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RESEARCH ARTICLE

Asymptotical Feedback Stabilization and Controllability of Probabilistic Logic Networks With State-Dependent Constraints

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ABSTRACT This study investigates the asymptotical feedback set stabilization and asymptotical feedback controllability of probabilistic logic control networks (PLCNs) with state-dependent constraints. First, based on the properties of the semi-tensor product (STP) of matrices and the vector representation of logic, a PLCN with state-dependent constraints is expressed as the algebraic form. Second, using a state-dependent input transformation, a PLCN with state-dependent constraints is transformed into one with free control input. The equivalence between the stabilizability and controllability of the original constrained PLCN and those of the resulting PLCN with free input is established. Based on these, we propose the necessary and sufficient conditions for both asymptotical feedback stabilizability and asymptotical feedback controllability. Finally, two examples are presented to demonstrate the application of the obtained results.

INDEX TERMS Probabilistic logic networks, semi-tensor product, state-dependent constraints, input transformation, asymptotical feedback stabilizability, asymptotical feedback controllability.

I. INTRODUCTION

Boolean networks (BNs) were first proposed by Kauffman [1] to describe genetic regulatory networks (GRNs) [2]–[4]. In the BN model, the state of each gene is quantized into two levels 1 and 0, corresponding to ON and OFF, respectively, and is updated depending on the states of other genes through a logical function. Owing to their effectiveness and convenience, BNs have attracted much interest from numerous scholars and have been applied to many fields such as game theory [5]–[7], cryptography [8], chemistry [9], social networks [10], robots [11], and biology [12]. To describe the uncertainties in GRNs, Shmulevich *et al*. proposed the probabilistic Boolean networks (PBNs) [13], [14]. A PBN switches among a collection of BNs randomly where the switching is usually assumed to be independent and identically distributed (i.i.d.). The assumption of i.i.d. switching implies that the analysis of PBNs can be framed within the theory of finite Markov chains [15].

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An important mathematical tool called the semi-tensor product (STP) of matrices was originally proposed by Cheng *et al*. [2]. The STP, together with the vector representation of logic, allows us to express a BN as a discrete-time linear dynamical system; thus, some classical control theories are applicable. The emergence of the STP has drawn much attention in the field of research into BNs, including Boolean control networks (BCNs) and probabilistic Boolean control networks (PBCNs), and promoted the solvability of a large number of challenging problems regarding BNs, such as observability [16], [17], controllability [18], [19], stability and stabilization [20]–[24], control of BNs with time-delays [25], [26], disturbance decoupling [27], [28], synchronization [29], [30], optimal control [31], [32], and pinning control [33], [34].

In practical applications, constraints on state and input are common and must be considered in control design. In GRNs, some undesirable states causing diseases should be avoided. For instance, the activation of gene WNT5A would increase the chance of melanoma metastasizing [35]. An example of logic systems with control constraints is the game of

Chinese chess, where at every step, the admissible control strategies of each player depend on the current state. Furthermore, constraints on state and control may exist alone or simultaneously [36] and may be state-independent [37] or state-dependent [38].

For a deterministic BCN, controllability avoiding undesirable states was investigated in [39]. In addition, the input-state incidence matrix technique [40] was applied to BCNs with state and control constraints characterized as fixed subsets [37]. In some practical situations the control constraints are state-dependent, as in Chinese chess. In order to deal with state-dependent constraints, the technique of input transformation was proposed to transform a constrained BCN into one with free control input [36], [38], [41], [42]. For probabilistic logic control networks (PLCNs) with state-independent constraints on state and control, controllability was investigated in [43]. To our knowledge, the analysis and control of PLCNs with state-dependent constraints has seldom been addressed in the existing literature.

In this study, we investigate the controllability and stabilization of PLCNs with state-dependent control constraints. The state-dependent constraint on control is described as a collection of admissible control subsets, each of which corresponds to a state in the state space. Whatever the state of the PLCN is, the control input is required to be confined within the corresponding subset of admissible controls. The definitions of asymptotical feedback stabilizability in distribution (AFSD) and asymptotical feedback controllability (AFC) are introduced. The main difference between these definitions and those for unconstrained PLCNs is that for the former, feedback is required to be admissible, in the sense that the given state-dependent constraints must be satisfied during the whole evolution process. It is worth pointing out that the method proposed in [43] is not directly applicable to these problems owing to the state-dependent feature of the constraint. In this work, we apply the technique of input transformation to PLCNs with state-dependent control constraints. The input transformation can also be viewed as a "pre-feedback", the input and output of which are the virtual control input and actual control input of the PLCN, respectively. The gain matrix of pre-feedback is a state-dependent logic matrix, the columns of which are equivalent to the correspondent admissible controls. This way of constructing the input transformation assures that the actual input to the PLCN always satisfies the given constraints no matter what the virtual control input is. Conversely, any admissible actual control sequence can be generated by pre-feedback driven by an elaborately designed virtual control sequence. By combining the pre-feedback and the original PLCN, we obtain a combined PLCN with the virtual input as its control input. Based on the nice properties of the pre-feedback, we prove that the asymptotical feedback controllability and asymptotical feedback stabilizability of the original PLCN under admissible feedback are equivalent to the respective properties of the combined PLCN. The combined PLCN is subject to no input constraints and the existing analysis techniques developed for unconstrained PLCNs [44], [45] are applied to derive the necessary and sufficient conditions for both AFSD and AFC. As far as we know, this is the first paper to report the criteria for the stabilizability and controllability for PLCNs with state-dependent constraints on control. For the problem of feedback design, we adopt the state-space partition technique proposed in [44] to design a stabilizing feedback for the combined PLCN first, after which an admissible stabilizing feedback for the original PLCN can be obtained by combining this feedback with the pre-feedback.

The remainder of this paper is organized as follows. In Section II, the preliminaries, problem formulation, and definitions of AFSD and AFC are introduced. Section III introduces the input transformation technique and proposes a necessary and sufficient condition for the existence of an admissible feedback control. In Section IV, the necessary and sufficient conditions for AFSD and AFC are proposed, and a procedure to design an admissible stabilizing set stabilizer is presented. Two illustrative examples are presented in Section V to support the obtained results. The conclusion of this work is presented in Section VI. The notation used throughout this paper is described in Table 1.

TABLE 1. Notations.

Notation	Description
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Set of real numbers
$\mathbb{Z}_{\geq a}$	Set $\{x \in \mathbb{Z} \| x \geq a, a \in \mathbb{Z}\}$
$[N : M]$	Set $\{k \in \mathbb{Z} \| N \leq k \leq M\}$
1_n	n-dimensional vector $[1\,1 \cdots 1]^T$
I_n	$n \times n$ identity matrix
Δ_n	Set of columns of I_n
δ_n^k	k-th column of I_n
$\delta_n[i_1 \cdots i_m]$	Matrix A with $\mathrm{Col}_j[A] = \delta_n^{i_j}$
$\mathscr{L}_{m\times n}$	Set of $m \times n$ logical matrices
$\mathscr{B}_{m\times n}$	Set of $m \times n$ Boolean matrices
$\mathbb{R}^{m\times n}$	Set of $m \times n$ real matrices
$[A]_{i,j}$	(i, j)-element of matrix A
$\mathrm{Col}_i[A]$	i-th column of matrix A
$\mathrm{Col}[A]$	Set of columns of matrix A
$\mathrm{Row}_i[A]$	i-th row of matrix A
$\vee$	Logical operators "OR"
$\|\mathcal{M}\|$	Cardinality of set $\mathcal{M}$
$\mathrm{Idx}[\mathcal{M}]$	Set $\{i \| \delta_n^i \in \mathcal{M}\}$
$(\mathscr{B})\sum$	Boolean summation

II. PROBLEM FORMULATION AND PRELIMINARIES

A. PRELIMINARIES

Definition 1 (STP of Matrices [2]): For two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, the STP of A and B is defined as

$$A \ltimes B := (A \otimes I_{\alpha/n})(B \otimes I_{\alpha/p}),$$

where $\alpha = \mathrm{lcm}(n, p)$ is the least common multiple of n and p, and "$\otimes$" denotes the Kronecker product.

Remark 1: In Definition 1, when $n = p$, the STP of matrices A and B degenerates to the conventional matrix product AB. Thus, the symbol "$\ltimes$" can be omitted without causing confusion.

Definition 2 (Boolean Addition [2]): For $A, B \in \mathscr{B}_{m\times n}$, the Boolean addition of A and B, denoted by $A +_{\mathscr{B}} B$, is defined as

$$[A +_{\mathscr{B}} B]_{i,j} := [A]_{i,j} \vee [B]_{i,j}, \quad i \in [1:m], j \in [1:n].$$

Definition 3 (Boolean Product [2]): For $A \in \mathscr{B}_{m\times n}$ and $B \in \mathscr{B}_{n\times p}$, the Boolean product of A and B, denoted by $A \times_{\mathscr{B}} B$, is defined as

$$[A \times_{\mathscr{B}} B]_{i,j} = \mathscr{B}\sum_{k=1}^{n}[A]_{i,k}[B]_{k,j}, \quad i \in [1:m], j \in [1:p].$$

For $A \in \mathscr{B}_{m\times n}$ and $B \in \mathscr{B}_{p\times q}$, the Boolean STP of A and B is defined as

$$A \ltimes_{\mathscr{B}} B := \left(A \otimes I_{\alpha/n}\right) \times_{\mathscr{B}} \left(B \otimes I_{\alpha/p}\right),$$

where $\alpha = \mathrm{lcm}(n, p)$ is the least common multiple of n and p. The k-order Boolean power of A is defined as

$$A^{(k)} := \underbrace{A \ltimes_{\mathscr{B}} A \ltimes_{\mathscr{B}} \cdots \ltimes_{\mathscr{B}} A}_{k}, \quad k \in Z_{\geq 1}.$$

In particular, if A is a square Boolean matrix of order n, then, $A^{(0)} := I_n$.

Lemma 1 (Swap Matrix [2]): For any positive integers m and n, the swap matrix $W_{[m,n]}$ is defined as

$$W_{[m,n]} := [I_n \otimes \delta_m^1\ I_n \otimes \delta_m^2 \cdots I_n \otimes \delta_m^m].$$

Then, for $x \in \mathbb{R}_{m\times 1}$, $y \in \mathbb{R}_{n\times 1}$, it holds that $W_{[m,n]}xy = yx$.

Lemma 2 (Power-Reducing Matrix [2]): For $x \in \Delta_n$, the power-reducing matrix $M_{r,n}$ is defined as

$$\mathrm{Col}_i[M_{r,n}] := \delta_{n^2}^{n(i-1)+i}, \quad i = 1, 2, \ldots, n.$$

Then it holds that $x^2 = M_{r,n}x$.

Lemma 3 [2]: Any logical function $f : \Delta_{n_1} \times \Delta_{n_2} \times \cdots \times \Delta_{n_n} \to \Delta_m$ with $x_i \in \Delta_{n_i}$, $i = 1, 2, \ldots, n$ can be expressed in a multilinear form as

$$f(x_1, x_2, \ldots, x_n) = L_f x_1 x_2 \ldots x_n,$$

where $L_f \in \mathscr{L}_{m\times N}$, $N = \prod_{i=1}^{n} n_i$, denotes the structure matrix of f, uniquely determined by f.

B. MODEL AND PROBLEM PORMULATION

A PLCN with state-dependent constraints on control is described as follow:

$$\begin{cases} x(t+1) = f_{\omega(t)}(u(t), x(t)), \\ u(t) \in \mathcal{U}_{\sigma(x(t))}, \end{cases} \tag{1}$$

where $x(t) = \ltimes_{i=1}^{n} x_i(t)$ and $u(t) = \ltimes_{j=1}^{m} u_j(t)$ respectively denote the n-dimensional state and the m-dimensional input at time t, $x_i \in \Delta_{n_i}$, $i \in [1:n]$, is the state of the ith state node and $u_j \in \Delta_{m_j}$, $j \in [1:m]$, is the state of the jth input node. For convenience, we denote $N = n_1 n_2 \cdots n_n$ and $M = m_1 m_2 \cdots m_m$. f_λ, $\lambda \in [1:r]$, are logic functions, and the switching signal $\omega(t) \in [1:r]$ is an i.i.d. process subject to the probability distribution

$$\mathbb{P}\{\omega(t) = \lambda\} = p_\lambda, \quad \lambda \in [1:r],$$

where p_λ, $\lambda \in [1:r]$ are nonnegative numbers satisfying $\sum_{\lambda=1}^{r} p_\lambda = 1$. Using $L_\lambda \in \mathcal{L}_{N\times MN}$ to denote the structural matrices of f_λ, it holds that

$$x(t+1) = L_{\omega(t)}u(t)x(t), \quad \omega(t) \in [1:r]. \tag{2}$$

$\sigma : \Delta_N \to \{1, 2, \ldots, N\}$ is a mapping defined as

$$\sigma(\delta_N^i) := i, \quad i \in [1:N].$$

The collection of subsets $\mathcal{U}_i \subseteq \Delta_M$, $1 \leq i \leq N$, characterizes the state-dependent control constraints. Specifically, if $x(t) = \delta_N^i$, then $\mathcal{U}_{\sigma(x(t))} = \mathcal{U}_i$, which means that the control is only permitted to take values from $\mathcal{U}_i$. The solution to PLCN (1) with an initial state $x_0 \in \Delta_N$ and a control sequence $\mathbf{u} = \{u(t)\}_{t\in Z_{\geq 0}}$ is denoted by $x(t; \mathbf{u}, x_0)$. A control sequence $\mathbf{u}$ is called an admissible control sequence if the state-dependent constraints are always satisfied when the PLCN (1) is controlled by $\mathbf{u}$.

In this study, we consider a feedback controller

$$u(t) = Gx(t) \tag{3}$$

for PLCN (1), where $G \in \mathscr{L}_{M\times N}$ is the state feedback gain matrix. For convenience, we use $\mathbf{u}_G$ to denote the state feedback of the form (3) determined by G, and use $x(t; \mathbf{u}_G, x_0)$ to denote the solution to the closed-loop system PLCN (1) controlled by feedback $\mathbf{u}_G$ with initial state x_0. If a feedback $\mathbf{u}_G$ satisfies the constraints, that is,

$$Gx \in \mathcal{U}_{\sigma(x)}, \quad \forall x \in \Delta_N, \tag{4}$$

then we call (3) an admissible feedback for PLCN (1).

Definition 4 (Asymptotical Feedback Stabilizability in Distribution (AFSD)):

1) For a given target state $x_d \in \Delta_N$, PLCN (1) is said to be asymptotically feedback x_d-stabilizable in distribution if there exists an admissible state feedback $\mathbf{u}_G$, such that

$$\lim_{t\to\infty} \mathbb{P}\{x(t; \mathbf{u}_G, x_0) = x_d\} = 1, \quad \forall x_0 \in \Delta_N.$$

2) For a given nonempty state set $\mathcal{M} \subseteq \Delta_N$, PLCN (1) is said to be asymptotically feedback $\mathcal{M}$-stabilizable in distribution if there exists an admissible state feedback $\mathbf{u}_G$, such that

$$\lim_{t\to\infty} \mathbb{P}\{x(t; \mathbf{u}_G, x_0) \in \mathcal{M}\} = 1, \quad \forall x_0 \in \Delta_N.$$

Definition 5 (Asymptotical Feedback Controllability (AFC)): Consider a PLCN (1).

1) $x_d \in \Delta_N$ is said to be asymptotically feedback reachable from $x_0 \in \Delta_N$ if there exists an admissible state feedback $\mathbf{u}_G$, such that

$$\lim_{t\to\infty} \mathbb{P}\{x(t; \mathbf{u}_G, x_0) = x_d\} = 1.$$

2) $x_d \in \Delta_N$ is said to be asymptotically feedback reachable if for any $x_0 \in \Delta_N$, x_d is asymptotically feedback reachable from $x_0 \in \Delta_N$.
3) PLCN (1) is said to be asymptotically feedback controllable if every $x_d \in \Delta_N$ is asymptotically feedback reachable.

Remark 2: The definitions of AFSD in Definition II-B and AFC in Definition II-B are similar to those in [44] and [45], respectively. The main difference is that for the former, the feedback is required to be admissible.

III. INPUT TRANSFORMATION AND ADMISSIBLE STATE FEEDBACK

To deal with the constraints, we convert a PLCN with state-dependent constraints on control into one with free control input by using the input transformation technique proposed in [36]. We define the input transformation matrix U of PLCN (1) as

$$U := [U_1\ U_2 \cdots U_N] \in \mathscr{L}_{M\times MN}, \tag{5}$$

where for each $i \in [1:N]$, $U_i \in \mathscr{L}_{M\times M}$ is a logic matrix satisfying

$$\mathrm{Col}[U_i] = \mathcal{U}_i.$$

The input transformation is defined as

$$u(t) := Ux(t)v(t), \tag{6}$$

where $v(t) \in \Delta_M$ is the virtual input variable. The input transformation (6) is also called a pre-feedback. The combination of the pre-feedback and the original PLCN is a PLCN with the virtual input v as its control input, called the combined PLCN. By inserting the pre-feedback (6) into (2), we obtain

$$\begin{aligned} x(t+1) &= L_{\omega(t)}Ux(t)v(t)x(t) \\ &= L_{\omega(t)}UW_{[N,MN]}x(t)x(t)v(t) \\ &= L_{\omega(t)}UW_{[N,MN]}M_{r,N}x(t)v(t) \\ &= L_{\omega(t)}UW_{[N,MN]}M_{r,N}W_{[M,N]}v(t)x(t). \end{aligned}$$

Thus, the combined PLCN is as follows:

$$x(t+1) = \tilde{L}_{\omega(t)}v(t)x(t), \tag{7}$$

where

$$\tilde{L}_{\omega(t)} := L_{\omega(t)}UW_{[N,MN]}M_{r,N}W_{[M,N]}.$$

For convenience, the solution to the combined PLCN (7) with initial state $x_0 \in \Delta_N$ and control sequence $\mathbf{v} = \{v(t)\}_{t\in Z_{\geq 0}}$ is denoted by $x_v(t; \mathbf{v}, x_0)$. We use the notation $\mathbf{v}_K$ to denote the feedback

$$v(t) = Kx(t)$$

for the combined PLCN (7), where $K \in \mathscr{L}_{M\times N}$.

Lemma 4:

1) For any control value $v \in \Delta_M$ and any state $x \in \Delta_N$, the control value $u := Uxv$ is always admissible in the sense that $u \in \mathcal{U}_{\sigma(x)}$.
2) For any $x \in \Delta_N$ and any admissible control value $u \in \mathcal{U}_{\sigma(x)}$, there exists a control value $v \in \Delta_M$ such that $u = Uxv$.

Proof: These claims are obviously true because $\mathrm{Col}[U_{\sigma(x)}] = \mathcal{U}_{\sigma(x)}$, $\forall x \in \Delta_N$. □

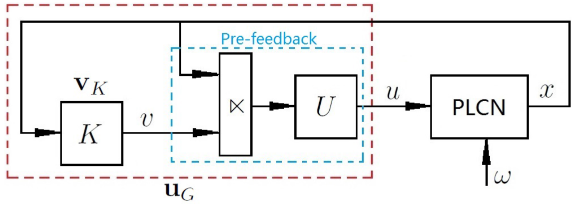

FIGURE 1. **Structure of an admissible feedback for a PLCN (Any admissible feedback can be decomposed into a pre-feedback and a constraint-free feedback $\mathbf{v}_K$).**

Proposition 1: A state feedback $\mathbf{u}_G$ for PLCN (1) is admissible if and only if there exists a feedback $\mathbf{v}_K$ for the combined PLCN (7) such that

$$G = UW_{[M,N]}KM_{r,N}. \tag{8}$$

Proof: [Sufficiency] For any feedback gain matrix K of the combined PLCN (7), the input transformation (6) determines a feedback for the original PLCN (1) as

$$u(t) = Ux(t)Kx(t).$$

By Lemma 4, this feedback always satisfies the state-dependent constraints, which can be rewritten in the form $u(t) = Gx(t)$ with G given by (8). In fact, by Lemmas 1 and 2,

$$\begin{aligned} u(t) &= Ux(t)v(t) \\ &= Ux(t)Kx(t) \\ &= UW_{[M,N]}Kx(t)x(t) \\ &= UW_{[M,N]}KM_{r,N}x(t) = Gx(t). \end{aligned}$$

This proves the sufficiency.

(Necessity) Suppose that $\mathbf{u}_G$ is an admissible feedback for PLCN (1). Thus, the state-dependent constraints (4) are satisfied. By Lemma 4, for any $j \in [1:N]$, there is a $v_j \in \Delta_M$ such that

$$G\delta_N^j = Uxv_j.$$

We define the logic matrix $K \in \mathscr{L}_{M\times N}$ as

$$\mathrm{Col}_j(K) = v_j, \quad \forall j \in [1:N].$$

By the definition of K, it obviously holds that $v_j = K\delta_N^j$, $\forall j \in [1:N]$. As a result,

$$Gx = UxKx = UW_{[M,N]}KM_{r,N}x, \quad \forall x \in \Delta_N.$$

Thus, (8) holds. □

Remark 3: Proposition 1 states that, in order to design an admissible feedback $\mathbf{u}_G$ for PLCN (1), we need only find a feedback for the combined PLCN (7) which is a PLCN without input constraints. Then, $\mathbf{u}_G$ is obtained by letting $G = UW_{[M,N]}KM_{r,N}$. In fact, by Proposition 1, any admissible feedback $\mathbf{u}_G$ for the original PLCN (1) can be regarded as the combination of a constraint-free feedback $\mathbf{v}_K$ for the combined PLCN (7) and the pre-feedback. Please see Fig. 1 for the structure of an admissible feedback.

In addition, the closed-loop system of PLCN (1) under the feedback $\mathbf{u}_G$ can be derived as follows:

$$\begin{aligned} x(t+1) &= L_{\omega(t)}Ux(t)v(t)x(t) \\ &= L_{\omega(t)}Ux(t)Kx(t)x(t) \\ &= L_{\omega(t)}UW_{[M,N]}Kx(t)x(t)x(t) \\ &= L_{\omega(t)}UW_{[M,N]}KM_{r,N}^2x(t) \\ &=: \hat{L}_{\omega(t)}x(t). \end{aligned} \tag{9}$$

Note that (9) is also the closed-loop system of the combined PLCN (7) under the feedback $\mathbf{v}_K$. Taking the expectation of both sides of the closed-loop system, we obtain

$$\mathbb{E}x(t+1) = \mathbf{P}_K\mathbb{E}x(t), \tag{10}$$

where $\mathbb{E}$ denotes the expectation and

$$\mathbf{P}_K := \sum_{\lambda=1}^{r} p_\lambda L_\lambda UW_{[M,N]}KM_{r,N}^2 \tag{11}$$

is the one-step transition probability matrix (TPM) of the closed-loop system. If we denote by $\mathbf{P}^u$ the control-dependent TPM of PLCN (1), that is,

$$\mathbf{P}^u := \sum_{\lambda=1}^{r} p_\lambda L_\lambda,$$

then it holds that

$$\mathbf{P}_K = \mathbf{P}^u UW_{[M,N]}KM_{r,N}^2.$$

Proposition 2: Suppose that $G \in \mathscr{L}_{M\times N}$ and $K \in \mathscr{L}_{M\times N}$ are feedback gain matrices for PLCNs (1) and (7), respectively, that are related via (8). Then, we have

$$x(t;\mathbf{u}_G,x_0) = x_v(t;\mathbf{v}_K,x_0). \tag{12}$$

Proof: The claim directly follows the fact that PLCNs (1) and (7) controlled by feedbacks $\mathbf{u}_G$ and $\mathbf{v}_K$, respectively, share the common closed-loop system. □

The following propositions are obtained directly from propositions 1 and 2.

Proposition 3:

1) PLCN (1) is asymptotically feedback $\mathcal{M}$-stabilizable in distribution if and only if PLCN (7) is asymptotically feedback $\mathcal{M}$-stabilizable in distribution.
2) Suppose that PLCN (7) is asymptotically $\mathcal{M}$-stabilized by the state feedback $\mathbf{v}_K$, then PLCN (1) is asymptotically $\mathcal{M}$-stabilized by the state feedback $\mathbf{u}_G$ with G being given by (8).

Proposition 4:

1) For PLCN (1) with state-dependent constraints, x_d is asymptotically feedback reachable from x_0 if and only if for PLCN (7), x_d is asymptotically feedback reachable from x_0.
2) PLCN (1) with state-dependent constraints is asymptotically feedback controllable if and only if PLCN (7) with free input is asymptotically feedback controllable.

Remark 4: Propositions 3 and 4 reveal the role of the input transformation, that is, it transfers the problems of feedback controllability and feedback stabilization of PLCNs with state-dependent input constraints into those of PLCNs with free input. In general, the input transformation matrix U is not unique. However, the results obtained in this paper do not depend on the choice of U.

IV. ASYMPTOTICAL FEEDBACK SET STABILIZATION AND CONTROLLABILITY OF PLCNs WITH STATE-DEPENDENT CONSTRAINTS

A. STABILIZABILITY AND CONTROLLABILITY

The control-dependent TPM of PLCN (7) is

$$\begin{aligned} \mathbf{P}^v &:= \sum_{\lambda=1}^{r} p_\lambda L_\lambda UW_{[N,MN]}M_{r,N}W_{[M,N]} \\ &= \mathbf{P}^u UW_{[N,MN]}M_{r,N}W_{[M,N]}, \end{aligned} \tag{13}$$

where $\mathbf{P}^u$ is the control-dependent TPM of PLCN (1). For any $j \in [1:M]$, the TPM of PLCN (7) under control $v(t) \equiv \delta_M^j$ is

$$\mathbf{P}_j^v := \mathbf{P}^v \ltimes \delta_M^j = \mathbf{P}^u UW_{[N,MN]}M_{r,N}(I_N \otimes \delta_M^j).$$

We define the Boolean matrix $\lceil \mathbf{P}^v \rceil$ as

$$\lceil \mathbf{P}^v \rceil := [(\mathscr{B})\sum_{\lambda=1}^{r} L_\lambda]UW_{[N,MN]}M_{r,N}W_{[M,N]}.$$

Then, the one-step reachability matrix of PLCN (7) is given by

$$\mathbf{R}^v := \lceil \mathbf{P}^v \rceil \ltimes_{\mathscr{B}} \mathbf{1}_M,$$

and the k-step reachability matrix of PLCN (7) is

$$\mathbf{R}^v[k] := (\mathscr{B})\sum_{j=1}^{k} \left(\mathbf{R}^v\right)^{(j)}.$$

Remark 5: There exists an admissible path of length k from $x(0) = x_0 = \delta_N^j$ to $x_d = \delta_N^i$ if and only if $[\mathbf{R}^v[k]]_{i,j} = 1$.

Definition 6 (Control Invariant Subset (CIS) [44]): A subset $\mathcal{C} \subseteq \Delta_N$ is called a control-invariant subset (CIS) of PLCN (7), if for any state $x_0 \in \mathcal{C}$, there exists a control $v_0 \in \Delta_M$, such that $\mathbb{P}\{x(1;v_0,x_0) \in \mathcal{C}\} = 1$. If a singleton $\{x_0\}$ is control invariant, then we call x_0 a control fixed point.

The largest control invariant subset (LCIS) of a given subset $\mathcal{M} \subseteq \Delta_N$ is denoted by $I_c(\mathcal{M})$, which is defined as the union of all CISs contained in $\mathcal{M}$.

Definition 7 (Invariant Subset [15]): A subset $\mathcal{C} \subseteq \Delta_N$ is said to be an invariant subset of the closed-loop system of PLCN (7) controlled by feedback $\mathbf{v}_K$ if

$$\mathbb{P}\{x_v(t;\mathbf{v}_K,x_0) \in \mathcal{C}\} = 1, \quad \forall x_0 \in \mathcal{C}, \forall t \in Z_{\geq 0}.$$

The largest invariant subset (LIS) of the closed-loop system of PLCN (7) controlled by $\mathbf{v}_K$ contained in a given subset $\mathcal{M} \subseteq \Delta_N$ is denoted by $I_{\mathbf{v}_K}(\mathcal{M})$, which is defined as the union of all invariant subsets contained in $\mathcal{M}$.

Remark 6: The LCIS $I_c(\mathcal{M})$ and LIS $I_{\mathbf{v}_K}(\mathcal{M})$ of PLCN (7) in a given subset $\mathcal{M}$ can be calculated using the TPMs $\mathbf{P}_j^v$, $j \in [1 : M]$, and $\mathbf{P}_K$, respectively. Please refer to [44] and [15] for details.

Theorem 1: Consider a PLCN (1) with state-dependent constraints and $\mathcal{M} \subseteq \Delta_N$ as a given subset. Then, PLCN (1) is asymptotically feedback $\mathcal{M}$-stabilizable in distribution if and only if $I_c(\mathcal{M}) \neq \emptyset$ and

$$(\mathscr{B}) \sum_{s\in \mathrm{Idx}[I_c(\mathcal{M})]} \mathrm{Row}_s(\mathbf{R}^v[\alpha]) = \mathbf{1}_N^\top, \tag{14}$$

where $\alpha := N- \mid I_c(\mathcal{M}) \mid$.

Proof: The claim follows directly from [44, Th. 1] and Proposition 3. □

Theorem 2: Given a PLCN (1), a state $x_d = \delta_N^r$ is asymptotically feedback reachable if and only if it is asymptotically feedback stabilizable in distribution, i.e., the following two conditions hold:

1) There exists an integer $j \in [1 : M]$ such that $[\mathbf{P}_j^v]_{r,r} = 1$.
2) It holds that

$$\mathrm{Row}_r(\mathbf{R}^v[N-1]) = \mathbf{1}_N^\top.$$

Proof: The claim follows directly from [45, Th. 1] and Proposition 4. □

Theorem 3: PLCN (1) is asymptotically feedback controllable if and only if every state is asymptotically feedback stabilizable, that is,

1) For any $r \in [1 : N]$, there exists an integer $j \in [1 : M]$ such that $[\mathbf{P}_j^v]_{r,r} = 1$.
2) For any $i, j \in [1 : N]$, $[\mathbf{R}^v[N-1]]_{i,j} = 1$.

Proof: This claim follows directly from [45, Th. 2]] and Proposition 4. □

B. DESIGN OF ADMISSIBLE FEEDBACK

If PLCN (1) is asymptotically feedback $\mathcal{M}$-stabilizable, then a procedure of designing an admissible set stabilizer for PLCN (1) can be summarized as follows:

1) Respecting the constraints, construct the pre-feedback as in (6).
2) Combine the pre-feedback and PLCN (1) to obtain the combined PLCN (7).
3) Calculate the TPM $\mathbf{P}^v$ of PLCN (7) and the LCIS $I_c(\mathcal{M})$ using the algorithm proposed in [44].
4) Design a stabilizing feedback $\mathbf{v}_K$ for PLCN (7) by the technique of partitioning the state space [44].
5) According to Proposition 1, an admissible stabilizing feedback $\mathbf{u}_G$ for PLCN (1) is obtained through $G = UW_{[M,N]}KM_{r,N}$.

V. EXAMPLES

Example 1: Consider a PLCN of the form (1) with $n = m = 2$, $n_i = m_j = 2$, $i, j \in [1 : 2]$, $r = 4$, where the structural matrices L_j and probability distribution p_j, $j \in [1 : 4]$, are given as follows:

$$\begin{aligned}
L_1 &= \delta_4[1\ 2\ 2\ 3\ 1\ 1\ 2\ 2\ 2\ 1\ 1\ 2\ 2\ 2\ 3\ 2], & p_1 &= 0.23,\\
L_2 &= \delta_4[2\ 3\ 3\ 3\ 1\ 2\ 3\ 3\ 3\ 2\ 1\ 3\ 3\ 3\ 3\ 2], & p_2 &= 0.25,\\
L_3 &= \delta_4[2\ 4\ 4\ 4\ 2\ 2\ 4\ 4\ 4\ 3\ 1\ 4\ 4\ 4\ 3\ 2], & p_3 &= 0.31,\\
L_4 &= \delta_4[2\ 1\ 1\ 4\ 2\ 1\ 1\ 1\ 1\ 4\ 2\ 1\ 1\ 1\ 4\ 4], & p_4 &= 0.21.
\end{aligned}$$

The sets of admissible control for all states are respectively

$$\begin{aligned}
\mathcal{U}_1 &= \{\delta_4^1, \delta_4^2\}, & \mathcal{U}_2 &= \{\delta_4^2, \delta_4^3\},\\
\mathcal{U}_3 &= \{\delta_4^3, \delta_4^4\}, & \mathcal{U}_4 &= \{\delta_4^1, \delta_4^4\}.
\end{aligned}$$

We construct a logic matrix U_i for each $\mathcal{U}_i$, $i \in [1 : 4]$, as

$$\begin{aligned}
U_1 &:= \delta_4[1\ 1\ 1\ 2], & U_2 &:= \delta_4[2\ 2\ 2\ 3],\\
U_3 &:= \delta_4[3\ 3\ 3\ 4], & U_4 &:= \delta_4[4\ 4\ 4\ 1].
\end{aligned}$$

Then, a pre-feedback $u(t) := Ux(t)v(t)$ is determined, where the transformation matrix U is $U := [U_1\ U_2\ U_3\ U_4]$ and $v(t) \in \Delta_4$ is the virtual input. The combination of the pre-feedback and the PLCN is of the form (7), where

$$\tilde{L}_{\omega(t)} = L_{\omega(t)}UW_{[4,16]}M_{r,4}W_{[4,4]} \tag{15}$$

with

$$\begin{aligned}
\tilde{L}_1 &= \delta_4[1\ 1\ 1\ 2\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 2\ 1\ 1\ 3\ 3],\\
\tilde{L}_2 &= \delta_4[2\ 2\ 1\ 2\ 2\ 2\ 1\ 2\ 2\ 2\ 1\ 2\ 1\ 2\ 3\ 3],\\
\tilde{L}_3 &= \delta_4[2\ 2\ 1\ 2\ 2\ 2\ 1\ 2\ 2\ 2\ 1\ 2\ 2\ 3\ 3\ 4],\\
\tilde{L}_4 &= \delta_4[2\ 1\ 2\ 4\ 2\ 1\ 2\ 4\ 2\ 1\ 2\ 4\ 2\ 4\ 4\ 4].
\end{aligned}$$

We obtain TPM $\mathbf{P}^v$ as

$$\mathbf{P}^v = \sum_{s=1}^{4} p_s\tilde{L}_s = \begin{bmatrix}\mathbf{P}_1^v & \mathbf{P}_2^v & \mathbf{P}_3^v & \mathbf{P}_4^v\end{bmatrix}, \tag{16}$$

where

$$\mathbf{P}_1^v = \mathbf{P}_2^v = \mathbf{P}_3^v = \begin{bmatrix} 0.23 & 0.44 & 0.79 & 0\\ 0.77 & 0.56 & 0.21 & 0.79\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0.21\end{bmatrix},$$

$$\mathbf{P}_4^v = \begin{bmatrix} 0.48 & 0.23 & 0 & 0\\ 0.52 & 0.25 & 0 & 0\\ 0 & 0.31 & 0.79 & 0.48\\ 0 & 0.21 & 0.21 & 0.52\end{bmatrix}.$$

The one-step and two-step reachability matrices as calculated as follows:

$$\mathbf{R}^v[1] = \lceil \mathbf{P}^v \rceil \ltimes_{\mathscr{B}} 1_4 = \begin{bmatrix} 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 1\\ 0 & 1 & 1 & 1\\ 0 & 1 & 1 & 1\end{bmatrix},$$

$$\mathbf{R}^v[2] = \mathbf{R}^v[1] +_{\mathscr{B}} (\mathbf{R}^v[1])^{(2)} = \begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix}. \tag{17}$$

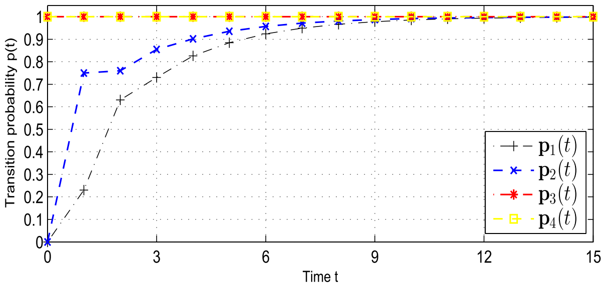

FIGURE 2. Curves of closed-loop transition probabilities from different initial states to $\mathcal{M}$.

In the following, we check the asymptotical stability with respect to $\mathcal{M} := \{\delta_4^1, \delta_4^3, \delta_4^4\}$. Using the algorithm in [44], it is easily confirmed that the LCIS in $\mathcal{M}$ is $I_c(\mathcal{M}) = \{\delta_4^3, \delta_4^4\}$. By (17), it obviously holds that

$$(\mathscr{B}) \sum_{i=3,4} \mathrm{Row}_i[\mathbf{R}^v[2]] = 1_4^T.$$

According to Theorem 1, we claim that this constrained PLCN is asymptotically feedback $\mathcal{M}$-stabilizable in distribution.

Now we design a state-feedback set stabilizer. First, using the technique of state-space partition proposed in [44], we design a state-feedback $\mathbf{v}_K$ with

$$K = \delta_4[1\ 4\ 4\ 4]$$

for PLCN (15) which asymptotically stabilizes the given subset $\mathcal{M}$. Then an admissible feedback $\mathbf{u}_G$ for the original PLCN is calculated as follow:

$$G = UW_{[4,4]}KM_{r,4} = \delta_4[1\ 3\ 4\ 1],$$

the admissibility of which can easily be checked. By (11), the TPM of the closed-loop system can be calculated as

$$\mathbf{P}_G = \sum_{\lambda=1}^{4} p_\lambda L_\lambda UW_{[4,4]}KM_{r,4}^2 = \begin{bmatrix} 0.23 & 0.23 & 0 & 0 \\ 0.77 & 0.25 & 0 & 0 \\ 0 & 0.31 & 0.79 & 0.48 \\ 0 & 0.21 & 0.21 & 0.52 \end{bmatrix}.$$

We denote by $\mathbf{p}_i(t)$ the transition probability from initial state δ_4^i to $\mathcal{M}$ under the feedback $\mathbf{u}_G$. The curves of $\mathbf{p}_i(t)$ for different initial states are shown in Fig. 2, which verifies the convergence of $\mathbf{p}_i(t)$. In the time-domain simulation, the sample state sequences and corresponding control sequences are obtained and shown in Fig. 3, which verifies the asymptotical stability with respect to $\mathcal{M}$.

Example 2: Consider a PLCN of the form (1) with $n = m = 2$, $n_i = m_j = 2$, $i, j \in [1:2]$, and $r = 4$. The structural matrices L_j and probability distribution p_j, $j \in [1:4]$, are as follows:

$$L_1 = \delta_4[4\ 2\ 3\ 4\ 1\ 2\ 2\ 4\ 1\ 3\ 4\ 4\ 4\ 3\ 3\ 1], \quad p_1 = 0.19,$$
$$L_2 = \delta_4[1\ 2\ 2\ 3\ 1\ 1\ 2\ 3\ 1\ 4\ 3\ 4\ 1\ 3\ 3\ 1], \quad p_2 = 0.33,$$

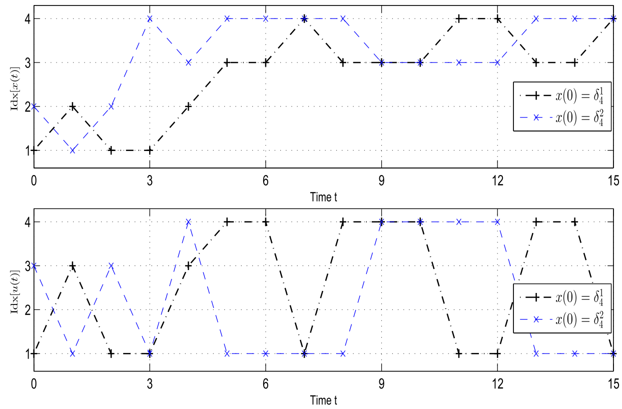

FIGURE 3. Sample sequences of state and control input under the feedback $\mathbf{u}_G$.

$$L_3 = \delta_4[4\ 2\ 2\ 1\ 2\ 2\ 3\ 4\ 1\ 2\ 3\ 4\ 4\ 3\ 3\ 1], \quad p_3 = 0.16,$$
$$L_4 = \delta_4[1\ 2\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 1\ 4\ 1\ 1\ 3\ 2], \quad p_4 = 0.32.$$

The control constraints are given as

$$\mathcal{U}_1 = \{\delta_4^1, \delta_4^3\}, \quad \mathcal{U}_2 = \{\delta_4^1, \delta_4^2\},$$
$$\mathcal{U}_3 = \{\delta_4^1, \delta_4^4\}, \quad \mathcal{U}_4 = \{\delta_4^2, \delta_4^3\}.$$

We define

$$U_1 := \delta_4[1\ 1\ 1\ 3], \quad U_2 := \delta_4[1\ 2\ 2\ 2],$$
$$U_3 := \delta_4[1\ 1\ 1\ 4], \quad U_4 := \delta_4[2\ 2\ 2\ 3].$$

Then, a pre-feedback $u(t) := Ux(t)v(t)$ is constructed with $U := [U_1\ U_2\ U_3\ U_4]$, which is combined with the original PLCN to obtain a PLCN with free input of the form (7) where

$$\tilde{L}_1 = \delta_4[4\ 2\ 3\ 4\ 4\ 2\ 3\ 4\ 4\ 2\ 3\ 4\ 1\ 2\ 3\ 4],$$
$$\tilde{L}_2 = \delta_4[1\ 2\ 2\ 3\ 1\ 1\ 2\ 3\ 1\ 1\ 2\ 3\ 1\ 1\ 3\ 4],$$
$$\tilde{L}_3 = \delta_4[4\ 2\ 2\ 4\ 4\ 2\ 2\ 4\ 4\ 2\ 2\ 4\ 1\ 2\ 3\ 4],$$
$$\tilde{L}_4 = \delta_4[1\ 2\ 3\ 3\ 1\ 1\ 3\ 3\ 1\ 1\ 3\ 3\ 1\ 1\ 3\ 4].$$

The TPM $\mathbf{P}^v$ of the combined PLCN is

$$\mathbf{P}^v = \sum_{s=1}^{4} p_s \tilde{L}_s = \begin{bmatrix} \mathbf{P}_1^v & \mathbf{P}_2^v & \mathbf{P}_3^v & \mathbf{P}_4^v \end{bmatrix}, \tag{18}$$

where

$$\mathbf{P}_1^v = \begin{bmatrix} 0.65 & 0 & 0 & 0 \\ 0 & 1 & 0.49 & 0 \\ 0 & 0 & 0.51 & 0.65 \\ 0.35 & 0 & 0 & 0.35 \end{bmatrix},$$

$$\mathbf{P}_2^v = \mathbf{P}_3^v = \begin{bmatrix} 0.65 & 0.65 & 0 & 0 \\ 0 & 0.35 & 0.49 & 0 \\ 0 & 0 & 0.51 & 0.65 \\ 0.35 & 0 & 0 & 0.35 \end{bmatrix},$$

$$\mathbf{P}_4^v = \begin{bmatrix} 1 & 0.65 & 0 & 0 \\ 0 & 0.35 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

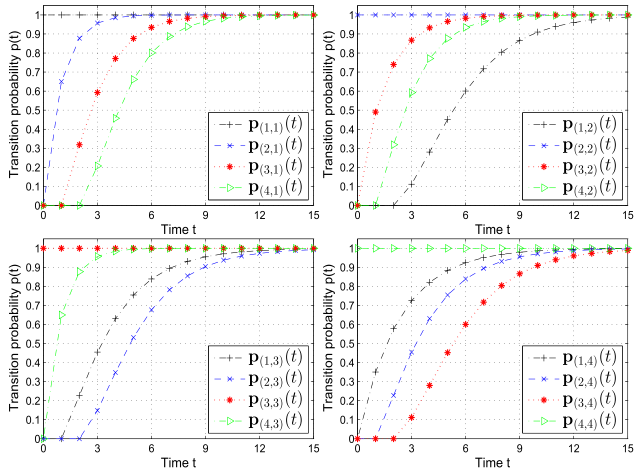

FIGURE 4. Curves of transition probabilities from different initial states to target state x_d under the different feedbacks.

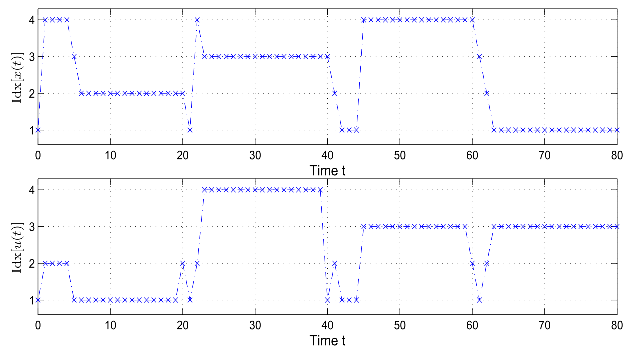

FIGURE 5. Sample sequences of state and control input.

Note that $[\mathbf{P}_1^\nu]_{2,2} = 1$, $[\mathbf{P}_4^\nu]_{r,r} = 1$, $r = 1, 3, 4$. Thus, every state is a control fixed point. In addition, the reachability matrices $\mathbf{R}^\nu[1]$ and $\mathbf{R}^\nu[3]$ are

$$\mathbf{R}^\nu[1] = \lceil \mathbf{P}^\nu \rceil \ltimes_{\mathscr{B}} 1_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{R}^\nu[3] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

According to Theorem 3, we claim that this PLCN is asymptotically feedback controllable. Thus, every state is asymptotically stabilizable. For each target state $x_d = \delta_4^d$, $d \in [1:4]$, we design an admissible feedback $\mathbf{u}_{G_d}$, where

$$G_1 = \delta_4[3\ 2\ 1\ 2], \quad G_2 = \delta_4[1\ 1\ 1\ 2],$$
$$G_3 = \delta_4[1\ 2\ 4\ 2], \quad G_4 = \delta_4[1\ 2\ 1\ 3].$$

We denote by $\mathbf{p}_{(i,d)}(t)$, $i, d \in [1:4]$, the closed-loop system transition probability from initial state δ_4^i to target state x_d under the feedback $\mathbf{u}_{G_d}$. The curves of the transition probabilities $\mathbf{p}_{(i,d)}(t)$ are shown in Fig. 4. In order to steer the PLCN from any state to a target state, one need only switch the controller to the feedback corresponding to the target state. It is easily confirmed that $\mathbf{p}_{(i,d)}(t) \geq 0.98$, $\forall t \geq 15$, $\forall i, d \in [1:4]$, which means that we can steer the PLCN to any target state within 15 steps with a probability of not less than 0.98. In the time-domain simulation, we steer the PLCN from the initial state $x(0) = \delta_4^1$ to the states δ_4^2, δ_4^3, δ_4^4, and δ_4^1 in order by switching between the feedbacks designed above. The dwell time of each feedback is set to $T = 20$. Thus, the switching feedback applied in the simulation is

$$u(t) = \begin{cases} G_2 x(t), & 0 \leq t < T, \\ G_3 x(t), & T \leq t < 2T, \\ G_4 x(t), & 2T \leq t < 3T, \\ G_1 x(t), & 3T \leq t < 4T. \end{cases} \tag{19}$$

The sample sequences of state and control input obtained in simulation are shown in Fig. 5. The results demonstrate that transitions between every pair of states are achievable, verifying the controllability of the PLCN.

VI. CONCLUSION

This study applied the technique of input transformation to deal with state-dependent constraints on control of PLCNs and proved that any admissible feedback can be decomposed into a pre-feedback and a constraint-free feedback, where the pre-feedback is constructed based on the constraints. This finding allowed us to transform the problems of controllability and stabilization of a PLCN with state-dependent constraints to those of a constraint-free PLCN obtained by combining the original PLCN and the pre-feedback. The necessary and sufficient conditions for AFC and AFSD were obtained. A procedure of designing an admissible stabilizing feedback for a PLCN with state-dependent constraints was provided. Finally, two examples were presented to illustrate the effectiveness of the obtained results.

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