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THEORY

Homogeneous $\mathcal{L}_p-$Stability for Homogeneous Systems

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ABSTRACT The motivation of this paper comes from the fact that $\mathcal{L}_p-$stability and $\mathcal{L}_p-$gain, using the classical signal norms, is not well-defined for arbitrary continuous weighted homogeneous systems. However, using homogeneous signal norms it is possible to show that every internally stable homogeneous system has a globally defined finite homogeneous $\mathcal{L}_p-$gain, for p sufficiently large. If the system has a homogeneous approximation, the homogeneous $\mathcal{L}_p-$gain is inherited locally. Homogeneous $\mathcal{L}_p-$stability can be characterized by a homogeneous dissipation inequality, which in the input affine case can be transformed to a homogeneous Hamilton-Jacobi inequality. An estimation of an upper bound for the homogeneous $\mathcal{L}_p-$gain can be derived from these inequalities. Homogeneous $\mathcal{L}_\infty-$stability is also considered and its strong relationship to Input-to-State stability is studied. These results are extensions to arbitrary homogeneous systems of the well-known situation for linear time-invariant systems, where the Hamilton-Jacobi inequality reduces to an algebraic Riccati inequality. A natural application of finite-gain homogeneous $\mathcal{L}_p-$stability is in the study of stability for interconnected systems. An extension of the small-gain theorem for negative feedback systems and results for systems in cascade are derived for different homogeneous norms. Previous results in the literature use classical signal norms, hence, they can only be applied to a restricted class of homogeneous systems. The results are illustrated by several examples.

INDEX TERMS Homogeneity, continuous weighted homogeneous system, non-linear system, homogeneous $\mathcal{L}_p-$norm, finite-gain homogeneous $\mathcal{L}_p-$stability, input-to-state stability, homogeneous small gain theorem.

I. INTRODUCTION

Input-Output Stability, e.g. $\mathcal{L}_p-$stability, is a rather intuitive concept. It implies that a small input causes a small output in the system, where "small" is related to some ways of measuring the size of the input and output signals. $\mathcal{L}_p-$stability and its related concept of $\mathcal{L}_p-$gain of a (dynamical) system are classical in systems and control theory [1], [2]. They can be used for example to design a controller that minimizes the effect of the perturbation to the controlled output variable, as in the classical $\mathcal{H}_\infty$ control [3]–[5] considering in particular the $\mathcal{L}_2-$gain. For linear systems these concepts are well understood [1], [3], [4]. Special attention has been paid in the literature to the particular case of $\mathcal{L}_2-$stability which for smooth non-linear systems can be characterized by a Hamilton-Jacobi Inequality [2], [6], [7]. In the linear case this inequality reduces to a more tractable matrix algebraic Riccati inequality. Furthermore, the powerful concept of Input-to-State Stability (ISS) is strongly related to $\mathcal{L}_\infty-$stability, and constitutes an important generalization [2], [6], [8]–[10].

Weighted homogeneous systems are an important class of non-linear systems since they generalize linear systems and can provide good local approximations for more general non-linear systems [11], [12]. Moreover, they have been used for the design of continuous and discontinuous controllers

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and observers, to induce e.g. convergence in finite time by imposing a negative homogeneity degree [9], [11], [13]–[24]. Apparently, it is important to understand the concepts of $\mathcal{L}_p-$stability and $\mathcal{L}_p-$gain for homogeneous systems. Since, in general, homogeneous systems may be non-smooth and their linearizations (when meaningful) are trivial, many of the standard results for non-linear systems [2], [6]–[8] cannot be used.

Despite of their importance, only few results about $\mathcal{L}_p-$stability of homogeneous systems have appeared in the literature, and they apply only to very specific classes of homogeneous systems. [9] shows that for smooth and standard homogeneous systems, with homogeneous weights equal to one and with non-negative homogeneity degree (see the formal definition below), having the state variable as output, the asymptotic stability of the origin of the unforced system implies $\mathcal{L}_p-$stability (for p sufficiently large) and ISS, both with linear gain. However, [9] does not discuss how to characterize and estimate the $\mathcal{L}_p-$gain and the ISS-gain. [25] considers a subclass of the systems treated in [9], which are affine in the input with constant input matrix, and have homogeneous output map. It is shown that internal Lyapunov stability of the unforced system implies $\mathcal{L}_2-$stability with finite $\mathcal{L}_2-$gain. The novelty consists in its characterization by means of a homogeneous Hamilton-Jacobi inequality.

Paper [17] covers a class of smooth weighted homogeneous systems of arbitrary homogeneity degree, which are affine in the input and have a homogeneous output map. It is shown that internal stability implies again $\mathcal{L}_2-$stability with finite $\mathcal{L}_2-$gain, and this can be characterized by using a homogeneous Hamilton-Jacobi Inequality. Moreover, internal Lyapunov stability is also shown to imply $\mathcal{L}_p-$stability (for sufficiently large p) with finite $\mathcal{L}_p-$gain (including $p = \infty$). However, these results are obtained by imposing strong restrictions on the homogeneous weights of the inputs, outputs and states. Again, estimation of the $\mathcal{L}_p-$gains is not discussed. [13] and [26] discuss ISS and other related properties for general weighted homogeneous systems, generalizing the results of [9] relating the internal stability of the unforced system and the ISS stability. However, [13], [26] do not consider $\mathcal{L}_p-$stability for any value of p. Also the linear ISS gain is not clearly stated. This latter issue is clarified in [27] for the more general version of geometric homogeneity.

The authors of [28], [29] use a homeomorphism to write the super-twisting algorithm (STA), a second order homogeneous non-linear system, in an almost linear form. Calculation of the $\mathcal{L}_2-$gain (i.e. $\mathcal{H}_\infty-$norm) for this almost linear system leads to a local gain, which is used to optimize the selection of parameters for the STA. In our previous work [30], the basic idea of the present paper is applied to the continuous super-twisting-like algorithm (CSTLA), which is extended in the examples of this paper. There the (globally defined) homogeneous $\mathcal{H}_\infty-$norm (homogeneous $\mathcal{L}_2-$gain) is calculated and used for the optimization of the gains of the CSTLA.

The objective of this paper is to consider $\mathcal{L}_p-$stability and ISS for arbitrary continuous weighted homogeneous systems, without imposing unnecessary restrictions on the homogeneous weights or degree. An interesting and surprising result of [9], [17], [25] is the linearity in the $\mathcal{L}_p-$stability and ISS for the class of homogeneous systems treated in those references, despite the fact that homogeneous systems can be highly non-linear. Our first observation is that for general homogeneous systems this is not possible, if the standard signal and vector norms are used. Therefore, we introduce *homogeneous vector and signal norms* and show that the induced homogeneous $\mathcal{L}_p-$stability and homogeneous ISS concepts are *linear* for general continuous homogeneous systems. This means that there are finite *constant* homogeneous $\mathcal{L}_p$ and homogeneous ISS gains, including the case of systems without memory. We characterize this homogeneous $\mathcal{L}_p-$stability and the homogeneous $\mathcal{L}_p-$gain in the general dynamic case by a homogeneous dissipation inequality. For systems affine in the input, this reduces to a homogeneous Hamilton-Jacobi inequality. Further, we show that for general dynamic homogeneous systems, asymptotic stability of the equilibrium point of the unforced system implies homogeneous $\mathcal{L}_p-$stability (for sufficiently large p) and ISS, obviously with finite linear gains. Using these results, we propose a method to estimate the value of the homogeneous $\mathcal{L}_p$ and ISS gains. We therefore extend all the results of [9], [17], [25], [30] to arbitrary continuous homogeneous systems.

In the present paper a new homogeneous $\mathcal{L}_p-$stability concept is introduced for an arbitrary continuous weighted homogeneous system, based on homogeneous $\mathcal{L}_p-$norms for input and output signals. Every stable homogeneous system has associated some homogeneous $\mathcal{L}_p-$gain with p sufficiently large that relates linearly and globally the homogeneous norms of input and output variables. This extends the well-known situation for linear systems. The so defined homogeneous $\mathcal{L}_p-$norms can then be used in the traditional manner for e.g. controller design to minimize the effect of perturbations, as it happens in the $\mathcal{H}_\infty-$control problem, or for parameter optimization, as illustrated in our previous work [30]. For non-homogeneous systems the homogeneous $\mathcal{L}_p-$norm can be calculated for the locally approximating homogeneous system, and its value corresponds to a local norm for the non-homogeneous system. This idea is a generalization of the use of linear systems and their gains to non-linear systems. Interestingly, the idea of using non-standard vector or signal norms to establish a relationship from ISS or iISS to $\mathcal{L}_2-$stability for general nonlinear systems has been used recently in [31]. However, the aims and results of that paper are rather different from our paper.

The structure of the paper is as follows: In Section II we introduce the weighted homogeneity for continuous homogeneous dynamics or input-output maps as well as some properties of homogeneous norms. Afterwards, we revisit the traditional concept of $\mathcal{L}_p-$stability and show that it is applicable to a homogeneous system only if the homogeneous weights of inputs and outputs are all equal. In order to extend the $\mathcal{L}_p-$stability to arbitrary homogeneous systems, in Section III we introduce the homogeneous $\mathcal{L}_p-$norm and the finite-gain homogeneous $\mathcal{L}_p-$stability. In Section IV we characterize homogeneous $\mathcal{L}_p-$stability with the help of a storage function and a corresponding homogeneous dissipation inequality. We further prove that every continuous homogeneous system is homogeneous $\mathcal{L}_p-$stable for some p large enough, if the unforced dynamics is asymptotically stable. Moreover, when the dynamics is affine in the input, the homogeneous dissipation inequality can be transformed into a homogeneous Hamilton-Jacobi Inequality. For a smooth non-linear system, if there exists a local homogeneous approximation, then the homogeneous $\mathcal{L}_p-$stability remains true locally. In Section V the special case of homogeneous $\mathcal{L}_\infty-$stability and homogeneous input-to-state stability (ISS) is brought up and proved. In Section VI the finite-gain homogeneous $\mathcal{L}_p-$stability of feedback and in cascade interconnected systems is studied for different values of p, and an extension of the classical small-gain theorem is obtained. In Section VII we compare our results with the previous literature involving $\mathcal{L}_p-$stability or ISS for homogeneous systems. In Section VIII we propose methods to calculate an upper estimate of the homogeneous $\mathcal{L}_p-$gain, using either the homogeneous dissipation inequality or the homogeneous Hamilton-Jacobi inequality. In Section IX we present three examples. First of all, we show that all continuous memoryless homogeneous input-output maps are finite-gain homogeneous $\mathcal{L}_p-$stable, and we provide a method to estimate its true value, which turns out to be independent of p. A numerical example is given to illustrate the procedure. The second example considers a scalar homogeneous system for which the homogeneous $\mathcal{L}_p-$gains can be derived analytically. In the third example we provide data collected for the continuous super-twisting like algorithm (CSTLA) as well as a comparison between the linear case, where the $\mathcal{H}_\infty$ norm is more thoroughly studied, and the non-linear cases. A detailed analysis of the homogeneous $\mathcal{H}_\infty-$norm (homogeneous $\mathcal{L}_2-$gain) for the CSTLA can be found in [30], whereas in this paper such results are extended for homogeneous $\mathcal{L}_p-$gain with $p \geq 2$. Finally, in Section X we draw our conclusions.

II. WEIGHTED HOMOGENEOUS SYSTEMS AND PROBLEM FORMULATION

A. CONTINUOUS WEIGHTED HOMOGENEOUS DYNAMICS

We consider a class of systems with inputs and outputs

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t), \quad u(t)) \\ y(t) = h(x(t), \quad u(t)) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ the input and $y(t) \in \mathbb{R}^o$ the output. We let $u(\cdot) \in \mathcal{U}$ where $\mathcal{U}$ denotes the space of measurable and locally essentially bounded functions from $\mathbb{R}$ to $\mathbb{R}^m$. We assume that the vector field $f(x, u)$ and the function $h(x, u)$ are continuous in x and u. Moreover, we assume that system (1) is homogeneous. To define this concept,

TABLE 1. Table of notations.

Symbols of functions:	
$\alpha(\cdot)$	$\mathcal{K}_\infty$ function, see Lemma 1
$\beta(\cdot,\cdot)$	$\mathcal{KL}$ function, see (36)
Symbols of weighted homogeneity:	
r	weight vector
$\mathbf{1}_n$	$n-$dimensional vector whose elements are all 1
κ	positive scaling number, used in a dilation
ℓ	positive number
$\nu_\kappa^{r_x}(x)$	dilation of vector x w.r.t. weight vector r_x, defined right before Definition 1
$d = -r_t$	homogeneous degree of a system, see Definition 1
Symbols for norms:	
$\lVert x \rVert_q$	classical $q-$norm of a vector x
$f \in \mathcal{L}_p^n$	$n-$dimensional signal $f(\cdot)$ in $\mathcal{L}_p-$space, see Section II-C
$f \in \mathcal{L}_{\text{pe}}^n$	$n-$dimensional signal $f(\cdot)$ in extended $\mathcal{L}_p-$space, see Section II-C
$\lVert f \rVert_{\mathcal{L}_p}$	classical $\mathcal{L}_p-$norm of a signal $f(\cdot)$, see (10)
$\lVert x \rVert_{r_x,q}$	homogeneous $q-$norm of a vector x w.r.t. the weight vector r_x, see (4)
$f \in \mathcal{L}_{r_f,p}^n$	$n-$dimensional signal $f(\cdot)$ in homogeneous $\mathcal{L}_p-$space, see Section III-A
$f \in \mathcal{L}_{r_f,\text{pe}}^n$	$n-$dimensional signal $f(\cdot)$ in extended homogeneous $\mathcal{L}_p-$space, see Section III-A
$\lVert f \rVert_{r_f,\mathcal{L}_p}$	homogeneous $\mathcal{L}_p-$norm of a signal $f(\cdot)$ w.r.t. the weight vector r_f
$\lVert f \rVert_{r_f,\mathcal{L}_\infty}$	homogeneous $\mathcal{L}_\infty-$norm of a signal $f(\cdot)$ w.r.t. the weight vector r_f, see (15)
Symbols for $\mathcal{L}_p-$gains:	
γ_p	upper estimate of the (homogeneous) $\mathcal{L}_p-$gain of a $\mathcal{L}_p-$stable system, see (11) (Definition 5)
$\gamma_p(G)$	$\mathcal{L}_p-$gain of an $\mathcal{L}_p-$stable system, see Section II-C
$\gamma_{\text{ph}}(G)$	homogeneous $\mathcal{L}_p-$gain of an $\mathcal{L}_p-$stable system, see Definition 5
γ_{iss}	upper estimate of the homogeneous ISS-gain of a system, see Definition 8
b_p	(homogeneous) bias, see (11) (Definition 5)
Γ	ratio of $\mathcal{L}_p-$norm of some particular output over input, see Example 1
ζ	function which allows to calculate an upper estimate of the $\mathcal{L}_p-$gain from dissipation inequality, see Proposition 2
$\mathcal{J}, \mathcal{Q}$	two functions which allow to calculate an upper estimate of the $\mathcal{L}_{\text{ph}}-$gain from dissipation inequality when the system is affine in input, see Proposition 3
$\gamma^\dagger$	local $\mathcal{L}_2-$gain, as designed in [28] for super-twisting algorithm
$\gamma^\star$	calculated upper estimate of the $\mathcal{L}_{\text{ph}}-$gain from dissipation inequality, see Proposition 2 or Proposition 3
Symbols for dissipation inequality:	
$V(x)$	homogeneous storage function, see Definition 6
J	homogeneous value function; $J \leq 0$ represents a homogeneous dissipation inequality, see Definition 6
$u^\star$	input that maximizes the homogeneous value function J, see Section IV-B
ϵ	a small positive number
Symbols in systems:	
Σ	state-space model of a system, as given in (1)
G	input-output map; for homogeneous G see Definition 2
$f(x,u)$	vector field of a homogeneous system Σ
$h(x,u)$	output of a homogeneous system Σ
$g(x)$	the input matrix when the system is affine in the input

with [15] introduce for a vector $x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$ the *weight vector* $r_x = (r_{x_1}, \ldots, r_{x_n})^\top$, which is an n-tuple of positive real numbers, the one-parameter family of *dilations* $\nu_\kappa^{r_x}$ (associated with weight r_x) for all $x \in \mathbb{R}^n$ and $\kappa \geq 0$, is given by

$$\nu_\kappa^{r_x}(x) \triangleq (\kappa^{r_{x_1}} x_1, \ldots, \kappa^{r_{x_n}} x_n)^\top .$$

This way, r_{x_i} is the *homogeneous weight* of x_i.

Definition 1 (Weighted Homogeneous System [15]): We call system (1) *homogeneous of degree d if there exists* $r_{x_i} > 0$, $r_{u_i} > 0$, *for* $i = 1, \ldots, n$, $d \in (-\min_i r_{x_i}, \infty)$ *and* $r_{y_i} > 0$, *for* $i = 1, \ldots, o$, *s.t.* $\forall x \in \mathbb{R}^n$, $\forall u \in \mathbb{R}^m$, $\forall \kappa > 0$

$$f_i(\nu_\kappa^{r_x}(x), \nu_\kappa^{r_u}(u)) = \kappa^{d+r_{x_i}} f_i(x,u), \quad \forall i = 1, \ldots, n,$$
$$h_j(\nu_\kappa^{r_x}(x), \nu_\kappa^{r_u}(u)) = \kappa^{r_{y_j}} h_j(x,u), \quad \forall j = 1, \ldots, o.$$

In the rest of the paper, $\min_i r_i$ will be denoted as $\min r$ and $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ as $(x,u) \in \mathbb{R}^{n+m}$ without ambiguity. This definition is an extension of the one given in [16] to systems with outputs (see also [17]). Homogeneity implies that $f(0,0) = 0$ and $h(0,0) = 0$, i.e. the origin is an equilibrium point when $u = 0$. From Definition 1 and the weight associated to the time variable $r_t = -d$, we obtain that

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(x,u) \quad \Leftrightarrow$$
$$\frac{\kappa^{r_{x_i}} \mathrm{d}x_i}{\kappa^{r_t} \mathrm{d}t} = \kappa^{d+r_{x_i}} f_i(x,u) = f_i(\nu_\kappa^{r_x}(x), \nu_\kappa^{r_u}(u)).$$

If we denote the trajectory of (1) by $x(t) = \varphi(t, x_0, u(\cdot))$, where φ is the *state transition map*, defined for each $x_0 \in \mathbb{R}^n$ and each $u(\cdot) \in \mathcal{U}$, and satisfying $\varphi(0, x_0, u(\cdot)) = x_0$, then the previous relation implies that [16], [18], [32]

$$\varphi(t, \nu_\kappa^{r_x}(x_0), \nu_\kappa^{r_u}(u(\kappa^{-r_t}\cdot))) = \nu_\kappa^{r_x}(\varphi(\kappa^{-r_t}t, x_0, u(\cdot)))$$
$$h_j(\nu_\kappa^{r_x}(x), \nu_\kappa^{r_u}(u)) = \kappa^{r_{y_j}} h_j(x,u). \tag{2}$$

This means that if the initial state x_0 is dilated as $\nu_\kappa^{r_x}(x_0)$ and the input signal $u(\cdot)$ is not only dilated as $\nu_\kappa^{r_u}(u(\kappa^{-r_t}\cdot))$ in amplitude, but also its time evolution is scaled, then the resulting state trajectory is scaled in amplitude as $\nu_\kappa^{r_x}(x(\kappa^{-r_t}t))$, with the same time scaling. A similar effect applies for the output signal $y(\cdot)$.

For every initial condition $x_0 \in \mathbb{R}^n$, system (1) defines, in principle, an input-output map G_{x_0}: Substituting an input signal $u(\cdot)$ and solving the differential equations for the initial condition x_0, one obtains the state trajectory $x(\cdot)$ and the corresponding output signal $y(\cdot)$. In general, additional conditions are necessary to ensure that for every input $u(\cdot)$ there exist a state trajectory $x(\cdot)$ and an output signal $y(\cdot)$ (see e.g. [2]).

In general, we can consider homogeneous (time-invariant and causal) input-output maps [2].

Definition 2 (Homogeneous Input-Output Map): An input-output map G is called homogeneous of degree $d = -r_t \in \mathbb{R}$, *if for each input* $u(\cdot) \in \mathcal{U}$ *and its corresponding output* $y(\cdot) = G(u(\cdot))$ *it satisfies*

$$G\left(\nu_\kappa^{r_u}\left(u\left(\kappa^{-r_t}\cdot\right)\right)\right) = \nu_\kappa^{r_y}\left(y\left(\kappa^{-r_t}\cdot\right)\right), \ \forall \kappa > 0. \tag{3}$$

In particular, linear input-output maps are homogeneous of degree $r_t = 0$ with weights $r_u = \mathbf{1}_m \triangleq (1, \ldots, 1) \in \mathbb{R}^m$, $r_y = \mathbf{1}_o$. Moreover, the input-output map G_{x_0} obtained from a homogeneous state space realization (1) is homogeneous only for $x_0 = 0$. The corresponding fact for linear time invariant systems $\dot{x} = Ax + Bu$, $y = Cx$ is well-known, since the output of the input-output map $y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-s)}Bu(s)\mathrm{d}s$ is linear in the input $u(\cdot)$ only for $x_0 = 0$.

B. HOMOGENEOUS NORMS FOR VECTORS

When working with (weighted) homogeneous systems, one is naturally led to consider homogeneous norms.

Definition 3 (r−Homogeneous q−Norm [15], [16]): A r_x−homogeneous q−norm *(qh−norm for short) for a vector* $x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$ *is a map* $\|\cdot\|_{r_x,q} : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$, *where for any* $q \in [1, \infty]$

$$\|x\|_{r_x,q} \triangleq \left(\sum_{i=1}^{n} |x_i|^{\frac{q}{r_{x_i}}}\right)^{\frac{1}{q}}, \tag{4}$$

in particular, $\|x\|_{r_x,\infty} \triangleq \max_i \left\{|x_i|^{\frac{1}{r_{x_i}}}\right\}$. *The set* $\mathcal{S}_{r_x,q} = \left\{x \in \mathbb{R}^n \mid \|x\|_{r_x,q} = 1\right\}$ *is the corresponding homogeneous unit sphere.*

The r_x−homogeneous norm $\|\cdot\|_{r_x,q}$ is r_x−homogeneous of degree 1 and it is positive definite. If $q \geq \max r_x$, the qh−norm $\|x\|_{r_x,q}$ is continuously differentiable on $\mathbb{R}^n \setminus \{0\}$. This is not needed in this paper. However, $\|\cdot\|_{r_x,q}$ is in general not a norm in the usual sense, since it is not $\mathbf{1}$−homogeneous in the classical sense, i.e. with $r_x = \mathbf{1}_n$. When $r_x = \mathbf{1}_n$ the r_x−homogeneous q−norm becomes the usual q−norm in $\mathbb{R}^n$. In that case, for $q \geq \max r_x = 1$ the triangle inequality is valid, i.e. $\|x+y\|_{\mathbf{1}_n,q} \leq \|x\|_{\mathbf{1}_n,q} + \|y\|_{\mathbf{1}_n,q}$. In the general case, as it will be shown later in Lemma 2, the triangle inequality will be replaced by the additive inequality (9).

Remark 1 (Effect of Homogeneous Weight Scaling on Homogeneous q−Norm): It is well-known that if system (1) *is homogeneous of degree d with weight vectors* r_x, r_u, r_y, *then it is also homogeneous of degree* λd *for any* $\lambda > 0$ *with weight vectors* λr_x, λr_u, λr_y. *For most studies about homogeneous systems, scaling the homogeneous weights by* $\lambda > 0$ *has no impact on the results. Yet in this paper, such scaling does matter. E.g. applying such scaling, the* λr−*homogeneous* q−*norm from* (4) *is related to the original* r−*homogeneous* q/λ−*norm by*

$$\|x\|_{\lambda r_x,q} = \|x\|_{r_x,q/\lambda}^{1/\lambda}, \tag{5}$$

for $0 \leq \lambda \leq q$. *Usually, there is no need to relate a* λr−*homogeneous* q−*norm back to a* r−*homogeneous* q/λ−*norm, so any scaling* $\lambda > 0$ *on weight vectors is allowed. Therefore, in this paper, it is important to fix the homogeneous weights and degree of the system at the outset, for the results to be consistent. In the main text of the paper, we will not allow such scaling. But we will discuss the effect of the scaling in some remarks for the interested reader.*

Remark 2 (Companion Vector and Relationship Between r−Homogeneous q−Norm and q−Norm): Note that for $x \in \mathbb{R}^n$ and (weight) $r_x \in \mathbb{R}^n_{>0}$, we may define the companion vector *as*

$$x^{\frac{1}{r_x}} \triangleq \left[\lceil x_1 \rfloor^{\frac{1}{r_{x_1}}}, \ldots, \lceil x_n \rfloor^{\frac{1}{r_{x_n}}} \right]^\top ,$$

where $\lceil \cdot \rfloor^s = |\cdot|^s \mathrm{sign}(\cdot)$, $s \in \mathbb{R}$, defines the sign preserving power. Note that the mapping $x \mapsto x^{\frac{1}{r_x}}$ is a homeomorphism, and if $\min r_x \leq 1$ it is a diffeomorphism. For homogeneous norms it is possible to associate the qh−norm (4) of x with the q−norm of its companion vector, i.e.

$$\|x\|_{r_x,q} = \left\| x^{\frac{1}{r_x}} \right\|_q . \quad (6)$$

This bears a relationship with the homeomorphic change of coordinates in [31, Definition 7].

Remark 3 (Equivalence Between r−Homogeneous q−Norms): All r−homogeneous q−norms, with the same weighting vector r, are equivalent *in the sense that if $\|\cdot\|_{r_x,\alpha}$ and $\|\cdot\|_{r_x,\beta}$ are two different homogeneous q−norms, with $\alpha \geq 1$, $\beta \geq 1$, then there exist positive constants c_1 and c_2 such that for all $x \in \mathbb{R}^n$*

$$c_1\|x\|_{r_x,\beta} \leq \|x\|_{r_x,\alpha} \leq c_2\|x\|_{r_x,\beta} . \quad (7)$$

This is easily seen using relation (6). Since q−norms are equivalent [1], i.e. $c_1\|x\|_\beta \leq \|x\|_\alpha \leq c_2\|x\|_\beta$, then

$$c_1\|x\|_{r_x,\beta} = c_1 \left\| x^{\frac{1}{r_x}} \right\|_\beta \leq \|x\|_{r_x,\alpha} = \left\| x^{\frac{1}{r_x}} \right\|_\alpha \leq c_2 \left\| x^{\frac{1}{r_x}} \right\|_\beta = c_2\|x\|_{r_x,\beta} .$$

Interestingly, the constants c_1 and c_2 relating two homogeneous q−norms are the same as the ones for the q−norms.

A natural question arises about the relationship between different homogeneous norms with possibly different weight vectors. The following Lemma clarifies this (and is a special case of [27, Lemma 9]).

Lemma 1 (Relationship Between r−Homogeneous q−Norm With Different r): Consider two homogeneous norms $\|\cdot\|_{r_1,\alpha}$ and $\|\cdot\|_{r_2,\beta}$ with (possibly) different weight vectors r_1 and r_2. Then, there exist two $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\alpha_1(\|x\|_{r_2,\beta}) \leq \|x\|_{r_1,\alpha} \leq \alpha_2(\|x\|_{r_2,\beta}) , \ \forall x \in \mathbb{R}^n. \quad (8)$$

Proof: Since homogeneous norms are continuous, positive definite and radially unbounded, it follows from a classical result [6, Lemma 4.3] that there exist $\mathcal{K}_\infty$ functions $\mu_{1,\alpha}(\cdot)$, $\mu_{2,\alpha}(\cdot)$ and $\mu_{1,\beta}(\cdot)$, $\mu_{2,\beta}(\cdot)$ such that $\forall x \in \mathbb{R}^n$

$$\mu_{1,\alpha}(\|x\|_\beta) \leq \|x\|_{r_1,\alpha} \leq \mu_{2,\alpha}(\|x\|_\beta) ,$$
$$\mu_{1,\beta}(\|x\|_\beta) \leq \|x\|_{r_2,\beta} \leq \mu_{2,\beta}(\|x\|_\beta) .$$

Using the properties of $\mathcal{K}_\infty$ functions it follows that

$$\mu_{1,\alpha} \circ \mu_{2,\beta}^{-1}(\|x\|_{r_2,\beta}) \leq \|x\|_{r_1,\alpha} \leq \mu_{2,\alpha} \circ \mu_{1,\beta}^{-1}(\|x\|_{r_2,\beta}) .$$

This establishes the result. □

In particular, a relationship between q−norms and r−homogeneous q−norms is obtained from (8) by setting $r_1 = \mathbf{1}_n$ or $r_2 = \mathbf{1}_n$. Relation (5) represents one example of this general relation, when $r_2 = \lambda r_1$ or vice versa. Note that r−homogeneous q−norms w.r.t. different weights r are usually not equivalent (different from Remark 3), since in general $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are not linear functions (e.g. (5) in Remark 1).

The r−homogeneous q−norm of a vector in $\mathbb{R}^n$ satisfies the triangle inequality when $r = \mathbf{1}_n$ and $q \geq 1$. For other cases we have instead the following additive inequality.

Lemma 2 (Additive Inequality for Homogeneous q−Norm): The r−homogeneous q−norm satisfies the following inequality for two vectors $x, y \in \mathbb{R}^n$ and $q \geq 1$:

$$\|x + y\|_{r,q} \leq \max\left\{1, 2^{\frac{1}{\min r} - \frac{1}{q}}\right\} \left(\|x\|_{r,q} + \|y\|_{r,q}\right) . \quad (9)$$

Proof: Using the definition, we have

$$\begin{aligned}
\|x + y\|_{r,q} &= \left(\sum_{i=1}^n |x_i + y_i|^{\frac{q}{r_i}}\right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^n (|x_i| + |y_i|)^{\frac{q}{r_i}}\right)^{\frac{1}{q}} \\
&\leq \left(\sum_{i=1}^n \max\left\{1, 2^{\frac{q}{r_i}-1}\right\} |x_i|^{\frac{q}{r_i}} + \sum_{i=1}^n \max\left\{1, 2^{\frac{q}{r_i}-1}\right\} |y_i|^{\frac{q}{r_i}}\right)^{\frac{1}{q}} \\
&\leq \left(\max\left\{1, 2^{\frac{q}{\min r}-1}\right\}\right)^{\frac{1}{q}} \left(\sum_{i=1}^n |x_i|^{\frac{q}{r_i}} + \sum_{i=1}^n |y_i|^{\frac{q}{r_i}}\right)^{\frac{1}{q}} \\
&\leq \max\left\{1, 2^{\frac{1}{\min r} - \frac{1}{q}}\right\} \left(\|x\|_{r,q} + \|y\|_{r,q}\right) ,
\end{aligned}$$

where the second inequality comes from (72) and the fourth inequality from (71). □

Remark 4 (Additive Inequality and Triangle Inequality for qh−Norm): If $\min r > 1$, then there exists some $q \in [1, \min r]$, s.t. inequality (9) is again in the form of the triangle inequality, i.e. $\|x + y\|_{r,q} \leq \|x\|_{r,q} + \|y\|_{r,q}$ for all $q \in [1, \min r]$. And this is always possible by scaling the weight vectors by $\lambda > \max\{1, 1/\min r\}$. Note that the value of the qh−norm is changed with such λ−scaling on weight vector r as shown in Remark 1.

C. $\mathcal{L}_p$−STABILITY AND FINITE $\mathcal{L}_p$−GAIN OF SYSTEMS

Usually, for analysis of the behavior of input-output maps, $\mathcal{L}_p$ signal spaces and their extensions are considered [1], [2]. For $p \geq 1$ the set $\mathcal{L}_p[0,\infty) = \mathcal{L}_p$ consists of all functions $f : \mathbb{R}^+ \to \mathbb{R}$ ($\mathbb{R}^+ = [0, \infty)$), which are measurable and satisfy $\int_0^\infty |f(t)|^p \mathrm{d}t < \infty$. Using the truncation f_T of f to the interval $[0, T]$, the extended $\mathcal{L}_{pe}$ space consists of all functions f such that $f_T \in \mathcal{L}_p$ for all $0 \leq T < \infty$. For multivariable signals $f : \mathbb{R}^+ \to \mathbb{R}^n$ the signal space $\mathcal{L}_p^n$ consists of all measurable signals such that

$$\int_0^\infty \|f(t)\|^p \mathrm{d}t < \infty ,$$

where $\|\cdot\|$ is any norm in $\mathbb{R}^n$. In this signal space

$$\|f\|_{\mathcal{L}_p} = \left(\int_0^\infty \|f(t)\|^p \mathrm{d}t\right)^{\frac{1}{p}} \tag{10}$$

defines a signal norm, and $\mathcal{L}_p^n$ becomes a Banach space for any $p \geq 1$. The extended space $\mathcal{L}_{\mathrm{pe}}^n$ is defined similarly. However, it is not a Banach space [1], [2].

If we consider an input-output map $G : \mathcal{L}_{\mathrm{pe}}^m \to \mathcal{L}_{\mathrm{pe}}^r$, we say that it is $\mathcal{L}_p$*−stable*[2] if

$$u \in \mathcal{L}_p^m \Rightarrow y \in \mathcal{L}_p^r .$$

The map G is said to have *finite* $\mathcal{L}_p$*−gain* if there exist non-negative constants γ_p and b_p such that

$$\|(G(u))_T\|_{\mathcal{L}_p} \leq \gamma_p \|u_T\|_{\mathcal{L}_p} + b_p , \quad \forall T \geq 0,\ u \in \mathcal{L}_{\mathrm{pe}}^m .$$

If G has finite $\mathcal{L}_p$−gain then it is automatically $\mathcal{L}_p$−stable [2]. Taking $T \to \infty$ and restricting $u \in \mathcal{L}_p^m \subset \mathcal{L}_{\mathrm{pe}}^m$ we have

$$\|(G(u))\|_{\mathcal{L}_p} \leq \gamma_p \|u\|_{\mathcal{L}_p} + b_p , \quad \forall u \in \mathcal{L}_p^m . \tag{11}$$

G is said to have *finite* $\mathcal{L}_p$*−gain with zero bias* if b_p in (11) can be taken equal to zero. If G has finite $\mathcal{L}_p$−gain, then the $\mathcal{L}_p$−gain of G is defined as

$$\gamma_p(G) \triangleq \inf\left\{\gamma_p \mid \exists\, b_p \text{ such that (11) holds}\right\} .$$

For the input-output maps G_{x_0} obtained from system (1) the previous definitions also apply [2], where for each G_{x_0} the constant $b_p(x_0)$ depends on the initial state x_0, but not on γ_p.

D. ON THE LIMITATION OF $\mathcal{L}_p$−STABILITY AND FINITE $\mathcal{L}_p$−GAIN FOR HOMOGENEOUS SYSTEMS

Consider a scalar (SISO) homogeneous input-output map $G : \mathcal{L}_{\mathrm{pe}} \to \mathcal{L}_{\mathrm{pe}}$, of homogeneous degree $d = -r_t$ and weights $r_u > 0$ and $r_y > 0$, for the input and output respectively. Select an input signal $u \in \mathcal{L}_p$ and suppose that $y = G(u) \in \mathcal{L}_p$. According to (3), applying the scaled input $\tilde{u}(\cdot) = \nu_\kappa^{r_u}(u(\kappa^{-r_t}\cdot))$ for any $\kappa > 0$, one obtains the scaled output $\tilde{y}(\cdot) = \nu_\kappa^{r_y}(y(\kappa^{-r_t}\cdot))$. The $\mathcal{L}_p$−norms of the scaled input and output are given by

$$\begin{aligned}\|\tilde{u}\|_{\mathcal{L}_p} &= \left(\int_0^\infty |\tilde{u}(t)|^p \mathrm{d}t\right)^{\frac{1}{p}} = \left(\int_0^\infty |\kappa^{r_u} u(\kappa^{-r_t} t)|^p \mathrm{d}t\right)^{\frac{1}{p}} \\ &= \kappa^{r_u + \frac{r_t}{p}} \left(\int_0^\infty |u(s)|^p \mathrm{d}s\right)^{\frac{1}{p}} = \kappa^{r_u + \frac{r_t}{p}} \|u\|_{\mathcal{L}_p} , \\ \|\tilde{y}\|_{\mathcal{L}_p} &= \kappa^{r_y + \frac{r_t}{p}} \|y\|_{\mathcal{L}_p} .\end{aligned}$$

If $r_u \neq r_y$, it follows that an inequality such as (11) cannot be satisfied globally. Suppose $u \in \mathcal{L}_p$. Then $\tilde{u} \in \mathcal{L}_p$ since $\|\tilde{u}\|_{\mathcal{L}_p} = \kappa^{r_u + \frac{r_t}{p}} \|u\|_{\mathcal{L}_p} < \infty$ for any finite $\kappa > 0$. Thus also (11) is satisifed with the dilated input $\tilde{u}$ and output $\tilde{y}$

$$\|\tilde{y}\|_{\mathcal{L}_p} \leq \gamma_p \|\tilde{u}\|_{\mathcal{L}_p} + b_p ,$$

implying

$$\kappa^{r_y + \frac{r_t}{p}} \|y\|_{\mathcal{L}_p} \leq \kappa^{r_u + \frac{r_t}{p}} \gamma_p \|u\|_{\mathcal{L}_p} + b_p ,$$

which gives

$$\|y\|_{\mathcal{L}_p} \leq \kappa^{r_u - r_y} \gamma_p \|u\|_{\mathcal{L}_p} + \kappa^{-r_y - \frac{r_t}{p}} b_p .$$

Compared with the original inequality (11), the effective gain and bias are both multiplied with κ−related terms. Now let $b_p = 0$ or $pr_y = d > 0$. Then when $r_y > r_u$, $\kappa \to \infty$ (large signal) or when $r_y < r_u$, $\kappa \to 0$ (small signal), quantity γ_p in the above inequality must be infinite such that the original inequality (11) stands. Consequently, no finite γ_p can satisfy (11) globally. On the other hand, when $b_p \neq 0$ and $pr_y \neq d$, for similar reasons there does not exist a global constant b_p, such that (11) holds.

In order to show that the classical $\mathcal{L}_p$−gain is not suitable for homogeneous systems, we inspect a simple homogeneous scalar system as an example.

Example 1: Consider the homogeneous scalar system

$$\dot{x} = -k\lceil x \rfloor^{\frac{1}{z}} + bu, \quad y = cx \tag{12}$$

with $z \in \mathbb{R}^+$*. The weight vectors are* $r_x = z$, $r_u = 1$, $r_y = z$, *and the homogeneous degree is* $d = -r_t = 1 - z$*. When the system evolves from the origin, an input* $u_e(\cdot)$ *not identical to zero will result in an output* $y_e(\cdot)$*, recording the ratio*

$$\Gamma_e = \frac{\|y_e\|_{\mathcal{L}_p}}{\|u_e\|_{\mathcal{L}_p}} .$$

As shown above, from homogeneity the output of system (12) *will be dilated to* $\tilde{y}_e(\tilde{t})$ *with the dilated input* $\tilde{u}_e(\tilde{t})$ *together with scaled time. Then the above ratio for this dilated input and output is*

$$\Gamma_{\kappa e} = \frac{\|\tilde{y}_e\|_{\mathcal{L}_p}}{\|\tilde{u}_e\|_{\mathcal{L}_p}} = \kappa^{z-1}\gamma_e$$

when

- $z < 1$ *Corresponding to* $d > 0$*, if the input is scaled smaller, i.e.* $\kappa < 1$*, the ratio* $\Gamma_{\kappa e}$ *will grow larger. And as* $\kappa \to 0$*, the ratio* $\Gamma_{\kappa e}$ *will grow unbounded.*
- $z > 1$ *Corresponding to* $d < 0$*, the ratio* $\Gamma_{\kappa e}$ *behaves conversely, namely being smaller with smaller input and vice versa.*
- $z = 1$ *The linear case corresponds to* $d = 0$*. The ratio* $\Gamma_{\kappa e}$ *is constant under linear scaling.*

The ratio $\Gamma_{\kappa e}$ *can simply be reflected as the gain constant in* (11)*, since*

$$\gamma_p \geq \sup_{u \in \mathcal{L}_p} \left(\frac{\|y\|_{\mathcal{L}_p}}{\|u\|_{\mathcal{L}_p}} - \frac{b_p(0)}{\|u\|_{\mathcal{L}_p}}\right) .$$

Thus the classical $\mathcal{L}_p$*−gain can be finite only if* $r_y = r_u$ *(*$z = 1$ *for system* (12)*). Note that for LTI systems and initial state at the origin (*$x_0 = 0$*), the bias becomes* $b_p(0) = 0$ *[2].*

Previous works, i.e. [17], [25], in the context of the H_∞−control problem have considered the finite $\mathcal{L}_2$−gain for the restricted class of homogeneous systems, for which $r_u = \ell \mathbf{1}_m$, $r_y = \ell \mathbf{1}_o$, for $\ell = d + \min r_x > 0$.

In view of a proper formulation and solution of, for example, the $\mathcal{H}_\infty$−control problem for homogeneous systems, the objective of the present paper is to show that it is possible to define an appropriate concept of finite $\mathcal{L}_p$−gain for every internally stable homogeneous system. Although this is possible for LTI systems, as it is well-known, it is not true for arbitrary non-linear systems. Moreover, we provide a characterization in terms of homogeneous storage functions, which also allow the calculation of the finite $\mathcal{L}_p$−gain.

III. HOMOGENEOUS $\mathcal{L}_p$−STABILITY FOR HOMOGENEOUS SYSTEMS

Similar to the situation in finite dimensional vector spaces, where homogeneity leads naturally to the introduction of homogeneous norms (see Section II-B), we introduce homogeneous norms for spaces of signals, which lead directly to the desired concept of finite gain of homogeneous systems.

A. SPACES OF SIGNALS WITH HOMOGENEOUS NORM

For multivariable signals $f : \mathbb{R}^+ \to \mathbb{R}^n$ consider the signal space $\mathcal{L}^n_{r_f,p}$ consisting of all measurable signals such that

$$\int_0^\infty \|f(t)\|^p_{r_f,q} \mathrm{d}t < \infty ,$$

where $\|\cdot\|_{r_f,q}$ is a homogeneous norm (4) in $\mathbb{R}^n$ with weight vector r_f, assuming $p \geq 1$, $q \geq 1$.

Lemma 3 $\mathcal{L}^n_{r,p}$−space is a linear signal space, i.e. if $f(\cdot), g(\cdot) \in \mathcal{L}^n_{r,p}$ then $af(\cdot) + bg(\cdot) \in \mathcal{L}^n_{r,p}$ for any $a, b \in \mathbb{R}$.

Proof: For the proof we will make use of the generalization of the weak triangle inequality [33], i.e. for all $x, y \in [0, \infty)$

$$\alpha(x+y) \leq \max\{\alpha(2x), \alpha(2y)\} \leq \alpha(2x) + \alpha(2y), \quad (13)$$

where α is a class $\mathcal{K}$ function defined in $[0, \infty)$. Consider

$$\begin{aligned}
&\int_0^\infty \|af(t) + bg(t)\|^p_{r,q} \mathrm{d}t \\
&= \int_0^\infty \left(\sum_{i=1}^n |af_i(t) + bg_i(t)|^{\frac{q}{r_i}}\right)^{\frac{p}{q}} \mathrm{d}t \\
&\leq \int_0^\infty \left(\sum_{i=1}^n (|af_i(t)| + |bg_i(t)|)^{\frac{q}{r_i}}\right)^{\frac{p}{q}} \mathrm{d}t \\
&\leq \int_0^\infty \left(\sum_{i=1}^n |2a|^{\frac{q}{r_i}} |f_i(t)|^{\frac{q}{r_i}} + \sum_{i=1}^n |2b|^{\frac{q}{r_i}} |g_i(t)|^{\frac{q}{r_i}}\right)^{\frac{p}{q}} \mathrm{d}t \\
&\leq \int_0^\infty \left(A\sum_{i=1}^n |f_i(t)|^{\frac{q}{r_i}} + B\sum_{i=1}^n |g_i(t)|^{\frac{q}{r_i}}\right)^{\frac{p}{q}} \mathrm{d}t \\
&\leq (2A)^{\frac{p}{q}} \int_0^\infty \left(\sum_{i=1}^n |f_i(t)|^{\frac{q}{r_i}}\right)^{\frac{p}{q}} \mathrm{d}t \\
&\quad + (2B)^{\frac{p}{q}} \int_0^\infty \left(\sum_{i=1}^n |g_i(t)|^{\frac{q}{r_i}}\right)^{\frac{p}{q}} \mathrm{d}t \\
&= (2A)^{\frac{p}{q}} \int_0^\infty \|f(t)\|^p_{r,q} \mathrm{d}t + (2B)^{\frac{p}{q}} \int_0^\infty \|g(t)\|^p_{r,q} \mathrm{d}t
\end{aligned}$$

is finite, where $A = \max_i \left\{|2a|^{\frac{q}{r_i}}\right\}$ and $B = \max_i \left\{|2b|^{\frac{q}{r_i}}\right\}$. Note that several times we have applied inequality (13) to the power function $|x|^r$, $r > 0$. This concludes the proof for finite p. For $p = \infty$ we have similarly

$$\begin{aligned}
&\sup_{t\geq 0} \|af(t) + bg(t)\|_{\tau,q} \\
&= \sup_{t\geq 0} \left(\sum_{i=1}^n |af_i(t) + bg_i(t)|^{\frac{q}{\tau_i}}\right)^{\frac{1}{q}} \\
&\leq \sup_{t\geq 0} \left(\sum_{i=1}^n (|af_i(t)| + |bg_i(t)|)^{\frac{q}{\tau_i}}\right)^{\frac{1}{q}} \\
&\leq \left(\sup_{t\geq 0}\left[\sum_{i=1}^n |2a|^{\frac{q}{\tau_i}} |f_i(t)|^{\frac{q}{\tau_i}}\right]\right. \\
&\quad \left. + \sup_{t\geq 0}\left[\sum_{i=1}^n |2b|^{\frac{q}{\tau_i}} |g_i(t)|^{\frac{q}{\tau_i}}\right]\right)^{\frac{1}{q}} \\
&\leq \left(A \sup_{t\geq 0}\left[\sum_{i=1}^n |f_i(t)|^{\frac{q}{\tau_i}}\right] + B \sup_{t\geq 0}\left[\sum_{i=1}^n |g_i(t)|^{\frac{q}{\tau_i}}\right]\right)^{\frac{1}{q}} \\
&= (2A)^{\frac{1}{q}} \sup_{t\geq 0} \|f(t)\|_{\tau,q} + (2B)^{\frac{1}{q}} \sup_{t\geq 0} \|g(t)\|_{\tau,q} < \infty .
\end{aligned}$$

□

Note that the signal space $\mathcal{L}^n_{r_f,\infty} = \mathcal{L}^n_\infty$. In this (linear) signal space we define a *homogeneous signal norm*

Definition 4 (r−Homogeneous $\mathcal{L}_p$−norm): An r_f−homogeneous $\mathcal{L}_p$−norm ($\mathcal{L}_{ph}$−norm for short) for the signal $f(\cdot) \in \mathcal{L}^n_{r_f,p}$, with $q \geq 1$, and $1 \leq p < \infty$, is given by

$$\|f(\cdot)\|_{r_f,\mathcal{L}_p} = \left(\int_0^\infty \|f(t)\|^p_{r_f,q} \mathrm{d}t\right)^{\frac{1}{p}} = \left(\int_0^\infty \left(\sum_{i=1}^n |f_i(t)|^{\frac{q}{r_{fi}}}\right)^{\frac{p}{q}} \mathrm{d}t\right)^{\frac{1}{p}} . \quad (14)$$

For $p = \infty$

$$\|f(\cdot)\|_{r_f,\mathcal{L}_\infty} = \sup_{t\geq 0} \|f(t)\|_{r_f,q} . \quad (15)$$

Note that $\|\cdot\|_{r,\mathcal{L}_p}$ is r−homogeneous of degree 1 with weight vector r, i.e. $\left\|\upsilon^{r_f}_\kappa(f(\cdot))\right\|_{r_f,\mathcal{L}_p} = \kappa\|f(\cdot)\|_{r_f,\mathcal{L}_p}$. Yet in general it is not a norm in the usual sense and the signal space $\mathcal{L}^n_{r_f,p}$ is not a Banach space for $r_f \neq c\mathbf{1}_n$, $c > 0$. The extended space $\mathcal{L}^n_{r_f,\mathrm{pe}}$ is defined similarly. Note that q in (14) does not need to be equal to p. This is also the case in the definition of a classical $\mathcal{L}_p$ norm [2]. Since finite dimensional r−homogeneous q−norms are equivalent for different q (see Remark 3), different choice of q does not alter the signal space $\mathcal{L}^n_{r_f,p}$.

Remark 5 (Effect of Weight Scaling on $\mathcal{L}_{ph}$−Norms): Similar to Remark 1, scaling the homogeneous degree $d = -r_t$ and the weight vectors r_x, r_u, r_y by a positive constant $\lambda \leq \min\{p, q\}$, affects the value of the homogeneous $\mathcal{L}_p$−norm

(14) *of the involved signals, i.e.*

$$\|f\|_{\lambda r_f,\mathcal{L}_p} = \left(\int_0^\infty \|f(t)\|^p_{\lambda r_f,q}\mathrm{d}t\right)^{\frac{1}{p}} = \left(\int_0^\infty \|f(t)\|^{p/\lambda}_{r_f,q/\lambda}\mathrm{d}t\right)^{\frac{1/\lambda}{p/\lambda}} = \|f\|^{1/\lambda}_{r_f,\mathcal{L}_{p/\lambda}}.$$

It implies that for any $\lambda \le \min\{p,q\}$ *the spaces* $\mathcal{L}^n_{\lambda r_f,p}$, $\mathcal{L}^n_{r_f,p/\lambda}$ *are identical, i.e.* $\mathcal{L}^n_{\lambda r_f,p} = \mathcal{L}^n_{r_f,p/\lambda}$. *Although the two spaces are identical, the power* $1/\lambda$ *indicates that the* λr_f−*homogeneous* $\mathcal{L}_p$−*norm for signal is* ***not*** *equivalent to the* r_f−*homogeneous* $\mathcal{L}_{p/\lambda}$−*norm.*

Remark 6 (Companion Signal and Relationship Between r−*Homogeneous* $\mathcal{L}_p$−*Norm and* $\mathcal{L}_p$−*Norm): Classically, the case with* $p = q = 2$ *is special [2], since the norm* $\|f\|_{\mathcal{L}_2}$ *given by* (10) *is associated with the inner product of signals*

$$\langle f,g\rangle_s = \int_0^\infty \langle f(t), g(t)\rangle \mathrm{d}t,\ f,g\in\mathcal{L}^n_2$$
$$\|f\|_{\mathcal{L}_2} = \langle f,f\rangle_s^{\frac{1}{2}},\ f\in\mathcal{L}^n_2,$$

and $\mathcal{L}^n_2$ *is a Hilbert space. Similar to Remark 2, for a multivariable signal* f, *define the companion signal*

$$f^{\frac{1}{r_f}}(t) \triangleq \left[\lceil f_1(t)\rfloor^{\frac{1}{r_{f_1}}},\dots,\lceil f_n(t)\rfloor^{\frac{1}{r_{f_n}}}\right]^\top.$$

For homogeneous norms it is possible to associate the $\mathcal{L}_{ph}$−*norm* (14) *of* $f(\cdot)$ *with the classical* $\mathcal{L}_{ph}$−*norm of its companion signal*

$$\|f\|_{r_f,\mathcal{L}_p} = \left\|f^{\frac{1}{r_f}}\right\|_{\mathcal{L}_p},\ f^{\frac{1}{r_f}}\in\mathcal{L}^n_p.$$

Analogous to Remark 2, this can be related to the ideas introduced in [31].

The triangle inequality is valid for r−homogeneous $\mathcal{L}_p$−norm when $r = \mathbf{1}_n$, i.e. for two signals $x,y\in\mathcal{L}^n_p$, $\|x+y\|_{\mathbf{1}_n,\mathcal{L}_p} \le \|x\|_{\mathbf{1}_n,\mathcal{L}_p} + \|y\|_{\mathbf{1}_n,\mathcal{L}_p}$. For other cases, the following more general additive inequality is valid.

Lemma 4 (Additive Inequality for r−*Homogeneous* $\mathcal{L}_p$−*Norm): The* r−*homogeneous* $\mathcal{L}_p$−*norm satisfies the following inequality for two signals* $x,y\in\mathcal{L}^n_{r,p}$, *with* $q\ge 1$, *for all* $p\ge 1$ *(including the case* $p=\infty$*)*

$$\|x+y\|_{r,\mathcal{L}_p} \le \max\left\{1, 2^{\frac{1}{\min r}-\frac{1}{q}}\right\}\left(\|x\|_{r,\mathcal{L}_p} + \|y\|_{r,\mathcal{L}_p}\right). \tag{16}$$

Proof: First of all, the signal $x+y\in\mathcal{L}_{r,p}$ from Lemma 3. Then similar to the classical proof of the triangle inequality for $\mathcal{L}_p$−norm, we have

$$\begin{aligned}
&\|x+y\|^p_{r,\mathcal{L}_p}\\
&= \int_0^\infty \|x(t)+y(t)\|_{r,q}\|x(t)+y(t)\|^{p-1}_{r,q}\mathrm{d}t\\
&\le \max\left\{1, 2^{\frac{1}{\min r}-\frac{1}{q}}\right\}\\
&\quad\times\int_0^\infty \left(\|x(t)\|_{r,q} + \|y(t)\|_{r,q}\right)\|x(t)+y(t)\|^{p-1}_{r,q}\mathrm{d}t\\
&\le \max\left\{1, 2^{\frac{1}{\min r}-\frac{1}{q}}\right\}\left(\int_0^\infty \|x(t)+y(t)\|^p_{r,q}\mathrm{d}t\right)^{\frac{p-1}{p}}\\
&\quad\times\left[\left(\int_0^\infty \|x(t)\|^p_{r,q}\mathrm{d}t\right)^{\frac{1}{p}} + \left(\int_0^\infty \|y(t)\|^p_{r,q}\mathrm{d}t\right)^{\frac{1}{p}}\right]\\
&= \max\left\{1, 2^{\frac{1}{\min r}-\frac{1}{q}}\right\}\left(\|x\|_{r,\mathcal{L}_p} + \|y\|_{r,\mathcal{L}_p}\right)\|x+y\|^{p-1}_{r,\mathcal{L}_p}
\end{aligned}$$

From here (16) follows immediately. The first inequality derives from Lemma 2, the second inequality comes Hölder's inequality (73). From (9) as well as from definition of homogeneous $\mathcal{L}_\infty$−norm (15), it is also easy to derive that

$$\begin{aligned}
&\|x+y\|_{r,\mathcal{L}_\infty}\\
&= \sup_{t\ge 0}\|x(t)+y(t)\|_{r,q}\\
&\le \max\left\{1, 2^{\frac{1}{\min r}-\frac{1}{q}}\right\}\left(\sup_{t\ge 0}\|x(t)\|_{r,q} + \sup_{t\ge 0}\|y(t)\|_{r,q}\right)\\
&\le \max\left\{1, 2^{\frac{1}{\min r}-\frac{1}{q}}\right\}\left(\|x\|_{r,\mathcal{L}_\infty} + \|y\|_{r,\mathcal{L}_\infty}\right).
\end{aligned}$$

Or simply (16) is also valid when $p=\infty$. □

Remark 7 (Additive Inequality and Triangle Inequality for $\mathcal{L}_{ph}$−*Norm): From Remark 4, by* λ−*scaling on the weight vector* r, *there always exists some* $q\in[1,\min r]$, *s.t.* (16) *appears in the form of triangle inequality, i.e.* $\|x+y\|_{r,\mathcal{L}_p} \le \|x\|_{r,\mathcal{L}_p} + \|y\|_{r,\mathcal{L}_p}$ *for all* $p\ge 1$, $q\in[1,\min r]$, $x,y\in\mathcal{L}^n_{r,p}$. *Note that with the* λ−*scaling on the weight vector* r *the value of the* $\mathcal{L}_{ph}$−*norm is also changed as shown in Remark 5.*

Remark 8 (Relationship Between $\mathcal{L}_{ph}$−*Spaces): It is well-known [1], [2] that classical* $\mathcal{L}_p$−*norms are not equivalent for different values of* p, *and this is also not the case for* r−*homogeneous* $\mathcal{L}_p$−*norms for different values of* p.

However, if a signal $f\in\mathcal{L}^n_1\cap\mathcal{L}^n_\infty$, *then* $f\in\mathcal{L}^n_p$ *for* $p\in[1,\infty]$ *[1, Fact 7]. This can be similarly derived for homogeneous* $\mathcal{L}_p$−*norms, i.e. if a signal* $f\in\mathcal{L}^n_{r_f,1}\cap\mathcal{L}^n_{r_f,\infty}$, *then* $f\in\mathcal{L}^n_{r_f,p}$ *for* $p\in[1,\infty]$ *by simply using its companion signal in Remark 6.*

Further, in extended $\mathcal{L}_p$−*space with finite* T, *we have* $\mathcal{L}^n_{\infty e}\subset\mathcal{L}^n_{pe}\subset\mathcal{L}^n_{1e}$. *[1, Exercise 4 in Page 17]. It is also true for extended* $\mathcal{L}_{ph}$−*norms by simply using* (7).

B. HOMOGENEOUS $\mathcal{L}_p$−STABILITY AND FINITE HOMOGENEOUS $\mathcal{L}_p$−GAIN OF SYSTEMS

Definition 5 (homogeneous $\mathcal{L}_p$−*stability): Consider an input-output map* $G:\mathcal{L}^m_{r_u,pe}\to\mathcal{L}^o_{r_y,pe}$. *We say that it is homogeneous* $\mathcal{L}_p$−*stable (*$\mathcal{L}_{ph}$−*stable) [2] if*

$$u\in\mathcal{L}^m_{r_u,p}\Rightarrow y\in\mathcal{L}^o_{r_y,p}.$$

The map G *is said to have* finite homogeneous $\mathcal{L}_p$−gain *(finite* $\mathcal{L}_{ph}$−*gain), if there exist non-negative constants* γ_p *and* b_q *such that for all* $T\ge 0$, $u\in\mathcal{L}^m_{r_u,pe}$

$$\|G(u)_T\|_{r_y,\mathcal{L}_p} \le \gamma_p\|u_T\|_{r_u,\mathcal{L}_p} + b_p. \tag{17}$$

G is said to have finite $\mathcal{L}_{\text{ph}}$−gain with zero bias *if* b_p *in* (17) *can be taken equal to zero. If G has finite* $\mathcal{L}_{ph}$−*gain, then the*

$\mathcal{L}_{\rm ph}$−gain *of G is defined as*

$$\gamma_{ph}(G) \triangleq \inf\left\{\gamma_p \mid \exists\, b_p \text{ such that (17) holds}\right\}. \quad (18)$$

For the input-output maps G_{x_0} obtained from system (1) the previous definitions also apply (cfr. [2]), where for each G_{x_0} the constant $b_p(x_0)$ depends on the initial condition x_0, but not on γ_p.

Similar to the classical situation [2], if G has finite $\mathcal{L}_{\rm ph}$−gain then it is automatically $\mathcal{L}_{\rm ph}$−stable. Indeed, taking $u \in \mathcal{L}^m_{r_u,p} \subset \mathcal{L}^m_{r_u,{\rm pe}}$ and letting $T \to \infty$ in (17), we obtain

$$\|G(u)\|_{r_y,\mathcal{L}_p} \le \gamma_p \|u\|_{r_u,\mathcal{L}_p} + b_p\,, \quad \forall\, u \in \mathcal{L}^m_{r_u,p}\,, \quad (19)$$

implying that $G(u) \in \mathcal{L}^r_{r_y,p}$ for all $u \in \mathcal{L}^m_{r_u,p}$. Moreover, for causal maps (19) implies (17).

Finally, we extend a well-known fact of linear input-output maps [2] to homogeneous input-output maps.

Proposition 1: Homogeneous input-output maps G with finite $\mathcal{L}_{ph}$−gain have zero bias.

Proof: Given (3) and (17), then for any $\kappa > 0$

$$\begin{aligned}
\kappa &\left\|\left(y\left(\kappa^{-r_t}\cdot\right)\right)_T\right\|_{r_y,\mathcal{L}_p} \\
&= \left\|\nu_\kappa^{r_y}\left(y\left(\kappa^{-r_t}\cdot\right)_T\right)\right\|_{r_y,\mathcal{L}_p} \\
&= \left\|\left(G\left(\nu_\kappa^{r_u}\left(u\left(\kappa^{-r_t}\cdot\right)\right)\right)\right)_T\right\|_{r_y,\mathcal{L}_p} \\
&\le \gamma_p \left\|\nu_\kappa^{r_u}\left(u_T\left(\kappa^{-r_t}\cdot\right)\right)\right\|_{r_u,\mathcal{L}_p} + b_p \\
&= \kappa\gamma_p \left\|u_T\left(\kappa^{-r_t}\cdot\right)\right\|_{r_u,\mathcal{L}_p} + b_p\,,
\end{aligned}$$

and the arbitrariness of κ implies $b_p = 0$. □

For a homogeneous input-output map G with finite $\mathcal{L}_{\rm ph}$−gain, dilating the input $u \in \mathcal{L}^m_{r_u,p}$ as in (3), i.e. $\tilde{u}(\cdot) = \nu_\kappa^{r_u}(u(\kappa^{-r_t}\cdot))$, leads to the dilated output $\tilde{y}(\cdot) = \nu_\kappa^{r_y}(y(\kappa^{-r_t}\cdot))$. Their corresponding $\mathcal{L}_{\rm ph}$−norms are

$$\begin{aligned}
\|\tilde{u}\|_{r_u,\mathcal{L}_p} &= \kappa \left\|u\left(\kappa^{-r_t}\cdot\right)\right\|_{r_u,\mathcal{L}_p} \\
&= \kappa \left(\int_0^\infty \left\|u\left(\kappa^{-r_t}t\right)\right\|^p_{r_u,q}\,{\rm d}t\right)^{\frac{1}{p}} \\
&= \kappa^{1+\frac{r_t}{p}} \left(\int_0^\infty \|u(s)\|^p_{r_u,q}\,{\rm d}s\right)^{\frac{1}{p}} \\
&= \kappa^{1+\frac{r_t}{p}} \|u\|_{r_u,\mathcal{L}_p} \\
\|\tilde{y}\|_{r_y,\mathcal{L}_p} &= \kappa^{1+\frac{r_t}{p}} \|y\|_{r_y,\mathcal{L}_p}\,. \quad (20)
\end{aligned}$$

Since $b_p = 0$, we see that the previous definition of finite $\mathcal{L}_{\rm ph}$−gain (17) is compatible with the dilation of the input/output signals, in contrast to the situation presented in Section II-D.

Remark 9 (Restriction on Homogeneous Weights Such That Classical $\mathcal{L}_p$−Stability is Possible): The relations (20)*, together with* (19)*, show that our definition of G having finite $\mathcal{L}_{ph}$−gain coincides with the classical one only if $r_u = \mathbf{1}_m$ and $r_y = \mathbf{1}_o$. If a system has $r_u = c\mathbf{1}_m$ and $r_y = c\mathbf{1}_o$, for some $c > 0$, then scaling with $\lambda = 1/c$ the weights and the degree of the input-output maps, one gets $\lambda r_u = \mathbf{1}_m$, $\lambda r_y = \mathbf{1}_o$, and the homogeneous norms $\|u\|_{\lambda r_u,\mathcal{L}_p} = \|u\|_{\mathcal{L}_p}$ and $\|G(u)\|_{\lambda r_y,\mathcal{L}_p} = \|G(u)\|_{\mathcal{L}_p}$ become standard norms. The restrictions imposed in [9], [17], [25] for the homogeneity weights imply that $r_u = c\mathbf{1}_m$ and $r_y = c\mathbf{1}_o$. In this case it is possible to use the classical $\mathcal{L}_p$−stability (see Section II-D). Note that in general, for arbitrary values of r_u and r_y, the inequality* (19) *cannot be converted into a linear inequality using classical norms.*

Remark 10 (Homogeneous Degree of $b_p(x_0)$): For the input-output maps G_{x_0} obtained from the state space system (1) *the previous derivations show that the function $b_p(x_0)$ is r_x−homogeneous of degree $1+\frac{r_t}{p}$, i.e. $b_p(\nu_\kappa^{r_x}(x_0)) = \kappa^{1+\frac{r_t}{p}} b_p(x_0)$. We also conclude that $b_p(0) = 0$, and so the input-output maps G_{x_0} is homogeneous for $x_0 = 0$. If $b_p(\cdot)$ is continuous on $\mathbb{R}^n \setminus \{0\}$, and $1+\frac{r_t}{p} > 0$, then it is continuous on $\mathbb{R}^n$ (see [14, Theorem 4.1]).*

For homogeneous and causal input-output maps, Proposition 1 allows us to characterize the $\mathcal{L}_{ph}$−*gain* of G defined by (18) as

$$\gamma_{\rm ph}(G) \triangleq \sup_{\|u\|_{r_u,\mathcal{L}_p}\neq 0} \frac{\|G(u)\|_{r_y,\mathcal{L}_p}}{\|u\|_{r_u,\mathcal{L}_p}}\,, \quad u \in \mathcal{L}^m_{r_u,p}\,. \quad (21)$$

Note that (21) makes apparent that $\gamma_{\rm ph}(G)$ is constant for homogeneous systems. Since the numerator and denominator are homogeneous of the same degree from (20), the ratio is homogeneous of degree zero. It is surprising that a non-linear system has a constant gain, valid for all inputs and outputs. This is a particularity not only due to the homogeneity of the system but also to the special way of measuring the size of input and output, provided by the homogeneous (signal) norms, introduced in this paper. Note that for LTI systems, whose degree $r_t = 0$, by choosing $r_u = \mathbf{1}_m$ and $r_y = \mathbf{1}_o$, the homogeneous $\mathcal{L}_p$−gain (21) corresponds to the traditional $\mathcal{L}_p$−gain. Recall also that for different values of p the value of $\mathcal{L}_p$−gains are not related.

IV. CHARACTERIZATION OF HOMOGENEOUS $\mathcal{L}_p$−STABILITY FOR HOMOGENEOUS STATE SPACE SYSTEMS

A. HOMOGENEOUS $\mathcal{L}_p$−STABILITY AND FINITE HOMOGENEOUS $\mathcal{L}_p$−GAIN FOR STATE SPACE SYSTEMS

For systems with a state-space representation (1) there exists a classical characterization of having a finite $\mathcal{L}_2$−gain by means of a dissipation inequality [2], which extends the input-output definition (11). In this section we extend this characterization to check finite homogeneous $\mathcal{L}_p$−gain for an arbitrary homogeneous system and some $p \geq 1$.

Definition 6 (Finite $\mathcal{L}_{ph}$−gain for Σ): The homogeneous state-space system Σ in (1) *has finite $\mathcal{L}_{ph}$−gain ≤ γ if there exists an r_x−homogeneous, positive definite and continuously differentiable storage function $V(x)$ of homogeneous degree $p-d > 0$, such that the following inequality is satisfied for some $\epsilon \geq 0$ and all $(x,u) \in \mathbb{R}^{n+m}$*

$$\frac{\partial V(x)}{\partial x} f(x,u) + \|y\|^p_{r_y,q} - \gamma^p \|u\|^p_{r_u,q} \le -\epsilon \|x\|^p_{r_x,q}\,. \quad (22)$$

The $\mathcal{L}_{ph}-$gain of Σ is defined as

$$\gamma_{ph}(\Sigma) \triangleq \inf\left\{\gamma \mid \Sigma \text{ has } \mathcal{L}_{ph} - \text{gain} \leq \gamma\right\}.$$

Defining the function $J(V_x, x, u)$ via $V_x = \frac{\partial V(x)}{\partial x}$ as

$$J(V_x, x, u) \triangleq \frac{\partial V(x)}{\partial x} f(x, u) + \|h(x, u)\|^p_{r_y,q} - \gamma^p \|u\|^p_{r_u,q} + \epsilon \|x\|^p_{r_x,q}, \tag{23}$$

inequality (22) *corresponds to* $J(V_x, x, u) \leq 0$, *for all* $(x, u) \in \mathbb{R}^{n+m}$.

Since all three homogeneous $q-$norms $\|\cdot\|_{r,q}$ are of homogeneous degree one and V_x is homogeneous of degree p, it follows that J is homogeneous of degree p, i.e. $J(V_x(\nu^{r_x}_\kappa(x)), \nu^{r_x}_\kappa(x), \nu^{r_u}_\kappa(u)) = \kappa^p J(V_x, x, u)$ for all $\kappa \geq 0$ and $(x, u) \in \mathbb{R}^{n+m}$. Building J homogeneous of degree p imposes a restriction on the homogeneity degree of V to be $p - d > 0$.

An inequality of the type of (22) with $p = q = 2$ is classically used to charaterize the $\mathcal{L}_2-$gain of a non-linear system [2], [6]–[8]. In the linear case this reduces to the well-known Riccati inequality, when choosing a quadratic $V(x)$. In [17] a particular case of (22), with $p = q = 2$, is used to characterize the $\mathcal{L}_2-$gain of a particular class of homogeneous systems. Here it is extended to characterize the $\mathcal{L}_{\text{ph}}-$norm for any $p > d$ and $p \geq 1$.

Definition 6 extends Definition 5, as shown in the following Theorem. It also gives conditions to assure internal stability of system Σ in (1) for zero input. Recall that, system Σ from (1) is said to be *zero-state detectable*, if $u(t) = 0, y(t) = 0, \forall t \geq 0$ implies $\lim_{t\to\infty} x(t) = 0$ [2].

Theorem 1: If system Σ in (1) *satisfies the conditions of Definition 6 for some $d < p < \infty$ and $p \geq 1$, then the associated input-output map G_{x_0} has finite $\mathcal{L}_{ph} - $gain $\leq \gamma$ according to Definition 5. If $\epsilon = 0$ and system Σ in* (1) *is further zero-state detectable, then the unperturbed system* (1) *with $u \equiv 0$ is globally asymptotically stable at the origin. If $\epsilon > 0$, the unperturbed system* (1) *with $u \equiv 0$ is globally asymptotically stable at the origin, without requiring the detectability condition.*

Proof: From Definition 6, ensuring (22) for all time leads to

$$\begin{aligned}\int_0^\infty & J(V_x(x(t)), x(t), u(t))\mathrm{d}t \\ &= V(x(t))|_{t\to\infty} - V(x_0) + \|y(\cdot)\|^p_{r_y,\mathcal{L}_p} \\ &\quad - \gamma^p \|u(\cdot)\|^p_{r_u,\mathcal{L}_p} + \epsilon \|x(\cdot)\|^p_{r_x,\mathcal{L}_p} \leq 0.\end{aligned} \tag{24}$$

Therefore for any input $u \in \mathcal{L}^m_{r_u,p}$ and its corresponding output y we have (since $\epsilon \geq 0$)

$$\begin{aligned}\|y(\cdot)\|^p_{r_y,\mathcal{L}_p} &\leq V(x(t))|_{t\to\infty} + \|y(\cdot)\|^p_{r_y,\mathcal{L}_p} \\ &\leq \gamma^p \|u(\cdot)\|^p_{r_u,\mathcal{L}_p} + V(x_0) - \epsilon \|x(\cdot)\|^p_{r_x,\mathcal{L}_p} \\ &\leq \gamma^p \|u(\cdot)\|^p_{r_u,\mathcal{L}_p} + V(x_0).\end{aligned}$$

The first inequality comes from the positive definiteness of $V(x)$, the second inequality results from (24), the third originates from the positive definiteness of $\mathcal{L}_p-$norm $\|x(\cdot)\|_{r_x,\mathcal{L}_p}$ and constant $\epsilon \geq 0$. Since (71) assures that for $p \geq 1$, $(|x|^p + |y|^p)^{\frac{1}{p}} \leq |x| + |y|$ for any $x, y \in \mathbb{R}$, we obtain

$$\begin{aligned}\|y(\cdot)\|_{r_y,\mathcal{L}_p} &\leq \left(\gamma^p \|u(\cdot)\|^p_{r_u,\mathcal{L}_p} + V(x_0)\right)^{\frac{1}{p}} \\ &\leq \gamma \|u(\cdot)\|_{r_u,\mathcal{L}_p} + V^{\frac{1}{p}}(x_0).\end{aligned}$$

Therefore, the input-output map G_{x_0} associated to system (1) has finite $\mathcal{L}_{\text{ph}}-$gain $\leq \gamma$ according to Definition 5, for every $x_0 \in \mathbb{R}^n$. The function $b_p(x_0)$ in (17) can be given by $b_p(x_0) = V^{\frac{1}{p}}(x_0)$, which has homogeneity degree $1 - \frac{d}{p} = 1 + \frac{r_t}{p} > 0$ (this follows from the assumption $p > d$ in Definition 6).

Further, inequality (22) with $u \equiv 0$ reads for all $x \in \mathbb{R}^n$

$$\frac{\partial V(x)}{\partial x} f(x, 0) \leq -\|h(x, 0)\|^p_{r_y,q} - \epsilon \|x\|^p_{r_x,q}, \tag{25}$$

such that for $\epsilon = 0$, $V(x)$ is a weak Lyapunov function, and because of homogeneity it is radially unbounded. Thus, LaSalle's Invariance principle, together with zero-state detectability, imply that the origin is a globally asymptotically stable equilibrium for the unperturbed system (1) [6], [34]. If instead $\epsilon > 0$, $V(x)$ is a strict Lyapunov function and Lyapunov's theorem implies global asymptotic stability of the origin $x = 0$ for the unperturbed system, without assuming detectability. □

Note that the value of γ obtained from (22) corresponds to an upper bound of the $\mathcal{L}_{\text{ph}}-$gain of Σ, that is

$$\gamma_{\text{ph}}(\Sigma) \leq \gamma.$$

When $p = q = 2$, Theorem 1 is well-known for LTI systems, which are homogeneous of degree $d = 0$, and with weights $r_x = \mathbf{1}_n$, $r_y = \mathbf{1}_o$, $r_u = \mathbf{1}_m$. Inequality (22) reduces to a Riccati inequality if a quadratic $V(x)$ is chosen, which can be converted to an LMI. It constitutes one of the main numerical tools to calculate the $\mathcal{L}_2-$gain of an LTI system.

For LTI systems a partial converse of Theorem 1 is also true, namely if the unperturbed LTI system is asymptotically stable at the origin then the system has finite $\mathcal{L}_p-$gain for any $1 \leq p \leq \infty$ [7, Theorem 4.18] [8, Theorem 10.9.1]. This is in general not true for non-linear systems (see e.g. the discussion in Chapter 5 of [6]). In particular, for (SISO) homogeneous systems Section II-D above shows that if $r_y \neq r_u$ finite $\mathcal{L}_p-$gain is in general not well-defined. The following Theorem shows that for homogeneous systems Σ (1) such a converse result to Theorem 1 is true with $\mathcal{L}_{\text{ph}}-$norm.

Theorem 2: Consider the homogeneous system Σ from (1). *Assume that*

$$p > d + \max r_x \tag{26}$$

and $p \geq 1$. If the unperturbed system, with $u = 0$, is locally asymptotically stable at the origin, then inequality (22) *is satisfied for some $\gamma > 0$ and some $\epsilon > 0$. Thus, system* (1) *has a finite $\mathcal{L}_{ph} - $gain $\leq \gamma$.*

Note that (26) implies $p > d$, since $\min r_x > 0$. For the proof of Theorem 2 we will also use the following well-known properties of continuous homogeneous functions.

Lemma 5 ([35], [36]): Let $\psi : \mathbb{R}^n \to \mathbb{R}$ and $\omega : \mathbb{R}^n \to \mathbb{R}$, and $\omega(x) \geq 0$, $\forall x \in \mathbb{R}^n$, be two continuous homogeneous functions with the same weight vector $r = (r_1, \ldots, r_n)$ and homogeneity degree s, such that

$$\{x \in \mathbb{R}^n \backslash \{0\} : \omega(x) = 0\} \subseteq \{x \in \mathbb{R}^n \backslash \{0\} : \psi(x) < 0\}.$$

Then there exist real numbers $\gamma^\star$ and $c > 0$ such that for all $\gamma \geq \gamma^\star$ and all $x \in \mathbb{R}^n \backslash \{0\}$ it holds

$$\psi(x) - \gamma\omega(x) < -c\|x\|^s_{r,p}.$$

Lemma 6 ([14], [35]): Suppose $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\chi : \mathbb{R}^n \to \mathbb{R}$ are continuous real-valued functions on $\mathbb{R}^n$, homogeneous with the same weight vector $r = (r_1, \ldots, r_n)$ and degrees $d_\phi > 0$ and $d_\chi > 0$, respectively, and ϕ is positive definite. Then for every $x \in \mathbb{R}^n$

$$\left(\min_{\mathcal{S}} \chi(x)\right) \phi^{\frac{d_\chi}{d_\phi}}(x) \leq \chi(x) \leq \left(\max_{\mathcal{S}} \chi(x)\right) \phi^{\frac{d_\chi}{d_\phi}}(x),$$

where $\mathcal{S} = \{x \in \mathbb{R}^n : \phi(x) = 1\}$.

Proof: By assumption, the homogeneous system $\dot{x} = f(x, 0)$ has $x = 0$ as an asymptotically stable equilibrium and $d < p - \max r_x$. The converse Lyapunov theorem [15], [22, Theorem 5.8] assures the existence of a homogeneous and continuously differentiable strict Lyapunov function $V_l(x)$ of homogeneity degree $p - d > \max r_x$, satisfying

$$\dot{V}_l(x) = \frac{\partial V_l(x)}{\partial x} f(x, 0) \leq -c\|x\|^p_{r_x,q}$$

for some constant $c > 0$. Recall that $\dot{V}_l(x)$ is homogeneous of degree p. Since for $u = 0$, $\|y\|^p_{r_y,q} = \|h(x, 0)\|^p_{r_y,q}$ is homogeneous of degree p, Lemma 6 assures the existence of a constant $\alpha > 0$ such that the following inequality is valid globally

$$\|y\|^p_{r_y,q} = \|h(x, 0)\|^p_{r_y,q} \leq \alpha\|x\|^p_{r_x,q}.$$

Now consider the homogeneous function

$$a\frac{\partial V_l(x)}{\partial x} f(x, 0) + \|h(x, 0)\|^p_{r_y,q} + \epsilon\|x\|^p_{r_x,q} \leq -(ac - \alpha - \epsilon)\|x\|^p_{r_x,q}, \quad (27)$$

with the strict Lyapunov function V_l obtained from the converse Lyapunov theorem, $a > 0$ a positive constant to be selected and $\epsilon > 0$ some constant. Choosing $a > \underline{a} \triangleq \frac{\alpha+\epsilon}{c}$, it follows that the last expression in (27) is negative definite. Define the functions

$$\omega(x, u) \triangleq \|u\|^p_{r_u,q},$$
$$\psi(x, u) \triangleq \frac{\partial V(x)}{\partial x} f(x, u) + \|h(x, u)\|^p_{r_y,q} + \epsilon\|x\|^p_{r_x,q},$$

where $V(x) = aV_l(x)$ for $a > \underline{a} \triangleq \frac{\alpha+\epsilon}{c}$. Both functions are continuous and homogeneous in $(x, u) \in \mathbb{R}^{n+m}$ of homogeneous degee p. Clearly $\omega(x, u) \geq 0$ and $\omega(x, u) = 0 \leftrightarrow u = 0$. Moreover, from (27) and by the selection of $a > \underline{a}$, $\psi(x, 0) < 0$ for all $x \neq 0$. Lemma 5 assures the existence of a value $\gamma^\star > 0$ such that for any $\gamma \geq \gamma^\star > 0$ the function $\psi(x, u) - \gamma^p\omega(x, u) < 0$ for all $(x, u) \neq 0$. We conclude the proof noticing that V and γ satisfy the inequality (22) with pre-chosen $\epsilon > 0$. Note that such value of ϵ is reflected in $\underline{a}$. □

Remark 11 (Effect of Weight Scaling on Dissipation Inequality): Recall that, as already observed in Remark 1 and Remark 5, scaling the homogeneity degree and weight vectors by a positive constant $\lambda > 0$ changes the values of the homogeneous vector norms as well as the homogeneous signal norms involved. Yet, we still have the identity between the λr−homogeneous $\mathcal{L}_p$ space and the r−homogeneous $\mathcal{L}_{p/\lambda}$ space (Remark 5). Performing the scaling, we can also relate the λr−homogeneous $\mathcal{L}_p$−gain with the r−homogeneous $\mathcal{L}_{p/\lambda}$−gain.

For example, from Theorem 2, there exists a finite γ for $p > \lambda(d + \max r_x)$ and $p \geq 1$ if the unperturbed continuous homogeneous system Σ in (1) is asymptotically stable (now with λ−scaled weight vector and degree). The dissipation inequality (22) is now for all $(x, u) \in \mathbb{R}^{n+m}$

$$\frac{\partial V(x)}{\partial x} f(x, u) + \|y\|^p_{\lambda r_y,q} - \gamma^p\|u\|^p_{\lambda r_u,q} \leq -\epsilon\|x\|^p_{\lambda r_x,q}.$$

From (5) in Remark 1, the previous inequality can be re-written as that for all $(x, u) \in \mathbb{R}^{n+m}$

$$\frac{\partial V(x)}{\partial x} f(x, u) + \|y\|^{p/\lambda}_{r_y,q/\lambda} - \tilde{\gamma}^{p/\lambda}\|u\|^{p/\lambda}_{r_u,q/\lambda} \leq -\epsilon\|x\|^{p/\lambda}_{r_x,q/\lambda}$$

with $\tilde{\gamma} = \gamma^\lambda$. With arguments as in Theorem 2 the r−homogeneous $\mathcal{L}_{p/\lambda}$−gain equals the λr−homogeneous $\mathcal{L}_p$−gain to the power of λ. However, we must be aware that the r−homogeneous $\mathcal{L}_{p_1}$−gain is not related to the r−homogeneous $\mathcal{L}_{p_2}$−gain for $p_1 \neq p_2$ with such scaling on weight vector and degree. The previous analysis only says that the r−homogeneous $\mathcal{L}_{p_2}$−gain can be derived from the λr−homogeneous $\mathcal{L}_{p_1}$−gain, with $\lambda = p_1/p_2$. Yet the latter has nothing to do with the r−homogeneous $\mathcal{L}_{p_1}$−gain.

Remark 12 (Construction of a Storage Function for the Dissipation Inequality): The dissipation inequality in Definition 6 depends on the construction of storage function. Although there is no general method to obtain storage functions, for certain classes of homogeneous systems the recent work [37] provides a methodology to construct them, using generalized homogeneous forms and the sums of squares technique.

For a smooth $V(x)$ of homogeneity degree $p - d$ satisfying inequality (22) it is possible to build another smooth storage function $V^{\frac{p'-d}{p-d}}(x)$ of homogeneous degree $p' - d$ for $p' > p$. On the contrary, for $p' < p$ differentiability of function $V^{\frac{p'-d}{p-d}}(x)$ at $x = 0$ might be lost.

B. SYSTEMS AFFINE IN THE INPUT

Theorems 1 and 2 show that homogeneous $\mathcal{L}_p$−stability can be characterized by the dissipation inequality (22). This inequality (22) depends on two independent variables, x and u, and it has also two unknowns: the function $V(x)$ and the (positive) real constant γ. Since (22) contains the partial derivative of $V(x)$, it is a Partial Differential Inequality. It is in general very hard to solve this kind of problems (for an overview see Chapter 11 of [2]).

It is therefore worthwhile to simplify (22), e.g. by eliminating the variable u. The idea is to find, for each value of x and $V_x = \frac{\partial V}{\partial x}$, the value of u that maximizes $J(V_x, x, u)$, i.e. a function $u^\star(x, V_x)$ such that

$$J(V_x, x, u) \le J(V_x, x, u^\star(x, V_x)), \ \forall (x, u) \in \mathbb{R}^{n+m}.$$

In this case the inequality (22) for all $x \in \mathbb{R}^n$ is equivalent to

$$\begin{aligned}&\frac{\partial V(x)}{\partial x} f(x, u^\star(x, V_x)) + \|h(x, u^\star(x, V_x))\|^p_{r_y,q}\\ &\qquad -\gamma^p \|u^\star(x, V_x)\|^p_{r_y,q} \le -\epsilon \|x\|^p_{r_x,q}.\end{aligned}$$

This inequality is a *Hamilton-Jacobi Inequality (HJI)* (see [2, Chapter 11]). Compared to (22) it is simpler to solve, since it does not depend on u.

Finding $u^\star(x, V_x)$ is simplified if system (1) is affine in u, and the output function h does not depend on u, i.e.

$$\dot{x} = f(x) + g(x)u, \quad y = h(x), \tag{28}$$

where $g(x) = [g_1(x), \ldots, g_m(x)] \in \mathbb{R}^{n\times m}$ is a matrix-valued function with columns $g_i(x) \in \mathbb{R}^n$. Homogeneity (see Definition 1) requires f and g_i to be r_x−homogeneous vector fields of degrees d and $d_i = d - r_{u_i}$, respectively, and h_j to be r_x−homogeneous functions of degree r_{y_i}, i.e.

$$\begin{aligned}&f(\nu^{r_x}_\kappa(x)) = \kappa^d \nu^{r_x}_\kappa(f(x)),\\ &g_i(\nu^{r_x}_\kappa(x)) = \kappa^{d-r_{u_i}} \nu^{r_x}_\kappa(g_i(x)),\ i = 1, \ldots, m,\\ &h_j(\nu^{r_x}_\kappa(x)) = \kappa^{r_{y_j}} h_j(x),\ j = 1, \ldots, o.\end{aligned}$$

Note that from assumption of continuity of the vector fields, we have $d > \max r_u$, and from the assumption $p > d$ in Definition 6, we have $p > \max r_u$. This shall be used in the next Lemma.

Lemma 7: For system (28)*, by selecting $q = p$ the dissipation inequality* (22) *in Definition 6 is equivalent to the homogeneous HJI for all $x \in \mathbb{R}^n$*

$$\begin{aligned}&\frac{\partial V(x)}{\partial x} f(x) + \|h(x)\|^p_{r_y,p}\\ &\quad + \sum_{i=1}^m \alpha_i(\gamma) \left|\frac{\partial V(x)}{\partial x} g_i(x)\right|^{\frac{p}{p-r_{u_i}}} \le -\epsilon \|x\|^p_{r_x,p},\end{aligned}$$

$$\alpha_i(\gamma) \triangleq \left(\frac{r_{u_i}}{p\gamma^p}\right)^{\frac{r_{u_i}}{p-r_{u_i}}} \left(1 - \frac{r_{u_i}}{p}\right). \tag{29}$$

Proof: In view of system (28), $J(V_x, x, u)$ expands into

$$\begin{aligned}J(V_x, x, u) = &\frac{\partial V(x)}{\partial x} f(x) + \|h(x)\|^p_{r_y,p} + \epsilon \|x\|^p_{r_x,p}\\ &+ \frac{\partial V(x)}{\partial x} g(x)u - \gamma^p \|u\|^p_{r_u,p}.\end{aligned}$$

Since function J is continuous (and differentiable in u due to $p > \max r_u$) and for each $x \in \mathbb{R}^n$ (by homogeneity) $\lim_{\|u\|\to\infty} J(V_x, x, u) = -\infty$, it follows by Weierstrass' theorem that there exists at least a global maximum (as a function of u). Taking the partial derivative

$$\frac{\partial J(V_x, x, u)}{\partial u_i} = \frac{\partial V(x)}{\partial x} g_i(x) - \frac{p\gamma^p}{r_{u_i}} \lceil u_i \rfloor^{\frac{p-r_{u_i}}{r_{u_i}}},$$

where

$$\frac{\partial \|u\|^p_{r_u,p}}{\partial u_i} = \frac{p}{r_{u_i}} \lceil u_i \rfloor^{\frac{p-r_{u_i}}{r_{u_i}}}$$

and setting $\frac{\partial J(V_x,x,u)}{\partial u_i} = 0$, we arrive at a unique critical point for all $i = 1, \ldots, m$

$$\begin{aligned}u_i^\star(x, V_x) &= \left(\frac{r_{u_i}}{p\gamma^p}\right)^{\frac{r_{u_i}}{p-r_{u_i}}} \lceil V_x g_i(x) \rfloor^{\frac{r_{u_i}}{p-r_{u_i}}}\\ &= \left(\frac{r_{u_i}}{p\gamma^p}\right)^{\frac{r_{u_i}}{p-r_{u_i}}} \left\lceil \frac{\partial V(x)}{\partial x} g_i(x) \right\rfloor^{\frac{r_{u_i}}{p-r_{u_i}}}.\end{aligned} \tag{30}$$

Replacing (30) into J we obtain inequality (29). Since $p > \max r_u$ the terms $\left(1 - \frac{r_{u_i}}{p}\right) > 0$, therefore, the last term on the left-hand side of (29) is non-negative. Clearly, inequality (29), which is a Hamilton-Jacobi Inequality, is equivalent to (22).

Also note that from (30), $u_i^\star(x, V_x)$ is r_x−homogeneous of degree r_u and the HJI (29) is a r_x−homogeneous HJI of degree p. □

For other cases of either $q \ne p$, system (1) not affine in u or the output function h depends on u, it is still possible to find $u^\star(x, V_x)$. In such case, $u^\star(x, V_x)$ will not have the simple form of (30), thus finding $u^\star(x, V_x)$ might be difficult in the sense of reducing computational effort.

C. EXAMPLE: THE LINEAR TIME INVARIANT CASE

For illustrating the previous procedure to reduce the dissipation inequality to the HJI, consider the case of an asymptotically stable LTI system (which is also homogeneous), given by its state space realization

$$\begin{aligned}\dot{x} &= Ax + Bu,\\ y &= Cx + Du\end{aligned}$$

where A, B, C, D are constant matrices of appropriate dimensions. Select $r_x = \mathbf{1}_n$, $r_u = \mathbf{1}_m$, $r_y = \mathbf{1}_o$, $d = 0$ and $p = 2$ for Theorem 2. Choose the storage function $V(x)$ of homogeneity degree $p - d = 2$ as in Definition 6, a quadratic function $V = x^\top Px$ serves as one option, with $P = P^\top > 0$ to be determined. Let $\epsilon = 0$ since A Hurwitz by assumption of asymptotical stability, then for all $(x, u) \in \mathbb{R}^{n+m}$ the dissipation inequality (22) becomes [4]

$$\begin{aligned}&x^\top(PA + A^\top P)x + 2x^\top PBu\\ &\quad + (Cx + Du)^\top(Cx + Du) - \gamma^2 u^\top u \le 0.\end{aligned}$$

Defining $R_\gamma \triangleq \gamma^2 I - D^\top D$, the latter inequality can be rewritten such that for all $(x, u) \in \mathbb{R}^{n+m}$

$$J(P, x, u) = x^\top(PA + A^\top P + C^\top C)x$$

$$+2x^\top(PB+C^\top D)u - u^\top R_\gamma u \le 0\,. \quad (31)$$

For $\gamma > \overline{\sigma}(D)$, R_γ is positive definite and invertible. Function J is concave in the input u. For given P and fixed x, the value of $u = u^\star(P,x)$ that maximizes the value of J can be obtained by setting $\frac{\partial J}{\partial u} = 0$, i.e.

$$\frac{\partial J(P,x,u)}{\partial u} = 2(PB+C^\top D)^\top x - 2R_\gamma u = 0\,.$$

Solving this equation we obtain

$$u^\star(P,x) = R_\gamma^{-1}(PB+C^\top D)^\top x\,.$$

Inserting $u^\star(P,x)$ in (31) for all $x \in \mathbb{R}^n$ we arrive at

$$x^\top\Big[P(A+BR_\gamma^{-1}D^\top C)+(A+BR_\gamma^{-1}D^\top C)^\top P + PBR_\gamma^{-1}B^\top P + C^\top(I+DR_\gamma^{-1}D^\top)C\Big]x \le 0\,,$$

which is a HJI, devoid of variable u. This inequality is equivalent to the *Algebraic Riccati Inequality (ARI)* [2], [4],

$$P(A+BR_\gamma^{-1}D^\top C)+(A+BR_\gamma^{-1}D^\top C)^\top P + PBR_\gamma^{-1}B^\top P + C^\top(I+DR_\gamma^{-1}D^\top)C \le 0\,, \quad (32)$$

where the unknowns are matrix P and the parameter γ, contained in matrix R_γ. In particular, a solution of the ARI (32) can be found by solving the *Algebraic Riccati Equation (ARE)*, obtained from (32) by replacing inequality by equality. Solutions of the ARE (and also of the ARI) can be characterized in terms of the *Hamiltonian matrix*, given by

$$H_\gamma \triangleq \begin{bmatrix} A+BR_\gamma^{-1}D^\top C & BR_\gamma^{-1}B^\top \\ -C^\top(I+DR_\gamma^{-1}D^\top)C & -(A+BR_\gamma^{-1}D^\top C)^\top \end{bmatrix}.$$

Whenever for a fixed $\gamma > \overline{\sigma}(D)$ matrix H_γ has no eigenvalues on the imaginary axis, the ARE has a positive semidefinite solution P. This matrix P can be calculated from the eigenvalues and their corresponding eigenvectors of H_γ [4]. Thus, to find appropriate values of γ, it is only necessary to check the eigenvalues of the the *Hamiltonian matrix* H_γ.

D. HOMOGENEOUS $\mathcal{L}_p$−STABILITY OF A NON-HOMOGENEOUS SYSTEM AND ITS HOMOGENEOUS APPROXIMATION

For smooth non-linear systems it is usual to consider its linearization to establish its (local) $\mathcal{L}_p$−stability and to calculate the (local) $\mathcal{L}_p$−gain [2], [8], [17]. This can be generalized to non-homogeneous systems which can be locally approximated by a homogeneous one.

Consider a continuous non-linear system (1) where f and h can be written as

$$f(x,u) = f_0(x,u) + \tilde{f}(x,u)\,,$$
$$h(x,u) = h_0(x,u) + \tilde{h}(x,u)\,.$$

We assume that the nominal system

$$\Sigma_0 : \begin{cases} \dot{x}(t) = f_0(x(t),u(t)) \\ y(t) = h_0(x(t),u(t)) \end{cases} \quad (33)$$

is continuous and homogeneous as per Definition 1 and $\tilde{f}$, $\tilde{h}$ are (possibly) non-homogeneous higher order terms. This means that near $(x,u) = 0$ the nominal functions f_0, h_0 dominate $\tilde{f}$, $\tilde{h}$. It is well-known, see e.g. [11]–[13], [15], [17], [22], [26], that this is the case if

$$\lim_{\kappa\to 0}\frac{\tilde{f}_i(\nu_\kappa^{r_x}(x),\nu_\kappa^{r_u}(u))}{\kappa^{r_{x_i}+d}} = 0\,,\ i = 1,\ldots,n\,,$$
$$\lim_{\kappa\to 0}\frac{\tilde{h}_i(\nu_\kappa^{r_x}(x),\nu_\kappa^{r_u}(u))}{\kappa^{r_{y_i}}} = 0\,,\ i = 1,\ldots,o\,, \quad (34)$$

for any $(x,u) \neq 0$.

We will require the following small-signal definition of homogeneous $\mathcal{L}_p$−stability [2], [6], [8].

Definition 7: An input-output map $G : \mathcal{L}^m_{r_u,pe} \to \mathcal{L}^r_{r_y,pe}$ is said to have small-signal finite homogeneous $\mathcal{L}_p$−gain *if there exists a positive constant U such that inequality* (17) *is satisfied for all $u(\cdot) \in \mathcal{L}^m_{r_u,pe}$ with $\sup_{0\le t\le T}\|u(t)\| \le U$.*

Theorem 3: Consider a system Σ as in (1) *with a continuous homogeneous approximation Σ_0* (33)*. Assume furthermore that $\tilde{f}$, $\tilde{h}$ are higher order terms, i.e. they satisfy* (34)*. Suppose that*

$$p > d + \max r_x$$

and $p \ge 1$. If the unperturbed system Σ_0 with $u = 0$ is locally asymptotically stable at the origin, then system (1) *is small-signal $\mathcal{L}_{ph}$−stable and has* small-signal finite homogeneous $\mathcal{L}_p$−gain *for all input-output maps G_{x_0} with $\|x_0\| \le \epsilon$ sufficiently small.*

The proof follows standard arguments and Theorem 2.

V. HOMOGENEOUS $\mathcal{L}_\infty$−STABILITY AND INPUT-TO-STATE STABILITY OF HOMOGENEOUS SYSTEMS

We have not considered explicitly the case $p = \infty$ in Theorems 1 and 2. If for system Σ in (1) the output $y = x$ is selected, then $\mathcal{L}_{\infty h}$−stability is intimately related to ISS (Input-to-State Stability). In fact, note that if inequality (22) is satisfied for some $p \ge 1$, $p > d$ and $\epsilon \ge 0$, then the storage function V qualifies as an *ISS Lyapunov function* [2], [6], [8], and therefore the system is ISS. Moreover, Theorem 2 assures that if the unperturbed system Σ in (1) has the origin as asymptotically stable equilibrium, then the perturbed system is ISS. This result is well-known, see [13], [26], [27]. It generalizes partial results obtained in [9] for classical homogeneous systems (explained in Section VII), and in [17] for a particular class of weighted homogeneous systems affine in the input. Our results make it explicit that the ISS Lyapunov function should satisfy inequality (22) in the homogeneous case, in contrast to the general form presented in [26], [27]. Moreover, the ISS inequality is given explicitly and an estimation of the linear ISS gain is provided using homogeneous norms, which are more appropriate in this context.

Furthermore, for homogeneous systems we provide a particular definition of homogeneous ISS, tailored to homogeneous systems, which is an extension of the homogeneous $\mathcal{L}_\infty$−stability with finite gain given in Definition 5.

Definition 8: The homogeneous system Σ in (1), with homogeneity degree d, is said to be homogeneous input-to-state stable (ISS) if there exist positive constants M, κ, γ_{iss} such that for any input $u(\cdot) \in \mathcal{L}_{\infty h}^m = \mathcal{L}_\infty^m$ and any $x_0 \in \mathbb{R}^n$, the state trajectory of (1) for all $t \geq 0$ satisfies

$$\|x(t)\|_{r_x,q} \leq \beta(\|x_0\|_{r_x,q}, t) + \gamma_{iss}\|u(\cdot)\|_{r_u,\mathcal{L}_\infty} \tag{35}$$

with $q \geq 1$ and β the following $\mathcal{KL}$ function

$$\beta(v, t) = \begin{cases} \begin{cases} M\left(v^{-d} - \kappa t\right)^{-\frac{1}{d}} & \text{when } t \leq \frac{1}{\kappa}v^{-d} \\ 0 & \text{when } t \geq \frac{1}{\kappa}v^{-d} \end{cases} & \text{if } d < 0, \\ M\exp(-\kappa t)v & \text{if } d = 0, \\ M\left(v^{-d} + \kappa t\right)^{-\frac{1}{d}} & \text{if } d > 0. \end{cases} \tag{36}$$

Note that the *homogeneous ISS gain function* is *linear* in $\|u(\cdot)\|_{r_u,\mathcal{L}_\infty}$ (cfr. (35)). This linearity follows from homogeneity and the use of homogeneous norms. In the classical norms, the relationship is non-linear (see e.g. [26]). This underscores the convenience of adopting homogeneous $\mathcal{L}_\infty$−norms for homogeneous systems.

It is possible to write an ISS inequality similar to (35) using classical norms. From the relationship between homogeneous norms given in Lemma 1 we obtain

$$\|x(t)\|_q \leq \tilde{\beta}(\|x_0\|_q, t) + \tilde{\gamma}_{\rm iss}(\|u(\cdot)\|_{\mathcal{L}_\infty}),$$

but with $\tilde{\gamma}_{\rm iss}(\cdot)$ a non–linear function, and $\tilde{\beta}$ different from (36). This is the form of the result presented in [26].

Theorem 4: Consider a homogeneous system Σ from (1). If the unperturbed system with $u = 0$ is locally asymptotically stable at the origin, then system (1) is homogeneous ISS with a linear gain and also has a finite $\mathcal{L}_{\infty h}$−gain.

Proof: In the proof of Theorem 2 it is shown that for any $p > d + \max r_x$, $p \geq 1$, and considering $y = x$, there exists a homogeneous storage function $V(x)$ such that inequality (22) is satisfied for some $\epsilon > 0$, i.e. for all $(x, u) \in \mathbb{R}^{n+m}$

$$\frac{\partial V(x)}{\partial x}f(x, u) \leq -(\epsilon + 1)\|x\|_{r_x,q}^p + \gamma^p\|u\|_{r_u,q}^p.$$

Then, for $\|x\|_{r_x,q} \geq \frac{\gamma k^{\frac{1}{p}}}{(\epsilon+1)^{\frac{1}{p}}}\|u\|_{r_u,q}$ with $k > 1$ inequality

$$\frac{\partial V(x)}{\partial x}f(x, u) \leq -\frac{k-1}{k}(\epsilon + 1)\|x\|_{r_x,q}^p \tag{37}$$

is satisfied. Note that, since $V(x)$ when $u = 0$ is a strict Lyapunov function, there are constants with $0 < \alpha_1 \leq \alpha_2$ s.t.

$$\alpha_1\|x\|_{r_x,q}^{p-d} \leq V(x) \leq \alpha_2\|x\|_{r_x,q}^{p-d}, \quad \forall x \in \mathbb{R}^n. \tag{38}$$

Define $b = \frac{\gamma k^{\frac{1}{p}}}{(\epsilon+1)^{\frac{1}{p}}}$ and $c = \alpha_2(b\|u(\cdot)\|_{r_u,\mathcal{L}_\infty})^{p-d}$ where $\|u(\cdot)\|_{r_u,\mathcal{L}_\infty} = \sup_{t\geq 0}\|u(t)\|_{r_u,q}$ is used. Then the set

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$$

is such that

$$\mathcal{B}_b = \{x \in \mathbb{R}^n : \|x\|_{r_x,q} < b\|u(\cdot)\|_{r_u,\mathcal{L}_\infty}\} \subset \Omega_c.$$

As a consequence, for each x on the boundary of Ω_c we have $\|x\|_{r_x,q} \geq b\|u(\cdot)\|_{r_u,\mathcal{L}_\infty}$. Therefore, at any $t \geq 0$ such that $x(t)$ is on the boundary of Ω_c we have $\|x(t)\|_{r_x,q} \geq b\|u(\cdot)\|_{r_u,\mathcal{L}_\infty} \geq \frac{\gamma k^{\frac{1}{p}}}{(\epsilon+1)^{\frac{1}{p}}}\|u(t)\|_{r_u,q}$. Then from (37) it follows that

$$\frac{\partial V(x(t))}{\partial x}f(x(t), u(t)) < 0$$

at any $t \geq 0$ for $x(t)$ on the boundary of Ω_c. It can be concluded that for any initial condition $\tilde{x}(0)$ in the interior of Ω_c, the solution $x = \tilde{x}(t)$ w.r.t. $\dot{\tilde{x}} = f(\tilde{x}, u)$ is defined for all $t \geq 0$ and $\tilde{x}(t) \in \Omega_c$ for all $t \geq 0$.

Two cases need to be studied separately. First of all, when $x(0) \in \Omega_c$ then $x(t)$ for all $t \geq 0$ satisfies

$$\|x(t)\|_{r_x,q}^{p-d} \leq \frac{1}{\alpha_1}V(x(t)) \leq \frac{c}{\alpha_1} = \frac{\alpha_2}{\alpha_1}(b\|u(\cdot)\|_{r_u,\mathcal{L}_\infty})^{p-d},$$

that is,

$$\|x(t)\|_{r_x,q} \leq \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{p-d}} b\|u(\cdot)\|_{r_u,\mathcal{L}_\infty}, \tag{39}$$

implying the solution $x = \tilde{x}(t)$ w.r.t. $\dot{x} = f(x, u)$ satisfies

$$\|\tilde{x}(\cdot)\|_{r_x,\mathcal{L}_\infty} = \sup_{t\geq 0}\|\tilde{x}(t)\|_{r_x,q} \leq \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{p-d}} b\|u(\cdot)\|_{r_u,\mathcal{L}_\infty}.$$

Secondly, when $x(0) \in \overline{\Omega}_c$ then since $\mathcal{B}_b \subset \Omega_c$, $x(0) \in \overline{\mathcal{B}}_b$, i.e. $\|x(0)\|_{r_x,q} \geq b\|u(\cdot)\|_{r_u,\mathcal{L}_\infty}$ and so long as $\|x(t)\|_{r_x,q} \geq b\|u(\cdot)\|_{r_u,\mathcal{L}_\infty} \geq \frac{\gamma k^{\frac{1}{p}}}{(\epsilon+1)^{\frac{1}{p}}}\|u(t)\|_{r_u,q}$ we have

$$\begin{aligned} \frac{dV(x(t))}{dt} &= \frac{\partial V(x(t))}{\partial x}f(x(t), u(t)) \\ &\leq -\frac{k-1}{k}(\epsilon + 1)\|x(t)\|_{r_x,q}^p < 0. \end{aligned}$$

In view of (38) this inequality implies

$$\frac{dV(x(t))}{dt} \leq -\frac{k-1}{k}\frac{(\epsilon+1)}{\alpha_2^{\frac{p}{p-d}}}V^{\frac{p}{p-d}}(x(t)).$$

Depending on $v \in \mathbb{R}_{\geq 0}$ and $t \in \mathbb{R}_{\geq 0}$ define the function

$$\Phi(v, t; l, k) = \begin{cases} \begin{cases} \left(v^{-l} + lkt\right)^{-\frac{1}{l}} & \text{when } t \leq -\frac{1}{lk}v^{-l} \\ 0 & \text{when } t \geq -\frac{1}{lk}v^{-l} \end{cases} & \text{if } l < 0, \\ \exp(-kt)v & \text{if } l = 0, \\ \left(v^{-l} + lkt\right)^{-\frac{1}{l}} & \text{if } l > 0. \end{cases} \tag{40}$$

which is a $\mathcal{KL}$ function. Indeed, $\Phi(0, t; l, k) = 0$ and Φ is monotonically increasing in v, decreasing in t, and $\lim_{t\to\infty}\Phi(v, t; l, k) = 0$. Since the solution to the differential equation $\dot{v} = -\kappa v^\ell$ for $\ell \in \mathbb{R}_{>0}$ is given by

$v(t) = \Phi(v(0),\, t;\, \ell - 1,\, \kappa)$ with Φ from (40), by using the comparison lemma [6] we conclude that $V(x(t))$ satisfies

$$V(x(t)) \le \Phi\left(V(x_0),\, t;\, \frac{d}{p-d},\, \frac{k-1}{k}\frac{(\epsilon+1)}{\alpha_2^{\frac{p}{p-d}}}\right).$$

Moreover,

$$\begin{aligned}\|x(t)\|_{r_x,q}^{p-d} &\le \frac{1}{\alpha_1}V(x(t))\\ &\le \frac{1}{\alpha_1}\Phi\left(V(x_0),\, t;\, \frac{d}{p-d},\, \frac{k-1}{k}\frac{(\epsilon+1)}{\alpha_2^{\frac{p}{p-d}}}\right)\\ &\le \frac{1}{\alpha_1}\Phi\left(\alpha_2\|x_0\|_{r_x,q}^{p-d},\, t;\, \frac{d}{p-d},\, \frac{k-1}{k}\frac{(\epsilon+1)}{\alpha_2^{\frac{p}{p-d}}}\right),\end{aligned}$$

and therefore

$$\|x(t)\|_{r_x,q} \le \beta(\|x_0\|_{r_x,q}, t)\,, \tag{41}$$

where

$$\begin{aligned}&\beta(\|x_0\|_{r_x,q}, t)\\ &= \left[\frac{1}{\alpha_1}\Phi\left(\alpha_2\|x_0\|_{r_x,q}^{p-d},\, t;\, \frac{d}{p-d},\, \frac{k-1}{k}\frac{(\epsilon+1)}{\alpha_2^{\frac{p}{p-d}}}\right)\right]^{\frac{1}{p-d}}.\end{aligned}$$

It can be shown that β coincides with (36) where

$$M = \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{p-d}},$$

$$\kappa = \frac{k-1}{k}\frac{(\epsilon+1)}{\alpha_2} \times \begin{cases}\frac{-d}{p-d} & \text{if } d < 0\,,\\ \frac{1}{p} & \text{if } d = 0\,,\\ \frac{d}{p-d} & \text{if } d > 0\,.\end{cases} \tag{42}$$

Thus, as long as $V(x(t)) > c$, function $V(x(t))$ is decreasing. This in particular shows that $x(t)$ is bounded and

$$\|x(t)\|_{r_x,q}^{p-d} \le \frac{1}{\alpha_1}V(x(t)) \le \frac{1}{\alpha_1}V(x(0))\,.$$

Furthermore, there is some finite time T such that $V(x(T)) = c$. For $t \ge T$, $x(t)$ obeys (39).

From the previous analysis we conclude that there exist a finite $T > 0$ s.t. (41) is satisfied for $t \in [0, T]$ and for $t \ge T$ (39) is observed. When $x(0) \in \Omega_c$, then $T = 0$. Combining both cases, we have for any $k > 1$ that the ISS inequality (35) is satisfied with the following constant linear gain

$$\gamma_{\text{iss}} = Mb = \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{p-d}} \frac{k^{\frac{1}{p}}}{(\epsilon+1)^{\frac{1}{p}}}\gamma\,. \tag{43}$$

To show that the system is $\mathcal{L}_{\infty h}$−stable with finite gain, recall that $h(x, u)$ is homogeneous and continuous, and therefore using Lemma 6 there is a positive constant $c_q > 0$ s.t.

$$\|h(x, u)\|_{r_y,q} \le c_q\|(x, u)\|_{(r_x,r_u),q}\,.$$

From Jensen's inequality for $q \ge 1$, for any positive real numbers a, b, we have $(a^q + b^q)^{1/q} \le a + b$. Thus we obtain

$$\begin{aligned}\|(x, u)\|_{(r_x,r_u),q} &= \left(\sum_{i=1}^{n}|x_i|^{\frac{q}{r_{x_i}}} + \sum_{j=1}^{m}|u_j|^{\frac{q}{r_{u_j}}}\right)^{\frac{1}{q}}\\ &\le \left(\sum_{i=1}^{n}|x_i|^{\frac{q}{r_{x_i}}}\right)^{\frac{1}{q}} + \left(\sum_{j=1}^{m}|u_j|^{\frac{q}{r_{u_j}}}\right)^{\frac{1}{q}}\\ &= \|x(t)\|_{r_x,q} + \|u(t)\|_{r_u,q}\,.\end{aligned}$$

Using the above expressions and the ISS inequality (35), for all $t \ge 0$ we arrive at

$$\begin{aligned}\|y(t)\|_{r_y,q} &= \|h(x(t), u(t))\|_{r_y,q}\\ &\le c_q\|x(t)\|_{r_x,q} + c_q\|u(t)\|_{r_u,q}\\ &\le c_q\left(\beta(\|x_0\|_{r_x,q}, t) + \gamma_{\text{iss}}\|u(\cdot)\|_{r_u,\mathcal{L}_\infty}\right)\\ &\quad + c_q\|u(\cdot)\|_{r_u,\mathcal{L}_\infty}\\ &\le c_q\beta(\|x_0\|_{r_x,q}, t) + c_q\left(\gamma_{\text{iss}} + 1\right)\|u(\cdot)\|_{r_u,\mathcal{L}_\infty}.\end{aligned} \tag{44}$$

From the previous expression, $\mathcal{L}_{\infty h}$−stability with finite gain follows immediately. □

Note that given $V(x)$ of homogeneity degree $p - d$ satisfying inequality (22) with some γ, γ_{iss} can be estimated via (43), while the parameters of function β in (36) are given by (42).

Further note that in (43) the value of γ corresponds to the value that satisfies (22) for some $p \ge 1$ taking $y = x$. Given V satisfying (22) the best (smallest) value of γ can be calculated using the results of Section VIII below. Then we can obtain an upper bound of the homogeneous ISS-gain from γ_{iss} in (43). The $\mathcal{L}_{\infty h}$−gain is also upper bounded by $c_q\left(\gamma_{\text{iss}} + 1\right)$ as in (44).

Remark 13: In this paper, the upper bound of the homogeneous ISS-gain can be found with the $\mathcal{L}_{ph}$−gain together with its storage function as shown in (43).

If the unperturbed homogeneous system Σ from (1) *with $u = 0$ is locally asymptotically stable at the origin, then from Theorem 2 it has finite $\mathcal{L}_{ph}$−gain (for $p > d + \max r_x$ and $p \ge 1$). From the proof of Theorem 4, system Σ in* (1) *also has a finite linear homogeneous ISS-gain and a finite $\mathcal{L}_{\infty h}$−gain.*

On the other hand, if system Σ from (1) *is homogeneous ISS, then an* ISS Lyapunov function *[2], [6], [8] may be found which can guarantee the existence of $\mathcal{L}_{ph}$−stability for some p, sufficiently larger. This will not further be discussed in this paper.*

VI. HOMOGENEOUS $\mathcal{L}_p$−STABILITY FOR INTERCONNECTED SYSTEMS

One of the most important applications of the classical $\mathcal{L}_p$−stability concept lies in the study of $\mathcal{L}_p$−stability for interconnected systems. A central result, with many consequences in robust control, is the *small-gain theorem*. It provides a sufficient condition for finite-gain $\mathcal{L}_p$−stability of the negative feedback interconnection of two finite-gain $\mathcal{L}_p$−stable systems, in terms of their $\mathcal{L}_p$−gains [1], [2], [6]. Moreover, it is well-known that two finite-gain $\mathcal{L}_p$−stable systems connected in cascade or in parallel leads to a finite-gain $\mathcal{L}_p$−stable system. Note that, since the classical $\mathcal{L}_p$−norms are not *equivalent*, there is a small-gain theorem for each p.

In this section we obtain the corresponding small-gain theorems for each p and the result for cascade systems derived from the homogeneous $\mathcal{L}_p$−stability concepts introduced in the previous sections for general homogeneous systems. Although there are already some small-gain theorems for homogeneous systems [13], [27], they are derived from the ISS stability, and thus correspond to $p = \infty$. Further [27] does consider neither an external input nor output maps.

A. HOMOGENEOUS SMALL-GAIN THEOREM FOR FEEDBACK INTERCONNECTED SYSTEMS

Since the additive inequality (16) is (slightly) different from the traditional triangle inequality for traditional $\mathcal{L}_p$−norms, the homogeneous small-gain theorem for negative feedback interconnected system also receives a different form.

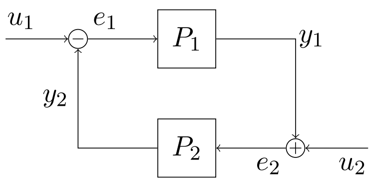

FIGURE 1. Homogeneous small-gain theorem.

Theorem 5: [Homogeneous Small-Gain Theorem] Consider the negative feedback interconnection of two homogeneous systems P_1, P_2 given in Figure 1. Suppose that $r_{y_1} = r_{u_2}$ and $r_{y_2} = r_{u_1}$ and that P_1 and P_2 have r−homogeneous $\mathcal{L}_p$−gains $\gamma_{ph}(P_1)$ and $\gamma_{ph}(P_2)$ for some $p \in [1, \infty]$, respectively. Then the closed-loop system has finite homogeneous $\mathcal{L}_p$−gain if

$$\gamma_{ph}(P_1)\gamma_{ph}(P_2) < \frac{1}{c_1 c_2} \tag{45}$$

where

$$c_1 = \max\left\{1, 2^{\frac{1}{\min r_{u_1}} - \frac{1}{q}}\right\}, \quad c_2 = \max\left\{1, 2^{\frac{1}{\min r_{u_2}} - \frac{1}{q}}\right\}.$$

Proof: As shown in Figure 1, the interconnected system has two inputs $u_1(t) \in \mathbb{R}^{n_1}$ and $u_2(t) \in \mathbb{R}^{n_2}$ whose weight vectors we may denote as $r_1 = r_{u_1}$ and $r_2 = r_{u_2}$. We have

$$e_1 = u_1 - y_2, \quad y_1 = P_1(e_1),$$
$$e_2 = u_2 + y_1, \quad y_2 = P_2(e_2).$$

From the additive inequality (16) in Lemma 4 and P_1, P_2 being $\mathcal{L}_{\rm ph}$−stable, when assuming $u_1 \in \mathcal{L}_{r_1,p}^{n_1}$, $u_2 \in \mathcal{L}_{r_2,p}^{n_2}$ the homogeneous $\mathcal{L}_p$−norm of e_1 is

$$\|e_1\|_{r_1,\mathcal{L}_{\rm ph}} = \|u_1 - y_2\|_{r_1,\mathcal{L}_{\rm ph}}$$
$$\leq \max\left\{1, 2^{\frac{1}{\min r_1} - \frac{1}{q}}\right\}\left(\|u_1\|_{r_1,\mathcal{L}_{\rm ph}} + \|y_2\|_{r_1,\mathcal{L}_{\rm ph}}\right)$$
$$\leq c_1\left(\|u_1\|_{r_1,\mathcal{L}_{\rm ph}} + \gamma_2\|e_2\|_{r_2,\mathcal{L}_{\rm ph}} + b_2\right)$$

with $\gamma_2 = \gamma_{\rm ph}(P_2)$. Similarly, with $\gamma_1 = \gamma_{\rm ph}(P_1)$, then

$$\|e_2\|_{r_2,\mathcal{L}_{\rm ph}} \leq c_2\left(\|u_2\|_{r_2,\mathcal{L}_{\rm ph}} + \gamma_1\|e_1\|_{r_1,\mathcal{L}_{\rm ph}} + b_1\right).$$

Combining both inequalities, we obtain

$$\begin{aligned}\|e_1\|_{r_1,\mathcal{L}_{\rm ph}} &\leq c_1\Big[\|u_1\|_{r_1,\mathcal{L}_{\rm ph}} + b_2 \\ &\quad + \gamma_2 c_2\left(\|u_2\|_{r_2,\mathcal{L}_{\rm ph}} + \gamma_1\|e_1\|_{r_1,\mathcal{L}_{\rm ph}} + b_1\right)\Big] \\ &= c_1\|u_1\|_{r_1,\mathcal{L}_{\rm ph}} + \gamma_2 c_1 c_2\|u_2\|_{r_2,\mathcal{L}_{\rm ph}} \\ &\quad + \gamma_1\gamma_2 c_1 c_2\|e_1\|_{r_1,\mathcal{L}_{\rm ph}} + \gamma_2 c_1 c_2 b_1 + c_1 b_2.\end{aligned}$$

If condition (45) is satisfied, this leads to

$$\begin{aligned}&\|e_1\|_{r_1,\mathcal{L}_{\rm ph}} \\ &\leq \frac{1}{1 - \gamma_1\gamma_2 c_1 c_2} \\ &\quad \times\left(c_1\|u_1\|_{r_1,\mathcal{L}_{\rm ph}} + \gamma_2 c_1 c_2\|u_2\|_{r_2,\mathcal{L}_{\rm ph}} + \gamma_2 c_1 c_2 b_1 + c_1 b_2\right).\end{aligned}$$

Similarly, the homogeneous $\mathcal{L}_2$−norm of e_2 is bounded by

$$\begin{aligned}&\|e_2\|_{r_2,\mathcal{L}_{\rm ph}} \\ &\leq \frac{1}{1 - \gamma_1\gamma_2 c_1 c_2} \\ &\quad \times\left(c_2\|u_2\|_{r_2,\mathcal{L}_{\rm ph}} + \gamma_1 c_1 c_2\|u_1\|_{r_1,\mathcal{L}_{\rm ph}} + \gamma_1 c_1 c_2 b_2 + c_2 b_1\right).\end{aligned}$$

This implies that the closed-loop system in Figure 1 has finite homogeneous $\mathcal{L}_p$−gain.

Further, from (35) as well as (9), the proof with homogeneous ISS-gain is similar. □

A classical use of the small-gain theorem leads to a robustness interpretation: If P_1 is the nominal system and P_2 an uncertainty, the stability of P_1 is preserved for any P_2 with sufficiently small gain satisfying (45). It is important to note that no restriction on the homogeneity degrees of P_1 and P_2 is imposed, i.e. they can be different.

Remark 14: Due to the scaling property of the weight vectors and the degree of homogeneity, it is always possible to achieve $\min\{r_{u_1}, r_{u_2}\} > 1$ as noted in Remark 4. Then choosing $q \in \left[1, \min\{r_{u_1}, r_{u_2}\}\right]$ we recover the well-known small-gain condition $\gamma_{ph}(P_1)\gamma_{ph}(P_2) < 1$ in (45). Note that in this case the value of p that satisfies the conditions of Theorem 2 also has to be scaled.

B. HOMOGENEOUS $\mathcal{L}_p$−GAIN OF A CASCADE OF TWO HOMOGENEOUS SYSTEMS

Similar to the classical case, the series interconnection of two homogeneous systems with finite $\mathcal{L}_{\rm ph}$−gain, see Figure 2, again is a system with finite $\mathcal{L}_{\rm ph}$−gain.

Theorem 6 (Homogeneous $\mathcal{L}_p$−Gain of Cascaded Homogeneous System): Consider the cascade interconnection of two homogeneous systems G_1, G_2 given in Figure 2. Suppose that $r_{y_1} = \ell r_{u_2}$ for some $\ell > 0$ and that

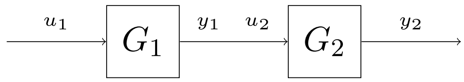

FIGURE 2. Cascade interconnection of homogeneous systems.

G_1 has r−homogeneous $\mathcal{L}_p$−gain $\gamma_{ph}(G_1)$ and G_2 has ℓr−homogeneous $\mathcal{L}_p$−gain $\gamma_{ph}(G_2)$ for some $p \in [1, \infty]$. Then the cascade system G_2G_1 has homogeneous $\mathcal{L}_p$−gain $\gamma_{ph}(G_2G_1) = \gamma_{ph}(G_1)\gamma_{ph}(G_2)$.

Note that the weight vectors of input and output of system G_2G_1 are r_{u_1} and ℓr_{y_2}, respectively.

Proof: Suppose the r−homogeneous $\mathcal{L}_p$−gain of G_1 is γ_1 with bias b_1 and the ℓr−homogeneous $\mathcal{L}_p$−gain of G_2 is γ_2 with bias b_2, then we have for $u_1 \in \mathcal{L}^m_{r_{u_1},p}$ that

$$\begin{aligned}\|y_2\|_{\ell r_{y_2},\mathcal{L}_p} &\leq \gamma_2\|u_2\|_{\ell r_{u_2},\mathcal{L}_p} + b_2 = \gamma_2\|y_1\|_{r_{y_1},\mathcal{L}_p} + b_2\\ &\leq \gamma_2\left(\gamma_1\|u_1\|_{r_{u_1},\mathcal{L}_p} + b_1\right) + b_2\\ &= \gamma_1\gamma_2\|u_1\|_{r_{u_1},\mathcal{L}_p} + \gamma_2 b_1 + b_2. \end{aligned} \tag{46}$$

Thus the $\mathcal{L}_{\text{ph}}$−gain of the cascaded system G_2G_1 is $\gamma_1\gamma_2$ with input weight vector r_{u_1} and output weight vector ℓr_{y_2}.

The proof for the case of homogeneous ISS-gain is similar, using (35) in (46). □

Note that for both interconnection results, the common signals have to have *compatible* weights. For the system in Figure 1 this means that $r_{y_2} = r_{e_1} = r_2$ and $r_{y_1} = r_{e_2} = r_1$. For the cascaded system in Figure 2 this is relaxed to $r_{y_1} = \ell r_{u_2}, \ell > 0$.

Remark 15: Note that for both Theorem 5 and Theorem 6, the homogeneous degree for subsystems can be different. This results from the fact that the $\mathcal{L}_{ph}$−gain is an input-output relationship, the behavior within each homogeneous subsystem does not matter from the input-output perspective. When P_1, P_2 have different homogeneous degrees (even of different sign), the closed loop system in Figure 1 does not behave homogeneously any longer. Yet the conclusions of Theorem 5 and Theorem 6 remain true. Such behavior is also observed in [27] where the homogeneous small gain theorem does not consider an input and allows different degrees for each sub-system.

Remark 16: Suppose the subsystems satisfy the condition of Theorem 5 or Theorem 6 for some $p = p_1$. Further if they have state space realization (1) *and meet the condition from Theorem 2, then the conclusion in Theorem 5 and Theorem 6 is also true for all $p \in [p_1, \infty]$.*

VII. COMPARISON TO PREVIOUS WORKS

In this section, differences to previously published works will be described in more detail.

Paper [9] considers a special class of system (1) which is homogeneous in the classical sense, i.e. $r_x = \mathbf{1}_n$, $r_u = \mathbf{1}_m$, with output $y = x$, and with non-negative homogeneity degree $d \geq 0$. The author shows that if the origin is asymptotically stable for the unforced system, then for $p \geq 1 + d$ the system (with output $y = x$) is $\mathcal{L}_p$−stable and has finite $\mathcal{L}_p$−gain with classical norms, i.e. relation (11) is satisfied. Our Theorem 2 generalizes these results to an arbitrary homogeneous system, but it shows that to obtain finite $\mathcal{L}_p$−gain it is necessary to consider homogeneous norms. Only in particular cases this is valid for classical signal norms, namely when $r_y = \ell\mathbf{1}_o$ and $r_u = \ell\mathbf{1}_m$, $\ell > 0$. Note that Theorem 2 requires $p > 1 + d$ which is stricter than the condition in [9]. This is a consequence of the converse Lyapunov theorem for homogeneous systems that assures a smooth Lyapunov function only if $p > 1 + d$. However, if $p = 1+d$ a Lyapunov function which is not differentiable at $x = 0$ can be constructed. Our Theorem 2 can be extended to cover also the case $p = 1+d$, but at the expense of a technical issue with non-differentiability at $x = 0$, as is done in [9]. We have not presented the details here of this extension.

Also in [9] ISS of the system is established (included in the case of $p = \infty$, $y = x$), with a linear gain and using classical norms, so inequality (35) is satisfied. Our Theorem 4 extends this result to arbitrary homogeneous systems and shows that the linear gain is valid only if homogeneous norms are considered. In some cases, the homogeneous $\mathcal{L}_p$−norms are equivalent to the usual $\mathcal{L}_p$−norms. Moreover, Theorem 4 considers also the case of $\mathcal{L}_{\infty h}$−stability for an arbitrary (homogeneous) output $y = h(x, u)$, which is not considered in [9].

Paper [25] deals with the special class of systems (1), $\dot{x} = f(x) + Bu$ with B a constant matrix, which are homogeneous in the classical sense, i.e. $r_x = \mathbf{1}_n$, $r_u = (d + 1)\mathbf{1}_m$, with output $y = h(x)$, $r_y = (d + 1)\mathbf{1}_o$, and with non-negative homogeneity degree $d \geq 0$. The main result of [25, Theorem 1] states that if the origin is asymptotically stable for the unforced system, then the system is $\mathcal{L}_2$−stable and has finite $\mathcal{L}_2$−gain with classical norms, i.e. relation (11) is satisfied for $p = 2$. This is characterized using a Hamilton-Jacobi Inequality. Our Theorem 2 generalizes these results to an arbitrary homogeneous system, but it shows that to obtain finite $\mathcal{L}_p$−gain it is necessary to consider homogeneous norms. Only in particular cases this is valid for classical signal norms, namely when $r_y = \ell\mathbf{1}_o$ and $r_u = \ell\mathbf{1}_m$, $\ell > 0$. In [25], $r_u = (d+1)\mathbf{1}_m$, $r_y = (d+1)\mathbf{1}_o$ are used, but actually $r_y = \ell\mathbf{1}_o$ and $r_u = \ell\mathbf{1}_m$ could also be considered. Neither $\mathcal{L}_p$−stabilty nor ISS stability is considered in [25]. We characterize all properties in a unified way using inequality (22), which is more general than the Hamilton-Jacobi inequality since the analytical form of $u^\star(x, V_x)$, as in (30), is not always easily attainable.

The author of [17] generalizes the results of [25] to the special class of weighted homogeneous systems that are affine in the input, i.e. $\dot{x} = f(x) + G(x)u$, $y = h(x)$, with $f(x)$ a r_x−homogeneous vector field of degree $d > -r_0 \triangleq -\min r_x$, $G(x)$ a matrix with columns being r_x−homogeneous vector fields of degree $s \geq -r_0$, and homogeneous weight of input and output being $r_u = (d - s)\mathbf{1}_m$, $r_y = (d - s)\mathbf{1}_o$. Since [17] deals with the H_∞ control problem, the following results on $\mathcal{L}_2$−stability are contained implicitely in the paper. In the particular case when $s = -r_0$, that strongly restricts matrix $G(x)$, [17] shows that if the origin $x = 0$ is asymptotically stable for the unforced system, then the system is $\mathcal{L}_2-$stable and has finite $\mathcal{L}_2-$gain with classical norms, i.e. relation (11) is satisfied for $p = 2$. This is characterized using a homogeneous Hamilton-Jacobi Inequality. For the relaxed condition on $G(x)$ that $s \geq -r_0$, it is shown that if the origin $x = 0$ is asymptotically stable for the unforced system, then the system is $\mathcal{L}_p-$stable and has finite $\mathcal{L}_p-$gain with classical norms, i.e. relation (11) is satisfied for p when $p \geq {}^{(d+\max r_x)}/_{(d-s)}$ (including $p = \infty$). Note that in the proof of [17, Theorem 6.1], the author quotes the converse Lyapunov Theorem, but wrongly sets the homogeneous degree of $V(x)$ to $p-d \geq r_0$ instead of $p-d \geq \max r_x$. [17] does actually not consider ISS stability for **weighted** homogeneous systems. It rather deals with $\mathcal{L}_\infty-$stability, which is different. Moreover, due to the restrictions imposed on the homogeneity of the output function $y = h(x)$, the particular case $y = h(x) = x$ cannot be considered, because then $r_y = (d - s)\mathbf{1}_n \leq (d + r_0) \neq r_x$, unless $r_x = r_0\mathbf{1}_n = (d - s)\mathbf{1}_n$. In such case, the homogeneous weight of states, input and output are all equal to r_0, which reduces the conclusion of ISS stability to **classical** homogeneous systems. In contrast to $\mathcal{L}_2-$stability, $\mathcal{L}_p-$stability is not characterized using a Hamilton-Jacobi Inequality, despite the fact that the system is affine in the input, and in that case this is still feasible.

Our Theorem 2 generalizes these results to an arbitrary homogeneous system, without the restrictions imposed in paper [17]: We propose a system not affine in the input, and the homogeneity weights and degree are strongly relaxed. We show that to obtain finite $\mathcal{L}_p-$gain it is necessary to consider homogeneous norms. Only in particular cases this is valid for classical signal norms, namely when $r_y = \ell\mathbf{1}_o$ and $r_u = \ell\mathbf{1}_m$. But even for the case with classical norms, our results are more general than those proposed in [17]. We characterize all properties in a unified way using the inequality (22), which is more general than the Hamilton-Jacobi inequality that is not always attainable in the non-affine input case. In fact, we have generalized the result of $\mathcal{H}_\infty$ norm from [17], [25] to be applicable to all continuous homogeneous systems with different homogeneous weights for input and output, as well as not necessarily being input affine. The conclusion of $\mathcal{L}_p-$stability from [17] is also included in our Theorem 2 for the $\lambda-$scaled weight and degree, with $\lambda = {}^1/_{(d-s)}$. In this case the weights are $\lambda r_u = \mathbf{1}_m$, $\lambda r_y = \mathbf{1}_o$, and the $\mathcal{L}_p-$stability is valid for $p - \lambda d \geq \lambda \max r_x$.

In [26] and [13], ISS and other related properties are studied for general weighted homogeneous systems (1), and they generalize the results of [9], [17] relating the internal stability of the unforced system and ISS stability. Our results on ISS reproduce those of [13] and [26], although we emphasize the linear relationship between the input and the state (35) when using homogeneous norms, which is not clear in [13] and [26]. This linearity issue is clarified in [27] for the more general version of geometric homogeneity by using solely homogeneous norms in contrast to the mixed use of homogeneous norms and Euclidean norms in [26]. However, [13], [26], [27] do not consider $\mathcal{L}_p-$stability for any value of p. In contrast to [13], [26], [27], we clarify the situation about $\mathcal{L}_p-$stability for general weighted homogeneous systems for arbitrary values of p and the linear homogeneous ISS gain is related to such homogeneous $\mathcal{L}_p-$gain in the proof of Theorem 4. Furthermore, the homogeneous small-gain theorem is considered in both [13], [27]. In [13], using the classical norms, the nonlinear ISS-gain is a $\mathcal{K}$ function. In [27] no external inputs are considered. The small-gain theorems derived in both [13], [27] use the (linear or nonlinear) ISS-gain to assure the closed loop stability, and thus are related to $p = \infty$. In contrast, the homogeneous small-gain theorem obtained in this paper adopts any homogeneous $\mathcal{L}_p-$gain (when it exists, including $\mathcal{L}_{\infty h}-$gain and also homogeneous ISS-gain) to verify closed loop stability.

In [30] the authors have introduced and studied the $\mathcal{L}_{2h}-$gain for the special case of the Continuous Super-Twisting Like algorithm (CSTLA), and it was used to optimize the parameter selection of the algorithm. That case is included in Theorem 2 for $p = 2$. A brief review and extension of the results for the CSTLA will be presented in Section IX-C.

In [31], the authors adopted a homeomorphic coordinate transformation that can be related to our paper. This is done using the companion signals introduced in Remark 6: A homogeneous $\mathcal{L}_p-$stable system, according to our Definition 5, is also $\mathcal{L}_p-$stable, using the traditional signal norms, from the transformed input $S(u) = u^{\frac{1}{\tau_u}}$ to transformed output $T(y) = y^{\frac{1}{\tau_y}}$ since (17) can also be written as

$$\left\| y^{\frac{1}{\tau_y}} \right\|_{\mathcal{L}_p} \leq \gamma_p \left\| u^{\frac{1}{\tau_u}} \right\|_{\mathcal{L}_p} + b_p \,, \quad \forall\, u \in \mathcal{L}_p^m \,.$$

In this sense, in the context of homogeneous systems, we generalize this idea to arbitrary $\mathcal{L}_p-$norms and not only to the $\mathcal{L}_2-$norm considered in [31].

However, beyond this connection with [31], the objectives and scope of both papers are rather different. [31] is concerned with the relationship between $\mathcal{L}_2-$stability and ISS or iISS for general nonlinear systems. Also they show the existence of appropriate input $S(\cdot)$ and output $T(\cdot)$ homeomorphisms, but without providing a way to obtain such functions.

Our paper is concerned specifically with general weighted homogeneous systems. It is shown that the homogeneous vector and signal norms are a natural setting for studying input-output and input-to-state stability. One obtains linear finite-gains in all cases, a clear relationship with the internal stability is derived and a method to calculate the gains is provided by means of a dissipation inequality. Moreover, arbitrary values of p are allowed, instead of only $p = 2$ as in [31].

VIII. ESTIMATION OF THE HOMOGENEOUS $\mathcal{L}_p$−GAIN FOR A STATE SPACE SYSTEM

Theorem 1 shows that $\mathcal{L}_{\text{ph}}$−stability can be characterized by the dissipation inequality (22) and that the $\mathcal{L}_{\text{ph}}$−gain can be estimated using (22). This inequality, in the unknown function $V(x)$ and the (positive) real constant γ, is a linear partial differential inequality in $\frac{\partial V}{\partial x}$ and in γ^p, but it depends on two variables $(x, u) \in \mathbb{R}^{n+m}$. As described in Section IV-B, for systems affine in the input it is possible to transform the dissipation inequality (22) into the HJI (29) which only depends on $x \in \mathbb{R}^n$. However, (29) is a highly non-linear partial differential inequality in $\frac{\partial V}{\partial x}$ and in γ^p. In any case, both kinds of partial differential inequalities are hard to solve (see for example [2, Chapter 11]).

For smooth dynamical systems there is a very complete theory [2], [6]–[8]. A geometric interpretation allows the use of the Hamiltonian dynamics that involves the linearization of the system. Also, using the linearization and quadratic storage function the HJI (29) becomes an Algebraic Riccati Inequality (refer to Section IV-C) and leads to approximate solutions in the non-linear case. However, in general (in particular for degrees different from zero), the linearization of homogeneous systems are not useful because they either do not exist or vanish. This failure of the linearization for continuous homogeneous systems has been already illustrated in Section II-D. Therefore, usual methods for non-linear systems are not appropriate for solving the homogeneous inequalities (22) and (29). Apparently, there are very few published results providing detailed methods to derive such solution for homogeneous systems. The previous works [9], [17], [25], dealing with $\mathcal{L}_p$−stability of homogeneous systems, do not provide methods to estimate the value of $\mathcal{L}_p$−gains, even for the special classes of homogeneous systems considered. [30] is a forerunner of the method to be proposed here, applied only to a second order system.

A simple (but important) observation about (22) is that for a solution $V(x)$ to exist it is necessary that (22) is fulfilled for $u = 0$, i.e. (25) is valid. This means that $V(x)$ is (for $\epsilon = 0$) at least a weak Lyapunov function for system $\dot{x} = f(x, 0)$. In fact, Theorem 2 reveals the nice and particular property of homogeneous systems that every strict (homogeneous) Lyapunov function of $\dot{x} = f(x, 0)$ is indeed a global solution to the inequality (22) and, if appropriate, the HJI (29), for some semi interval of values $\gamma \in [\gamma^\star, \infty)$.

This observation leads to a useful estimation method for a given non-linear homogeneous system with values of γ satisfying inequality (22) or (29). The method consists of two steps:

1) Determine a strict, homogeneous, and continuously differentiable Lyapunov function $V_l(x)$ of homogeneity degree $p - d$ for system $\dot{x} = f(x, 0)$. The converse Lyapunov Theorem for homogeneous systems [15], [22, Theorem 5.8] assures the existence of such a function if $p > d + \max r_x$.
2) Using this function $V_l(x)$ find the minimal value of γ that satisfies (22) or (29). As shown in the proof of Theorem 2 this is always feasible by selecting an appropriate $a > 0$ and setting $V(x) = aV_l(x)$.

The optimal value of γ corresponding to the given V will be termed $\gamma^\star(V) = \min\{\gamma \mid (22) \text{ or } (29) \text{ is satisfied with } V(x)\}$. $\gamma^\star(V)$ provides an upper bound of the gain $\gamma_{\text{ph}}(\Sigma)$, i.e. $\gamma_{\text{ph}}(\Sigma) \le \gamma^\star(V)$.

The first step requires the construction of a smooth homogeneous Lyapunov function, which is not a simple task. Recently, there have been some advances in obtaining Lyapunov functions for homogeneous systems (see e.g. [37] and the references therein).

For the rest of this section the second step of the procedure will be presented, i.e. given $V(x)$ how to find the best estimate of γ. We provide two strategies for the calculation, based either on the dissipation inequality (22) or on the HJI (29). Clearly, the second path is only valid for some classes of systems, e.g. systems affine in the input and whose output $y = h(x)$ is devoid of u.

A. CALCULATION OF γ FROM THE DISSIPATION INEQUALITY

In this case we use the dissipation inequality (22) directly to calculate the smallest value of γ w.r.t. the given V, i.e. $\gamma^\star(V) = \min\{\gamma \mid (22) \text{ is satisfied with } V\}$.

Proposition 2: Assume the condition of Definition 6 is met, the smallest value of γ w.r.t. V that satisfies (22) is

$$\gamma^{\star p}(V) = \max_{\|(x,u)\|=1,\, \|u\| \neq 0} \{\zeta(x, u)\} \tag{47}$$

where $\zeta(x, u)$ for $\|u\| \neq 0$ is the function

$$\zeta(x, u) \triangleq \frac{\frac{\partial V(x)}{\partial x} f(x, u) + \|h(x, u)\|^p_{r_y,q} + \epsilon \|x\|^p_{r_x,q}}{\|u\|^p_{r_u,q}}.$$

Recall that from the proof of Theorem 2, the required function $V(x)$ for the proposition can be obtained as $V(x) = aV_l(x)$, for $a > \underline{a} \triangleq \frac{\alpha+\epsilon}{c}$. $V_l(x)$ is any strict, homogeneous, and continuously differentiable Lyapunov function of degree $p - d > \max r_x$, whose existence is assured by the converse Lyapunov Theorem for the continuous asymptotically stable system $\dot{x} = f(x, u)$ when $u = 0$.

Proof: In the proof of Theorem 2 the existence of $\gamma^\star(V)$ is assured, such that for any $\gamma \ge \gamma^\star(V)$, $J(x, u) \le 0$. Clearly, such γ has to satisfy $\gamma^p \ge \zeta(x, u)$ for all $(x, u) \in \mathbb{R}^{n+m}$, and therefore $\gamma^{\star p}(V) = \max_{(x,u)\in\mathbb{R}^{n+m}}\{\zeta(x, u)\}$. (47) states that this maximum is attained on the unit sphere $\{\|(x, u)\| = 1\}$. This follows from the observation that the function $\zeta(x, u)$ is homogeneous of degree zero, i.e. $\zeta(\nu_\kappa^{r_x}(x), \nu_\kappa^{r_u}(u)) = \kappa^0\zeta(x, u) = \zeta(x, u)$, and all values of a degree zero function can be attained on any unit sphere w.r.t. some norm. Finally, ζ has degree zero since its numerator and denominator are both homogeneous of degree p, because V is homogeneous of degree $p - d$ as demanded in Definition 6. □

The search procedure for $\max_{\|(x,u)\|=1}\{\zeta(x, u)\}$ is described in Algorithm 1. We use the $\|(x, u)\|_2$−norm as example.

Algorithm 1 Procedure of Search for $\max_{\|(x,u)\|=1} \zeta(x,u)$ in Proposition 2

$\zeta_c = -10^{300}$
for $j_1 = 1, \ldots, \upsilon$ **do** ▷ Rough search
  $x_1 = -1 + \frac{2j_1}{\upsilon+2}$
  for $j_2 = 1, \ldots, \upsilon$ **do**
    $x_2 = \sqrt{|1-|x_1|^2|}\left(-1+\frac{2j_2}{\upsilon+2}\right)$
    …
    for $j_{n+1} = 1, \ldots, \upsilon$ **do**
      $u_1 = \sqrt{|1-\|x\|_2^2|}\left(-1+2\frac{j_{n+1}-1}{\upsilon-1}\right)$
      …
      for $j_{n+m-1} = 1, \ldots, \upsilon$ **do**

$$u_{m-1} = \sqrt{\left|1-\|x\|_2^2 - \sum_{i=1}^{m-2}|u_i|^2\right|} \times \left(-1+2\frac{j_{n+m-1}-1}{\upsilon-1}\right)$$

$$u_m = \pm\sqrt{1-\sum_{i=1}^{m-1}|u_i|^2 - \|x\|_2^2}$$

        Evaluate $\zeta(x,u)$ in (47)
        if $\zeta(x,u) > \zeta_c$ **then** ▷ Record the maximal $\zeta(x,u)$ in rough search
          $\zeta_c = \zeta(x,u)$
          $(x_r, u_r) = (x,u)$ ▷ Record the point for refined search
        end if
      end for
      …
    end for
    …
  end for
end for
repeat ▷ Several rounds of refined search
  $\zeta_p = \zeta_c$ ▷ Record ζ_p as the previous maximal ζ
  for Divide the neighborhood of (x_r, u_r) similarly as Rough search **do**
    Evaluate $\zeta(x,u)$
    if $\zeta(x,u) > \zeta_c$ **then**
      $\zeta_c = \zeta(x,u)$
      $(x_r, u_r) = (x,u)$ ▷ Record for next round
    end if
  end for
until $(\zeta_c - \zeta_p)/\zeta_p \leq 10^{-7}$ ▷ maximal value of two rounds are close enough
$\gamma_c = \sqrt{\zeta_c}$

After the rough search shown in Algorithm 1, several rounds of refined search around the point of (x,u) recorded in the last round should be carried out. The maximal value of $\zeta(x,u)$ on unit sphere should happen in a neighborhood of such point of (x,u), since the function $\zeta(x,u)$ is continuous on the surface of the unit sphere w.r.t. (x,u). Note that in practice, several local maxima might occur. In such case, refined searches around each local maximum need to be carried out.

Therefore, with each fixed V the rough search for $\max_{\|(x,u)\|=1}\{\zeta(x,u)\} \leq 0$ takes υ^{n+m-1} steps evaluating the value of $\zeta(x,u)$. The refined search of step 6 around several local maximal points collected in the rough search usually takes less steps. Thus the computational complexity is of $O(\upsilon^{n+m-1})$.

B. CALCULATION OF γ FROM THE HAMILTON-JACOBI INEQUALITY

For a homogeneous system affine in the input, given by (28), instead of the dissipation inequality (22) we can consider the simpler HJI (29). These are equivalent as shown in Lemma 7.

Proposition 3: Consider a continuous homogeneous system affine in the input (28)*. Assume the condition of Definition 6 is met and select $q = p$. The smallest value of γ w.r.t. V that satisfies* (29) *can be obtained by solving the optimization problem*

$$\gamma^\star(V) = \arg\min_{\gamma\geq 0}\left\{\max_{\|x\|=1} \mathcal{J}(x;\gamma) \leq 0\right\} \tag{48}$$

where

$$\mathcal{J}(x;\gamma) = \frac{\partial V(x)}{\partial x} f(x) + \|h(x)\|_{r_y,p}^p + \epsilon\|x\|_{r_x,p}^p + \mathcal{Q}(x;\gamma) \tag{49}$$

and

$$\mathcal{Q}(x;\gamma) \triangleq \sum_{i=1}^m \left(\frac{r_{u_i}}{p\gamma^p}\right)^{\frac{r_{u_i}}{p-r_{u_i}}} \left(1-\frac{r_{u_i}}{p}\right)\left|\frac{\partial V(x)}{\partial x} g_i(x)\right|^{\frac{p}{p-r_{u_i}}}.$$

Proof: Note that the HJI (29) can be written as

$$\mathcal{J}(x;\gamma) \leq 0$$

where

$$\mathcal{J}(x;\gamma) \triangleq J(V_x(x), x, u^\star(x))$$

and $V_x(x) = \frac{\partial V(x)}{\partial x}$ for the given function $V(x)$. Function $\mathcal{J}(x;\gamma)$ is continuous and homogeneous in x of degree p. Moreover, by the hypothesis on V, $\lim_{\gamma\to\infty}\mathcal{J}(x;\gamma) < 0$, since $p > \max r_u$ and the terms $\left(1-\frac{r_{u_i}}{p}\right) > 0$ and $\mathcal{Q}(x;\gamma) \geq 0$ shown in Lemma 7. In the proof of Theorem 2 the existence of $\gamma^\star(V)$ is assured, such that for any $\gamma \geq \gamma^\star(V)$, $J(V_x, x, u) \leq 0$, or equivalently, $\mathcal{J}(x;\gamma) \leq 0$. The value $\gamma^\star(V)$ satisfies

$$\max_{x\in\mathbb{R}^n} \mathcal{J}(x;\gamma^\star(V)) = 0\,.$$

Due to the homogeneity of $\mathcal{J}(x;\gamma)$ it suffices to restrict the maximization to the unit sphere, i.e. (48). □

The result of Proposition 3 suggests the Algorithm 2 to find an estimate of $\gamma^\star(V)$: $\max_{\|x\|=1}\mathcal{J}(x;\gamma_i)$.

Note that this procedure is analogous to the search performed to estimate the $\mathcal{L}_2$−gain using the Hamiltonian matrix (see the end of Section IV-C). For the Hamiltonian matrix we need to check whether its eigenvalues stay on the imaginary axis with each γ. Here we need to check the sign of $\max_{\|x\|=1}\mathcal{J}(x;\gamma)$ for each V that satisfies (22).

Algorithm 2 Search Procedure for Proposition 3

$\gamma_u = 10^{300}$ ▷ Initialize the upper limit of γ
$\gamma_l = 10^{-300}$ ▷ Initialize the lower limit of γ
repeat
  Choose one $\gamma \in (\gamma_l, \gamma_u)$, e.g. $\gamma = (\gamma_u + \gamma_l)/2$
  Evaluate $\max_{\|x\|=1} \mathcal{J}(x;\gamma)$ in (48)
  if $\max_{\|x\|=1} \mathcal{J}(x;\gamma) \leq 0$ **then**
    $\gamma_u = \gamma$
  else
    $\gamma_l = \gamma$
  end if
until $(\gamma_u - \gamma_l)/\gamma_l \leq 10^{-7}$
$\gamma^\star = \gamma_u$

Remark 17: The search of $\max_{\|x\|=1} \mathcal{J}(x;\gamma_i)$ *is similar to that described in Section VIII-A. The difference is that we only need to search on* $\|x\| = 1$ *instead of* $\|(x, u)\| = 1$. *Besides that, a further iteration on* γ *is necessary. Thus by using the search of* (48) *we can reduce the computational complexity from* $O(\upsilon^{n+m-1})$ *in* (47) *to* $O(\upsilon^n)$. *When* $r_u = \ell \mathbf{1}_m$, *the computational complexity can be further reduced to* $O(\upsilon^{n-1})$, *with a similar method in Proposition 2, i.e.*

$$\gamma^\star(V) = \left(\max_{\|x\|=1} \frac{\left(\frac{\ell}{p}\right)^{\frac{\ell}{p-\ell}} \left(1-\frac{\ell}{p}\right) \sum_{i=1}^m \left|\frac{\partial V(x)}{\partial x} g_i(x)\right|^{\frac{p}{p-\ell}}}{-\frac{\partial V(x)}{\partial x} f(x) - \|h(x)\|^p_{r_y,p}} \right)^{\frac{p-\ell}{p\ell}}$$
$$= \left(\max_{\|x\|=1} \frac{\left(\frac{\ell}{p}\right)^{\frac{\ell}{p-\ell}} \left(1-\frac{\ell}{p}\right) \sum_{i=1}^m \left|\frac{\partial V(x)}{\partial x} g_i(x)\right|^{\frac{p}{p-\ell}}}{-J(V_x, x, 0)} \right)^{\frac{p-\ell}{p\ell}}. \tag{50}$$

Refer to the example of CSTLA in [30].

Note that different Lyapunov functions lead to different gain estimated of the system, so that we can eventually improve the quality of the estimation. A further strategy to improve the estimate of $\gamma_{\text{ph}}(\Sigma)$ is to use a parametrized family of Lyapunov functions $V_l(x, \lambda)$, $\lambda \in \Lambda$, with Λ a set of values of λ, and such that V_l satisfies (25). In that case we can solve an optimization problem to find the minimal value of $\gamma^\star$ in this family of Lyapunov functions. This is basically what can be done in the LTI case, where the Lyapunov functions are quadratic and parametrized by the symmetric and positive definite matrix P. In any case, this procedure does not guarantee that the actual value of $\gamma_{\text{ph}}(\Sigma)$ can be calculated. Only an upper bound will be obtained.

Both methods proposed in Propositions 2 and 3 also apply to LTI systems, for which we can fix $r_x = \mathbf{1}_n$, $r_y = \mathbf{1}_o$, $r_u = \mathbf{1}_m$, and $d = 0$. For $p = 2$, the results are recalled in Section IV-C. The HJI (29) reduces to the ARI (32) (for $D = 0$), which is even independent of x. For $p \neq 2$, function $\mathcal{J}(x;\gamma)$, given by (49) and appearing in the HJI (29), is not anymore a quadratic form since $\mathcal{J}(x;\gamma)$ is $\mathbf{1}_n-$homogeneous of degree $p \neq 2$. Furthermore, the HJI does not lead to a matrix inequality and thus is much harder to solve. However, the HJI provides a method to estimate an upper bound of the $\mathcal{L}_p-$gain of the LTI system. In fact, there are not many methods to calculate such a gain for LTI systems. Note that for $p \neq 2$ the required Lyapunov function for the HJI is required to be $\mathbf{1}_n-$homogeneous of degree p. It is not easy to construct such Lyapunov functions, even for LTI systems. One possibility is to calculate a quadratic one and then construct $V(x) = \left(x^T P x\right)^{\frac{p}{2}}$, which satisfies the homogeneity condition for the HJI. Such Lyapunov function is smooth only if $p \geq 2$.

Therefore, by using Proposition 2 or 3, we have contributed a method to calculate the $\mathcal{L}_{\text{ph}}-$gain defined by (21). This is an extension from [30] where it is applied only to a special continuous homogeneous system, the CSTLA, for $p = 2$, and it is also an extension from [17], [25], where it applies only to the case $r_u = \ell \mathbf{I}_m$, $r_y = \ell \mathbf{I}_o$ and vector field being affine in the input. For LTI systems, the proposed method provides an upper bound of the $\mathcal{L}_p-$gain for any $p > 1$ ($r_x = \mathbf{1}_n$, $d = 0$).

C. UPPER BOUND OF HOMOGENEOUS ISS GAIN AND $\mathcal{L}_{\infty h}-$GAIN

In contrast to the $\mathcal{L}_{\text{ph}}-$gain, which can be derived through dissipation inequality or HJI by using the Proposition 2 or 3, in Section V the upper bound of the homgoeneous ISS gain depends on the value of some $\mathcal{L}_{\text{ph}}-$gain in (43). Further, the upper bound of $\mathcal{L}_{\infty h}-$gain depends again on γ_{iss} in (44).

In order to calculate an upper bound of γ_{iss}, constants α_1 and α_2 in (43) need to be derived first. From Lemma 6, since both $\|x\|^{p-d}_{r_x,q}$ and $V(x)$ are positive definite continuous r_x-homogeneous functions of degree $p - d$, we have from (38) for any $q \geq 1$ that

$$\alpha_1 = \min_{\|x\|_{r_x,q}=1} V(x) = \min_{\|x\|=1} \frac{V(x)}{\|x\|_{r_x,q}},$$
$$\alpha_2 = \max_{\|x\|_{r_x,q}=1} V(x) = \max_{\|x\|=1} \frac{V(x)}{\|x\|_{r_x,q}}.$$

Then combined with any $\mathcal{L}_p-$gain, relation (43) provides an upper bound of the ISS gain.

Finally, an upper bound for the $\mathcal{L}_{\infty h}-$gain can be derived similarly. The constant c_q in (44) can be calculated as

$$c_q = \max_{\|(x,u)\|_{(r_x,r_u),q}=1} \|h(x,u)\|_{r_y,q}$$
$$= \max_{\|(x,u)\|=1} \frac{\|h(x,u)\|_{r_y,q}}{\|(x,u)\|_{(r_x,r_u),q}}.$$

Thus $c_q(\gamma_{\text{iss}} + 1)$ serves as an upper bound of the $\mathcal{L}_{\infty h}-$gain from (44). All three constants can be derived from the method shown in Section VIII-A, among which c_q is independent of the choice of storage function $V(x)$.

IX. EXAMPLES

In this section we show how the homogeneous $\mathcal{L}_p-$gain can be derived analytically referring to two examples, namely a continuous memoryless input-output map and a continuous scalar homogeneous system. In both examples, the input whose ratio of the $\mathcal{L}_p-$norm of output over input (denoted as Γ) being equal to the analytical homogeneous $\mathcal{L}_p-$gain, can be found in closed terms. Thus, the analytical result is the homogeneous $\mathcal{L}_p-$gain.

In the third example of continuous super-twisting like algorithm, we can only find some upper bound of the homogeneous $\mathcal{L}_p-$gain under different parameters. Fortunately, some inputs, whose Γ is equal to the collected homogeneous $\mathcal{L}_2-$gain, can also be found under some parameter ranges. Thus these collected upper bounds are also tight bounds, where the analytical form of the homogeneous $\mathcal{L}_2-$gain can be found.

As analyzed in Section VII, our approach coincides with previous works under some particular assumption on weight vectors. However, breaking such assumptions is not allowed in the previous works. Thus in this section, no comparisons to previous works are provided.

A. CONTINUOUS MEMORYLESS INPUT-OUTPUT MAPS

A function $g:\mathbb{R}^m\to\mathbb{R}^o$ can be viewed as an operator G that assigns to every input signal $u(\cdot)$ the output signal $y(\cdot)=G(u(\cdot))$ given by $y(t)=g(u(t))$, i.e. the output $y(t)$ depends only on the present value of the input $u(t)$, and not on its past or future, and thus it is called memoryless. It is homogeneous if $g(\nu_\kappa^{r_u}u)=\nu_\kappa^{r_y}(g(u))$ for every $u\in\mathbb{R}^m$ and $\kappa>0$. This means that each component function $g_i:\mathbb{R}^m\to\mathbb{R}$, $i=1,\ldots,o$, is r_u-homogeneous of degree r_{y_i}. According to Definition 2 the operator G is of arbitrary homogeneous degree d. We assume that the function g is continuous w.r.t. u.

Theorem 7: A continuous memoryless Input-Output Map $y=g(u)$ is finite-gain $\mathcal{L}_{ph}-$stable with zero bias, for every $1\le p\le\infty$. Its $\mathcal{L}_{ph}-$gain is the same for all values of $1\le p\le\infty$, i.e. $\gamma_{ph}=\gamma$, and it can be calculated as

$$\gamma=\max_{\|u\|_{r_u,q}=1}\|g(u)\|_{r_y,q}\,. \tag{51}$$

Proof: Since g is continuous w.r.t. u, by setting $\chi(u)=\|g(u)\|_{r_y,q}$ and $\phi(u)=\|u\|_{r_u,q}$, whose homogeneous degrees are both 1 and both being continuous w.r.t. u, it follows from Lemma 6 that

$$\|y\|_{r_y,q}=\|g(u)\|_{r_y,q}\le\left(\max_{\|u\|_{r_u,q}=1}\|g(u)\|_{r_y,q}\right)\|u\|_{r_u,q}\,. \tag{52}$$

Thus, the operator G has finite $\mathcal{L}_{\infty h}-$gain $\gamma_{\infty h}=\gamma$ as in (51). Furthermore, it follows from (52) that for every $p\ge 1$ and $u\in\mathcal{L}^m_{r_u,p}$

$$\begin{aligned}\|y(\cdot)\|_{r_y,\mathcal{L}_p}&=\left(\int_0^\infty\|y(t)\|^p_{r_y,q}\mathrm{d}t\right)^{\frac{1}{p}}\\&\le\gamma\left(\int_0^\infty\|u(t)\|^p_{r_u,q}\mathrm{d}t\right)^{\frac{1}{p}}=\gamma\|u(\cdot)\|_{r_u,\mathcal{L}_p}\,.\end{aligned}$$

And therefore the operator G is $\mathcal{L}_{\rm ph}-$stable with $\mathcal{L}_{\rm ph}-$gain given by $\gamma_{\rm ph}=\gamma$ with zero bias. Such $\mathcal{L}_{\rm ph}-$gain can be achieved by the particular $u^\star$ that maximizes the value $\max_{\|u\|_{r_u,q}=1}\|g(u)\|_{r_y,q}$. Thus the $\mathcal{L}_{\rm ph}-$gain equals γ for all $1\le p<\infty$. For the case $p=\infty$, (52) suffices since the operator G is memoryless. $\square$

Take a simple example of a continuous memoryless homogeneous Input-Output Map given by

$$y(t)=g(u(t))=\left\lceil c_1\lceil u_1(t)\rfloor^3+c_2\lceil u_2(t)\rfloor^2\right\rfloor^{\frac{1}{3}}$$

with $m=2$, $r=1$, and weight vectors $r_u=(2,3)$, $r_y=2$, and c_1, c_2 some constants.

First note that, if we set (for simplicity) $u_1\equiv 0$, then the function becomes $y=g(u_2)=\lceil c_2\rfloor^{\frac{1}{3}}\lceil u_2\rfloor^{\frac{2}{3}}$. Since $\frac{dg(u_2)}{du_2}=\frac{2}{3}\frac{\lceil c_2\rfloor^{\frac{1}{3}}}{|u_2|^{\frac{1}{3}}}$, the "linearization" at $u_2=0$ is not well-defined because the derivative is unbounced at this point. This illustrates the fact that linearization for homogeneous systems is usually not well-defined. Thus, the function $g(u_2)$ cannot be well approximated by a linear function, consequently, the classical "gain" fails.

From Theorem 7, it is finite-gain $\mathcal{L}_{\rm ph}-$stable for all $p\ge 1$ and its $\mathcal{L}_{\rm ph}-$gain is given by

$$\begin{aligned}\gamma&=\max_{\|u\|_{r_u,q}=1}|g(u)|^{\frac{1}{2}}\\&=\max_{\left\{|u_1|^{\frac{q}{2}}+|u_2|^{\frac{q}{3}}=1\right\}}\left|c_1\lceil u_1\rfloor^3+c_2\lceil u_2\rfloor^2\right|^{\frac{1}{6}}\end{aligned} \tag{53}$$

To solve this maximization problem with constraints we can use the method of Lagrange multipliers, to find the (restricted) extremal points of the function $c_1\lceil u_1\rfloor^3+c_2\lceil u_2\rfloor^2$, i.e. we can solve the condition

$$\begin{aligned}&\frac{\partial}{\partial u}\left[c_1\lceil u_1\rfloor^3+c_2\lceil u_2\rfloor^2+\lambda\left(|u_1|^{\frac{q}{2}}+|u_2|^{\frac{q}{3}}-1\right)\right]\\&\quad=\left[3c_1|u_1|^2+\lambda\frac{q}{2}\lceil u_1\rfloor^{\frac{q-2}{2}},\ 2c_2|u_2|+\lambda\frac{q}{3}\lceil u_2\rfloor^{\frac{q-3}{3}}\right]=0\,.\end{aligned} \tag{54}$$

This equation has three non-trivial solutions (we assume $q>6$)

$$\bar{u}^{(1)}=-\begin{bmatrix}\left\lceil\frac{6c_1}{q\lambda}\right\rfloor^{\frac{2}{q-6}}\\\left\lceil\frac{6c_2}{q\lambda}\right\rfloor^{\frac{3}{q-6}}\end{bmatrix},$$

$$\bar{u}^{(2)}=-\begin{bmatrix}0\\\left\lceil\frac{6c_2}{q\lambda}\right\rfloor^{\frac{3}{q-6}}\end{bmatrix},$$

$$\bar{u}^{(3)}=-\begin{bmatrix}\left\lceil\frac{6c_1}{q\lambda}\right\rfloor^{\frac{2}{q-6}}\\0\end{bmatrix}.$$

Each one of these solutions satisfies the restriction $|u_1|^{\frac{q}{2}}+|u_2|^{\frac{q}{3}}=1$, if the Lagrange parameter is selected as

$$\lambda^{(1)}=\pm\left(\left|\frac{6c_1}{q}\right|^{\frac{q}{q-6}}+\left|\frac{6c_2}{q}\right|^{\frac{q}{q-6}}\right)^{\frac{q-6}{q}},$$

$$\lambda^{(2)} = \pm\left|\frac{6c_2}{q}\right|,$$
$$\lambda^{(3)} = \pm\left|\frac{6c_1}{q}\right|,$$

respectively. Replacing the two possible values of $\lambda^{(i)}$ into the corresponding solution $\bar{u}^{(i)}$, for $i = 1, 2, 3$, we obtain six extremal points

$$\bar{u}^{(1)} = \pm\begin{bmatrix} \dfrac{\left\lceil \frac{6c_1}{q}\right\rfloor^{\frac{2}{q-6}}}{\left(\left|\frac{6c_1}{q}\right|^{\frac{q}{q-6}}+\left|\frac{6c_2}{q}\right|^{\frac{q}{q-6}}\right)^{\frac{2}{q}}} \\ \dfrac{\left\lceil \frac{6c_2}{q}\right\rfloor^{\frac{3}{q-6}}}{\left(\left|\frac{6c_1}{q}\right|^{\frac{q}{q-6}}+\left|\frac{6c_2}{q}\right|^{\frac{q}{q-6}}\right)^{\frac{3}{q}}} \end{bmatrix},$$
$$\bar{u}^{(2)} = \pm\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$\bar{u}^{(3)} = \pm\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Evaluating $\left|c_1\lceil u_1\rfloor^3 + c_2\lceil u_2\rfloor^2\right|$ at these (six) points, we obtain three values, from which we have to choose

$$\max_{\left\{|u_1|^{\frac{q}{2}}+|u_2|^{\frac{q}{3}}=1\right\}} \left|c_1\lceil u_1\rfloor^3 + c_2\lceil u_2\rfloor^2\right|$$
$$= \left\{\left(|c_1|^{\frac{q}{q-6}} + |c_2|^{\frac{q}{q-6}}\right)^{\frac{q-6}{q}}, |c_2|, |c_1|\right\}$$

and thus the gain is given by

$$\gamma = \max\left\{\left(|c_1|^{\frac{q}{q-6}} + |c_2|^{\frac{q}{q-6}}\right)^{\frac{q-6}{6q}}, |c_2|^{\frac{1}{6}}, |c_1|^{\frac{1}{6}}\right\}.$$

Note that the $\mathcal{L}_{\mathrm{ph}}$−gain does not depend on p, but it depends on the selection of the qh−norm (which affects the value of $\mathcal{L}_{\mathrm{ph}}$−norm). Fig. 3 shows the value of the following homogeneous function

$$\Gamma(u) = \frac{\left|c_1\lceil u_1\rfloor^3 + c_2\lceil u_2\rfloor^2\right|^{\frac{1}{6}}}{\left||u_1|^{\frac{q}{2}} + |u_2|^{\frac{q}{3}}\right|^{\frac{1}{q}}}$$

of degree zero along the unit circle $\|u\|_2 = 1$ in the coordination of $u_1 = \sin\theta, u_2 = \cos\theta, \theta \in [0, 2\pi]$ with $c_1 = 1, c_2 = 2$ (only $\theta \in [0, \pi]$ is shown in Fig. 3 since $\Gamma(-u) = \Gamma(u)$). Note that $\gamma = \max_u \Gamma(u) = \max_\theta \Gamma(\theta)$. For $q = 2$, $\gamma = c_2^{\frac{1}{6}} = 1.1225$. For $q = 20$, $\gamma = \left(|c_1|^{\frac{q}{q-6}} + |c_2|^{\frac{q}{q-6}}\right)^{\frac{q-6}{6q}} = 1.1646$.

Besides Theorem 7, a discontinuous memoryless homogeneous Input-Output Map G could also have a finite $\mathcal{L}_{\mathrm{ph}}$−gain. Take another example of memoryless homogeneous Input-Output Map, $y(t) = u_1(t)^2\mathrm{sign}\,(u_2(t))$, with homogeneous weight of $r_u = (1, 1), r_y = 2$. Note that y is discontinuous along $u_1 \neq 0, u_2 = 0$. Yet, with an approach similar to shown above, it is easily verified that $\gamma = 1$.

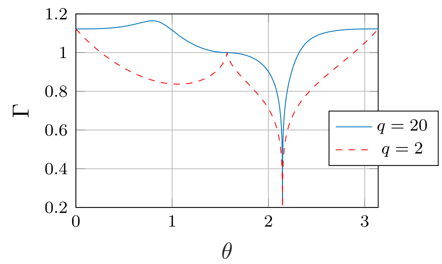

FIGURE 3. Γ for (53) along unit sphere, with $c_1 = 1, c_2 = 2$.

B. SCALAR HOMOGENEOUS SYSTEM

We would like to show how to derive the $\mathcal{L}_{\mathrm{ph}}$−gain analytically for SISO scalar homogeneous dynamics by using the methods in Theorem 2, i.e. for

$$\dot{x} = -k\lceil x\rfloor^{\frac{1}{z_3}} + b\lceil u\rfloor^{\frac{1}{z_1}}, \quad y = c\lceil x\rfloor^{\frac{z_2}{z_3}}, \tag{55}$$

where $z_1, z_2, z_3, k \in \mathbb{R}^+$, $b, c \in \mathbb{R}\setminus\{0\}$, with homogeneous weight as $r_x = \lambda z_3, r_u = \lambda z_1, d = \lambda(1 - z_3), r_y = \lambda z_2$, for any $\lambda > 0$. We leave the free parameter $\lambda > 0$ in order to illustrate the effect of scaling on the homogeneous weight and the degree of the system.

Function J from (23) is now

$$J(V_x, x, u) = V_x\left(-k\lceil x\rfloor^{\frac{1}{z_3}} + b\lceil u\rfloor^{\frac{1}{z_1}}\right) + |c|^{\frac{p}{\lambda z_2}}|x|^{\frac{p}{\lambda z_3}}$$
$$-\gamma^p|u|^{\frac{p}{\lambda z_1}} + \epsilon|x|^{\frac{p}{\lambda z_3}}.$$

Since (55) is a scalar SISO system, $\|\cdot\|_q = |\cdot|$ here, and $J(V_x, x, u)$ is independent of q. For general continuous homogeneous dynamics, $J(V_x, x, u)$ will be dependent on both p and q. To render this function homogeneous we can select

$$V_x = \frac{\partial V(x)}{\partial x} = \alpha\lceil x\rfloor^{\frac{p-\lambda}{\lambda z_3}}, \ \alpha > 0,$$

which indicate the storage function is now

$$V(x) = \alpha\frac{\lambda z_3}{p - \lambda + \lambda z_3}|x|^{\frac{p-\lambda+\lambda z_3}{\lambda z_3}}.$$

V is differentiable if

$$\frac{p - \lambda + \lambda z_3}{\lambda z_3} > 1 \Leftrightarrow p > \lambda.$$

This actually comes from $p > d + \max r_x = \lambda$. So it is possible to select a smooth function V for any $p \geq 1$ by choosing $\lambda < 1$. With this, replacing the variable V_x by $\alpha\lceil x\rfloor^{\frac{p-\lambda}{\lambda z_3}}$, and abusing the notation to keep the name J, we get

$$J(x, u) = -\left(k\alpha - |c|^{\frac{p}{\lambda z_2}} - \epsilon\right)|x|^{\frac{p}{\lambda z_3}} + b\alpha\lceil x\rfloor^{\frac{p-\lambda}{\lambda z_3}}\lceil u\rfloor^{\frac{1}{z_1}}$$
$$-\gamma^p|u|^{\frac{p}{\lambda z_1}}.$$

Inequality (22) corresponds to $J(x, u) \leq 0$. Since this requires that for $u = 0$ the inequality $J(x, 0) \leq 0$ is satisfied, we need

$$\alpha \geq \frac{|c|^{\frac{p}{\lambda z_2}} + \epsilon}{k}.$$

Although the system is not affine in the input, it is still possible to calculate the value $u^\star(x)$ that maximizes $J(x,u) \leq J(x, u^\star(x))$. If we select $p > r_u = \lambda z_1$, then the latter term $|u|^{\frac{p}{\lambda z_1}}$ in $J(x,u)$ is differentiable. Hence, calculating

$$\frac{\partial J(x,u)}{\partial u} = b\alpha \frac{1}{z_1} \lceil x \rfloor^{\frac{p-\lambda}{\lambda z_3}} |u|^{\frac{1}{z_1}-1} - \frac{p}{\lambda z_1}\gamma^p \lceil u \rfloor^{\frac{p}{\lambda z_1}-1} = 0$$

$$\Updownarrow$$

$$u^\star(x) = \left(\frac{\alpha \lambda z_1}{p z_1 \gamma^p}\right)^{\frac{\lambda z_1}{p-\lambda}} \lceil b \rfloor^{\frac{\lambda z_1}{p-\lambda}} \lceil x \rfloor^{\frac{z_1}{z_3}} \tag{56}$$

and therefore when $p > \lambda \max\{1, z_1\}$ we have

$$J(x, u^\star(x)) = \Bigg\{ -\alpha k + |c|^{\frac{p}{\lambda z_2}} + \epsilon + \alpha \left(\frac{\alpha\lambda}{p\gamma^p}\right)^{\frac{\lambda}{p-\lambda}} |b|^{\frac{p}{p-\lambda}} - \gamma^p \left(\frac{\alpha\lambda}{p\gamma^p}\right)^{\frac{p}{p-\lambda}} |b|^{\frac{p}{p-\lambda}} \Bigg\} |x|^{\frac{p}{\lambda z_3}}.$$

Inequality (22) is fulfilled for $\gamma \geq \gamma^\star(\alpha)$, given by

$$\gamma^\star(\alpha) = \frac{\alpha^{\frac{1}{\lambda}} |b|^{\frac{1}{\lambda}} \left(\frac{\lambda}{p}\right)^{\frac{1}{p}} \left(1 - \frac{\lambda}{p}\right)^{\frac{p-\lambda}{p\lambda}}}{\left(\alpha k - |c|^{\frac{p}{\lambda z_2}} - \epsilon\right)^{\frac{p-\lambda}{p\lambda}}}.$$

Since α is a parameter that we can select, we choose it such that $\gamma^\star(\alpha)$ attains its minimal value. Since the difference between the power on α in numerator and denominator, $\frac{1}{\lambda} - \frac{p-\lambda}{p\lambda} = \frac{1}{p} > 0$, it is clear that function $\gamma^\star(\alpha)$ has a minimal value in the interval $\alpha \in \left[\frac{|c|^{\frac{p}{\lambda z_2}}+\epsilon}{k}, \infty\right)$, obtained by solving $\frac{d\gamma^\star(\bar\alpha)}{d\alpha} = 0$,

$$\bar\alpha = \frac{p}{k\lambda}\left(|c|^{\frac{p}{\lambda z_2}} + \epsilon\right).$$

Function V is differentiable when $p > \lambda \max\{1, z_1\} \geq \lambda$ and $J(x,0) \leq 0$ from $\bar\alpha \geq \frac{|c|^{\frac{p}{\lambda z_2}}+\epsilon}{k}$. Therefore

$$\tilde\gamma = \gamma^\star(\bar\alpha) = \left(\frac{|b|}{k}\right)^{\frac{1}{\lambda}} \left(|c|^{\frac{p}{\lambda z_2}} + \epsilon\right)^{\frac{1}{p}}.$$

Selecting $\epsilon = 0$ we obtain the minimal value

$$\gamma_{\text{ph}} = \left(\frac{|b||c|^{\frac{1}{z_2}}}{k}\right)^{\frac{1}{\lambda}}, \tag{57}$$

for $p > \lambda \max\{1, z_1\}$ and $p \geq 1$. Note that this value depends on the selected (scaling) λ, as discussed in Remark 11. Remarkably, it is independent of p.

If we use control law (56) with $\bar\alpha$ plugged back, with the minimal value of γ_{ph}, in the system (55) then we get

$$\dot x = 0,$$
$$y = c\lceil x \rfloor^{\frac{z_2}{z_3}},$$
$$u^\star(x) = \left\lceil \frac{k}{b} \right\rfloor^{z_1} \lceil x \rfloor^{\frac{z_1}{z_3}}.$$

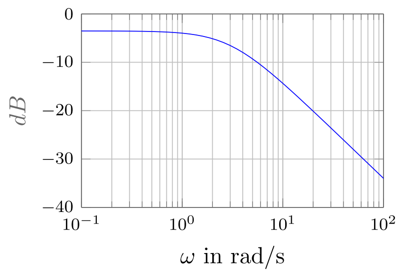

FIGURE 4. **Bode plot of system** (55) **with $z_1 = z_2 = z_3 = 1$ (linear case) and $k = 3$, $b = 2$, $c = 1$.**

The state trajectory for initial state x_0, the input and the output are constant, given by

$$x(\cdot) = x_0$$
$$u^\star(\cdot) = \left\lceil \frac{k}{b} \right\rfloor^{z_1} \lceil x_0 \rfloor^{\frac{z_1}{z_3}}$$
$$y(\cdot) = c\lceil x_0 \rfloor^{\frac{z_2}{z_3}}$$

for all $t \in [0, \infty)$. We calculate For this particular input and output we calculate the ratio of $\mathcal{L}_{\text{ph}}$−norm of output over input

$$\frac{\|(y(\cdot))_T\|_{r_y,\mathcal{L}_p}}{\|(u^\star(\cdot))_T\|_{r_u,\mathcal{L}_p}} = \frac{\left(\int_0^T \|y(t)\|^p_{r_y,q}\right)^{\frac{1}{p}}}{\left(\int_0^T \|u^\star(t)\|^p_{r_u,q}\right)^{\frac{1}{p}}} = \frac{T^{\frac{1}{p}} \left\| c\lceil x_0 \rfloor^{\frac{z_2}{z_3}} \right\|_{r_y,q}}{T^{\frac{1}{p}} \left\| \left\lceil \frac{k}{b} \right\rfloor^{z_1} \lceil x_0 \rfloor^{\frac{z_1}{z_3}} \right\|_{r_u,q}} = \left(\frac{|b||c|^{\frac{1}{z_2}}}{k}\right)^{\frac{1}{\lambda}}$$

where $(u(\cdot))_T$ is the truncated input signal. This value coincides with the estimated value of the gain (57). Therefore, since for all $1 \leq p \leq \infty$ and $p > \lambda \max\{1, z_1\}$ the gain γ_{ph} is upper and lower bounded by the same number, we conclude that $\gamma_{\text{ph}}(\Sigma) = \left(\frac{|b||c|^{\frac{1}{z_2}}}{k}\right)^{\frac{1}{\lambda}}$, and the bound is tight in this case. Consequently, we obtain

$$\gamma_{\text{ph}}(\Sigma) = \left(\frac{|b||c|^{\frac{1}{z_2}}}{k}\right)^{\frac{1}{\lambda}}.$$

This agrees with Remark 11, that is, the λr−homogeneous gain equals the r−homogeneous gain to the power λ. Interestingly, the $\mathcal{L}_{\text{ph}}$−gain is independent of p, z_1 and z_3. However, the worst input $u^\star(\cdot)$ depends on z_1 and z_3, but not on p. When $z_1 = z_2 = z_3 = 1$, the system is linear and $V(x) = \alpha\frac{\lambda}{p}|x|^{\frac{p}{\lambda}}$ is the quadratic storage function if we select $\frac{p}{\lambda} = 2$. The Bode plot, showing the gain between harmonic signals of different frequencies, e.g. $u(t) = \sin(\omega t)$, is plotted in Fig. 4. The $\mathcal{L}_2$−gain is the maximum of all these gains, and it is clear that for this LTI system the maximum is achieved for $\omega = 0$, that is, for a constant input. This agrees with the calculated worst input signal $u^\star(\cdot)$.

C. CONTINUOUS SUPER-TWISTING LIKE ALGORITHM

In [30], a detailed analysis of the $\mathcal{L}_{2h}$−gain for the continuous super-twisting-like algorithm (CSTLA) is provided. We shall only briefly introduce the system and extend it to include the analysis of the $\mathcal{L}_{\mathrm{ph}}$−gain.

The CSTLA has the (closed loop) form [30], [38]

$$\begin{aligned}\dot{x}_1 &= -k_1\lceil x_1\rfloor^{\frac{1}{1-d}} + x_2\\ \dot{x}_2 &= -k_2\lceil x_1\rfloor^{\frac{1+d}{1-d}} + b\,u\\ y &= \left(E_1x_1\ E_2x_2\right)^\top \end{aligned} \tag{58}$$

with homogeneous degree $d \in (-1, 1)$, where $k_1, k_2 > 0$ are tunable parameters. The homogeneous weights are $r_y = r_x = (1-d, 1)$, $r_u = 1+d$. If $d = 0$ then (58) is a linear system and if $d = -1$ then it is the super-twisting algorithm (STA) [39]. Yet, since the STA has a discontinuous vector field and $r_u = 0$ when $d = -1$, it will not be considered for the $\mathcal{L}_{\mathrm{ph}}$−gain analysis.

From Theorem 2, we must have $p > \max r_x + d$. When $d > 0$, it is $p > 1 + d$, and when $d < 0$, it is $p > 1$. Similar to [30], [37], [40], we construct the following homogeneous storage function of degree $p - d$

$$V(x) = \alpha_1 V_l(x) = \alpha_1\left(\frac{1-d}{p-d}|x_1|^{\frac{p-d}{1-d}} - \alpha_{12}x_1\lceil x_2\rfloor^{p-1} + \frac{\alpha_2}{p-d}|x_2|^{p-d}\right) \tag{59}$$

with $\alpha_1, \alpha_{12}, \alpha_2 > 0$. It is continuously differentiable for $p \geq 2$ and it is positive definite if

$$\alpha_{12} < \left(\frac{\alpha_2}{p-1}\right)^{\frac{p-1}{p-d}}$$

which is obtained by Young's inequality. Since its gradient is

$$V_x^\top = \alpha_1\begin{bmatrix}\lceil x_1\rfloor^{\frac{p-1}{1-d}} - \alpha_{12}\lceil x_2\rfloor^{p-1}\\ -(p-1)\alpha_{12}x_1|x_2|^{p-2} + \alpha_2\lceil x_2\rfloor^{p-d-1}\end{bmatrix},$$

its derivative along the trajectories of (58) is

$$\begin{aligned}\dot{V} = \alpha_1\Big(&-k_1|x_1|^{\frac{p}{1-d}} - \alpha_{12}|x_2|^p + k_1\alpha_{12}\lceil x_1\rfloor^{\frac{1}{1-d}}\lceil x_2\rfloor^{p-1}\\ &+\lceil x_1\rfloor^{\frac{p-1}{1-d}}x_2 - k_2\alpha_2\lceil x_1\rfloor^{\frac{1+d}{1-d}}\lceil x_2\rfloor^{p-d-1}\\ &+k_2(p-1)\alpha_{12}|x_1|^{\frac{2}{1-d}}|x_2|^{p-2} + b\alpha_2\lceil x_2\rfloor^{p-d-1}u\\ &-b(p-1)\alpha_{12}x_1|x_2|^{p-2}u\Big).\end{aligned}$$

Clearly, with $p \geq 2$ the condition of Theorem 2 is met (ehen $d \in (0, 1)$, $p > 1 + d$, and when $d(-1, 0]$, $p > 1$). For negative definiteness of $\dot{V}$ it is necessary that $k_1, \alpha_{12} > 0$, which is satisfied by the assumption. If $p = 2$ it is further required that $\alpha_{12} < k_1/k_2$, since the sixth term is combined into the first term. The homogeneous dissipation inequality for this storage function is

$$J(V_x, x, u) = \dot{V} + \|y\|^p_{r_y,p} - \gamma^p\|u\|^p_{r_u,p} - \epsilon\|x\|^p_{r_x,p} \leq 0.$$

For $J(V_x, x, 0) < 0$ when $x \in \mathbb{R}^2\backslash\{0\}$, we need at least for $p > 2$ that

$$\alpha_1 > \frac{|E_1|^{\frac{p}{1-d}} - \epsilon}{k_1},\ \alpha_{12} > \frac{|E_2|^p - \epsilon}{\alpha_1},$$

and for $p = 2$ that

$$\alpha_1 > \frac{|E_1|^{\frac{2}{1-d}} - \epsilon}{k_1 - k_2\alpha_{12}},\ \alpha_{12} > \frac{|E_2|^2 - \epsilon}{\alpha_1}.$$

For the LTI system, obtained when $d = 0$, we have $r_y = r_u = 1$, and thus γ is the classical $\mathcal{L}_p$−gain. With the choice of $p = 2$ it provides an upper bound for the traditional $\mathcal{L}_2$−gain. When $p = 3$ or $p = 4$ it provides an upper bound for the $\mathcal{L}_3$−gain or $\mathcal{L}_4$−gain, respectively. If $p = 2$, the storage function (59) is in a quadratic form and we can compare our approach with the traditional ARE solution for linear systems, presented in Section IV-C. For the LTI case we will also include the results for $\mathcal{L}_3 - gain$ and $\mathcal{L}_4$−gain.

System (58) is affine in u, so using the proposed V equation (30) in Lemma 7 becomes

$$\begin{aligned}u^\star(x, V_x) &= \left|\frac{(1+d)}{p\gamma^p}\right|^{\frac{1+d}{p-1-d}}\\ &\times\left\lceil b\alpha_1\left(-(p-1)\alpha_{12}x_1|x_2|^{p-2} + \alpha_2\lceil x_2\rfloor^{p-d-1}\right)\right\rfloor^{\frac{1+d}{p-1-d}}.\end{aligned}$$

Since there is only one input, instead of (50) we can use

$$\begin{aligned}\gamma^\star &= \left|\frac{1+d}{p}\right|^{\frac{1}{p}}\left(\max_{\|x\|=1}\left(1-\frac{1+d}{p}\right)\right.\\ &\times\left.\frac{\left|b\alpha_1\left(-(p-1)\alpha_{12}x_1|x_2|^{p-2}+\alpha_2\lceil x_2\rfloor^{p-d-1}\right)\right|^{\frac{p}{p-1-d}}}{-J(V_x, x, 0)}\right)^{\frac{p-1-d}{p(1+d)}}.\end{aligned} \tag{60}$$

The choice of E_1, E_2 depends on the use of the algorithm. For controller design we shall pick $E_1 = 1, E_2 = 0$ and for observer design $E_1 = 0, E_2 = 1$, see [28].

1) LINEAR CASE

For $d = 0$ (linear case) the storage function in (59) is the quadratic form $V(x) = x^\top Px, P = P^\top > 0$ used in traditional $\mathcal{L}_2$−gain analysis [5]. Therefore, we can verify our approach of the homogeneous search by comparing with the ARI in (32). For $d = 0$ and $p = 2$, a simple mapping from (59) to P is

$$P_{11} = \frac{\alpha_1}{2},\ P_{12} = -\frac{\alpha_1\alpha_{12}}{2},\ P_{22} = \frac{\alpha_1\alpha_2}{2}.$$

We find that, the numerical search (not shown here) in Proposition 2 or 3 and the calculation of the optimal storage function using ARI (32) lead to the same $\gamma^\star$ and to the same storage function that ensure (22) or (32) when $\gamma = \gamma^\star$.

With weight matrix $E = \text{diag}\{E_1, E_2\}$, quantity $\gamma^\dagger$ designed in [28], i.e.

$$\gamma^\dagger = \sup_{u \neq 0} \frac{\|Ex\|_{r_x, \mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}}$$

has homogeneous degree 0 in the linear case ($d = 0$), which is equivalent to (21) since $r_x = (1, 1)$, $r_u = 1$ for $d = 0$.

After carrying out similar derivations as in [28] the analytical expression for $\gamma^\dagger$ for $k_1 < k_c$ results in

$$\gamma^{\dagger 2}(k_1, k_2) = \frac{b^2\left(k_1^2|E_2|+2k_2|E_2|+2|E_1|\right)}{k_1^2(4k_2 - k_1^2)} + \frac{b^2\sqrt{\left(k_1^2|E_2|+2k_2|E_2|+2|E_1|\right)^2 + k_1^2(4k_2 - k_1^2)|E_2|^2}}{k_1^2(4k_2 - k_1^2)}$$

and for $k_1 \geq k_c$ in

$$\gamma^{\dagger 2}(k_1, k_2) = \frac{b^2}{k_2^2}\left(k_1^2|E_2| + |E_1|\right)$$

where

$$k_c = 2k_2 + \frac{\sqrt{(|E_1| + 2k_2|E_2|)^2 + 4k_2^2|E_2|^2} - (|E_1| + 2k_2|E_2|)}{2|E_2|}.$$

As expected, the $\mathcal{L}_2$−gain $\gamma^\dagger$ is a function of k_1, k_2. Considering system (58) as the closed loop form for an observer or a controller design, adjusting the gains k_1 and k_2 can help us achieve a smaller $\mathcal{L}_2$−gain for the closed loop system (58). For each fixed k_2, the $\mathcal{L}_2$−gain $\gamma^\dagger(k_1, k_2)$ is convex in k_1, with a global minimum w.r.t. k_1 given by

$$k_1^\dagger(k_2) = \sqrt{2k_2 - \frac{k_2^2|E_2|}{2k_2|E_2| + |E_1|}}, \tag{61}$$

$$\gamma^\dagger(k_1^\dagger, k_2) = \frac{b}{k_2}\sqrt{2k_2|E_2| + |E_1|}. \tag{62}$$

Apparently, when $k_1 = k_1^\dagger(k_2)$ and $k_2 \to \infty$, $\gamma^\dagger(k_1^\dagger, k_2) \to 0$. So there is no global minimum w.r.t. the pair of (k_1, k_2) for the $\mathcal{L}_2$−gain. Since k_1, k_2 cannot be selected infinitely large, we are left with a conditioned optimization option for the $\mathcal{L}_2$−gain: fix k_2 first and then choose $k_1 = k_1^\dagger(k_2)$.

By taking the extremals of $|E_1/E_2|$, we obtain the $\mathcal{L}_2$−gain optimal range as

$$\underline{k}_1^\dagger \triangleq \sqrt{\frac{3}{2}k_2}, \quad \overline{k}_1^\dagger \triangleq \sqrt{2k_2}.$$

Fig. 5 shows $\gamma^\dagger$ as a function of k_1. The upper sub-figure shows $\gamma^\dagger$ from u to x_1, while the lower sub-figure to x_2. Fig. 5 shows that in the linear case, with a fixed k_2, the $\mathcal{H}_\infty$ norm $\gamma^\dagger$ to x_1 will remain constant for $k_1 > \overline{k}_1^\dagger$ (cross in upper sub-figure), and $\gamma^\dagger$ to x_2 is convex for $k_1 = \underline{k}_1^\dagger$ (cross in lower sub-figure). Since system (58) has a single input u, the Bode plot's maximum gain also shows the $\mathcal{H}_\infty$ norm to x_1 and x_2, reflecting Fig. 5 in the frequency domain. The upper sub-figure in Fig. 6 shows that with $k_1 < \overline{k}_1^\dagger$ the gain has a peak above 0 dB at mid-frequency. With $k_1 \geq \overline{k}_1^\dagger$ the gain is reduced for higher frequency, yet the DC gain is not improved. The lower sub-figure in Fig. 6 shows that the maximum gain gets to a minimum at $k_1 = \underline{k}_1^\dagger$, where larger k_1 leads to a larger DC gain, and smaller k_1 at higher frequency.

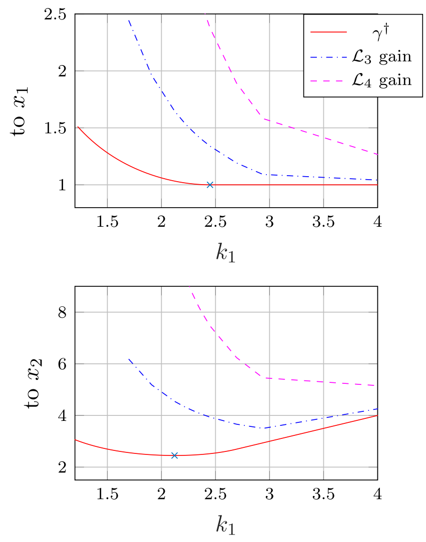

FIGURE 5. $\mathcal{L}_2$**−gain (derived solving ARI) and upper bound of** $\mathcal{L}_3$**−gain and** $\mathcal{L}_4$**−gain (from** (60)**) for system** (58) **when** $d = 0$ **(linear case) and** $k_2 = b = 3$**.**

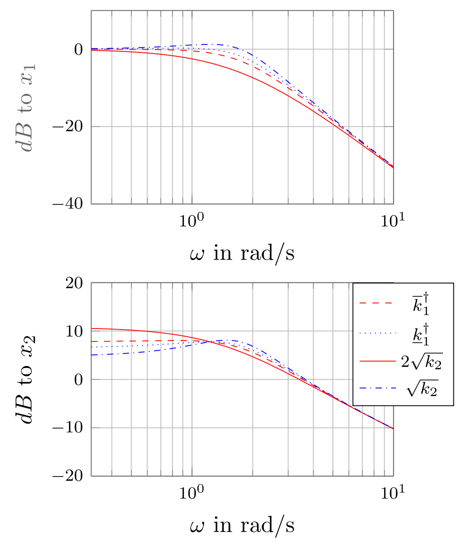

FIGURE 6. **Bode plot for** $y = x_1$ **(upper) and** $y = x_2$ **(lower) (**$\mathcal{L}_2$**−gain for sinusoidal input) of system** (58) **with** $d = 0$ **(linear case) and** $k_2 = b = 3$**.**

Also in Fig. 5, the upper bounds of the $\mathcal{L}_3$−gain and $\mathcal{L}_4$−gain are plotted. Note that such value is derived from the storage function (59). For $d = 0$, $p = 3$, it is

$$V(x) = \alpha_1\left(\frac{1}{3}|x_1|^3 - \alpha_{12}x_1\lceil x_2\rfloor^2 + \frac{\alpha_2}{3}|x_2|^3\right)$$

and for $d = 0, p = 4$, it is

$$V(x) = \alpha_1 \left(\frac{1}{4}|x_1|^4 - \alpha_{12} x_1 \lceil x_2 \rfloor^3 + \frac{\alpha_2}{4}|x_2|^4 \right).$$

There are apparently more possible candidates than the above two constructions of storage function, especially the cross term can be designed differently. This has highlighted the importance of the choice of storage function, which will affect how good may be the upper bound of the $\mathcal{L}_p$−gain, collected from dissipation inequality or HJI using Proposition 2 or 3.

2) NON-LINEAR CASE

For negative d, the upper bound of $\mathcal{L}_{2h}$−gain $\gamma^\star$ collected from using Proposition 2 or 3 for the CSTLA is displayed in Fig. 7, as presented in [30] with $\underline{k}_1^\star \triangleq \sqrt{1.5(1-d)k_2}$ and $\overline{k}_1^\star \triangleq \sqrt{2(1-d)k_2}$. It shows the behavior of the $\mathcal{L}_2$−gains to x_1 and x_2, respectively, as k_1 varies, for 4 negative values of $d = \{-0.5, -0.75, -0.9, -0.99\}$ (in different colors), and for each d three values of k_2 were used, given by $k_2 = \{0.99b, b, 1.01b\}$. There is a clear similarity to Fig. 5. A more detailed analysis can be found in [30].

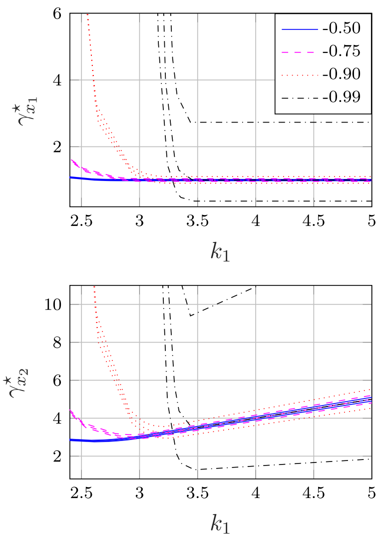

FIGURE 7. $\gamma^\star$ for $y = x_1$ (upper) and $y = x_2$ (lower) of system (58) with negative $d = \{-0.5, -0.75, -0.9, -0.99\}$ and $k_2 = \{0.99b, b, 1.01b\}$, $p = 2$.

After analyzing the collected $\gamma^\star$ for $k_1 \geq \overline{k}_1^\star$, the analytical expressions of $\gamma^\star$ can be summarized as

$$\gamma_{x_1} = \left(\frac{b}{k_2}\right)^{\frac{1}{1+d}}, \quad \gamma_{x_2} = k_1 \left(\frac{b}{k_2}\right)^{\frac{1}{1+d}} \tag{63}$$

which is achievable from a constant input [30]. By means of simulations we have also found an input signal that attains such $\mathcal{L}_2$−gains, given by

$$u(t) = W(D\,\mathrm{sign}\,(\sin(\omega t)) + (1-D)\sin(\omega t)) \tag{64}$$

where W is the amplitude of u, ω is the frequency in rad/s of the sine component, and D is the ratio between the signum function and sine function. The resulting number

$$\Gamma = \frac{\|Ex_T\|_{r_x,\mathcal{L}_2}}{\|u_T\|_{r_u,\mathcal{L}_2}}, \; T = \frac{4\pi}{\omega}, \text{ for} u \text{ from (64)} \tag{65}$$

are listed in Table 2. When $E_1 = 1, E_2 = 0$, Γ in (65) is denoted as Γ_{x_1}, and when $E_1 = 0, E_2 = 1$ as Γ_{x_2}. This number records the actual ratio of $\mathcal{L}_{2h}$−norm of output over input for the period of $t \in [0, T]$ with the u from (64). They agree with all $\gamma^\star$ collected for all $k_1 \geq \overline{k}_1^\star$ [30], whose expression is (63). Therefore, we can say that when the parameter satisfies $k_1 \geq \overline{k}_1^\star(k_2)$, $k_2 > 0$, we have $\gamma^\star = \Gamma_{2h}$.

TABLE 2. Achieved $\mathcal{L}_{2h}$ gain for $k_2 = b = 3$, $T = 10^{-5}s$ with u as (64).

d	$k_1 = 10\overline{k}_1^\star(d)$	W	D	ω	Γ_{x_1}	Γ_{x_2}
0.75	12.25	1	0	0.002	0.9998	12.25
0.50	17.32	1	0	0.002	0.9998	17.32
0.25	21.21	1	0	0.002	0.9999	21.21
−0.75	32.40	0.5	0.45	0.002	0.9999	32.40
−0.90	33.76	0.7	0.65	0.002	0.9999	33.76
−0.99	34.55	0.98	0.96	0.002	0.9561	33.04

The upper and lower plots of Fig. 8 for $d = -0.5$ and $k_1 = 3, k_2 = b = 3$ present the ratio of the amplitude of the outputs $y = x_1$ and $y = x_2$, respectively, and the amplitude of the input signal $u = \kappa \sin(\omega t)$, for $\kappa = \{0.5, 1, 2, 3\}$ and ω from 0.001 rad/s to 100 rad/s. From Fig. 7 we obtain that the value of $\gamma^\star$ to x_1 is 1, and to x_2 is 3. From Fig. 8, the following function

$$\Gamma(\omega) = \frac{\|Ex_T\|_{r_x,\mathcal{L}_2}}{\|(\kappa \sin(\omega t))_T\|_{r_u,\mathcal{L}_2}}, \; T = \frac{4\pi}{\omega} \tag{66}$$

is plotted, it is apparent that these gains are attained at a low frequency of the input signal.

Fig. 8 illustrates another interesting phenomenon. Recall that for homogeneous systems a dilated input signal (2) causes a dilated output not only in amplitude, but also in time. Thus, with negative d, if the amplitude of the input u is increased, and if the same value of Γ from (66) is to be resulted, then the frequency of the increased-magnitude input u has to be diminished, as is observed in Fig. 8. Note that when the amplitudes of input are increased, the frequency plot is shifted to the left, i.e. an input with lower frequency and larger amplitude achieve the same Γ in (66) with negative d.

On the other hand, this is reversed with positive d, since $r_t = -d < 0$. Namely, when the amplitudes of input are increased, the frequency plot is shifted to the right, i.e. an input with higher frequency and larger amplitude achieve the same Γ in (66) with positive d.

The simulations were carried out with forward Euler method and sampling period of $10^{-5}s$.

At last, the behavior of $\gamma^\star$ for positive values of $d = \{0.25, 0.5, 0.75\}$, $k_2 = b$, which is not studied in [30], are plotted in Fig. 9, the cases of $k_2 = \{0.99b, 1.01b\}$ are omitted, since the difference are too small and thus negligible. All three $\gamma^\star$ converge also to (63), yet with a larger k_1 as d grows larger.

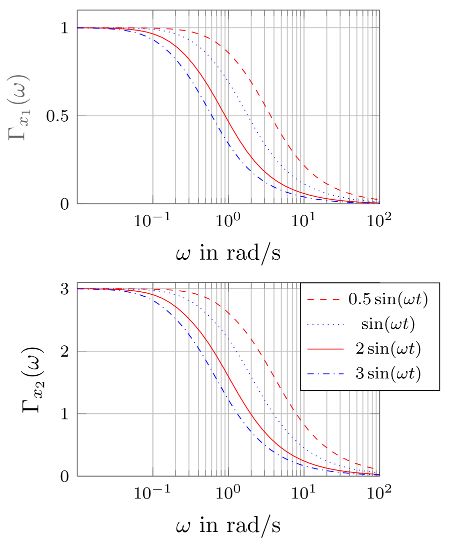

FIGURE 8. $\Gamma(\omega)$ in (66) for $y = x_1$ (upper) and $y = x_2$ (lower) in response to harmonic input $u = \kappa \sin(\omega t)$, for different amplitudes and range of frequencies. The system parameters are $d = -0.5$ and $k_1 = 3, k_2 = b = 3$.

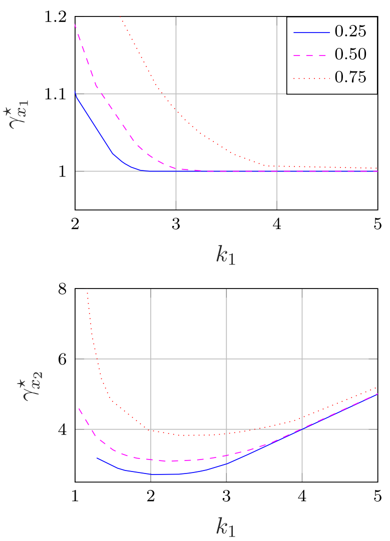

FIGURE 9. $\gamma^\star$ for $y = x_1$ (upper) and $y = x_2$ (lower) of system (58) with positive $d = \{0.25, 0.5, 0.75\}$ and $k_2 = b$, $p = 2$.

X. CONCLUSION

In this paper, motivated by the fact that the traditional concept of $\mathcal{L}_p$−stability is not well-defined for arbitrary homogeneous systems, a novel definition of homogeneous $\mathcal{L}_p$−stability and $\mathcal{L}_p$−gain has been introduced which is applicable to arbitrary homogeneous systems. This requires the definition of homogeneous signal $\mathcal{L}_p$−norms that allow to arrive at a globally defined homogeneous $\mathcal{L}_p$−gain for any homogeneous system. It is shown that previous results, using classical signal norms, are only possible under severe restrictions of the homogeneous system class. The novel concept of homogeneous $\mathcal{L}_p$−stability is applicable to $p = \infty$, and its related concept of ISS. Homogeneous $\mathcal{L}_p$−stability is characterized by a homogeneous dissipation inequality, which, for system affine in control input, can be reduced to a homogeneous Hamilton-Jacobi inequality. This allows to show that any homogeneous system, having an asymptotically stable equilibrium point for zero input, is homogeneous $\mathcal{L}_p$−stable and has a finite homogeneous $\mathcal{L}_p$−gain, for every p sufficiently large. Moreover, using either the dissipation inequality or the Hamilton-Jacobi inequality, and finding a homogeneous Lyapunov function, an upper bound of the homogeneous $\mathcal{L}_p$−gain can be estimated. Remarkable is that this upper bound is not only possible for $p = 2$, as it is well-known, but for an arbitrary p. With the tool of finite-gain homogeneous $\mathcal{L}_p$−stability the closed loop stability can be studied with the homogeneous small gain theorem for interconnected homogeneous systems. The results of the paper are illustrated by some examples. A natural consequence of the present work is the use of homogeneous $\mathcal{L}_p$−stability and gain to assure the stability of feedback interconnected systems. Further, homogeneous control or observer design can be performed to optimize the homogeneous $\mathcal{L}_p$−gain, as e.g. the H_∞−norm. This will be reported in the future.

APPENDIX

A. JENSEN's INEQUALITY

Lemma 8 (Jensen's inequality [41]): If I is an interval in $\mathbb{R}$ on which $f(x)$ is convex, if $n \geq 2$, w is a positive n-tuple with $\sum_{i=1}^n w_i = 1$, x an n-tuple elements in I, then

$$f\left(\sum_{i=1}^n w_i x_i\right) \leq \sum_{i=1}^n w_i f(x_i)$$

If f is strictly convex, the inequality is strict unless $x = \ell \mathbf{1}_n$.

First, choose $f(\cdot) = |\cdot|^p$, $p \geq 1$, which is a convex function, and choose $w = (0.5, 0.5)^\top$, $x = (2\,a, 2\,b)^\top$ where a, b non-negative. Then we have

$$(a + b)^p \leq 2^{p-1}(a^p + b^p)\,. \tag{67}$$

Furthermore, from [42, Theorem 5.26] in the section of Jensen-Petrović' Inequality, we extract

Lemma 9 ([42]): Let w, x be two non-negative n-tuples, suppose $x_i \in [0, a]$, $i = 1, \ldots, n$ and

$$\sum_{i=1}^n w_i x_i \geq x_j$$

for all $j = 1, \ldots, n$, as well as

$$\sum_{i=1}^n w_i x_i \in [0, a]\,.$$

If $f(x)/x$ is a decreasing function, then

$$f\left(\sum_{i=1}^n w_i x_i\right) \leq \sum_{i=1}^n w_i f(x_i)\,. \tag{68}$$

If $f(x)/x$ is an increasing function, then the reverse of inequality (68) holds.

Let $p_1 > p_2 > 0$ and $f(\cdot) = |\cdot|^{\frac{p_1}{p_2}}$, then the function

$$\frac{f(x)}{x} = \lceil x \rfloor^{\frac{p_1-p_2}{p_2}}$$

is increasing w.r.t. $x \in \mathbb{R}^n$ (since $p_1 > p_2$). Using Lemma 9 with choice of $w = \mathbf{1}_n$ (the two conditions are met with $a = \infty$), we have

$$\left(\sum_{i=1}^{n} |x_i|^{p_2}\right)^{\frac{p_1}{p_2}} = f\left(\sum_{i=1}^{n} |x_i|^{p_2}\right) \geq \sum_{i=1}^{n} f(|x_i|^{p_2}) = \sum_{i=1}^{n} (|x_i|^{p_2})^{\frac{p_1}{p_2}} = \sum_{i=1}^{n} |x_i|^{p_1},$$

i.e.

$$\left(\sum_{i=1}^{n} |x_i|^{p_2}\right)^{\frac{1}{p_2}} \geq \left(\sum_{i=1}^{n} |x_i|^{p_1}\right)^{\frac{1}{p_1}}. \tag{69}$$

This implies that for a vector $x \in \mathbb{R}^n$, $p_1 > p_2 > 0$, the classical $p-$norm satisfies $\|x\|_{p_2} \geq \|x\|_{p_1}$. In other words, for $p \geq 1$ and positive a, b we have

$$(a^p + b^p)^{\frac{1}{p}} \leq a + b. \tag{70}$$

Combined with (67), we obtain

$$a^p + b^p \leq (a+b)^p \leq 2^{p-1}(a^p + b^p).$$

On the other hand, for $0 < p < 1$ and positive a, b, we have

$$(a^p + b^p)^{\frac{1}{p}} \geq a + b$$

which is

$$(a+b)^p \leq (a^p + b^p). \tag{71}$$

Thus combining the case of $0 < p < 1$ and $p \geq 1$, that is $p > 0$, we have for positive a, b that

$$(a+b)^p \leq \max\left\{1, 2^{p-1}\right\}(a^p + b^p). \tag{72}$$

B. HÖLDER's INEQUALITY

A special form of [42, Theorem 4.12] is given by

Lemma 10 (Hölder's inequality [42]): If f, g are measurable real functions, then the following inequality holds

$$\int_0^\infty |f(t)g(t)|\mathrm{d}t \leq \left(\int_0^\infty |f(t)|^p \mathrm{d}t\right)^{\frac{1}{p}} \left(\int_0^\infty |f(t)|^q \mathrm{d}t\right)^{\frac{1}{q}} \tag{73}$$

for positive real numbers p, q satisfying $p^{-1} + q^{-1} = 1$.

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