p_1, p_2 \leq 1$. Because if $p_1 = 0 (p_2 = 1)$, then the interrelationship between attributes is not taken into account, which is usually inconsistent with the reality. In Table 7, we can find the similar phenomenon, i.e., if any two of the parameters p_1, p_2, and p_3 are fixed, then the increase of the other parameter leads to the increase of the score values of each alternative. In addition, the parameters p_1, p_2, and p_3 should not be assigned zero, otherwise the proposed method fails to consider the interrelationship among multiple attributes.

C. ADVANTAGES AND SUPERIORITIES ANALYSIS

In this section, we attempt to demonstrate the advantages and superiorities of our proposed MAGDM method by comparing to some existing PFL sets based MAGDM method. These methods involve that proposed by Peng and Yang [30] based on the PFL weighted average (PFLWA) operator, that developed by Liu *et al.* [31] based on PFL weighted Muirhead mean (PFLWMM) operator, and that presented by Teng *et al.* [32] based on PFL power weighted MSM (PFLPWMSM) operator.

1) THE RATIONALITY AND FLEXIBILITY OF THE PROPOSED OPERATIONAL RULES

It is noted that the MAGDM methods presented in [30]–[32] are based on the basic algebraic operational rules. However, as pointed out in Introduction, the main shortcoming of these operational rules is that they are not closed. In other word, the calculation process may exceed the bound of the predefined LTS. In addition, these operational rules may cause

TABLE 7. The score values and ranking orders of Example 1 with different parameters ($k = 3$).

p_1	p_2	p_3	Scores $S(\alpha_i) = (i = 1,2,3,4)$				Ranking results
0.5	0.5	0.5	$S(\alpha_1)=0.2407$	$S(\alpha_2)=0.2146$	$S(\alpha_3)=0.1768$	$S(\alpha_4)=0.2278$	$A_1 \succ A_4 \succ A_2 \succ A_3$
1	0	0	$S(\alpha_1)=0.2798$	$S(\alpha_2)=0.3641$	$S(\alpha_3)=0.2418$	$S(\alpha_4)=0.4421$	$A_4 \succ A_2 \succ A_1 \succ A_3$
0	1	0	$S(\alpha_1)=0.1789$	$S(\alpha_2)=0.2004$	$S(\alpha_3)=0.1626$	$S(\alpha_4)=0.1463$	$A_2 \succ A_1 \succ A_3 \succ A_4$
0	0	1	$S(\alpha_1)=0.3475$	$S(\alpha_2)=0.1871$	$S(\alpha_3)=0.1700$	$S(\alpha_4)=0.2246$	$A_1 \succ A_4 \succ A_2 \succ A_3$
1	1	2	$S(\alpha_1)=0.2619$	$S(\alpha_2)=0.2113$	$S(\alpha_3)=0.1755$	$S(\alpha_4)=0.2278$	$A_1 \succ A_4 \succ A_2 \succ A_3$
1	1	3	$S(\alpha_1)=0.2759$	$S(\alpha_2)=0.2091$	$S(\alpha_3)=0.1756$	$S(\alpha_4)=0.2285$	$A_1 \succ A_4 \succ A_2 \succ A_3$
1	1	4	$S(\alpha_1)=0.2867$	$S(\alpha_2)=0.2090$	$S(\alpha_3)=0.1767$	$S(\alpha_4)=0.2301$	$A_1 \succ A_4 \succ A_2 \succ A_3$
1	2	1	$S(\alpha_1)=0.2243$	$S(\alpha_2)=0.2203$	$S(\alpha_3)=0.1750$	$S(\alpha_4)=0.2055$	$A_1 \succ A_2 \succ A_4 \succ A_3$
1	3	1	$S(\alpha_1)=0.2153$	$S(\alpha_2)=0.2273$	$S(\alpha_3)=0.1768$	$S(\alpha_4)=0.1932$	$A_2 \succ A_1 \succ A_4 \succ A_3$
1	4	1	$S(\alpha_1)=0.2103$	$S(\alpha_2)=0.2356$	$S(\alpha_3)=0.1803$	$S(\alpha_4)=0.1859$	$A_2 \succ A_1 \succ A_4 \succ A_3$
2	1	1	$S(\alpha_1)=0.2513$	$S(\alpha_2)=0.2500$	$S(\alpha_3)=0.1919$	$S(\alpha_4)=0.2669$	$A_4 \succ A_1 \succ A_2 \succ A_3$
3	1	1	$S(\alpha_1)=0.2598$	$S(\alpha_2)=0.2700$	$S(\alpha_3)=0.2009$	$S(\alpha_4)=0.2947$	$A_4 \succ A_2 \succ A_1 \succ A_3$
4	1	1	$S(\alpha_1)=0.2673$	$S(\alpha_2)=0.2844$	$S(\alpha_3)=0.2073$	$S(\alpha_4)=0.3162$	$A_4 \succ A_2 \succ A_1 \succ A_3$

contrary to the subjective intuition of the decision makers in the process of MAGDM. Our proposed method is based on new operational rules for PFLNs, i.e., LSF based operations. Advantages of these new operational rules are obvious. First of all, the new operational laws proposed in this study have properties of closure and can solve the cross-border problems of the operational rules used in [30]–[32]. Second, our proposed operational rules can flexibly adapt to the semantic changes of DMs, which is consistent with realistic decision-making process. Hence, our proposed method is more powerful, useful and flexible than those MAGDM approaches presented in [30]–[32].

2) THE ABILITY OF CAPTURING THE INTERRELATIONSHIP AMONG ANY NUMBERS OF ATTRIBUTES

Our proposed method is based on the PFLPWGMSM operator and hence it can considers the interrelationship that widely exists in practical MAGDM problems. In addition, our method is capable to consider the different important of different aggregated values of attributes. Given these advantages, our proposed method is more powerful than some existing PFL sets based decision-making methods. First, Peng and Yang's [30] method is based on the simply weighted average operator. As it is known that the simple weighted operator fails to consider the interrelationship among attributes. In other word, Peng and Yang's [30] method is based on the assumption that attributes are independent, which is somewhat inconsistent with the reality. In most real MAGDM problems, attributes are correlated and such interrelationship among attributes should be taken into consideration. Hence, the MAGDM method introduced by Peng and Yang's [30] is somewhat defective. Our proposed method is able to consider the interrelationship when calculating the final decision-making results and hence our proposed method is better and more powerful than that proposed by Peng and Yang. In addition, Teng *et al.*'s [32] method based on the PFLPWMSM operators can also take the interrelationship among attributes into account, which is the same as our method. However, its flaw is also obvious, i.e., it assumes that all input evaluation values have the same importance, which is somewhat inconsistent with real cases. Our proposed method is effective to consider the different importance of input aggregated values. By assigning different values in the vector P, the importance degrees of aggregated values are manipulated. Moreover, the PMSM operator is a special case of our proposed PGMSM operator. Hence, our method is more powerful than Teng *et al.*'s [32] method.

3) THE ABILITY OF REDUCING THE BAD INFLUENCE OF EXTREME EVALUATION VALUES

In many practical MAGDM problems, in order to make a smart decision, a group of DMs instead of only one are invited to evaluate the performance of alternatives. DMs usually come from different fields and have different expertise and experience. In addition, due to time shortage and complexity of decision-making problems, it is difficult for DMs to acquire all information related to decision-making problems. Hence it is common that DMs maybe provide unduly high or low evaluation values, which obviously have negative impact on the final decision results. To make the final decision results reasonable and acceptable, such kind of bad influence of extreme evaluation values should be reduced or eliminated. Peng and Yang's [30] simple weighted average operator fails to effectively handle DMs' extremely high or low evaluation values. In other word, the reliability of decision results produced by Peng and Yang's [30] method maybe not reliable if DMs provide unreasonable evaluation values. Moreover, Liu *et al.*'s [31] cannot effectively cope with DMs' extreme evaluation values, either. Hence, our proposed method more powerful and reasonable than Peng and Yang's [30] and Lu *et al.*'s [31] methods.

VI. CONCLUSION

Recently PFLSs are have been regarded as an efficient tool to express fuzzy information, which have been extensively investigated and applied in MAGDM procedures. The main contribution of this paper is to propose a new MAGDM method wherein attribute values are given in the forms of PFLNs. In order to do this, we firstly introduced new operational rules of PFLNs based on LSFs. Then, we presented novel AOs to fuse Pythagorean fuzzy linguistic information, i.e. PFLPA, PFLPWA, PFLPGMSM, and PFLPWGMSM operators. Thirdly, we introduced an approached to objectively determine the weighs information. Finally, based on the newly developed AOs and weights determination approach we presented a novel MAGDM method. Afterwards, we proved the effectiveness and advantages of our method via numerical examples. In the future, we shall continue our study form the following two aspects. First, we shall investigate applications our proposed method in more practical MAGDM problems, such as evaluation of offshore oil spill response waste management strategies [42], sustainable supplier evaluation [43], plan selection of urban integrated energy systems [44], evaluation of groundwater quality [45], etc. Second, our study does not consider whether the final decision results are accepted by DMs. Actually, consensus reaching process is an important and interesting research topic in group decision-making and large scale group decision-making [46]–[50]. Hence, we shall study consensus reaching process in group decision-making under PLF sets.

REFERENCES

[1] R. R. Yager and A. M. Abbasov, "Pythagorean membership grades, complex numbers, and decision making," *Int. J. Intell. Syst.*, vol. 28, no. 5, pp. 436–452, Mar. 2013.

[2] R. Zhang, J. Wang, X. Zhu, M. Xia, and M. Yu, "Some generalized Pythagorean fuzzy Bonferroni mean aggregation operators with their application to multiattribute group decision-making," *Complexity*, vol. 2017, pp. 1–16, Aug. 2017.

[3] D. Liang, Z. Xu, and A. P. Darko, "Projection model for fusing the information of Pythagorean fuzzy multicriteria group decision making based on geometric Bonferroni mean," *Int. J. Intell. Syst.*, vol. 32, no. 2, pp. 966–987, Apr. 2017.

[4] J. Qin, "Generalized Pythagorean fuzzy Maclaurin symmetric means and its application to multiple attribute SIR group decision model," *Int. J. Fuzzy Syst.*, vol. 20, no. 3, pp. 943–957, Mar. 2018.

[5] G. Wei and M. Lu, "Pythagorean fuzzy Maclaurin symmetric mean operators in multiple attribute decision making," *Int. J. Intell. Syst.*, vol. 33, no. 5, pp. 1043–1070, 2018.

[6] G. Wei and M. Lu, "Pythagorean fuzzy power aggregation operators in multiple attribute decision making," *Int. J. Intell. Syst.*, vol. 33, no. 1, pp. 169–186, 2018.

[7] D. Liang, A. P. Darko, and Z. Xu, "Pythagorean fuzzy partitioned geometric Bonferroni mean and its application to multi-criteria group decision making with grey relational analysis," *Int. J. Fuzzy Syst.*, vol. 21, no. 1, pp. 115–128, Feb. 2019.

[8] L. Li, R. Zhang, J. Wang, X. Zhu, and Y. Xing, "Pythagorean fuzzy power Muirhead mean operators with their application to multi-attribute decision making," *J. Intell. Fuzzy. Syst.*, vol. 35, no. 2, pp. 1–16, Jan. 2018.

[9] X. Peng and H. Yuan, "Fundamental properties of Pythagorean fuzzy aggregation operators," *Fundam. Inform.*, vol. 147, no. 4, pp. 415–446, Nov. 2016.

[10] Z. Ma and Z. Xu, "Symmetric Pythagorean fuzzy weighted geometric/averaging operators and their application in multicriteria decision-making problems," *Int. J. Intell. Syst.*, vol. 31, no. 12, pp. 1198–1219, May 2016.

[11] X. Peng and Y. Yang, "Pythagorean fuzzy Choquet integral based MABAC method for multiple attribute group decision making," *Int. J. Intell. Syst.*, vol. 31, no. 10, pp. 989–1020, Feb. 2016.

[12] Z. Li, G. Wei, and M. Lu, "Pythagorean fuzzy Hamy mean operators in multiple attribute group decision making and their application to supplier selection," *Symmetry*, vol. 10, no. 10, p. 505, Oct. 2018.

[13] H. Garg, "A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making," *Int. J. Intell. Syst.*, vol. 31, no. 9, pp. 886–920, Feb. 2016.

[14] H. Garg, "Generalized Pythagorean fuzzy geometric aggregation operators using Einstein t-norm and t-conorm for multicriteria decision-making process," *Int. J. Intell. Syst.*, vol. 32, no. 6, pp. 597–630, Jun. 2017.

[15] H. Gao, "Pythagorean fuzzy Hamacher prioritized aggregation operators in multiple attribute decision making," *J. Intell. Fuzzy Syst.*, vol. 35, no. 2, pp. 2229–2245, Aug. 2018.

[16] Y. Xing, R. Zhang, J. Wang, and X. Zhu, "Some new Pythagorean fuzzy Choquet-frank aggregation operators for multi-attribute decision making," *Int. J. Intell. Syst.*, vol. 33, no. 11, pp. 2189–2215, Jul. 2018.

[17] M. Akram, W. A. Dudek, and J. M. Dar, "Pythagorean Dombi fuzzy aggregation operators with application in multicriteria decision-making," *Int. J. Intell. Syst.*, vol. 34, no. 11, pp. 3000–3019, 2019.

[18] Y. Yang, K. Chin, H. Ding, H. Lv, and Y. Li, "Pythagorean fuzzy Bonferroni means based on t-norm and its dual t-conorm," *Int. J. Intell. Syst.*, vol. 34, no. 6, pp. 1303–1336, Jan. 2019.

[19] M. Akram, W. A. Dudek, and F. Ilyas, "Group decision-making based on Pythagorean fuzzy TOPSIS method," *Int. J. Intell. Syst.*, vol. 34, no. 7, pp. 1455–1475, Mar. 2019.

[20] L. Pérez-Domínguez, L. A. Rodríguez-Picón, A. Alvarado-Iniesta, D. L. Cruz, and Z. Xu, "MOORA under Pythagorean fuzzy set for multiple criteria decision making," *Complexity*, vol. 2018, pp. 1–10, Mar. 2018.

[21] T.-Y. Chen, "Remoteness index-based Pythagorean fuzzy VIKOR methods with a generalized distance measure for multiple criteria decision analysis," *Inf. Fusion*, vol. 41, pp. 129–150, May 2018.

[22] T.-Y. Chen, "An outranking approach using a risk attitudinal assignment model involving Pythagorean fuzzy information and its application to financial decision making," *Appl. Soft Comput.*, vol. 71, pp. 460–487, Oct. 2018.

[23] Z.-X. Zhang, W.-N. Hao, X.-H. Yu, G. Chen, S.-J. Zhang, and J.-Y. Chen, "Pythagorean fuzzy preference ranking organization method of enrichment evaluations," *Int. J. Intell. Syst.*, vol. 34, no. 7, pp. 1416–1439, Mar. 2019.

[24] R. Verma and J. M. Merigó, "On generalized similarity measures for Pythagorean fuzzy sets and their applications to multiple attribute decision-making," *Int. J. Intell. Syst.*, vol. 34, no. 10, pp. 2556–2583, Aug. 2019.

[25] W. Zeng, D. Li, and Q. Yin, "Distance and similarity measures of Pythagorean fuzzy sets and their applications to multiple criteria group decision making," *Int. J. Intell. Syst.*, vol. 33, no. 11, pp. 2236–2254, Jul. 2018.

[26] J. Wang, H. Gao, and G. W. Wei, "The generalized Dice similarity measures for Pythagorean fuzzy multiple attribute group decision making," *Int. J. Intell. Syst.*, vol. 34, no. 6, pp. 1158–1183, Jun. 2019.

[27] G. Wei and Y. Wei, "Similarity measures of Pythagorean fuzzy sets based on the cosine function and their applications," *Int. J. Intell. Syst.*, vol. 33, no. 3, pp. 634–652, Jan. 2018.

[28] M.-S. Yang and Z. Hussain, "Fuzzy entropy for Pythagorean fuzzy sets with application to multicriterion decision making," *Complexity*, vol. 2018, pp. 1–14, Nov. 2018.

[29] T.-Y. Chen, "An effective correlation-based compromise approach for multiple criteria decision analysis with Pythagorean fuzzy information," *J. Intell. Fuzzy Syst.*, vol. 35, no. 3, pp. 3529–3541, Oct. 2018.

[30] X. D. Peng and Y. Yang, "Multiple attribute group decision making methods based on Pythagorean fuzzy linguistic set," *Comput. Eng.*, vol. 52, no. 23, pp. 50–54, Oct. 2018.

[31] Y. Liu, J. Liu, and Y. Qin, "Pythagorean fuzzy linguistic Muirhead mean operators and their applications to multiattribute decision-making," *Int. J. Intell. Syst.*, vol. 35, no. 2, pp. 300–332, 2020.
[32] F. Teng, Z. Liu, and P. Liu, "Some power Maclaurin symmetric mean aggregation operators based on Pythagorean fuzzy linguistic numbers and their application to group decision making," *Int. J. Intell. Syst.*, vol. 33, no. 9, pp. 1949–1985, May 2018.
[33] P. Liu and H. Xu, "Group decision making method based on hybrid aggregation operator for intuitionistic uncertain linguistic variables," *J. Intell. Fuzzy Syst.*, vol. 36, no. 2, pp. 1879–1898, Mar. 2019.
[34] Z. Liu, H. Xu, P. Liu, L. Li, and X. Zhao, "Interval-valued intuitionistic uncertain linguistic multi-attribute decision-making method for plant location selection with partitioned Hamy mean," *Int. J. Fuzzy Syst.*, vol. 22, no. 6, pp. 1993–2010, Sep. 2020.
[35] S. Xian, Y. Cheng, and Z. Liu, "A novel picture fuzzy linguistic Muirhead mean aggregation operators and their application to multiple attribute decision making," *Soft Comput.*, vol. 25, no. 23, pp. 14741–14756, Aug. 2021.
[36] R. R. Yager, "The power average operator," *IEEE Trans. Syst., Man, Cybern. A, Syst. Humans*, vol. 31, no. 6, pp. 724–731, Nov. 2001.
[37] J. Q. Wang, Y. Yang, and L. Li, "Multi-criteria decision-making method based on single-valued neutrosophic linguistic Maclaurin symmetric mean operators," *Neural Comput. Appl.*, vol. 30, no. 5, pp. 1529–1547, 2018.
[38] P. Liu and Y. Wang, "Multiple attribute decision making based on q-rung orthopair fuzzy generalized Maclaurin symmetic mean operators," *Inf. Sci.*, vol. 518, pp. 181–210, May 2020.
[39] H. Garg and R. Arora, "Generalized Maclaurin symmetric mean aggregation operators based on Archimedean t-norm of the intuitionistic fuzzy soft set information," *Artif. Intell. Rev.*, vol. 54, no. 4, pp. 3173–3213, Apr. 2021.
[40] P. D. Liu and Y. Li, "Multi-attribute decision making method based on generalized Maclaurin symmetric mean aggregation operators for probabilistic linguistic information," *Comput. Ind. Eng.*, vol. 131, pp. 282–294, May 2019.
[41] J.-Q. Wang, J.-T. Wu, J. Wang, H.-Y. Zhang, and X.-H. Chen, "Interval-valued hesitant fuzzy linguistic sets and their applications in multi-criteria decision-making problems," *Inf. Sci.*, vol. 288, pp. 55–72, Dec. 2014.
[42] S. Saleem, G. Hu, J. Li, K. Hewage, and R. Sadiq, "Evaluation of offshore oil spill response waste management strategies: A lifecycle assessment-based framework," *J. Hazardous Mater.*, vol. 432, Jun. 2022, Art. no. 128659.
[43] C. Bai, S. Kusi-Sarpong, H. B. Ahmadi, and J. Sarkis, "Social sustainable supplier evaluation and selection: A group decision-support approach," *Int. J. Prod. Res.*, vol. 57, no. 22, pp. 7046–7067, Jan. 2019.
[44] Y. Ke, J. Liu, J. Meng, S. Fang, and S. Zhuang, "Comprehensive evaluation for plan selection of urban integrated energy systems: A novel multi-criteria decision-making framework," *Sustain. Cities Soc.*, vol. 81, Jun. 2022, Art. no. 103837.
[45] I. Mukherjee, U. K. Singh, and S. Chakma, "Evaluation of groundwater quality for irrigation water supply using multi-criteria decision-making techniques and GIS in an agroeconomic tract of Lower Ganga Basin, India," *J. Environ. Manage.*, vol. 309, May 2022, Art. no. 114691.
[46] C. Zheng, Y. Zhou, L. Zhou, and H. Chen, "Clustering and compatibility-based approach for large-scale group decision making with hesitant fuzzy linguistic preference relations: An application in e-waste recycling," *Expert Syst. Appl.*, vol. 197, Jul. 2022, Art. no. 116615.
[47] P. Liu, Y. Li, X. Zhang, and W. Pedrycz, "A multiattribute group decision-making method with probabilistic linguistic information based on an adaptive consensus reaching model and evidential reasoning," *IEEE Trans. Cybern.*, early access, Apr. 29, 2022, doi: 10.1109/TCYB.2022.3165030.
[48] H. Zhang, W. Zhu, X. Chen, Y. Wu, H. Liang, C.-C. Li, and Y. Dong, "Managing flexible linguistic expression and ordinal classification-based consensus in large-scale multi-attribute group decision making," *Ann. Oper. Res.*, vol. 358, pp. 1–54, Apr. 2022, doi: 10.1007/s10479-022-04687-3.
[49] M. Zhou, J. L. Li, Y. W. Chen, Z. P. Zhou, and J. Wu, "Consensus reaching process for group decision making with distributed preference relations under fuzzy uncertainty," *Int. J. Intell. Syst.*, vol. 24, pp. 1–19, Apr. 2022.
[50] X. Jian, M. Cai, Y. Wang, and Y. Gao, "A trust-enhanced consensus reaching model based on interaction among decision-makers with incomplete preferences," *Kybernetes*, Apr. 2022, doi: 10.1108/K-12-2021-1294.

JUNHUI CHEN was born in Nanyang, Henan, China, in 1981. She received the B.S. degree from the Henan University of Economics and Law, Zhengzhou, Henan, in 2003, and the M.S. degree from Beijing Jiaotong University, Beijing, China, in 2006, where she is currently pursuing the Ph.D. degree with the School of Economics and Management. She is also an Instructor with the Henan University of Economics and Law. Her research interests include decision making, fuzzy set theory, health informatics, and data-driven healthcare management.

RUNTONG ZHANG (Senior Member, IEEE) received the B.S. degree from Dalian Maritime University, in 1985, and the Ph.D. degree from the Technical University of Crete, in 1996. He is currently a Professor and the Head of the Department of Information Management, Beijing Jiaotong University, China. His current research interests include big data, decision making health-care management, operations research, and artificial intelligence.