Received 23 June 2022, accepted 23 July 2022, date of publication 28 July 2022, date of current version 3 August 2022.
Digital Object Identifier 10.1109/ACCESS.2022.3194562

METHODS

Improvement of Approximation Spaces Using Maximal Left Neighborhoods and Ideals

TAREQ M. AL-SHAMI[1] AND MONA HOSNY[2,3]
[1]Department of Mathematics, Sana'a University, Sana'a, Yemen
[2]Department of Mathematics, Faculty of Science for Girls, King Khalid University, Abha 61471, Saudi Arabia
[3]Department of Mathematics, Faculty of Education, Ain Shams University, Cairo 11341, Egypt

Corresponding author: Tareq M. Al-Shami (t.alshami@su.edu.ye)

The work of Mona Hosny was supported by the Deanship of Scientific Research at King Khalid University through Research Groups Program under Grant RGP2/144/43.

ABSTRACT Rough set theory was introduced by Pawlak, in 1982, as a methodology to discover structural relationships within imprecise and uncertain data. This theory has been generalized using the idea of neighborhood systems to be more efficient to get rid of uncertainty and deal with a wide scope of practical applications. Motivated by this idea, in this work, we initiate novel generalized rough set models using the concepts of "maximal left neighborhoods and ideals." Their basic features are studied and the relationships between them are revealed. The main merits of these models, as we prove, are first to preserve almost all major properties of approximation operators with respect to the Pawlak model. Second, they keep the monotonic property, which leads to an efficient evaluation of the uncertainty in the data, and third, these models enlarge the knowledge gotten from the information systems because they minimize the vagueness regions more than some previous models. We complete this manuscript by applying the proposed approach to analyze educational data and illustrate its role to improve the obtained classifications of objects and show the great performance of the present approach against other ones. Elucidative examples that support the obtained results are provided.

INDEX TERMS Approximation space, ideals, maximal left neighborhoods, rough set.

I. INTRODUCTION

Uncertainty is presented in several practical decision-making issues and real-life problems due to the incompleteness of knowledge. There are various approaches to handle uncertainty in these areas such as rough set theory, introduced by Pawlak [28], [29]. Rough set theory contributed to solve some issues such as characterization of a subset in terms of attribute values, finding dependency between the attributes, reduction of superfluous attributes, determining the most important attributes and decision rule generation.

This theory has rapidly progressed since it was advent in several directions; one of them is to reproduce the approximations operators and their related notions from neighborhoods generated by different binary relations instead of equivalence classes inspired by equivalence relations. This trend started by Yao [35], [36] in the nineties of the last century. He defined new approximation spaces induced from the right and left neighborhoods with respect to an arbitrary relation. These approximation spaces relax the strict condition of an equivalence relation and expand the scope of applications. On the other hand, these models lead to evaporate some properties of the original model given by Pawlak as well as the measures of accuracy exceed one in some cases, which requires some treatments [11]. Following the pioneering works of Yao, many researchers and scholars interested in the rough set theory introduced novel sorts of neighborhood systems and applied to establish new generalized rough set paradigms. Among these neighborhood systems, minimal left neighborhoods and minimal right neighborhoods [2], intersection of minimal left and right neighborhoods [25], and maximal neighborhoods [4], [9].

Mareay [27] familiarized four kinds of approximation spaces using new neighborhoods defined by the equality

The associate editor coordinating the review of this manuscript and approving it for publication was Hualong Yu.

relation between Yao's neighborhoods. Further types of these sorts of neighborhoods were established in [8]. Recently, Al-shami [3] discussed new approximation spaces inspired by containment neighborhoods which are defined using the inclusion relation. Then, he and Ciucci [6] initiated novel rough set models induced from subset neighborhoods which are defined using the superset relation. These neighborhoods and their approximation spaces were exploited to rank suspected individuals of COVID-19. In fact, many characterizations of Pawlak's models are still valid by these rough paradigms [3], [6]; especially, those are related to the properties of Pawlak's approximation operators and the property of monotonicity. These rough paradigms were debated under arbitrary binary relations. In contrast, investigation of some rough paradigms was conducted under specific type of relations such as quasiorder [31] and similarity [32].

In 2013, Kandi *et al.* [23] proposed a new technique to study approximation spaces by using "ideal structure" to high the accuracy measure which refers to the completeness of knowledge. This development helps to deal with uncertainty and get rid of obstacles in decision-making problems as shown in [13], [22], [33]. Topological spaces are another significant approach to investigate approximation operators [5], [10], [24], [26], [37]. These approaches applied with ideal to construct new rough set models. In this trend, Hosny [14]–[16] has recently displayed topological approximation spaces via ideals. Afterwards, Güler *et al.* [12] studied rough approximations induced from containment neighborhoods via ideals. Another technique of studying rough set paradigms is presented by Abu-Donia [1]. He explored approximation spaces using a class of binary relations instead of one binary relation. This idea was exploited to build the previous approximation spaces in terms of finite number of binary relations and ideals; see, for example [7], [18], [20].

The aim of this study is to provide another interesting and novel version of approximation spaces induced from "maximal left neighborhoods and ideals." The main motivation for us to introduce and study this version is to improve the approximation operators and increase the accuracy values of subsets, and to preserve as many properties as possible of Pawlak's approximation spaces and the property of monotonicity.

This article has been structured in the following manner. Section 2 presents an overview of rough neighborhood systems and ideals, which is required for the understanding of this work. The aim of Section 3 is to establish four rough set models and discuss their essential characterizations. The currently proposed models are compared in Section 4 and shown their advantages in comparison with the previous models. After that, a numerical example is shown that the current approach can be effectively applied to some practical issues in Section 5. Finally, Section 6 concludes with a summary of this manuscript and a suggestion for further research.

II. PRELIMINARIES

This section is dedicated to mentioning the main notions and ideas that will be used in the coming sections.

Definition 1 [21]: We call a non-empty family $\mathcal{P}$ of the power set of $U \neq \phi$ an "ideal" over U if it is closed under finite unions and subsets. That is, $V, W \in \mathcal{P} \Rightarrow V \cup W \in \mathcal{P}$, and if $V \in \mathcal{P}$ then every subset of V is a member of $\mathcal{P}$.

Definition 2 [22]: Assume that $\mathcal{P}_1, \mathcal{P}_2$ are ideals on a set $U \neq \phi$. The smallest collection generated by $\mathcal{P}_1, \mathcal{P}_2$, denoted by $\mathcal{P}_1 \vee \mathcal{P}_2$, is defined as

$$\mathcal{P}_1 \vee \mathcal{P}_2 = \{G \cup F : G \in \mathcal{P}_1, F \in \mathcal{P}_2\}. \tag{1}$$

Proposition 3 [22]: The collection $\mathcal{P}_1 \vee \mathcal{P}_2$ has the following properties.

(1) $\mathcal{P}_1 \vee \mathcal{P}_2 \neq \phi$,
(2) $V \in \mathcal{P}_1 \vee \mathcal{P}_2,\ W \subseteq V \Rightarrow W \in \mathcal{P}_1 \vee \mathcal{P}_2$,
(3) $V, W \in \mathcal{P}_1 \vee \mathcal{P}_2 \Rightarrow V \cup W \in \mathcal{P}_1 \vee \mathcal{P}_2$.

That is, the collection $\mathcal{P}_1 \vee \mathcal{P}_2$ is an ideal on U.

Definition 4 [28]: Consider δ as an equivalence relation on a universe U and let $[\nu]_\delta$ be the equivalence class containing ν. It can be associated each subset V of U with two other sets called "lower approximation $\underline{apr}(V)$" and "upper approximation $\overline{apr}(V)$" given as follows.

$$\underline{apr}(V) = \{\nu \in U : [\nu]_\delta \subseteq V\}. \tag{2}$$
$$\overline{apr}(V) = \{\nu \in U : [\nu]_\delta \cap V \neq \phi\}. \tag{3}$$

The main characterizations of these approximation operators are listed in the following.

($\mathcal{L}_1$) $\underline{apr}(V^c) = [\overline{apr}(V)]^c$, where V^c is the complement of V.
($\mathcal{L}_2$) $\underline{apr}(U) = U$.
($\mathcal{L}_3$) $\underline{apr}(\phi) = \phi$.
($\mathcal{L}_4$) $\underline{apr}(V) \subseteq V$.
($\mathcal{L}_5$) $\underline{apr}(V \cap W) = \underline{apr}(V) \cap \underline{apr}(W)$
($\mathcal{L}_6$) $\underline{apr}(V \cup W) \supseteq \underline{apr}(V) \cup \underline{apr}(W)$
($\mathcal{L}_7$) $V \subseteq W \Rightarrow \underline{apr}(V) \subseteq \underline{apr}(W)$.
($\mathcal{L}_8$) $\underline{apr}(\underline{apr}(V)) = \underline{apr}(V)$.
($\mathcal{L}_9$) $\overline{apr}(V) \subseteq \underline{apr}(\overline{apr}(V))$.
($\mathcal{U}_1$) $\overline{apr}(V^c) = [\underline{apr}(V)]^c$.
($\mathcal{U}_2$) $\overline{apr}(U) = U$.
($\mathcal{U}_3$) $\overline{apr}(\phi) = \phi$.
($\mathcal{U}_4$) $V \subseteq \overline{apr}(V)$.
($\mathcal{U}_5$) $\overline{apr}(V \cup W) = \overline{apr}(V) \cup \overline{apr}(W)$.
($\mathcal{U}_6$) $\overline{apr}(V \cap W) \subseteq \overline{apr}(V) \cap \overline{apr}(W)$.
($\mathcal{U}_7$) $V \subseteq W \Rightarrow \overline{apr}(V) \subseteq \overline{apr}(W)$.
($\mathcal{U}_8$) $\overline{apr}(\overline{apr}(V)) = \overline{apr}(V)$.
($\mathcal{U}_9$) $\overline{apr}(\underline{apr}(V)) \subseteq \underline{apr}(V)$.

Definition 5 [28]: Let δ be an equivalence relation on a universe U. Then accuracy measure $Acc_R(V)$ of any nonempty subset V is defined as follows: $Acc_R(V) = \frac{|\underline{apr}(V)|}{|\overline{apr}(V)|}$. If δ_1 and δ_2 are equivalence relations on a universe U such that $\delta_1 \subseteq \delta_2$. Then the approximations induced from these relations have the monotonic property if $Acc_{\delta_2}(V) \leq Acc_{\delta_1}(V)$.

Definition 6 [4], [9], [35], [36]: Take δ as an arbitrary binary relation on a finite set $U \neq \phi$ and let $\nu \in U$. Then,

1) the right neighborhood of ν, denoted by $N_r(\nu)$ is given by $N_r(\nu) = \{\lambda \in U : (\nu, \lambda) \in \delta\}$.
2) the left neighborhood of ν, denoted by $N_l(\nu)$ is given by $N_r(\nu) = \{\lambda \in U : (\lambda, \nu) \in \delta\}$.
3) $\theta_r(\nu)$ is the union of all right neighborhoods containing ν.
4) $\theta_l(\nu)$ is the union of all left neighborhoods containing ν.
5) $\theta_u(\nu) = \theta_r(\nu) \cup \theta_l(\nu)$.

Theorem 7 [4]: Let U be a universal set and δ_1, δ_2 be two binary relations on U. If $\delta_1 \subseteq \delta_2$, then $\theta_{1l}(\nu) \subseteq \theta_{2l}(\nu)$, $\forall \nu \in U$.

Definition 8 [4]: Let δ be a binary relation on a nonempty set U. For any subset $\phi \neq V \subseteq U$. The lower and upper approximations, boundary regions, accuracy and roughness of V induced from maximal left neighborhoods according to δ are defined respectively by:

$$L^\delta(V) = \{\nu \in U : \theta_l(\nu) \subseteq V\}. \tag{4}$$
$$U^\delta(V) = \{\nu \in U : \theta_l(\nu) \cap V \neq \phi\}. \tag{5}$$
$$Bnd_\delta^\delta(V) = U^\delta(V) - L^\delta(V). \tag{6}$$
$$Acc^\delta(V) = |\frac{L^\delta(V) \cap V}{U^\delta(V) \cup V}|. \tag{7}$$
$$Rough^\delta(V) = 1 - Acc^\delta(V). \tag{8}$$

Definition 9 [4]: Let δ be a binary relation on a nonempty set U. For any subset $\phi \neq V \subseteq U$. The lower and upper approximations, boundary regions, accuracy and roughness of V induced from maximal union neighborhoods according to δ are defined respectively by:

$$Low^\delta(V) = \{\nu \in U : \theta_u(\nu) \subseteq V\}. \tag{9}$$
$$Upp^\delta(V) = \{\nu \in U : \theta_u(\nu) \cap V \neq \phi\}. \tag{10}$$
$$Boundary_\delta^\delta(V) = Upp^\delta(V) - Low^\delta(V). \tag{11}$$
$$Accuracy^\delta(V) = |\frac{Low^\delta(V) \cap V}{Upp^\delta(V) \cup V}|. \tag{12}$$
$$Roughness^\delta(V) = 1 - Accuracy^\delta(V). \tag{13}$$

Definition 10 [19]: Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The first form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal union neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$Low_1{}^{\mathcal{P}\delta}(V) = \{\nu \in U : \theta_u(\nu) \cap V^c \in \mathcal{P}\}. \tag{14}$$
$$Upp_1{}^{\mathcal{P}\delta}(V) = \{\nu \in U : \theta_u(\nu) \cap V \notin \mathcal{P}\}. \tag{15}$$
$$Boundary_1{}^{\mathcal{P}\delta}(V) = Upp_1{}^{\mathcal{P}\delta}(V) - Low_1{}^{\mathcal{P}\delta}(V). \tag{16}$$
$$Accuracy_1{}^{\mathcal{P}\delta}(V) = \frac{|Low_1{}^{\mathcal{P}\delta}(V) \cap V|}{|Upp_1{}^{\mathcal{P}\delta}(V) \cup V}|. \tag{17}$$
$$Roughness_1{}^{\mathcal{P}\delta}(V) = 1 - Accuracy_1{}^{\mathcal{P}\delta}(V). \tag{18}$$

Definition 11 [19]: Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The second form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal union neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$Low_2{}^{\mathcal{P}\delta}(V) = \{\nu \in A : \theta_u(\nu) \cap V^c \in \mathcal{P}\}. \tag{19}$$
$$Upp_2{}^{\mathcal{P}\delta}(V) = V \cup Upp_1{}^{\mathcal{P}\delta}(V). \tag{20}$$
$$Boundary_2{}^{\mathcal{P}\delta}(V) = Upp_2{}^{\mathcal{P}\delta}(V) - Low_2{}^{\mathcal{P}\delta}(V). \tag{21}$$
$$Accuracy_2{}^{\mathcal{P}\delta}(V) = \frac{|Low_2{}^{\mathcal{P}\delta}(V)|}{|Upp_2{}^{\mathcal{P}\delta}(V)|}, \quad Upp_2{}^{\mathcal{P}\delta}(V) \neq \phi. \tag{22}$$
$$Roughness_2{}^{\mathcal{P}\delta}(V) = 1 - Accuracy_2{}^{\mathcal{P}\delta}(V). \tag{23}$$

Definition 12 [19]: Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The third form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal union neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$Low_3{}^{\mathcal{P}\delta}(V) = \cup\{\theta_u(\nu) : \theta_u(\nu) \cap V^c \in \mathcal{P}\}. \tag{24}$$
$$Upp_3{}^{\mathcal{P}\delta}(V) = (Low_3{}^{\mathcal{P}\delta}(V^c))^c. \tag{25}$$
$$Boundary_3{}^{\mathcal{P}\delta}(V) = Upp_3{}^{\mathcal{P}\delta}(V) - Low_3{}^{\mathcal{P}\delta}(V). \tag{26}$$
$$Accuracy_3{}^{\mathcal{P}\delta}(V) = \frac{|Low_3{}^{\mathcal{P}\delta}(V) \cap V|}{|Upp_3{}^{\mathcal{P}\delta}(V) \cup V}|. \tag{27}$$
$$Roughness_3{}^{\mathcal{P}\delta}(V) = 1 - Accuracy_3{}^{\mathcal{P}\delta}(V). \tag{28}$$

Definition 13 [19]: Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The fourth form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal union neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$Upp_4{}^{\mathcal{P}\delta}(V) = \cup\{\theta_u(\nu) : \theta_u(\nu) \cap V \notin \mathcal{P}\}. \tag{29}$$
$$Low_4{}^{\mathcal{P}\delta}(V) = (Upp_4{}^{\mathcal{P}\delta}(V^c))^c. \tag{30}$$
$$Boundary_4{}^{\mathcal{P}\delta}(V) = Upp_4{}^{\mathcal{P}\delta}(V) - Low_4{}^{\mathcal{P}\delta}(V). \tag{31}$$
$$Accuracy_4{}^{\mathcal{P}\delta}(V) = \frac{|Low_4{}^{\mathcal{P}\delta}(V) \cap V|}{|Upp_4{}^{\mathcal{P}\delta}(V) \cup V|}. \tag{32}$$
$$Roughness_4{}^{\mathcal{P}\delta}(V) = 1 - Accuracy_4{}^{\mathcal{P}\delta}(V). \tag{33}$$

III. SOME NEW ROUGH SET MODELS INDUCED FROM $\theta_l(\nu)$-NEIGHBORHOODS AND IDEALS

In this section, we display four types of rough set models defined by maximal left neighborhoods and ideals under any arbitrary relation. Their main features and characterizations are scrutinized and some counterexamples are provided to clarify the obtained facts and relationships.

A. FIRST TECHNIQUE TO GENERATE GENERALIZED ROUGH SETS VIA IDEALS

Definition 14: Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The first form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal left neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$L_1{}^{\mathcal{P}\delta}(V) = \{\nu \in U : \theta_l(\nu) \cap V^c \in \mathcal{P}\}. \tag{34}$$

$$U_1{}^{\mathcal{P}\delta}(V) = \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P}\}. \tag{35}$$

$$Bnd_1{}^{\mathcal{P}\delta}(V) = U_1{}^{\mathcal{P}\delta}(V) - L_1{}^{\mathcal{P}\delta}(V). \tag{36}$$

$$Acc_1{}^{\mathcal{P}\delta}(V) = \frac{|L_1{}^{\mathcal{P}\delta}(V) \cap V|}{|U_1{}^{\mathcal{P}\delta}(V) \cup V}|. \tag{37}$$

$$Rough_1{}^{\mathcal{P}\delta}(V) = 1 - Acc_1{}^{\mathcal{P}\delta}(V). \tag{38}$$

Proposition 15: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,

(1) $U_1{}^{\mathcal{P}\delta}(\phi) = \phi$.
(2) $V \subseteq W \Rightarrow U_1{}^{\mathcal{P}\delta}(V) \subseteq U_1{}^{\mathcal{P}\delta}(W)$.
(3) $U_1{}^{\mathcal{P}\delta}(V \cap W) \subseteq U_1{}^{\mathcal{P}\delta}(V) \cap U_1{}^{\mathcal{P}\delta}(W)$.
(4) $U_1{}^{\mathcal{P}\delta}(V \cup W) = U_1{}^{\mathcal{P}\delta}(V) \cup U_1{}^{\mathcal{P}\delta}(W)$.
(5) $U_1{}^{\mathcal{P}\delta}(V) = (L_1{}^{\mathcal{P}\delta}(V^c))^c$.
(6) If $V \in \mathcal{P}$, then $U_1{}^{\mathcal{P}\delta}(V) = \phi$.
(7) If $\mathcal{P} \subseteq \mathcal{T}$, then $U_1{}^{\mathcal{T}\delta}(V) \subseteq U_1{}^{\mathcal{P}\delta}(V)$.
(8) If $\mathcal{P} = P(U)$, then $U_1{}^{\mathcal{P}\delta}(V) = \phi$.
(9) $U_1{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = U_1{}^{\mathcal{P}\delta}(V) \cup U_1{}^{\mathcal{T}\delta}(V)$.
(10) $U_1{}^{(\mathcal{P}\vee\mathcal{T})\delta}(V) = U_1{}^{\mathcal{P}\delta}(V) \cap U_1{}^{\mathcal{T}\delta}(V)$.

Proof:

(1) $U_1{}^{\mathcal{P}\delta}(\phi) = \{\nu \in U : \theta_l(\nu) \cap \phi \notin \mathcal{P}\} = \phi$.
(2) Let $\nu \in U_1{}^{\mathcal{P}\delta}(V)$. Then, $\theta_l(\nu) \cap V \notin \mathcal{P}$. Since $V \subseteq W$ and $\mathcal{P}$ is an ideal. It follows that $\theta_l(\nu) \cap W \notin \mathcal{P}$. Therefore, $\nu \in U_1{}^{\mathcal{P}\delta}(W)$. Hence, $U_1{}^{\mathcal{P}\delta}(V) \subseteq U_1{}^{\mathcal{P}\delta}(W)$.
(3) It directly comes from (2).
(4) $U_1{}^{\mathcal{P}\delta}(V) \cup U_1{}^{\mathcal{P}\delta}(W) \subseteq U_1{}^{\mathcal{P}\delta}(V \cup W)$ according to (2). Let $\nu \in U_1{}^{\mathcal{P}\delta}(V \cup W)$. Then, $\theta_l(\nu) \cap (V \cup W) \notin \mathcal{P}$. It follows that $((\theta_l(\nu) \cap V) \cup (\theta_l(\nu) \cap W)) \notin \mathcal{P}$. Therefore, $\theta_l(\nu) \cap V \notin I$ or $\theta_l(\nu) \cap W \notin \mathcal{P}$, which gives $\nu \in U_1{}^{\mathcal{P}\delta}(V)$ or $\nu \in U_1{}^{\mathcal{P}\delta}(W)$. Then, $\nu \in U_1{}^{\mathcal{P}\delta}(V) \cup U_1{}^{\mathcal{P}\delta}(W)$. Thus, $U_1{}^{\mathcal{P}\delta}(V \cup W) \subseteq U_1{}^{\mathcal{P}\delta}(V) \cup U_1{}^{\mathcal{P}\delta}(W)$. Hence, $U_1{}^{\mathcal{P}\delta}(V \cup W) = U_1{}^{\mathcal{P}\delta}(V) \cup U_1{}^{\mathcal{P}\delta}(W)$.
(5) $(L_1{}^{\mathcal{P}\delta}(V^c))^c = (\{\nu \in U : \theta_l(\nu) \cap V \in \mathcal{P}\})^c = \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P}\} = U_1{}^{\mathcal{P}\delta}(V)$.
(6) The proof is straightforward by Definition 14.
(7) Let $\nu \in U_1{}^{\mathcal{T}\delta}(V)$. Then, $\theta_l(\nu) \cap V \notin \mathcal{T}$. Since $\mathcal{P} \subseteq \mathcal{T}$. So, $\theta_l(\nu) \cap V \notin \mathcal{P}$. Therefore, $\nu \in U_1{}^{\mathcal{P}\delta}(V)$. Hence, $U_1{}^{\mathcal{T}\delta}(V) \subseteq U_1{}^{\mathcal{P}\delta}(V)$.
(8) The proof is straightforward by Definition 14.
(9) $U_1{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P} \cap \mathcal{T}\}$
$= \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P}\}$ *or* $\{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{T}\}$
$= \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P}\} \cup \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{T}\}$
$= U_1{}^{(\mathcal{P}\cup\mathcal{T})\delta}(V)$.
(10) $U_1{}^{(\mathcal{P}\vee\mathcal{T})\delta}(V) = \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P} \vee \mathcal{T}\}$
$= \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P} \cup \mathcal{T}\}$
$= \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P}\}$ *and* $\{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{T}\}$
$= \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{P}\} \cap \{\nu \in U : \theta_l(\nu) \cap V \notin \mathcal{T}\}$
$= U_1{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V)$.

Proposition 16: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,

(1) $L_1{}^{\mathcal{P}\delta}(U) = U$.
(2) $V \subseteq W \Rightarrow L_1{}^{\mathcal{P}\delta}(V) \subseteq L_1{}^{\mathcal{P}\delta}(W)$.
(3) $L_1{}^{\mathcal{P}\delta}(V) \cup L_1{}^{\mathcal{P}\delta}(W) \subseteq L_1{}^{\mathcal{P}\delta}(V \cup W)$.
(4) $L_1{}^{\mathcal{P}\delta}(V \cap W) = L_1{}^{\mathcal{P}\delta}(V) \cap L_1{}^{\mathcal{P}\delta}(W)$.
(5) $L_1{}^{\mathcal{P}\delta}(V) = (U_1{}^{\mathcal{P}\delta}(V^c))^c$.
(6) If $V^c \in \mathcal{P}$, then $L_1{}^{\mathcal{P}\delta}(V) = U$.
(7) If $\mathcal{P} \subseteq \mathcal{T}$, then $L_1{}^{\mathcal{P}\delta}(V) \subseteq L_1{}^{\mathcal{T}\delta}(V)$.
(8) If $\mathcal{P} = P(U)$, then $L_1{}^{\mathcal{P}\delta}(V) = U$.
(9) $L_1{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = L_1{}^{\mathcal{P}\delta}(V) \cap L_1{}^{\mathcal{T}\delta}(V)$.

Proof:

(1) $L_1{}^{\mathcal{P}\delta}(U) = \{\nu \in U : \theta_l(\nu) \cap \phi \in \mathcal{P}\} = U$.
(2) Let $\nu \in L_1{}^{\mathcal{P}\delta}(V)$. Then, $\theta_l(\nu) \cap V^c \in \mathcal{P}$. Since $W^c \subseteq V^c$ and $\mathcal{P}$ is an ideal. So, $\theta_l(\nu) \cap W^c \in \mathcal{P}$. Therefore, $\nu \in L_1{}^{\mathcal{P}\delta}(W)$. Hence, $L_1{}^{\mathcal{P}\delta}(V) \subseteq L_1{}^{\mathcal{P}\delta}(W)$.
(3) It directly comes from (2).
(4) $L_1{}^{\mathcal{P}\delta}(V) \cap L_1{}^{\mathcal{P}\delta}(W) \supseteq L_1{}^{\mathcal{P}\delta}(V \cap W)$ according to (2). Let $\nu \in L_1{}^{\mathcal{P}\delta}(V) \cap L_1{}^{\mathcal{P}\delta}(W)$. Then, $\theta_l(\nu) \cap V^c \in \mathcal{P}$ and $\theta_l(\nu) \cap W^c \in \mathcal{P}$. It follows that $(\theta_l(\nu) \cap (V^c \cup W^c)) \in \mathcal{P}$. So, $(\theta_l(\nu) \cap (V \cap W)^c) \in \mathcal{P}$. Therefore, $\nu \in L_1{}^{\mathcal{P}\delta}(V \cap W)$. Thus, $L_1{}^{\mathcal{P}\delta}(V) \cap L_1{}^{\mathcal{P}\delta}(W) \subseteq L_1{}^{\mathcal{P}\delta}(V \cap W)$. Hence, $L_1{}^{\mathcal{P}\delta}(V) \cap L_1{}^{\mathcal{P}\delta}(W) = L_1{}^{\mathcal{P}\delta}(V \cap W)$.
(5) $(U_1{}^{\mathcal{P}\delta}(V^c))^c = (\{\nu \in U : \theta_l(\nu) \cap V^c \notin \mathcal{P}\})^c = \{\nu \in U : \theta_l(\nu) \cap V^c \in \mathcal{P}\} = L_1{}^{\mathcal{P}\delta}(V)$.
(6) The proof is straightforward by Definition 14.
(7) Let $\nu \in L_1{}^{\mathcal{P}\delta}(V)$. Then, $\theta_l(\nu) \cap V^c \in \mathcal{P}$. Since $\mathcal{P} \subseteq \mathcal{T}$. It follows that $\theta_l(\nu) \cap V^c \in \mathcal{T}$. Therefore, $\nu \in L_1{}^{\mathcal{P}\delta}(V)$. Hence, $L_1{}^{\mathcal{P}\delta}(V) \subseteq L_1{}^{\mathcal{T}\delta}(V)$.
(8) The proof is straightforward by Definition 14.
(9) $L_1{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = \{\nu \in U : \theta_l(\nu) \cap V^c \in \mathcal{P} \cap \mathcal{T}\}$
$= \{\nu \in U : \theta_l(\nu) \cap V^c \in \mathcal{P}\}$ *and* $\{\nu \in U : \theta_l(\nu) \cap V^c \in \mathcal{T}\}$
$= \{\nu \in U : \theta_l(\nu) \cap V^c \in \mathcal{P}\} \cap \{\nu \in U : \theta_l(\nu) \cap V^c \in \mathcal{T}\}$
$= L_1{}^{\mathcal{P}\delta}(V) \cap L_1{}^{\mathcal{T}\delta}(V)$.

With the help of the next counterexample, we elucidate that the converse of (2), (6), (7) and (8) of Proposition 15 and Proposition 16 is generally false. Also, we illustrate that the inclusion relations of (3) in Proposition 15 and Proposition 16 are proper, in general.

Example 17:

(i) Let

$$U = \{a, b, c, d\},$$
$$\mathcal{P} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

and

$$\delta = \{(a, c), (b, a), (c, a), (c, b), (d, b), (d, c)\}$$

be a binary relation defined on U thus $\theta_l(a) = \{a, d\}, \theta_l(b) = \{b, c\}, \theta_l(c) = \{b, c, d\}$ and $\theta_l(d) = \{a, c, d\}$. For (2), take

(a) $V = \{a\}$ and $W = \{d\}$; then, $U_1{}^{\mathcal{P}\delta}(V) = \phi$ and $U_1{}^{\mathcal{P}\delta}(W) = \{a, c, d\}$. Therefore, $U_1{}^{\mathcal{P}\delta}(V) \subseteq U_1{}^{\mathcal{P}\delta}(W)$, but $V \not\subseteq W$.
(b) $V = \{b\}$ and $W = \{a, c, d\}$; then, $L_1{}^{\mathcal{P}\delta}(V) = \{b\}$ and $L_1{}^{\mathcal{P}\delta}(W) = U$. Therefore, $L_1{}^{\mathcal{P}\delta}(V) \subseteq L_1{}^{\mathcal{P}\delta}(W)$, but $V \not\subseteq W$.

(ii) Let $U = \{a, b, c, d\}$, $\mathcal{T} = \{\phi, \{a\}\}$, $\mathcal{P} = \{\phi, \{d\}\}$ and $\delta = \{(a, a), (b, b), (c, c)\}$ be a binary relation defined on U; thus, $\theta_l(a) = \{a\}, \theta_l(b) = \{b\}, \theta_l(c) = \{c\}$ and $\theta_l(d) = \phi$.

(1) For (6), take
   (a) $V = \{a, d\}$; then, $U_1{}^{\mathcal{T}\delta}(V) = \phi$. Therefore, $U_1{}^{\mathcal{T}\delta}(V) = \phi$, but $V \notin \mathcal{T}$.
   (b) $V = \{b, c\}$; then, $L_1{}^{\mathcal{T}\delta}(V) = U$. Therefore, $L_1{}^{\mathcal{T}\delta}(V) = U$, but $V^c \notin \mathcal{T}$.

(2) For (7), take
   (a) $V = \{a, d\}$; then, $U_1{}^{\mathcal{P}\delta}(V) = \{a\}$ and $U_1{}^{\mathcal{T}\delta}(V) = \phi$. Therefore, $U_1{}^{\mathcal{T}\delta}(V) \subseteq U_1{}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
   (b) $V = \{b, c\}$; then, $L_1{}^{\mathcal{P}\delta}(V) = \{b, c, d\}$ and $L_1{}^{\mathcal{T}\delta}(V) = U$. Therefore, $L_1{}^{\mathcal{P}\delta}(V) \subseteq L_1{}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

(3) For (8), take
   (a) $V = \{a, d\}$; then, $U_1{}^{\mathcal{T}\delta}(V) = \phi$, but $\mathcal{T} \neq P(U)$.
   (b) $V = \{b, c\}$; then, $L_1{}^{\mathcal{T}\delta}(V) = U$, but $\mathcal{T} \neq P(U)$.

(iii) Let $U = \{a, b, c, d\}, \mathcal{P} = \{\phi, \{d\}\}$ and $\delta = \Delta \cup \{(a, b), (a, c), (a, d)\}$ thus, $\theta_l(a) = U, \theta_l(b) = \{a, b\}, \theta_l(c) = \{a, c\}$ and $\theta_l(d) = \{a, d\}$. For (3), take $V = \{a, d\}, W = \{b, c\}$ and
   (a) $V \cap W = \phi$; then, $U_1{}^{\mathcal{P}\delta}(V) = U, U_1{}^{\mathcal{P}\delta}(W) = \{a, b, c\}$ and $U_1{}^{\mathcal{P}\delta}(V \cap W) = \phi$. Therefore, $U_1{}^{\mathcal{P}\delta}(V) \cap U_1{}^{\mathcal{P}\delta}(W) = \{a, b, c\} \neq \phi = U_1{}^{\mathcal{P}\delta}(V \cap W)$.
   (b) $V \cup W = U$; then, $L_1{}^{\mathcal{P}\delta}(V) = \{d\}, L_1{}^{\mathcal{P}\delta}(W) = \phi$ and $U_1{}^{\mathcal{P}\delta}(V \cup W) = U$. Therefore, $L_1{}^{\mathcal{P}\delta}(V) \cup L_1{}^{\mathcal{P}\delta}(W) = \{d\} \neq U = L_1{}^{\mathcal{P}\delta}(V \cup W)$.

Remark 18: Some properties of Pawlak are not satisfy by this type as we show in the following.

(i) Considering Example 17 (i), take
   (1) $V = \{a\}$; then, $U_1{}^{\mathcal{P}\delta}(V) = \phi$. Hence, $V \nsubseteq U_1{}^{\mathcal{P}\delta}(V)$.
   (2) $V = \{b, c, d\}$; then, $L_1{}^{\mathcal{P}\delta}(V) = U$. Hence, $L_1{}^{\mathcal{P}\delta}(V) \nsubseteq V$.
   (3) $V = U$; then, $U_1{}^{\mathcal{P}\delta}(U) = \{a, c, d\}$. Hence, $U_1{}^{\mathcal{P}\delta}(U) \neq U$.
   (4) $V = \phi$; then, $L_1{}^{\mathcal{P}\delta}(\phi) = \{b\}$. Hence, $L_1{}^{\mathcal{P}\delta}(\phi) \neq \phi$.

(ii) Considering Example 17 (iii), take
   (1) $V = \{b, c\}$; then, $U_1{}^{\mathcal{P}\delta}(V) = \{a, b, c\}$ and $U_1{}^{\mathcal{P}\delta}(U_1{}^{\mathcal{P}\delta}(V)) = U$. Hence, $U_1{}^{\mathcal{P}\delta}(V) \neq U_1{}^{\mathcal{P}\delta}(U_1{}^{\mathcal{P}\delta}(V))$.
   (2) $V = \{a, d\}$; then, $L_1{}^{\mathcal{P}\delta}(V) = \{d\}$ and $L_1{}^{\mathcal{P}\delta}(L_1{}^{\mathcal{P}\delta}(V)) = \phi$. Hence, $L_1{}^{\mathcal{P}\delta}(V) \neq L_1{}^{\mathcal{P}\delta}(L_1{}^{\mathcal{P}\delta}(V))$.

(iii) *Example 19:* Let $U = \{a, b, c, d\}, \mathcal{P} = \{\phi, \{a\}\}$ and $\delta = \Delta \cup \{(a, b), (a, c), (a, d), (b, a), (b, c), (b, d)\}$. Then $\theta_l(a) = \theta_l(b) = U, \theta_l(c) = \{a, b, c\}$ and $\theta_l(d) = \{a, b, d\}$. It is clear that, if
   (1) $V = \{c\}$; then, $U_1{}^{\mathcal{P}\delta}(V) = \{a, b, c\}$ and $L_1{}^{\mathcal{P}\delta}(U_1{}^{\mathcal{P}\delta}(V)) = \{c\}$. Hence, $U_1{}^{\mathcal{P}\delta}(V) \nsubseteq L_1{}^{\mathcal{P}\delta}(U_1{}^{\mathcal{P}\delta}(V))$.
   (2) $V = \{a, b, d\}$; then, $L_1{}^{\mathcal{P}\delta}(V) = \{d\}$ and $U_1{}^{\mathcal{P}\delta}(L_1{}^{\mathcal{P}\delta}(V)) = \{a, b, d\}$. Hence,
$$U_1{}^{\mathcal{P}\delta}(L_1{}^{\mathcal{P}\delta}(V)) \nsubseteq L_1{}^{\mathcal{P}\delta}(V).$$

Proposition 20: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that V is a nonempty subset of U. Then,

1) $0 \leq Acc_1^{\mathcal{P}\delta}(V) \leq 1$.
2) $Acc_1^{\mathcal{P}\delta}(U) = 1$.

Proof: We prove (1) only and (2) is straightforward. Since, $\phi \neq V \subseteq U$, then $U_1^{\mathcal{P}\delta}(V) \cup V \neq \phi$. Hence, $\phi \subseteq L_1^{\mathcal{P}\delta}(V) \cap V \subseteq U_1^{\mathcal{P}\delta}(V) \cup V$. Therefore, $0 \leq |L_1^{\mathcal{P}\delta}(V) \cap V| \leq |U_1^{\mathcal{P}\delta}(V) \cup V|$. So, $0 \leq \frac{|L_1^{\mathcal{P}\delta}(V) \cap V|}{|U_1^{\mathcal{P}\delta}(V) \cup V|} \leq 1$. It means that, $0 \leq Acc_1^{\mathcal{P}\delta}(V) \leq 1$.

Theorem 21: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $\mathcal{P} \subseteq \mathcal{T}$. Then,

(1) $Bnd_1{}^{\mathcal{T}\delta}(V) \subseteq Bnd_1{}^{\mathcal{P}\delta}(V)$.
(2) $Acc_1{}^{\mathcal{P}\delta}(V) \leq Acc_1{}^{\mathcal{T}\delta}(V)$.
(3) $Rough_1{}^{\mathcal{T}\delta}(V) \leq Rough_1{}^{\mathcal{P}\delta}(V)$.

Proof:

(1) Let $\nu \in Bnd_1{}^{\mathcal{T}\delta}(V)$. Then, $\nu \in U_1{}^{\mathcal{T}\delta}(V) - L_1{}^{\mathcal{T}\delta}(V)$. So, $\nu \in U_1{}^{\mathcal{T}\delta}(V)$ and $\nu \in (L_1{}^{\mathcal{T}\delta}(V))^c$. Hence, $\nu \in U_1{}^{\mathcal{P}\delta}(V)$ and $\nu \in (L_1{}^{\mathcal{P}\delta}(V))^c$ according to (7) of Propositions 15 and 16. It follows that $\nu \in Bnd_1{}^{\mathcal{P}\delta}(V)$. Therefore, $Bnd_1{}^{\mathcal{T}\delta}(V) \subseteq Bnd_1{}^{\mathcal{P}\delta}(V)$.
(2) $Acc_1{}^{\mathcal{P}\delta}(V) = |\frac{L_1{}^{\mathcal{P}\delta}(V) \cap V}{U_1{}^{\mathcal{P}\delta}(V) \cup V}| \leq |\frac{L_1{}^{\mathcal{T}\delta}(V) \cap V}{U_1{}^{\mathcal{T}\delta}(V) \cup V}| = Acc_1{}^{\mathcal{T}\delta}(V)$.
(3) Straightforward by (2).

Remark 22: In Theorem 21 the converse of (1) and (2) is generally false. To validate this consider Example 17 (ii) and let $V = \{b, c\}$. Then,

(1) $Bnd_1{}^{\mathcal{T}\delta}(V) = \phi \subseteq \phi = Bnd_1{}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(2) $Acc_1{}^{\mathcal{P}\delta}(V) = 1 \leq 1 = Acc_1{}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(3) $Rough_1{}^{\mathcal{T}\delta}(V) = 0 \leq 0 = Rough_1{}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

Theorem 23: Let $\phi \neq V \subseteq U$, $\mathcal{P}$ be an ideal on U and δ_1, δ_2 be two binary relations on U. If $\delta_1 \subseteq \delta_2$, then

(1) $U_1{}^{\mathcal{P}\delta_1}(V) \subseteq U_1{}^{\mathcal{P}\delta_2}(V)$.
(2) $L_1{}^{\mathcal{P}\delta_2}(V) \subseteq L_1{}^{\mathcal{P}\delta_1}(V)$.
(3) $Bnd_1{}^{\mathcal{P}\delta_1}(V) \subseteq Bnd_1{}^{\mathcal{P}\delta_2}(V)$.
(4) $Acc_1{}^{\mathcal{P}\delta_2}(V) \leq Acc_1{}^{\mathcal{P}\delta_1}(V)$.
(5) $Rough_1{}^{\mathcal{P}\delta_1}(V) \leq Rough_1{}^{\mathcal{P}\delta_2}(V)$.

Proof:

(1) Let $\nu \in U_1{}^{\mathcal{P}\delta_1}(V)$. Then, $\theta_{1l}(\nu) \cap V \notin \mathcal{P}$. Since $\theta_{1l}(\nu) \subseteq \theta_{2l}(\nu)$ (by Theorem 7 [4]). It follows that $\theta_{2l}(\nu) \cap V \notin \mathcal{P}$. Thus, $\nu \in U_1{}^{\mathcal{P}\delta_2}(V)$. Hence, $U_1{}^{\mathcal{P}\delta_1}(V) \subseteq U_1{}^{\mathcal{P}\delta_2}(V)$.
(2) Let $\nu \in L_1{}^{\mathcal{P}\delta_2}(V)$. Then, $\theta_{2l}(\nu) \cap V^c \in \mathcal{P}$. Since $\theta_{1l}(\nu) \subseteq \theta_{2l}(\nu)$ (by Theorem 7 [4]). It follows that $\theta_{1l}(\nu) \cap V^c \in \mathcal{P}$. Thus, $\nu \in L_1{}^{\mathcal{P}\delta_1}(V)$. Hence, $L_1{}^{\mathcal{P}\delta_2}(V) \subseteq L_1{}^{\mathcal{P}\delta_1}(V)$.
(3) Let $\nu \in Bnd_1{}^{\mathcal{P}\delta_1}(V)$. Then, $\nu \in U_1{}^{\mathcal{P}\delta_1}(V) - L_1{}^{\mathcal{P}\delta_1}(V)$. So, $\nu \in U_1{}^{\mathcal{P}\delta_1}(V)$ and $\nu \in (L_1{}^{\mathcal{P}\delta_1}(V))^c$. Thus, $\nu \in U_1{}^{\mathcal{P}\delta_2}(V)$ and $\nu \in (L_1{}^{\mathcal{P}\delta_2}(V))^c$ according to (1) and (2). Hence, $\nu \in Bnd_1{}^{\mathcal{P}\delta_2}(V)$. Therefore, $Bnd_1{}^{\mathcal{P}\delta_1}(V) \subseteq Bnd_1{}^{\mathcal{P}\delta_2}(V)$.

(4) $Acc_1{}^{\mathcal{P}\delta_2}(V) = |\frac{L_1{}^{\mathcal{P}\delta_2}(V)\cap V}{U_1{}^{\mathcal{P}\delta_2}(V)\cup V}| \leq |\frac{L_1{}^{\mathcal{P}\delta_1}(V)\cap V}{U_1{}^{\mathcal{P}\delta_1}(V)\cup V}| = Acc_1{}^{\mathcal{P}\delta_1}(V)$.
(5) Straightforward by (4).

To show that the inclusion and less than relation in Theorem 23 is proper, we provide the next example.

Example 24: Let

$$U = \{a, b, c, d\},$$
$$\mathcal{P} = \{\phi, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$$
$$\delta_1 = \Delta\cup\{(a, b), (b, a)\} \text{ and } \delta_2 = \Delta\cup\{(a, b), (b, a), (c, a), (a, c)\}$$

be two relations defined on U; thus,

$$\theta_{1l}(a) = \theta_{1l}(b) = \{a, b\}, \theta_{1l}(c) = \{c\}, \theta_{1l}(d) = \{d\},$$
$$\theta_{2l}(a) = \theta_{2l}(b) = \theta_{2l}(c) = \{a, b, c\} \text{ and } \theta_{2l}(d) = \{d\}.$$

Take

(i) $V = \{a, d\}$; then,
  (1) $U_1{}^{\mathcal{P}\delta_1}(V) = \{a, b\} \neq \{a, b, c\} = U_1{}^{\mathcal{P}\delta_2}(V)$.
  (2) $Acc_1{}^{\mathcal{P}\delta_1}(V) = \frac{2}{3} \neq \frac{1}{2} = Acc_1{}^{\mathcal{P}\delta_2}(V)$.
  (3) $Rough_1{}^{\mathcal{P}\delta_1}(V) = \frac{1}{3} \neq \frac{1}{2} = Rough_1{}^{\mathcal{P}\delta_2}(V)$.
(ii) $V = \{b, c\}$; then, $L_1{}^{\mathcal{P}\delta_1}(V) = \{c, d\} \neq \{d\} = L_1{}^{\mathcal{P}\delta_2}(V)$.

B. SECOND TECHNIQUE TO GENERATE GENERALIZED ROUGH SETS VIA IDEALS

Definition 25: Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The second form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal union neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$L_2{}^{\mathcal{P}\delta}(V) = \{\nu \in A : \theta_l(\nu) \cap V^c \in \mathcal{P}\}. \quad (39)$$
$$U_2{}^{\mathcal{P}\delta}(V) = V \cup U_1{}^{\mathcal{P}\delta}(V). \quad (40)$$
$$Bnd_2{}^{\mathcal{P}\delta}(V) = U_2{}^{\mathcal{P}\delta}(V) - L_2{}^{\mathcal{P}\delta}(V). \quad (41)$$
$$Acc_2{}^{\mathcal{P}\delta}(V) = \frac{|L_2{}^{\mathcal{P}\delta}(V)|}{|U_2{}^{\mathcal{P}\delta}(V)|}, U_2{}^{\mathcal{P}\delta}(V) \neq \phi. \quad (42)$$
$$Rough_2{}^{\mathcal{P}\delta}(V) = 1 - Acc_2{}^{\mathcal{P}\delta}(V). \quad (43)$$

Proposition 26: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,
(1) $V \subseteq U_2{}^{\mathcal{P}\delta}(V)$ equality holds if $V = \phi$ or U.
(2) $V \subseteq W \Rightarrow U_2{}^{\mathcal{P}\delta}(V) \subseteq U_2{}^{\mathcal{P}\delta}(W)$.
(3) $U_2{}^{\mathcal{P}\delta}(V) \subseteq U_2{}^{\mathcal{P}\delta}(U_2{}^{\mathcal{P}\delta}(V))$.
(4) $U_2{}^{\mathcal{P}\delta}(V \cap W) \subseteq U_2{}^{\mathcal{P}\delta}(V) \cap U_2{}^{\mathcal{P}\delta}(W)$.
(5) $U_2{}^{\mathcal{P}\delta}(V \cup W) = U_2{}^{\mathcal{P}\delta}(V) \cup U_2{}^{\mathcal{P}\delta}(W)$.
(6) $U_2{}^{\mathcal{P}\delta}(V) = (L_2{}^{\mathcal{P}\delta}(V^c))^c$.
(7) If $V \in \mathcal{P}$, then $U_2{}^{\mathcal{P}\delta}(V) = V$.
(8) If $\mathcal{P} \subseteq \mathcal{T}$, then $U_2{}^{\mathcal{T}\delta}(V) \subseteq U_2{}^{\mathcal{P}\delta}(V)$.
(9) If $\mathcal{P} = P(U)$, then $U_2{}^{\mathcal{P}\delta}(V) = V$.
(10) $U_2{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = U_2{}^{\mathcal{P}\delta}(V) \cup U_2{}^{\mathcal{T}\delta}(V)$.
(11) $U_2{}^{(\mathcal{P}\vee\mathcal{T})\delta}(V) = U_2{}^{\mathcal{P}\delta}(V) \cap U_2{}^{\mathcal{T}\delta}(V)$.

Proof: Similar to Proposition 15.

Proposition 27: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,
(1) $L_2{}^{\mathcal{P}\delta}(V) \subseteq V$ equality holds if $V = \phi$ or U.
(2) $V \subseteq W \Rightarrow L_2{}^{\mathcal{P}\delta}(V) \subseteq L_2{}^{\mathcal{P}\delta}(W)$.
(3) $L_2{}^{\mathcal{P}\delta}(L_2{}^{\mathcal{P}\delta}(V)) \subseteq L_2{}^{\mathcal{P}\delta}(V)$.
(4) $L_2{}^{\mathcal{P}\delta}(V) \cup L_2{}^{\mathcal{P}\delta}(W) \subseteq L_2{}^{\mathcal{P}\delta}(V \cup W)$.
(5) $L_2{}^{\mathcal{P}\delta}(V \cap W) = L_2{}^{\mathcal{P}\delta}(V) \cap L_2{}^{\mathcal{P}\delta}(W)$.
(6) $L_2{}^{\mathcal{P}\delta}(V) = (U_2{}^{\mathcal{P}\delta}(V^c))^c$.
(7) If $V^c \in \mathcal{P}$, then $L_2{}^{\mathcal{P}\delta}(V) = V$.
(8) If $\mathcal{P} \subseteq \mathcal{T}$, then $L_2{}^{\mathcal{P}\delta}(V) \subseteq L_2{}^{\mathcal{T}\delta}(V)$.
(9) If $\mathcal{P} = P(U)$, then $L_2{}^{\mathcal{P}\delta}(V) = V$.
(10) $L_2{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = L_2{}^{\mathcal{P}\delta}(V) \cap L_2{}^{\mathcal{T}\delta}(V)$.

Proof: Similar to Proposition 16.

Remark 28: (i) It follows from Example 17 (i) that the converse of (2), (7) and (9) of Proposition 26 and Proposition 27 is generally false.
  (a) For (2), take
    (1) $V = \{a\}$ and $W = \{d\}$; then, $U_2{}^{\mathcal{P}\delta}(V) = \{a\} \subseteq \{a, c, d\} = U_2{}^{\mathcal{P}\delta}(W)$, but $V \nsubseteq W$.
    (2) $V = \{a, b, c\}$ and $W = \{b, c, d\}$; then, $L_2{}^{\mathcal{P}\delta}(V) = \{b\} \subseteq \{b, c, d\} = L_2{}^{\mathcal{P}\delta}(W)$, but $V \nsubseteq W$.
  (b) For (7), take
    (1) $V = \{a, c, d\}$; then, $U_2{}^{\mathcal{P}\delta}(V) = V$, but $V \notin \mathcal{P}$.
    (2) $V = \{b\}$; then, $L_2{}^{\mathcal{P}\delta}(V) = V$, but $V^c \notin \mathcal{P}$.
  (c) For (9), take
    (1) $V = \{a, c, d\}$; then, $U_2{}^{\mathcal{P}\delta}(V) = V$, but $\mathcal{P} \neq P(U)$.
    (2) $V = \{b\}$; then, $L_2{}^{\mathcal{P}\delta}(V) = V$, but $\mathcal{P} \neq P(U)$.
(ii) It can be seen from Example 17 (ii) that the converse of (8) of Proposition 26 and Proposition 27 is generally false. To show that, let
  (1) $V = \{a, d\}$. Then, $U_2{}^{\mathcal{T}\delta}(V) = V \subseteq V = U_2{}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
  (2) $V = \{b, c\}$; then, $L_2{}^{\mathcal{P}\delta}(V) = V \subseteq V = L_2{}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(iii) Example 17 (iii) illustrates that the inclusion relations of (3) and (4) of Proposition 26 and Proposition 27 are proper.
  (a) For (3), take
    (1) $V = \{b, c\}$; then, $U_2{}^{\mathcal{P}\delta}(V) = \{a, b, c\}$ and $U_2{}^{\mathcal{P}\delta}(U_2{}^{\mathcal{P}\delta}(V)) = U$. Therefore, $U_2{}^{\mathcal{P}\delta}(V) = \{a, b, c\} \neq U = U_2{}^{\mathcal{P}\delta}(U_2{}^{\mathcal{P}\delta}(V))$.
    (2) $V = \{a, d\}$; then, $L_2{}^{\mathcal{P}\delta}(V) = \{d\}$ and $L_2{}^{\mathcal{P}\delta}(L_2{}^{\mathcal{P}\delta}(V)) = \phi$. Therefore, $L_2{}^{\mathcal{P}\delta}(V) = \{d\} \neq \phi = L_2{}^{\mathcal{P}\delta}(L_2{}^{\mathcal{P}\delta}(V))$.
  (b) For (4), take $V = \{a, d\}, B = \{b, c\}$ and
    (1) $V \cap W = \phi$. Hence, $U_2{}^{\mathcal{P}\delta}(V) = U$ and $U_2{}^{\mathcal{P}\delta}(W) = \{a, b, c\}$. Therefore, $U_2{}^{\mathcal{P}\delta}(V) \cap U_2{}^{\mathcal{P}\delta}(W) = \{a, b, c\} \neq \phi = U_2{}^{\mathcal{P}\delta}(V \cap W)$.
    (2) $V \cup W = U$. Hence, $L_2{}^{\mathcal{P}\delta}(V) = \{d\}$ and $L_2{}^{\mathcal{P}\delta}(W) = \phi$. Therefore, $L_2{}^{\mathcal{P}\delta}(V) \cup L_2{}^{\mathcal{P}\delta}(W) = \{d\} \neq U = L_2{}^{\mathcal{P}\delta}(V \cup W)$.

Remark 29: Some properties given in the first type are not hold by this type as we show in the following.

(i) Considering Example 17 (i), take

(1) $V = \{a\} \in \mathcal{P}$; then, $U_2{}^{\mathcal{P}\delta}(V) = V$. Hence, if $V \in \mathcal{P} \not\Rightarrow U_2{}^{\mathcal{P}\delta}(V) = \phi$.

(2) $V^c = \{a\} \in \mathcal{P}$; then, $L_2{}^{\mathcal{P}\delta}(V) = V$. Hence, if $V^c \in \mathcal{P} \not\Rightarrow L_2{}^{\mathcal{P}\delta}(V) = U$.

(ii) Considering Example 17 (ii), take

(1) $\mathcal{T} = P(U)$ and $V = \{a, d\}$; then, $U_2{}^{\mathcal{T}\delta}(V) = V$. Hence, if $\mathcal{T} = P(U) \not\Rightarrow U_2{}^{\mathcal{T}\delta}(V) = \phi$.

(2) $\mathcal{T} = P(U)$ and $V = \{b, c\}$; then, $L_2{}^{\mathcal{T}\delta}(V) = V$. Hence, if $\mathcal{T} = P(U) \not\Rightarrow L_2{}^{\mathcal{T}\delta}(V) = U$.

Remark 30: Some properties of Pawlak are not satisfy by this type as we show in the following. In Example 19, take

(1) $V = \{c\}$; then, $U_2{}^{\mathcal{P}\delta}(V) = \{a, b, c\}$ and $L_2{}^{\mathcal{P}\delta}(U_2{}^{\mathcal{P}\delta}(V)) = \{c\}$. Therefore, $U_2{}^{\mathcal{P}\delta}(V) = \{a, b, c\} \nsubseteq \{c\} = L_2{}^{\mathcal{P}\delta}(U_2{}^{\mathcal{P}\delta}(V))$.

(2) $V = \{a, b, d\}$; then, $L_2{}^{\mathcal{P}\delta}(V) = \{d\}$ and $U_2{}^{\mathcal{P}\delta}(L_2{}^{\mathcal{P}\delta}(V)) = \{a, b, d\}$. Therefore, $U_2{}^{\mathcal{P}\delta}(L_2{}^{\mathcal{P}\delta}(V)) = \{a, b, d\} \nsubseteq \{d\} = L_2{}^{\mathcal{P}\delta}(V)$.

Proposition 31: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that V is a nonempty subset of U. Then,

1) $0 \leq Acc_2^{\mathcal{P}\delta}(V) \leq 1$.
2) $Acc_2^{\mathcal{P}\delta}(U) = 1$.

Proof: It is similar to Proposition 20.

Theorem 32: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $\mathcal{P} \subseteq \mathcal{T}$. Then,

(1) $Bnd_2{}^{\mathcal{T}\delta}(V) \subseteq Bnd_2{}^{\mathcal{P}\delta}(V)$.
(2) $Acc_2{}^{\mathcal{P}\delta}(V) \leq Acc_2{}^{\mathcal{T}\delta}(V)$.
(3) $Rough_2{}^{\mathcal{T}\delta}(V) \leq Rough_2{}^{\mathcal{P}\delta}(V)$.

Proof: Similar to the proof of Theorem 21.

Remark 33: In Theorem 32 the converse of (1) and (2) is generally false as illustrated in (ii) of Example 17. To show that let $V = \{b, c\}$. Then,

(1) $Bnd_2{}^{\mathcal{T}\delta}(V) = \phi \subseteq \phi = Bnd_2{}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(2) $Acc_2{}^{\mathcal{P}\delta}(V) = 1 \leq 1 = Acc_2{}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(3) $Rough_2{}^{\mathcal{T}\delta}(V) = 0 \leq 0 = Rough_2{}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

Theorem 34: Let $\phi \neq V \subseteq U$, $\mathcal{P}$ be an ideal on U and δ_1, δ_2 be two binary relations on U. If $\delta_1 \subseteq \delta_2$, then

(1) $U_2{}^{\mathcal{P}\delta_1}(V) \subseteq U_2{}^{\mathcal{P}\delta_2}(V)$.
(2) $L_2{}^{\mathcal{P}\delta_2}(V) \subseteq L_2{}^{\mathcal{P}\delta_1}(V)$.
(3) $Bnd_2{}^{\mathcal{P}\delta_1}(V) \subseteq Bnd_2{}^{\mathcal{P}\delta_2}(V)$.
(4) $Acc_2{}^{\mathcal{P}\delta_2}(V) \leq Acc_2{}^{\mathcal{P}\delta_1}(V)$.
(5) $Rough_2{}^{\mathcal{P}\delta_1}(V) \leq Rough_2{}^{\mathcal{P}\delta_2}(V)$.

Proof: Similar to Theorem 23.

Remark 35: In Theorem 34 the inclusion and less than relation is proper as showed in Example 24. To validate that let $V = \{a, d\}$. Then,

(1) $U_2{}^{\mathcal{P}\delta_1}(V) = \{a, b, d\} \neq U = U_2{}^{\mathcal{P}\delta_2}(V)$.
(2) $Bnd_2{}^{\mathcal{P}\delta_1}(V) = \{b\} \neq \{b, c\} = Bnd_2{}^{\mathcal{P}\delta_2}(V)$.
(3) $Acc_2{}^{\mathcal{P}\delta_1}(V) = \frac{2}{3} \neq \frac{1}{2} = Acc_2{}^{\mathcal{P}\delta_2}(V)$.
(4) $Rough_2{}^{\mathcal{P}\delta_1}(V) = 0.3 \neq 0.5 = Rough_2{}^{\mathcal{P}\delta_2}(V)$.

C. THIRD TECHNIQUE TO GENERATE GENERALIZED ROUGH SETS VIA IDEALS

Definition 36: Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The third form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal union neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$L_3{}^{\mathcal{P}\delta}(V) = \bigcup_{\nu \in U}\{\theta_l(\nu) : \theta_l(\nu) \cap V^c \in \mathcal{P}\}. \quad (44)$$

$$U_3{}^{\mathcal{P}\delta}(V) = (L_3{}^{\mathcal{P}\delta}(V^c))^c. \quad (45)$$

$$Bnd_3{}^{\mathcal{P}\delta}(V) = U_3{}^{\mathcal{P}\delta}(V) - L_3{}^{\mathcal{P}\delta}(V). \quad (46)$$

$$Acc_3{}^{\mathcal{P}\delta}(V) = \frac{|L_3{}^{\mathcal{P}\delta}(V) \cap V|}{|U_3{}^{\mathcal{P}\delta}(V) \cup V}|. \quad (47)$$

$$Rough_3{}^{\mathcal{P}\delta}(V) = 1 - Acc_3{}^{\mathcal{P}\delta}(V). \quad (48)$$

Proposition 37: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,

(1) $V \subseteq W \Rightarrow L_3{}^{\mathcal{P}\delta}(V) \subseteq L_3{}^{\mathcal{P}\delta}(W)$.
(2) $L_3{}^{\mathcal{P}\delta}(V) \cup L_3{}^{\mathcal{P}\delta}(W) \subseteq L_3{}^{\mathcal{P}\delta}(V \cup W)$.
(3) $L_3{}^{\mathcal{P}\delta}(V \cap W) \subseteq L_3{}^{\mathcal{P}\delta}(V) \cap L_3{}^{\mathcal{P}\delta}(W)$.
(4) $L_3{}^{\mathcal{P}\delta}(V) = (U_3{}^{\mathcal{P}\delta}(V^c))^c$.
(5) If $\mathcal{P} \subseteq \mathcal{T}$, then $L_3{}^{\mathcal{P}\delta}(V) \subseteq L_3{}^{\mathcal{T}\delta}(V)$.
(6) $L_3{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = L_3{}^{\mathcal{P}\delta}(V) \cap L_3{}^{\mathcal{T}\delta}(V)$.

Proof:

(1) Let $V \subseteq W$ and $\nu \in L_3{}^{\mathcal{P}\delta}(V)$. Then, $\exists\ y \in U$ such that $\nu \in \theta_l(y) \cap V^c \in \mathcal{P}$. Hence, $\nu \in \theta_l(y) \cap W^c \in \mathcal{P}$ (by $W^c \subseteq V^c$, and the properties of an ideal). Thus, $\nu \in L_3{}^{\mathcal{P}\delta}(W)$. Therefore, $L_3{}^{\mathcal{P}\delta}(V) \subseteq L_3{}^{\mathcal{P}\delta}(W)$.
(2) It is directly obtained by (1).
(3) It is directly obtained by (1).
(4) It immediately follows from Definition 36.
(5) Let $\mathcal{P} \subseteq \mathcal{T}$ and $\nu \in L_3{}^{\mathcal{P}\delta}(V)$. Then, $\exists\ y \in U$ such that $\nu \in \theta_l(y) \cap V^c \in \mathcal{P} \subseteq \mathcal{T}$. So, $\nu \in L_3{}^{\mathcal{T}\delta}(V)$, and hence $L_3{}^{\mathcal{P}\delta}(V) \subseteq L_3{}^{\mathcal{T}\delta}(V)$.
(6) $L_3{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = \bigcup_{\nu \in U}\{\theta_l(\nu) : \theta_l(\nu) \cap V^c \in \mathcal{P} \cap \mathcal{T}\}$
$= (\bigcup_{\nu \in U}\{\theta_l(\nu) : \theta_l(\nu) \cap V^c \in \mathcal{P}\})$ and $(\bigcup_{\nu \in U}\{\theta_l(\nu) : \theta_l(\nu) \cap V^c \in \mathcal{T}\})$
$= (\bigcup_{\nu \in U}\{\theta_l(\nu) : \theta_l(\nu) \cap V^c \in \mathcal{P}\}) \cap (\bigcup_{\nu \in U}\{\theta_l(\nu) : \theta_l(\nu) \cap V^c \in \mathcal{T}\})$
$= L_3{}^{\mathcal{P}\delta}(V) \cap L_3{}^{\mathcal{T}\delta}(V)$.

Proposition 38: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,

(1) $V \subseteq W \Rightarrow U_3{}^{\mathcal{P}\delta}(V) \subseteq U_3{}^{\mathcal{P}\delta}(W)$.
(2) $U_3{}^{\mathcal{P}\delta}(V \cap W) \subseteq U_3{}^{\mathcal{P}\delta}(V) \cap U_3{}^{\mathcal{P}\delta}(W)$.
(3) $U_3{}^{\mathcal{P}\delta}(V) \cup U_3{}^{\mathcal{P}\delta}(W) \subseteq U_3{}^{\mathcal{P}\delta}(V \cup W)$.
(4) $U_3{}^{\mathcal{P}\delta}(V) = (L_3{}^{\mathcal{P}\delta}(V^c))^c$.
(5) If $\mathcal{P} \subseteq \mathcal{T}$, then $U_3{}^{\mathcal{T}\delta}(V) \subseteq U_3{}^{\mathcal{P}\delta}(V)$.
(6) $U_3{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = U_3{}^{\mathcal{P}\delta}(V) \cup U_3{}^{\mathcal{T}\delta}(V)$.

Proof:

(1) Let $V \subseteq W$. Thus, $W^c \subseteq V^c$, and $L_3{}^{\mathcal{P}\delta}(W^c) \subseteq L_3{}^{\mathcal{P}\delta}(V^c)$ (by (1) in Proposition 37). So, $(L_3{}^{\mathcal{P}\delta}(V^c))^c \subseteq (L_3{}^{\mathcal{P}\delta}(W^c))^c$. Consequently, $U_3{}^{\mathcal{P}\delta}(V) \subseteq U_3{}^{\mathcal{P}\delta}(W)$.
(2) The proof directly follows by (1).

(3) The proof directly follows by (1).
(4) The proof is straightforward by Definition 36.
(5) Let $\mathcal{P} \subseteq \mathcal{T}$ and $\nu \in U_3{}^{\mathcal{T}\delta}(V)$. Then, $\nu \in (L_3{}^{\mathcal{T}\delta}(V^c))^c \subseteq (L_3{}^{\mathcal{P}\delta}(V^c))^c$, (by (5) in Proposition 37). Thus, $\nu \in (L_3{}^{\mathcal{P}\delta}(V^c))^c = U_3{}^{\mathcal{P}\delta}(V)$. Therefore, $U_3{}^{\mathcal{T}\delta}(V) \subseteq U_3{}^{\mathcal{P}\delta}(V)$.
(6) $U_3{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = (L_3{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V^c))^c$
$= (L_3{}^{\mathcal{P}\delta}(V^c) \cap L_3{}^{\mathcal{T}\delta}(V^c))^c$ (by (6) in Proposition 37)
$= (L_3{}^{\mathcal{P}\delta}(V^c))^c \cup (L_3{}^{\mathcal{T}\delta}(V^c))^c$
$= U_3{}^{\mathcal{P}\delta}(V) \cup U_3{}^{\mathcal{T}\delta}(V)$.

Remark 39: (1) In Proposition 37 and Proposition 38 the converse of (1) is generally false. To elucidate that consider Example 17 (i) and let

(a) $V = \{a\}$ and $W = \{d\}$; then, $U_3{}^{\mathcal{P}\delta}(V) = \phi$ and $U_3{}^{\mathcal{P}\delta}(W) = \{a, d\}$. Therefore, $U_3{}^{\mathcal{P}\delta}(V) \subseteq U_3{}^{\mathcal{P}\delta}(W)$, but $V \nsubseteq W$.
(b) $V = \{b\}$ and $W = \{a, c, d\}$; then, $L_3{}^{\mathcal{P}\delta}(V) = \{b, c\}$ and $L_3{}^{\mathcal{P}\delta}(W) = U$. Therefore, $L_3{}^{\mathcal{P}\delta}(V) \subseteq L_3{}^{\mathcal{P}\delta}(W)$, but $V \nsubseteq W$.

(2) In Proposition 37 and Proposition 38 the inclusion relation of (2) is proper as (iii) of Example 17 shows. For this, let $V = \{a, d\}$ and $W = \{b, c\}$. Then

(a) $U_3{}^{\mathcal{P}\delta}(V) = U, U_3{}^{\mathcal{P}\delta}(W) = W$ and $U_3{}^{\mathcal{P}\delta}(V \cap W) = \phi$. Therefore, $U_3{}^{\mathcal{P}\delta}(V) \cap U_3{}^{\mathcal{P}\delta}(W) = W \neq \phi = U_3{}^{\mathcal{P}\delta}(V \cap W)$.
(b) $L_3{}^{\mathcal{P}\delta}(V) = V, L_3{}^{\mathcal{P}\delta}(W) = \phi$ and $L_3{}^{\mathcal{P}\delta}(V \cup W) = U$. Therefore, $L_3{}^{\mathcal{P}\delta}(V) \cup L_3{}^{\mathcal{P}\delta}(W) = V \neq U = L_3{}^{\mathcal{P}\delta}(V \cup W)$.

(3) *Example 40:* Let $U = \{a, b, c, d\}, \mathcal{P} = \{\phi, \{a\}\}$ and $\delta = \{(a, a), (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, a)\}$. Then $\theta_l(a) = \theta_l(c) = U, \theta_l(b) = \{a, b, c\}$ and $\theta_l(d) = \{a, c, d\}$. To show that the inclusion relations of (3) of Proposition 37 and Proposition 38 are proper, take

(a) $V = \{a, c, d\}, W = \{a, b, c\}$ and $V \cap W = \{a, c\}$; then, $L_3{}^{\mathcal{P}\delta}(V) = V, L_3{}^{\mathcal{P}\delta}(W) = W$ and $L_3{}^{\mathcal{P}\delta}(V \cap W) = \phi$. Therefore, $L_3{}^{\mathcal{P}\delta}(V) \cap L_3{}^{\mathcal{P}\delta}(W) = \{a, c\} \neq \phi = L_3{}^{\mathcal{P}\delta}(V \cap W)$.
(b) $V = \{b\}, W = \{d\}$ and $V \cup W = \{b, d\}$; then, $U_3{}^{\mathcal{P}\delta}(V) = V, U_3{}^{\mathcal{P}\delta}(W) = W$ and $U_3{}^{\mathcal{P}\delta}(V \cup W) = U$. Therefore, $U_3{}^{\mathcal{P}\delta}(V) \cup U_3{}^{\mathcal{P}\delta}(W) = \{b, d\} \neq U = U_3{}^{\mathcal{P}\delta}(V \cup W)$.

(4) The converse of (5) in Proposition 37 and Proposition 38 is generally false. To elucidate this consider (ii) of Example 17 and let

(a) $V = \{a, d\}$; then, $U_3{}^{\mathcal{P}\delta}(V) = \{a, d\}$ and $U_3{}^{\mathcal{T}\delta}(V) = \{d\}$. Therefore, $U_3{}^{\mathcal{T}\delta}(V) \subseteq U_3{}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(b) $V = \{b, c\}$; then, $L_3{}^{\mathcal{P}\delta}(V) = \{b, c\}$ and $L_3{}^{\mathcal{T}\delta}(V) = \{a, b, c\}$. Therefore, $L_3{}^{\mathcal{P}\delta}(V) \subseteq L_3{}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

Remark 41: Some properties in the second type are not satisfy by this type as we show in the following.

(i) Considering Example 17 (i), take

(1) $V = \{a\}$; then, $U_3{}^{\mathcal{P}\delta}(V) = \phi$. Hence, $V \nsubseteq U_3{}^{\mathcal{P}\delta}(V)$.
(2) $V = \{b, c, d\}$; then, $L_3{}^{\mathcal{P}\delta}(V) = U$. Hence, $L_3{}^{\mathcal{P}\delta}(V) \nsubseteq V$.
(3) $V = U$; then, $U_3{}^{\mathcal{P}\delta}(U) = \{a, d\}$. Hence, $U_3{}^{\mathcal{P}\delta}(U) \neq U$.
(4) $V = \phi$; then, $L_3{}^{\mathcal{P}\delta}(\phi) = \{b, c\}$. Hence, $L_3{}^{\mathcal{P}\delta}(\phi) \neq \phi$.

(ii) *Example 42:* Let $U = \{a, b, c, d\}, \mathcal{P} = \{\phi, \{a\}\}$ and $\delta = \{(a, a)\}$ be a binary relation defined on U; thus, $\theta_l(a) = \{a\}$ and $\theta_l(b) = \theta_l(c) = \theta_l(d) = \phi$. Take

(1) $V = U$; then, $L_3{}^{\mathcal{P}\delta}(U) = \{a\}$. Hence, $L_3{}^{\mathcal{P}\delta}(U) \neq U$.
(2) $V = \phi$; then, $U_3{}^{\mathcal{P}\delta}(\phi) = \{b, c, d\}$. Hence, $U_3{}^{\mathcal{P}\delta}(\phi) \neq \phi$.

Remark 43: Some properties given in the firrst/second type are not satisfy by this type as we show in the following. In Example 42, take

(1) $V = \{b, c, d\}$; then, $V^c \in \mathcal{P}$ and $L_3{}^{\mathcal{P}\delta}(V) = \{a\}$. Hence, if $V^c \in \mathcal{P} \nRightarrow L_3{}^{\mathcal{P}\delta}(V) = U$ or V.
(2) $V = \{a\} \in \mathcal{P}$; then, $U_3{}^{\mathcal{P}\delta}(V) = \{b, c, d\}$. Hence, if $V \in \mathcal{P} \nRightarrow U_3{}^{\mathcal{P}\delta}(V) = \phi$ or V.
(3) $V = \{b, c, d\}$ and $\mathcal{P} = P(U)$; then, $L_3{}^{\mathcal{P}\delta}(V) = \{a\}$. Hence, if $\mathcal{P} = P(U) \nRightarrow L_3{}^{\mathcal{P}\delta}(V) = U$, or V.
(4) $V = \{a\}$ and $\mathcal{P} = P(U)$; then, $U_3{}^{\mathcal{P}\delta}(V) = \{b, c, d\}$. Hence, if $\mathcal{P} = P(U) \nRightarrow U_3{}^{\mathcal{P}\delta}(V) = \phi$, or V.

Proposition 44: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that V is a nonempty subset of U. Then,

1) $0 \leq Acc_3^{\mathcal{P}\delta}(V) \leq 1$.
2) $Acc_3^{\mathcal{P}\delta}(U) = 1$.

Proof: It is similar to Proposition 20.

Theorem 45: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $\mathcal{P} \subseteq \mathcal{T}$. Then,

(1) $Bnd_3{}^{\mathcal{T}\delta}(V) \subseteq Bnd_3{}^{\mathcal{P}\delta}(V)$.
(2) $Acc_3{}^{\mathcal{P}\delta}(V) \leq Acc_3{}^{\mathcal{T}\delta}(V)$.
(3) $Rough_3{}^{\mathcal{T}\delta}(V) \leq Rough_3{}^{\mathcal{P}\delta}(V)$.

Proof: Similar to Theorem 21.

Remark 46: It follows from (ii) of Example 17 that the converse of (1) and (2) of Theorem 45 is generally false. To demonstrate that let $V = \{b, c\}$. Then,

(1) $Bnd_3{}^{\mathcal{T}\delta}(V) = \{d\} \subseteq \{d\} = Bnd_3{}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(2) $Acc_3{}^{\mathcal{P}\delta}(V) = \frac{2}{3} \leq \frac{2}{3} = Acc_3{}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(3) $Rough_3{}^{\mathcal{T}\delta}(V) = \frac{1}{3} \leq \frac{1}{3} = Rough_3{}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.

The approximations operators, boundary-regions, measures of accuracy and roughness induced from the this type do not have monotonicity. The next example illustrates this fact.

Example 47: Let $U = \{a, b, c, d, e, f, g\}, \mathcal{P} = \{\phi, \{a\}\}$ and δ_1, δ_2 be two relations on U where $\delta_1 = \Delta \cup \{(a, c), (c, a), (c, g), (d, f), (e, g), (f, d), (g, c), (g, e)\}$. and $\delta_2 = \Delta \cup \{(a, c), (a, d), (a, e), (b, f), (c, a), (c, g), (d, a), (d, f), (e, a), (e, g), (f, b), (f, d), (g, c), (g, e)\}$. Thus, $\theta_{1l}(a) = \{a, c, g\}, \theta_{1l}(b) = \{b\}, \theta_{1l}(c) = \theta_{1l}(g) = \{a, c, e, g\}, \theta_{1l}(d) = \theta_{1l}(f) = \{d, f\}, \theta_{1l}(e) = \{c, e, g\}, \theta_{2l}(a) = \{a, c, d, e, f, g\}, \theta_{2l}(b) = \{b, d, f\}, \theta_{2l}(c) = \theta_{2l}(e) = \{a, c, d, e, g\}, \theta_{2l}(d) = \{a, b, c, d, e, f\}, \theta_{2l}(f) = \{a, b, d, f\}$ and $\theta_{2l}(g) = \{a, c, e, g\}$. Take

(1) $V = \{a, b, c, d, e, f\}$; then, $L_3{}^{\mathcal{P}\delta_1}(V) = \{b, d, f\}$ and $L_3{}^{\mathcal{P}\delta_2}(V) = V$. Therefore, $L_3{}^{\mathcal{P}\delta_1}(V) \not\supseteq L_3{}^{\mathcal{P}\delta_2}(V)$.
(2) $V = \{g\}$; then, $U_3{}^{\mathcal{P}\delta_1}(V) = \{a, c, e, g\}$ and $U_3{}^{\mathcal{P}\delta_2}(V) = \{g\}$. Therefore, $U_3{}^{\mathcal{P}\delta_1}(V) \nsubseteq U_3{}^{\mathcal{P}\delta_2}(V)$.
(3) $V = \{a, b, c, d, e, f\}$; then, $L_3{}^{\mathcal{P}\delta_1}(V) = \{b, d, f\}$, $U_3{}^{\mathcal{P}\delta_1}(V) = U, L_3{}^{\mathcal{P}\delta_2}(V) = V$ and $U_3{}^{\mathcal{P}\delta_2}(V) = U$. Therefore,
 (a) $Bnd_3{}^{\mathcal{P}\delta_1}(V) = \{a, c, e, g\} \nsubseteq \{g\} = Bnd_3{}^{\mathcal{P}\delta_2}(V)$.
 (b) $Acc_3{}^{\mathcal{P}\delta_1}(V) = \frac{3}{7} < \frac{6}{7} = Acc_3{}^{\mathcal{P}\delta_2}(V)$.
 (c) $Rough_3{}^{\mathcal{P}\delta_1}(V) = \frac{4}{7} > \frac{1}{7} = Rough_3{}^{\mathcal{P}\delta_2}(V)$.

Although, $\delta_1 \subseteq \delta_2$.

D. FOURTH TECHNIQUE TO GENERATE GENERALIZED ROUGH SETS VIA IDEALS

Definition 48: Let δ and $\mathcal{P}$ be binary relation and ideal on a set $U \neq \phi$. The fourth form of generalized approximations (lower and upper), boundary-regions, accuracy and rough values of a nonempty subset V of U produced by maximal union neighborhoods according to δ and $\mathcal{P}$ are respectively given by

$$U_4{}^{\mathcal{P}\delta}(V) = \bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu) \cap V \notin \mathcal{P}\}. \quad (49)$$

$$L_4{}^{\mathcal{P}\delta}(V) = (U_4{}^{\mathcal{P}\delta}(V^c))^c. \quad (50)$$

$$Bnd_4{}^{\mathcal{P}\delta}(V) = U_4{}^{\mathcal{P}\delta}(V) - L_4{}^{\mathcal{P}\delta}(V). \quad (51)$$

$$Acc_4{}^{\mathcal{P}\delta}(V) = \frac{|L_4{}^{\mathcal{P}\delta}(V) \cap V|}{|U_4{}^{\mathcal{P}\delta}(V) \cup V|}. \quad (52)$$

$$Rough_4{}^{\mathcal{P}\delta}(V) = 1 - Acc_4{}^{\mathcal{P}\delta}(V). \quad (53)$$

Proposition 49: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,
(1) $U_4{}^{\mathcal{P}\delta}(\phi) = \phi$.
(2) $V \subseteq W \Rightarrow U_4{}^{\mathcal{P}\delta}(V) \subseteq U_4{}^{\mathcal{P}\delta}(W)$.
(3) $U_4{}^{\mathcal{P}\delta}(V \cap W) \subseteq U_4{}^{\mathcal{P}\delta}(V) \cap U_4{}^{\mathcal{P}\delta}(W)$.
(4) $U_4{}^{\mathcal{P}\delta}(V \cup W) = U_4{}^{\mathcal{P}\delta}(V) \cup U_4{}^{\mathcal{P}\delta}(W)$.
(5) $U_4{}^{\mathcal{P}\delta}(V) = (L_4{}^{\mathcal{P}\delta}(V^c))^c$.
(6) If $V \in \mathcal{P}$, then $U_4{}^{\mathcal{P}\delta}(V) = \phi$.
(7) If $\mathcal{P} \subseteq \mathcal{T}$, then $U_4{}^{\mathcal{T}\delta}(V) \subseteq U_4{}^{\mathcal{P}\delta}(V)$.
(8) If $\mathcal{P} = P(U)$, then $U_4{}^{\mathcal{P}\delta}(V) = \phi$.
(9) $U_4{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = U_4{}^{\mathcal{P}\delta}(V) \cup U_4{}^{\mathcal{T}\delta}(V)$.
(10) $U_4{}^{(\mathcal{P}\vee\mathcal{T})\delta}(V) = U_4{}^{\mathcal{P}\delta}(V) \cap U_4{}^{\mathcal{T}\delta}(V)$.

Proof:
(1) $U_4{}^{\mathcal{P}\delta}(\phi) = \cup\{\theta_l(\nu) : \theta_l(\nu) \cap \phi \notin \mathcal{P}\} = \phi$.
(2) Let $V \subseteq W$ and $\nu \in U_4{}^{\mathcal{P}\delta}(V)$. Then, $\exists\, y \in U$ such that $\nu \in \theta_l(y)$ and $\theta_l(y) \cap V \notin \mathcal{P}$. Thus, $\theta_l(y) \cap W \notin \mathcal{P}$. So, $\nu \in U_4{}^{\mathcal{P}\delta}(W)$. Consequently, $U_4{}^{\mathcal{P}\delta}(V) \subseteq U_4{}^{\mathcal{P}\delta}(W)$.
(3) It is immediately obtained by (2).
(4) $U_4{}^{\mathcal{P}\delta}(V \cup W) = \bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu) \cap (V \cup W) \notin \mathcal{P}\}$.
$= (\bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu) \cap V \notin \mathcal{P}\}) \cup (\bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu) \cap W \notin \mathcal{P}\})$.
$= (\bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu) \cap V \notin \mathcal{P}\})\ or\ (\bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu) \cap W \notin \mathcal{P}\})$.
$= U_4{}^{\mathcal{P}\delta}(V) \cup U_4{}^{\mathcal{P}\delta}(W)$.
(5)

$$(L_4{}^{\mathcal{P}\delta}(V^c))^c = ((U_4{}^{\mathcal{P}\delta}(V))^c)^c. \\ = U_4{}^{\mathcal{P}\delta}(V).$$

(6) The proof is straightforward by Definition 48.
(7) Let $\mathcal{P} \subseteq \mathcal{T}$, $\nu \in U_4{}^{\mathcal{T}\delta}(V)$. Then, $\exists\, y \in U$ such that $\nu \in \theta_l(y)$ and $\theta_l(y) \cap V \notin \mathcal{T}$. Thus, $\theta_l(y) \cap V \notin \mathcal{P}$ as $\mathcal{P} \subseteq \mathcal{T}$. So, $\nu \in U_4{}^{\mathcal{P}\delta}(V)$. Hence, $U_4{}^{\mathcal{T}\delta}(V) \subseteq U_4{}^{\mathcal{P}\delta}(V)$.
(8) The proof is straightforward by Definition 48.
(9) $U_4{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = \bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu) \cap V \notin \mathcal{P} \cap \mathcal{T}\}$
$= (\bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu) \cap V \notin \mathcal{P}\})\ or\ (\bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu) \cap V \notin \mathcal{T}\})$
$= (\bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu) \cap V \notin \mathcal{P}\}) \cup (\bigcup_{\nu\in U}\{\theta_l(\nu) : \theta_l(\nu) \cap V \notin \mathcal{T}\})$
$= U_4{}^{\mathcal{P}\delta}(V) \cup U_4{}^{\mathcal{T}\delta}(V)$.
(10) $U_4{}^{(\mathcal{P}\vee\mathcal{T})\delta}(V) = \cup\{\theta_l(\nu) : \theta_l(\nu) \cap V \notin \mathcal{P} \vee \mathcal{T}\}$
$= \cup\{\theta_l(\nu) : \theta_l(\nu) \cap V \notin \mathcal{P} \cup \mathcal{T}\}$
$= (\cup\{\theta_l(\nu) : \theta_l(\nu) \cap V \notin \mathcal{P}\})\ and\ (\cup\{\theta_l(\nu) : \theta_l(\nu) \cap V \notin \mathcal{T}\})$
$= (\cup\{\theta_l(\nu) : \theta_l(\nu) \cap V \notin \mathcal{P}\}) \cap (\cup\{\theta_l(\nu) : \theta_l(\nu) \cap V \notin \mathcal{T}\})$
$= U_4{}^{\mathcal{P}\delta}(V) \cap U_4{}^{\mathcal{T}\delta}(V)$.

Proposition 50: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $V, W \subseteq U$ Then,
(1) $L_4{}^{\mathcal{P}\delta}(U) = U$.
(2) $V \subseteq W \Rightarrow L_4{}^{\mathcal{P}\delta}(V) \subseteq L_4{}^{\mathcal{P}\delta}(W)$.
(3) $L_4{}^{\mathcal{P}\delta}(V) \cup L_4{}^{\mathcal{P}\delta}(W) \subseteq L_4{}^{\mathcal{P}\delta}(V \cup W)$.
(4) $L_4{}^{\mathcal{P}\delta}(V \cap W) = L_4{}^{\mathcal{P}\delta}(V) \cap L_4{}^{\mathcal{P}\delta}(W)$.
(5) $L_4{}^{\mathcal{P}\delta}(V) = (U_4{}^{\mathcal{P}\delta}(V^c))^c$.
(6) If $V^c \in \mathcal{P}$, then $L_4{}^{\mathcal{P}\delta}(V) = U$.
(7) If $\mathcal{P} \subseteq \mathcal{T}$, then $L_4{}^{\mathcal{P}\delta}(V) \subseteq L_4{}^{\mathcal{T}\delta}(V)$.
(8) If $\mathcal{P} = P(U)$, then $L_4{}^{\mathcal{P}\delta}(V) = U$.
(9) $L_4{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = L_4{}^{\mathcal{P}\delta}(V) \cap L_4{}^{\mathcal{T}\delta}(V)$.
(10) $L_4{}^{(\mathcal{P}\vee\mathcal{T})\delta}(V) = L_4{}^{\mathcal{P}\delta}(V) \cup L_4{}^{\mathcal{T}\delta}(V)$.

Proof:
(1) $L_4{}^{\mathcal{P}\delta}(U) = (U_4{}^{\mathcal{P}\delta}(\phi))^c = \phi^c = U$ by (1) in Proposition 49.
(2) Let $V \subseteq W$. Thus, $W^c \subseteq V^c$ and $U_4{}^{\mathcal{P}\delta}(W^c) \subseteq U_4{}^{\mathcal{P}\delta}(V^c)$ (by (2) in Proposition 49). Then, $(U_4{}^{\mathcal{P}\delta}(V^c))^c \subseteq (U_4{}^{\mathcal{P}\delta}(W^c))^c$. So, $L_4{}^{\mathcal{P}\delta}(V) \subseteq L_4{}^{\mathcal{P}\delta}(W)$.
(3) The proof is directly by (2).
(4) $L_4{}^{\mathcal{P}\delta}(V \cap W) = (U_4{}^{\mathcal{P}\delta}(V \cap W)^c)^c$
$= (U_4{}^{\mathcal{P}\delta}(V^c \cup W^c))^c$
$= (U_4{}^{\mathcal{P}\delta}(V^c) \cup U_4{}^{\mathcal{P}\delta}(W^c))^c$ (by (4) in Proposition 49)
$= (U_4{}^{\mathcal{P}\delta}(V^c))^c \cap (U_4{}^{\mathcal{P}\delta}(W^c))^c$
$= L_4{}^{\mathcal{P}\delta}(V) \cap L_4{}^{\mathcal{P}\delta}(W)$.
(5) The proof is straightforward by Definition 48.
(6) Let $V^c \in \mathcal{P}$; then, $L_4{}^{\mathcal{P}\delta}(V) = (U_4{}^{\mathcal{P}\delta}(V^c))^c = (\phi)^c = U$ according to Proposition 49 (6).
(7) Let $\mathcal{P} \subseteq \mathcal{T}$. Then, $U_4{}^{\mathcal{T}\delta}(V^c) \subseteq U_4{}^{\mathcal{P}\delta}(V^c)$ according to Proposition 49 (7). Thus, $(U_4{}^{\mathcal{P}\delta}(V^c))^c \subseteq (U_4{}^{\mathcal{T}\delta}(V^c))^c$. Hence, $L_4{}^{\mathcal{P}\delta}(V) \subseteq L_4{}^{\mathcal{T}\delta}(V)$.
(8) Let $\mathcal{P} = P(U)$; then, $L_4{}^{\mathcal{P}\delta}(V) = (U_4{}^{\mathcal{P}\delta}(V^c))^c = (\phi)^c = U$ according to Proposition 49 (8).
(9) $L_4{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V) = (U_4{}^{(\mathcal{P}\cap\mathcal{T})\delta}(V^c))^c$
$= (U_4{}^{\mathcal{P}\delta}(V^c) \cup U_4{}^{\mathcal{T}\delta}(V^c))^c$ (by (9) in Proposition 49)
$= (U_4{}^{\mathcal{P}\delta}(V^c))^c \cap (U_4{}^{\mathcal{T}\delta}(V^c))^c$
$= L_4{}^{\mathcal{P}\delta}(V) \cap L_4{}^{\mathcal{T}\delta}(V)$.

(10) ${L_4}^{(\mathcal{P}\vee\mathcal{T})\delta}(V) = ({U_4}^{(\mathcal{P}\vee\mathcal{T})\delta}(V^c))^c$
$= ({U_4}^{\mathcal{P}\delta}(V^c) \cap {U_4}^{\mathcal{T}\delta}(V^c))^c$ by (10) in Proposition 49)
$= ({U_4}^{\mathcal{P}\delta}(V^c))^c \cup ({U_4}^{\mathcal{T}\delta}(V^c))^c$
$= {L_4}^{\mathcal{P}\delta}(V) \cup {L_4}^{\mathcal{T}\delta}(V)$.

Remark 51: (1) To show that the converse of (2) of Proposition 49 and Proposition 50 is generally false take (i) of Example 17 and let
(a) $V = \{a\}$ and $W = \{d\}$; then, ${U_4}^{\mathcal{P}\delta}(V) = \phi$ and ${U_4}^{\mathcal{P}\delta}(W) = U$. Therefore, ${U_4}^{\mathcal{P}\delta}(V) \subseteq {U_4}^{\mathcal{P}\delta}(W)$, but $V \nsubseteq W$.
(b) $V = \{b\}$ and $W = \{a, c, d\}$; then, ${L_4}^{\mathcal{P}\delta}(V) = \phi$ and ${L_4}^{\mathcal{P}\delta}(W) = U$. Therefore, ${L_4}^{\mathcal{P}\delta}(V) \subseteq {L_4}^{\mathcal{P}\delta}(W)$, but $V \nsubseteq W$.

(2) In Proposition 49 and Proposition 50 the converse of (6), (7) and (8) is generally false as illustrated by ii) of Example 17
(i) For (6), take
(a) $V = \{a, d\}$; then, ${U_4}^{\mathcal{T}\delta}(V) = \phi$. Therefore, ${U_4}^{\mathcal{T}\delta}(V) = \phi$, but $V \notin \mathcal{T}$.
(b) $V = \{b, c\}$; then, ${L_4}^{\mathcal{T}\delta}(V) = U$. Therefore, ${L_4}^{\mathcal{T}\delta}(V) = U$, but $V^c \notin \mathcal{T}$.
(ii) For (7), take
(a) $V = \{a, d\}$; then, ${U_4}^{\mathcal{P}\delta}(V) = \{a\}$ and ${U_4}^{\mathcal{T}\delta}(V) = \phi$. Therefore, ${U_4}^{\mathcal{T}\delta}(V) \subseteq {U_4}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(b) $V = \{b, c\}$; then, ${L_4}^{\mathcal{P}\delta}(V) = \{b, c, d\}$ and ${L_4}^{\mathcal{T}\delta}(V) = U$. Therefore, ${L_4}^{\mathcal{P}\delta}(V) \subseteq {L_4}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(iii) For (8), take
(a) $V = \{a, d\}$; then, ${U_4}^{\mathcal{T}\delta}(V) = \phi$, but $\mathcal{T} \neq P(U)$.
(b) $V = \{b, c\}$; then, ${L_4}^{\mathcal{T}\delta}(V) = U$, but $\mathcal{T} \neq P(U)$.

(3) Example 17 (iii) elaborates that the inclusion relations given in (3) of Proposition 49 and Proposition 50 are proper. To illustrate that, let $V = \{a, d\}$ and $W = \{b, c\}$. Then,
(a) ${U_4}^{\mathcal{P}\delta}(V) = {U_4}^{\mathcal{P}\delta}(W) = U$ and ${U_4}^{\mathcal{P}\delta}(V \cap W) = \phi$. Therefore, ${U_4}^{\mathcal{P}\delta}(V) \cap {U_4}^{\mathcal{P}\delta}(W) = U \neq \phi = {U_3}^{\mathcal{P}\delta}(V \cap W)$.
(b) ${L_4}^{\mathcal{P}\delta}(V) = {L_4}^{\mathcal{P}\delta}(W) = \phi$ and ${L_4}^{\mathcal{P}\delta}(V \cup W) = U$. Therefore, ${L_4}^{\mathcal{P}\delta}(V) \cup {L_4}^{\mathcal{P}\delta}(W) = \phi \neq U = {L_4}^{\mathcal{P}\delta}(V \cup W)$.

Remark 52: Some properties given in the second type are not satisfy by this type as we show in the following.
(i) Considering Example 17 (i), take
(1) $V = \{a\}$; then, ${U_4}^{\mathcal{P}\delta}(V) = \phi$. Hence, $V \nsubseteq {U_4}^{\mathcal{P}\delta}(V)$.
(2) $V = \{b, c, d\}$; then, ${L_4}^{\mathcal{P}\delta}(V) = U$. Hence, ${L_4}^{\mathcal{P}\delta}(V) \nsubseteq V$.
(ii) Considering Example 17 (ii), take
(1) $V = U$; then, ${U_4}^{\mathcal{P}\delta}(U) = \{a, b, c\}$. Hence, ${U_4}^{\mathcal{P}\delta}(U) \neq U$.
(2) $V = \phi$; then, ${L_4}^{\mathcal{P}\delta}(\phi) = \{d\}$. Hence, ${L_4}^{\mathcal{P}\delta}(\phi) \neq \phi$.

Proposition 53: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that V is a nonempty subset of U. Then,
1) $0 \leq Acc_4^{\mathcal{P}\delta}(V) \leq 1$.
2) $Acc_4^{\mathcal{P}\delta}(U) = 1$.

Proof: It is similar to Proposition 20.

Theorem 54: Let $\mathcal{P}$ and $\mathcal{T}$ be ideals and δ be a binary relation on U such that $\mathcal{P} \subseteq \mathcal{T}$. Then,
(1) ${Bnd_4}^{\mathcal{T}\delta}(V) \subseteq {Bnd_4}^{\mathcal{P}\delta}(V)$.
(2) ${Acc_4}^{\mathcal{P}\delta}(V) \leq {Acc_4}^{\mathcal{T}\delta}(V)$.
(3) ${Rough_4}^{\mathcal{T}\delta}(V) \leq {Rough_4}^{\mathcal{P}\delta}(V)$.

Proof: Similar to Theorem 21.

Remark 55: According to (ii) of Example 17, the converse of (1) and (2) in Theorem 54 is generally false. To clarify that, let $V = \{b, c\}$. Then,
(1) ${Bnd_4}^{\mathcal{T}\delta}(V) = \phi \subseteq \phi = {Bnd_4}^{\mathcal{P}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(2) ${Acc_4}^{\mathcal{P}\delta}(V) = 1 \leq 1 = {Acc_4}^{\mathcal{T}\delta}(V)$, but $\mathcal{P} \nsubseteq \mathcal{T}$.
(3) ${Rough_4}^{\mathcal{T}\delta}(V) = 0 \leq 0 = {Rough_4}^{\mathcal{P}\delta}(V)$.

Theorem 56: Let $\phi \neq V \subseteq U$, $\mathcal{P}$ be an ideal on U and δ_1, δ_2 be two binary relations on U. If $\delta_1 \subseteq \delta_2$, then
(1) ${U_4}^{\mathcal{P}\delta_1}(V) \subseteq {U_4}^{\mathcal{P}\delta_2}(V)$.
(2) ${L_4}^{\mathcal{P}\delta_2}(V) \subseteq {L_4}^{\mathcal{P}\delta_1}(V)$.
(3) ${Bnd_4}^{\mathcal{P}\delta_1}(V) \subseteq {Bnd_4}^{\mathcal{P}\delta_2}(V)$.
(4) ${Acc_4}^{\mathcal{P}\delta_2}(V) \leq {Acc_4}^{\mathcal{P}\delta_1}(V)$.
(5) ${Rough_4}^{\mathcal{P}\delta_1}(V) \leq {Rough_4}^{\mathcal{P}\delta_2}(V)$.

Proof:
(1) Let $\nu \in {U_4}^{\mathcal{P}\delta_1}(V)$. Then, $\exists\ y \in U$ such that $\nu \in \theta_{1l}(y) \cap V \notin \mathcal{P}$. Since $\theta_{1l}(y) \subseteq \theta_{2l}(y)$ (by Theorem 7 [4]), it follows that $\nu \in \theta_{2l}(y) \cap V \notin \mathcal{P}$. Thus, $\nu \in {U_4}^{\mathcal{P}\delta_2}(V)$. Hence, ${U_4}^{\mathcal{P}\delta_1}(V) \subseteq {U_4}^{\mathcal{P}\delta_2}(V)$.
(2) $\nu \in {L_4}^{\mathcal{P}\delta_2}(V) = ({U_4}^{\mathcal{P}\delta_2}(V^c))^c \subseteq ({U_4}^{\mathcal{P}\delta_1}(V^c))^c$ (according to (1)) $= {L_4}^{\mathcal{P}\delta_1}(V)$.
(3) Let $\nu \in {Bnd_4}^{\mathcal{P}\delta_1}(V)$. Then, $\nu \in {U_4}^{\mathcal{P}\delta_1}(V) - {L_4}^{\mathcal{P}\delta_1}(V)$. So, $\nu \in {U_4}^{\mathcal{P}\delta_1}(V)$ and $\nu \in ({L_4}^{\mathcal{P}\delta_1}(V))^c$. Thus, $\nu \in {U_4}^{\mathcal{P}\delta_2}(V)$ and $\nu \in ({L_4}^{\mathcal{P}\delta_2}(V))^c$ according to (1) and (2). Hence, $\nu \in {Bnd_4}^{\mathcal{P}\delta_2}(V)$. Therefore, ${Bnd_4}^{\mathcal{P}\delta_1}(V) \subseteq {Bnd_4}^{\mathcal{P}\delta_2}(V)$.
(4) ${Acc_4}^{\mathcal{P}\delta_2}(V) = \left|\frac{{L_4}^{\mathcal{P}\delta_2}(V)\cap V}{{U_4}^{\mathcal{P}\delta_2}(V)\cup V}\right| \leq \left|\frac{{L_4}^{\mathcal{P}\delta_1}(V)\cap V}{{U_4}^{\mathcal{P}\delta_1}(V)\cup V}\right| = {Acc_4}^{\mathcal{P}\delta_1}(V)$.
(5) Straightforward by (4).

Remark 57: According to Example 24, the inclusion and less than relation in Theorem 56 is proper. To clarify that, take
(i) $V = \{a, d\}$; then,
(1) ${U_4}^{\mathcal{P}\delta_1}(V) = \{a, b\} \neq \{a, b, c\} = {U_4}^{\mathcal{P}\delta_2}(V)$.
(2) ${Acc_4}^{\mathcal{P}\delta_1}(V) = 1 \neq \frac{2}{3} = {Acc_4}^{\mathcal{P}\delta_2}(V)$.
(3) ${Rough_4}^{\mathcal{P}\delta_1}(V) = 0 \neq \frac{1}{3} = {Rough_4}^{\mathcal{P}\delta_2}(V)$.
(ii) $V = \{b, c\}$; then, ${L_4}^{\mathcal{P}\delta_1}(V) = \{c, d\} \neq \{d\} = {L_4}^{\mathcal{P}\delta_2}(V)$.

IV. COMPARISON THE PROPOSED METHODS AND THEIR ADVANTAGES COMPARED TO THE PREVIOUS ONES

A. COMPARISON THE PROPOSED METHODS IN TERMS APPROXIMATIONS AND ACCURACY MEASURES OF SUBSETS

Theorem 58: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that $V \subseteq U$ Then,

(1) ${U_1}^{\mathcal{P}\delta}(V) \subseteq {U_2}^{\mathcal{P}\delta}(V)$.
(2) ${L_2}^{\mathcal{P}\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V)$.
(3) ${Bnd_1}^{\mathcal{P}\delta}(V) \subseteq {Bnd_2}^{\mathcal{P}\delta}(V)$.
(4) ${Acc_2}^{\mathcal{P}\delta}(V) = {Acc_1}^{\mathcal{P}\delta}(V)$.
(5) ${Rough_1}^{\mathcal{P}\delta}(V) = {Rough_2}^{\mathcal{P}\delta}(V)$.

Proof: directly follows from Definitions 14 and 25.

Remark 59: The inclusion and less than relations in the above theorem are proper. To elaborates that consider Example 24 and let $V = \{a, c, d\}$. Then,
(1) ${U_1}^{\mathcal{P}\delta_1}(V) = \{a, b\} \neq U = {U_2}^{\mathcal{P}\delta_1}(V)$.
(2) ${L_2}^{\mathcal{P}\delta_1}(V) = \{a, c, d\} \neq U = {L_1}^{\mathcal{P}\delta_1}(V)$.
(3) ${Bnd_1}^{\mathcal{P}\delta_1}(V) = \phi \neq \{b\} = {Bnd_2}^{\mathcal{P}\delta_1}(V)$.

Theorem 60: Let $\mathcal{P}$ be an ideal and δ be a reflexive relation on U such that $V \subseteq U$ Then,
1) ${L_2}^{\mathcal{P}\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V) \subseteq {L_3}^{\mathcal{P}\delta}(V)$.
2) ${U_3}^{\mathcal{P}\delta}(V) \subseteq {U_1}^{\mathcal{P}\delta}(V) \subseteq {U_2}^{\mathcal{P}\delta}(V)$.
3) ${Bnd^3}^{\mathcal{P}\delta}(V) \subseteq {Bnd^1}^{\mathcal{P}\delta}(V) \subseteq {Bnd^2}^{\mathcal{P}\delta}(V)$.
4) ${Acc^2}^{\mathcal{P}\delta}(V) \leq {Acc^1}^{\mathcal{P}\delta}(V) \leq {Acc^3}^{\mathcal{P}\delta}(V)$.
5) ${Rough^3}^{\mathcal{P}\delta}(V) \leq {Rough^1}^{\mathcal{P}\delta}(V) \leq {Rough^2}^{\mathcal{P}\delta}(V)$.

Proof:
(1) By Theorem 58, we have ${L_2}^{\mathcal{P}\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V)$. To prove, ${L_1}^{\mathcal{P}\delta}(V) \subseteq {L_3}^{\mathcal{P}\delta}(V)$. Let $\nu \in {L_1}^{\mathcal{P}\delta}(V)$, then $\theta_l(\nu) \cap V^c \in \mathcal{P}$. Hence, $\theta_l(\nu) \subseteq {L_3}^{\mathcal{P}\delta}(V)$. Since, δ is a reflexive relation, thus $\nu \in \theta_l(\nu) \subseteq {L_3}^{\mathcal{P}\delta}(V)$. Therefore, $\nu \in {L_3}^{\mathcal{P}\delta}(V)$.
(2) To prove, ${U_3}^{\mathcal{P}\delta}(V) \subseteq {U_1}^{\mathcal{P}\delta}(V)$. Let $\nu \in {U_3}^{\mathcal{P}\delta}(V) = ({L_3}^{\mathcal{P}\delta}(V^c))^c$, then $\nu \notin {L_3}^{\mathcal{P}\delta}(V^c)$. Hence, by Definition 36, we get $\theta_l(\nu) \cap V \notin \mathcal{P}$. It follows that $\nu \in {U_1}^{\mathcal{P}\delta}(V)$. By Theorem 58, we have ${U_1}^{\mathcal{P}\delta}(V) \subseteq {U_2}^{\mathcal{P}\delta}(V)$.
(3)-(5) Straightforward from (1) and (2).

Remark 61: In Theorem 60 the inclusion and less than relations are proper. To demonstrate that consider (iii) of Example 17 and let $V = \{b, c\}$. Then, ${U_3}^{\mathcal{P}\delta}(V) = \{b, c\} \subsetneq \{a, b, c\} = {U_1}^{\mathcal{P}\delta}(V)$. Moreover, take $V = \{a, d\}$, then
(1) ${L_1}^{\mathcal{P}\delta}(V) = \{d\} \subsetneq \{a, d\} = {L_3}^{\mathcal{P}\delta}(V)$.
(2) ${Bnd_3}^{\mathcal{P}\delta}(V) = \{b, c\} \subsetneq \{a, b, c\} = {Bnd_1}^{\mathcal{P}\delta}(V)$.
(3) ${Acc_1}^{\mathcal{P}\delta}(V) = \frac{1}{4} \lneq \frac{1}{2} = {Acc_3}^{\mathcal{P}\delta}(V)$.
(4) ${Rough^3}^{\mathcal{P}\delta}(V) = \frac{1}{2} \lneq \frac{3}{4} = {Rough^1}^{\mathcal{P}\delta}(V)$.

Theorem 62: Let $\mathcal{P}$ be an ideal and δ be a reflexive relation on U such that $V \subseteq U$ Then,
1) ${L_4}^{\mathcal{P}\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V) \subseteq {L_3}^{\mathcal{P}\delta}(V)$.
2) ${U_3}^{\mathcal{P}\delta}(V) \subseteq {U_1}^{\mathcal{P}\delta}(V) \subseteq {U_4}^{\mathcal{P}\delta}(V)$.
3) ${Bnd^3}^{\mathcal{P}\delta}(V) \subseteq {Bnd^1}^{\mathcal{P}\delta}(V) \subseteq {Bnd^4}^{\mathcal{P}\delta}(V)$.
4) ${Acc^4}^{\mathcal{P}\delta}(V) \leq {Acc^1}^{\mathcal{P}\delta}(V) \leq {Acc^3}^{\mathcal{P}\delta}(V)$.
5) ${Rough^3}^{\mathcal{P}\delta}(V) \leq {Rough^1}^{\mathcal{P}\delta}(V) \leq {Rough^4}^{\mathcal{P}\delta}(V)$.

Proof:
(1) By Theorem 60, we have ${L_1}^{\mathcal{P}\delta}(V) \subseteq {L_3}^{\mathcal{P}\delta}(V)$. To prove, ${L_4}^{\mathcal{P}\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V)$, let $\nu \in {L_4}^{\mathcal{P}\delta}(V) = {U_4}^{\mathcal{P}\delta}(V^c)^c$. Then, $\nu \notin {U_4}^{\mathcal{P}\delta}(V^c)$. Thus, by Definition 48, $\theta_l(\nu) \cap V^c \in \mathcal{P}$. It follows that $\theta_l(\nu) \subseteq {L_1}^{\mathcal{P}\delta}(V)$. Since, δ is a reflexive relation, then $\nu \in \theta_u(\nu) \subseteq {L_1}^{\mathcal{P}\delta}(V)$. Therefore, $\nu \in {L_1}^{\mathcal{P}\delta}(V)$.
(2) By Theorem 60, we have ${U_3}^{\mathcal{P}\delta}(V) \subseteq {U_1}^{\mathcal{P}\delta}(V)$. To prove ${U_1}^{\mathcal{P}\delta}(V) \subseteq {U_4}^{\mathcal{P}\delta}(V)$, let $\nu \in {U_1}^{\mathcal{P}\delta}(V)$. Then $\theta_l(\nu) \cap V \notin \mathcal{P}$. It follows that $\theta_u(\nu) \subseteq {U_4}^{\mathcal{P}\delta}(V)$. Since, δ is a reflexive relation, then $\nu \in \theta_l(\nu) \subseteq {U_4}^{\mathcal{P}\delta}(V)$. Therefore, $\nu \in {U_4}^{\mathcal{P}\delta}(V)$.
(3)-(5) Straightforward from (1) and (2).

TABLE 1. Comparison between the first and second methods in terms of the properties given in Definition 4.

	The first approach	The second approach
$\mathcal{L}_1$	•	•
$\mathcal{L}_2$	•	•
$\mathcal{L}_3$	∘	•
$\mathcal{L}_4$	∘	•
$\mathcal{L}_5$	•	•
$\mathcal{L}_6$	•	•
$\mathcal{L}_7$	•	•
$\mathcal{L}_8$	∘	∘
$\mathcal{L}_9$	∘	∘
$\mathcal{U}_1$	•	•
$\mathcal{U}_2$	∘	•
$\mathcal{U}_3$	•	•
$\mathcal{U}_4$	∘	•
$\mathcal{U}_5$	•	•
$\mathcal{U}_6$	•	•
$\mathcal{U}_7$	•	•
$\mathcal{U}_8$	∘	∘
$\mathcal{U}_9$	∘	∘

Remark 63: To clarify that the inclusion and less than relations in Theorem 62 are proper, consider (iii) of Example 17 and take $V = \{b, c\}$. Then, ${U_1}^{\mathcal{P}\delta}(V) = \{a, b, c\} \subsetneq U = {U_4}^{\mathcal{P}\delta}(V)$. Moreover, take $V = \{a, d\}$; then,
(1) ${L_4}^{\mathcal{P}\delta}(V) = \phi \subsetneq \{d\} = {L_1}^{\mathcal{P}\delta}(V)$.
(2) ${Bnd_1}^{\mathcal{P}\delta}(V) = \{a, b, c\} \subsetneq U = {Bnd_4}^{\mathcal{P}\delta}(V)$.
(3) ${Acc_4}^{\mathcal{P}\delta}(V) = 0 \lneq \frac{1}{4} = {Acc_1}^{\mathcal{P}\delta}(V)$.
(4) ${Rough_1}^{\mathcal{P}\delta}(V) = \frac{3}{4} \lneq 1 = {Rough_4}^{\mathcal{P}\delta}(V)$.

Remark 64: It can be seen from the above findings that there are several techniques to study approximation operators and measures of the subsets. The third technique, displayed in Section III-C, is the best one to approximate the subsets because it reduces (or cancels) the boundary regions, and increases the measures of accuracy more than the other types displayed in the other sections.

In Tables 1 and 2, we compare the proposed four approaches in terms of Pawlak properties. In these tables • means that the property holds and ∘ means that the property does not hold.

B. COMPARISON THE PROPOSED METHODS WITH THE PREVIOUS ONES

Theorem 65: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that $V \subseteq U$ Then,
(1) ${U_1}^{\mathcal{P}\delta}(V) \subseteq U^{\delta}(V)$.
(2) $L^{\delta}(V) \subseteq {L_1}^{\mathcal{P}\delta}(V)$.
(3) ${Bnd_1}^{\mathcal{P}\delta}(V) \subseteq Bnd^{\delta}(V)$.
(4) $Acc^{\delta}(V) \leq {Acc_1}^{\mathcal{P}\delta}(V)$.
(5) ${Rough_1}^{\mathcal{P}\delta}(V) \leq Rough^{\delta}(V)$.

Proof:
(1) Let $\nu \in {U_1}^{\mathcal{P}\delta}(V)$. Then, $\theta_l(\nu) \cap V \notin \mathcal{P}$. Hence, $\theta_l(\nu) \cap V \neq \phi$. Therefore, $\nu \in U^{\delta}(V)$. So, ${U_1}^{\mathcal{P}\delta}(V) \subseteq U^{\delta}(V)$.

TABLE 2. Comparison between the third and fourth approaches in terms of the properties in Definition 4.

	The third approach	The fourth approach
$\mathcal{L}_1$	$\bullet$	$\bullet$
$\mathcal{L}_2$	$\circ$	$\bullet$
$\mathcal{L}_3$	$\circ$	$\circ$
$\mathcal{L}_4$	$\circ$	$\circ$
$\mathcal{L}_5$	$\circ$	$\bullet$
$\mathcal{L}_6$	$\bullet$	$\bullet$
$\mathcal{L}_1$	$\bullet$	$\bullet$
$\mathcal{U}_1$	$\bullet$	$\bullet$
$\mathcal{U}_2$	$\circ$	$\circ$
$\mathcal{U}_3$	$\circ$	$\bullet$
$\mathcal{U}_4$	$\circ$	$\circ$
$\mathcal{U}_5$	$\circ$	$\bullet$
$\mathcal{U}_6$	$\bullet$	$\bullet$
$\mathcal{U}_7$	$\bullet$	$\bullet$

(2) Let $\nu \in L^{\delta}(V)$. Then, $\theta_l(\nu) \subseteq V$. Hence, $\theta_l(\nu) \cap V^c \in \mathcal{P}$. Therefore, $\nu \in L_1{}^{\mathcal{P}\delta}(V)$. So,$L^{\delta}(V) \subseteq L_1{}^{\mathcal{P}\delta}(V)$.

(3)-(5) It is immediately obtained by (1) and (2).

Remark 66: In Theorem 65 the inclusion and less than relations are proper as illustrated by Example 24. To this end, Take $V = \{a, d\}$. Then,

(1) $U_1{}^{\mathcal{P}\delta_1}(V) = \{a, b\} \neq \{a, b, d\} = U_{\delta_1}(V)$.
(2) $L_1{}^{\mathcal{P}\delta_1}(V) = U \neq \{d\} = L_{\delta_1}(V)$.
(3) $Bnd_1{}^{\mathcal{P}\delta_1}(V) = \phi \neq \{a, b\} = Bnd_{\delta_1}(V)$.
(4) $Acc_1{}^{\mathcal{P}\delta_1}(V) = \frac{2}{3} \neq \frac{1}{3} = Acc_{\delta_1}(V)$.
(5) $Rough_1{}^{\mathcal{P}\delta_1}(V) = \frac{1}{3} \neq \frac{2}{3} = Rough_{\delta_1}(V)$.

Theorem 67: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that $V \subseteq U$ Then,

(1) $U_1{}^{\mathcal{P}\delta}(V) \subseteq Upp^{\delta}(V)$.
(2) $Low^{\delta}(V) \subseteq L_1{}^{\mathcal{P}\delta}(V)$.
(3) $Bnd_1{}^{\mathcal{P}\delta}(V) \subseteq Boundary^{\delta}(V)$.
(4) $Accuracy^{\delta}(V) \leq Acc_1{}^{\mathcal{P}\delta}(V)$.
(5) $Rough_1{}^{\mathcal{P}\delta}(V) \leq Roughness^{\delta}(V)$.

Proof:

(1) Let $\nu \in U_1{}^{\mathcal{P}\delta}(V)$. Then, $\theta_l(\nu) \cap V \notin \mathcal{P}$. Hence, $\theta_u(\nu) \cap V \notin \mathcal{P}$. So, $\theta_u(\nu) \cap V \neq \phi$. Therefore, $\nu \in Upp^{\delta}(V)$. So, $U_1{}^{\mathcal{P}\delta}(V) \subseteq Upp^{\delta}(V)$.

(2) Let $\nu \in Low^{\delta}(V)$. Then, $\theta_u(\nu) \subseteq V$. Hence, $\theta_u(\nu) \cap V^c \in \mathcal{P}$. Since, $\theta_l(\nu) \subseteq \theta_u(\nu)$. So, $\theta_l(\nu) \cap V^c \in \mathcal{P}$. Therefore, $\nu \in L_1{}^{\mathcal{P}\delta}(V)$. So,$Low^{\delta}(V) \subseteq L_1{}^{\mathcal{P}\delta}(V)$.

(3)-(5) The proof is immediately by (1) and (2).

Remark 68: In Theorem 67 the inclusion and less than relations are proper. To clarify that consider Example 24 and let $V = \{a, d\}$. Then,

(1) $U_1{}^{\mathcal{P}\delta_1}(V) = \{a, b\} \neq \{a, b, d\} = Upp_{\delta_1}(V)$.
(2) $L_1{}^{\mathcal{P}\delta_1}(V) = U \neq \{d\} = Low_{\delta_1}(V)$.
(3) $Bnd_1{}^{\mathcal{P}\delta_1}(V) = \phi \neq \{a, b\} = Boundary_{\delta_1}(V)$.
(4) $Accu_1{}^{\mathcal{P}\delta_1}(V) = \frac{2}{3} \neq \frac{1}{3} = Accuracy_{\delta_1}(V)$.
(5) $Rough_1{}^{\mathcal{P}\delta_1}(V) = \frac{1}{3} \neq \frac{2}{3} = Roughness_{\delta_1}(V)$.

According to Theorems 65 and 67 it can be seen that the present methods reduce the boundary region with the comparison of Al-shami's methods [4]. This means that the current approximation spaces are proper generalizations of Al-shami's approximations [4].

One can easily prove the next result which shows that Al-shami's approximations [4] are special cases of the current approximations.

Proposition 69:

(1) If the ideal $\mathcal{P}$ is the empty set, then the approximation spaces given herein and the approximation spaces given in Definition 8 [4] are identical.
(2) If the ideal $\mathcal{P}$ is the empty set and binary relation is a similarity relation, then the approximation spaces given herein and the approximation spaces given in Definition 9 [4] are identical.

Theorem 70: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that $V \subseteq U$ Then,

(1) $U_1{}^{\mathcal{P}\delta}(V) \subseteq Upp_1{}^{\mathcal{P}\delta}(V)$.
(2) $Low_1{}^{\mathcal{P}\delta}(V) \subseteq L_1{}^{\mathcal{P}\delta}(V)$.
(3) $Bnd_1{}^{\mathcal{P}\delta}(V) \subseteq Boundary_1{}^{\mathcal{P}\delta}(V)$.
(4) $Accuracy_1{}^{\mathcal{P}\delta}(V) \leq Acc_1{}^{\mathcal{P}\delta}(V)$.
(5) $Rough_1{}^{\mathcal{P}\delta}(V) \leq Roughness_1{}^{\mathcal{P}\delta}(V)$.

Proof:

(1) Let $\nu \in U_1{}^{\mathcal{P}\delta}(V)$. Then, $\theta_l(\nu) \cap V \notin \mathcal{P}$. Since, $\theta_l(\nu) \subseteq \theta_u(\nu)$. Hence, $\theta_u(\nu) \cap V \notin \mathcal{P}$. Therefore, $\nu \in Upp_1{}^{\mathcal{P}\delta}(V)$. So, $U_1{}^{\mathcal{P}\delta}(V) \subseteq U^{\delta}(V)$.

(2) Let $\nu \in Low_1{}^{\mathcal{P}\delta}(V)$. Then, $\theta_u(\nu) \cap V^c \in \mathcal{P}$. Hence, $\theta_l(\nu) \cap V^c \in \mathcal{P}$. Therefore, $\nu \in L_1{}^{\mathcal{P}\delta}(V)$. So, $Low_1{}^{\mathcal{P}\delta}(V) \subseteq L_1{}^{\mathcal{P}\delta}(V)$.

(3)-(5) It is immediately obtained by (1) and (2).

Remark 71: The inclusion and the less than in Theorem 70 can not be replaced by equality relation in general. In Example 17 (i), take $\mathcal{P} = \{\phi, \{d\}\}$ and $V = \{b\}$, then $U_1{}^{\mathcal{P}\delta}(V) = \{b, c\} \neq \{a, b, c\} = Upp_1{}^{\mathcal{P}\delta}(V)$. Additionally, in Example 17 (i), take $\mathcal{P} = \{\phi, \{a\}\}$ and $V = \{a, c, d\}$, then

(1) $Low_1{}^{\mathcal{P}\delta}(V) = \{d\} \neq \{a, d\} = L_1{}^{\mathcal{P}\delta}(V)$.
(2) $Bnd_1{}^{\mathcal{P}\delta}(V) = \{b, c\} \neq \{a, b, c\} = Boundary_1{}^{\mathcal{P}\delta}(V)$.
(3) $Accuracy_1{}^{\mathcal{P}\delta}(V) = \frac{1}{4} \neq \frac{1}{2} = Acc_1{}^{\mathcal{P}\delta}(V)$.
(4) $Rough_1{}^{\mathcal{P}\delta}(V) = \frac{1}{2} \neq \frac{3}{4} = Roughness_1{}^{\mathcal{P}\delta}(V)$.

Theorem 72: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that $V \subseteq U$ Then,

(1) $U_2{}^{\mathcal{P}\delta}(V) \subseteq Upp_2{}^{\mathcal{P}\delta}(V)$.
(2) $Low_2{}^{\mathcal{P}\delta}(V) \subseteq L_2{}^{\mathcal{P}\delta}(V)$.
(3) $Bnd_2{}^{\mathcal{P}\delta}(V) \subseteq Boundary_2{}^{\mathcal{P}\delta}(V)$.
(4) $Accuracy_2{}^{\mathcal{P}\delta}(V) \leq Acc_2{}^{\mathcal{P}\delta}(V)$.
(5) $Rough_2{}^{\mathcal{P}\delta}(V) \leq Roughness_2{}^{\mathcal{P}\delta}(V)$.

Proof: Similar to the proof of Theorem 70.

Remark 73: In Theorem 72 the inclusion and the less than relations are proper as (i) of Example 17 shows. To this end, let $\mathcal{P} = \{\phi, \{d\}\}$ and $V = \{b\}$. Then $U_2{}^{\mathcal{P}\delta}(V) = \{b, c\} \neq \{a, b, c\} = Upp_2{}^{\mathcal{P}\delta}(V)$. Additionally, in Example 17 (i), take $\mathcal{P} = \{\phi, \{a\}\}$ and $V = \{a, c, d\}$, then

(1) $Low_2{}^{\mathcal{P}\delta}(V) = \{d\} \neq \{a, d\} = L_2{}^{\mathcal{P}\delta}(V)$.
(2) $Bnd_2{}^{\mathcal{P}\delta}(V) = \{b, c\} \neq \{a, b, c\} = Boundary_2{}^{\mathcal{P}\delta}(V)$.
(3) $Accuracy_2{}^{\mathcal{P}\delta}(V) = \frac{1}{4} \neq \frac{1}{2} = Acc_2{}^{\mathcal{P}\delta}(V)$.
(4) $Rough_2{}^{\mathcal{P}\delta}(V) = \frac{1}{2} \neq \frac{3}{4} = Roughness_2{}^{\mathcal{P}\delta}(V)$.

Theorem 74: Let $\mathcal{P}$ be an ideal and δ be a binary relation on U such that $V \subseteq U$ Then,
(1) $U_3{}^{\mathcal{P}\delta}(V) \subseteq Upp_3{}^{\mathcal{P}\delta}(V)$.
(2) $Low_3{}^{\mathcal{P}\delta}(V) \subseteq L_3{}^{\mathcal{P}\delta}(V)$.
(3) $Bnd_3{}^{\mathcal{P}\delta}(V) \subseteq Boundary_3{}^{\mathcal{P}\delta}(V)$.
(4) $Accuracy_3{}^{\mathcal{P}\delta}(V) \leq Acc_3{}^{\mathcal{P}\delta}(V)$.
(5) $Rough_3{}^{\mathcal{P}\delta}(V) \leq Roughness_3{}^{\mathcal{P}\delta}(V)$.

Proof: Similar to the proof of Theorem 70.

Remark 75: In Theorem 74 the inclusion and the less than relations are proper as illustrated by (iii) of Example 17. To this end, let $V = \{a, d\}$. Then,
(1) $U_3{}^{\mathcal{P}\delta}(V) = \{a, b, c\} \neq U = Upp_3{}^{\mathcal{P}\delta}(V)$.
(2) $Low_3{}^{\mathcal{P}\delta}(V) = \phi \neq \{d\} = L_3{}^{\mathcal{P}\delta}(V)$.
(3) $Bnd_3{}^{\mathcal{P}\delta}(V) = \{a, b, c\} \neq U = Boundary_3{}^{\mathcal{P}\delta}(V)$.
(4) $Accuracy_3{}^{\mathcal{P}\delta}(V) = 0 \neq \frac{1}{4} = Acc_3{}^{\mathcal{P}\delta}(V)$.
(5) $Rough_3{}^{\mathcal{P}\delta}(V) = \frac{3}{4} \neq 1 = Roughness_3{}^{\mathcal{P}\delta}(V)$.

Theorem 76: Let $\phi \neq V \subseteq U$, $\mathcal{P}$ be an ideal on U and δ be a binary relation on a non-empty set U. Then,
(1) $U_4{}^{\mathcal{P}\delta}(V) \subseteq Upp_4{}^{\mathcal{P}\delta}(V)$.
(2) $Low_4{}^{\mathcal{P}\delta}(V) \subseteq L_4{}^{\mathcal{P}\delta}(V)$.
(3) $Bnd_4{}^{\mathcal{P}\delta}(V) \subseteq Boundary_4{}^{\mathcal{P}\delta}(V)$.
(4) $Accuracy_4{}^{\mathcal{P}\delta}(V) \leq Acc_4{}^{\mathcal{P}\delta}(V)$.
(5) $Rough_4{}^{\mathcal{P}\delta}(V) \leq Roughness_4{}^{\mathcal{P}\delta}(V)$.

Proof: The proof is similar to that of Theorem 70.

Remark 77: In Theorem 76 the inclusion and the less than relations are proper. To illustrate that consider (i) of Example 17. Then,
(i) Take $\mathcal{P} = \{\phi, \{a\}\}$ and $V = \{b\}$, then
   (1) $U_4{}^{\mathcal{P}\delta}(V) = \{b, c, d\} \neq U = Upp_4{}^{\mathcal{P}\delta}(V)$.
   (2) $Bnd_4{}^{\mathcal{P}\delta}(V) = \{b, c, d\} \neq U = Boundary_4{}^{\mathcal{P}\delta}(V)$.
(ii) Take $\mathcal{P} = \{\phi, \{a\}\}$ and $V = \{a, c, d\}$, then
   (1) $Low_4{}^{\mathcal{P}\delta}(V) = \phi \neq \{a\} = L_4{}^{\mathcal{P}\delta}(V)$.
   (2) $Accuracy_4{}^{\mathcal{P}\delta}(V) = 0 \neq \frac{1}{4} = Acc_4{}^{\mathcal{P}\delta}(V)$.
   (3) $Rough_4{}^{\mathcal{P}\delta}(V) = \frac{3}{4} \neq 1 = Roughness_4{}^{\mathcal{P}\delta}(V)$.

One can easily prove the next result which shows that Hosny's and Al-shami's approximations [19] are special cases of the current approximations.

Proposition 78: If the binary relation is a similarity relation, then the approximation spaces given herein and the approximation spaces given in [19] are identical.

Finally, we draw attention to that the maximal left neighborhoods and maximal right neighborhoods are independent of each other. In Example 40, the maximal left neighborhood of b is $\theta_l(b) = \{a, b, c\} \neq \{a, b, d\} = \theta_r(b)$ the maximal right neighborhood of b. Consequently, the approximation spaces generated by the maximal left neighborhoods and maximal right neighborhoods via ideals are independent of each other. To validate that, in Example 40 take $V = \{b\}$, then the lower, upper approximations and boundary by the previous third method in [17] are ϕ, ϕ and ϕ, respectively. Meantime, the lower, upper approximations and boundary by the present third method are ϕ, $\{c\}$ and $\{c\}$, respectively. Therefore, the previous boundary $= \phi \nsupseteq \{c\} =$ the present boundary. If we take another set say $W = \{a, c, d\}$, then the lower, upper approximations and boundary by the previous third method in [17] are ϕ, $\{c\}$ and $\{c\}$, respectively. While, the lower, upper approximations and boundary by the present third method are $\{a, c, d\}$, ϕ and ϕ, respectively. So, the present boundary $= \phi \nsupseteq \{c\} =$ the previous boundary. That is, the proposed approaches and their counterparts introduced in [17] are different in general.

TABLE 3. Information system of students' rank for each subject.

U	biology	chemistry	mathematics	physics
s_1	good	fair	excellent	excellent
s_2	very good	good	excellent	fair
s_3	very good	good	failed	good
s_4	excellent	fair	very good	excellent
s_5	very good	fair	very good	excellent
s_6	good	good	excellent	fair

V. NUMERICAL EXAMPLE

In this section, we analysis the obtained computations from six students exams in four subjects in terms of the current first approach and those were introduced in two recent manuscripts [4], [19]. We illustrate this matter by considering six students $S = \{s_i : i = 1, 2, \ldots, 6\}$ have been examined in four subjects; say, biology, chemistry, mathematics, and physics. The evaluation of students' performance is given by five ranks or levels as follows.

$$\begin{aligned} &Rank1: \ excellent \\ &Rank2: \ very\ good \\ &Rank3: \ good \\ &Rank4: \ fair \\ &Rank5: \ failed \end{aligned}$$

These ranks are ordered as follows: excellent $\succ$ very good $\succ$ good $\succ$ fair $\succ$ failed, where $\succ$ means "greater than."

The relation that associates between students is defined by: xRy iff student x has at least two subjects with a rank greater than the rank of the corresponding subjects of student y. For instance, $s_4\delta s_3$ because the student's ranks s_4 in biology, mathematics, and physics are greater than the student's ranks s_3 in these subjects. But, $(s_3, s_4) \notin \delta$ because the student s_3 has only one subject's rank greater than student s_4.

The approximation space of the students' information system is constructed firstly by converting Table 3 to the following binary relation: $\delta = \{(s_1, s_3), (s_2, s_1), (s_2, s_4), (s_2, s_5), (s_4, s_2), (s_4, s_3), (s_4, s_6), (s_5, s_3), (s_5, s_6), (s_6, s_4)\}$. It should be noted that this relation is irreflexive because $(s, s) \notin \delta$ for each $s \in S$, not symmetry because $(s_3, s_1) \notin \delta$ in spite of $(s_1, s_3) \in \delta$, also, it is not transitive because $(s_5, s_4) \notin \delta$ in spite of $(s_5, s_6) \in \delta$ and $(s_6, s_4) \in \delta$.

Secondly, the left neighborhood N_l and maximal left neighborhood M_l are calculated for each $s \in S$

$$\begin{aligned} &N_l(s_1) = N_l(s_5) = \{s_2\} \quad N_l(s_4) = \{s_2, s_6\} \\ &N_l(s_2) = \{s_4\} \quad N_l(s_6) = \{s_4, s_5\} \\ &N_l(s_3) = \{s_1, s_4, s_5\} \end{aligned}$$

$$M_l(s_1) = M_l(s_4) = M_l(s_5) = \{s_1, s_4, s_5\}$$
$$M_l(s_2) = M_l(s_6) = \{s_2, s_6\}$$
$$M_l(s_3) = \phi$$

Let $\mathcal{T} = \{\phi, \{s_4\}\}$ be an ideal on S. Then, we examine the performance of the present approach in Definition 14 and their counterparts approaches studied in [4], [19]. To this end, let $F = \{s_3, s_4, s_6\}$. In what follows, we calculate its lower and upper approximations, boundary regions and accuracy values utilizing a method of maximal left neighborhoods given in [4] and the first method given herein.

- It follows from Al-shami's approach [4](see, Definition 8) that:

$$\begin{cases} L^{\delta}(F) = & \{s_3\}; \\ U^{\delta}(F) = & \{s_1, s_2, s_4, s_5, s_6\}; \\ Bnd^{\delta}(F) = & \{s_1, s_2, s_4, s_5, s_6\}; \\ Acc^{\delta}(F) = & \frac{1}{6}; \\ Rough^{\delta}(F) = & \frac{5}{6}. \end{cases} \quad (54)$$

- It follows from our approach given in Definition 14 that

$$\begin{cases} L_1{}^{\mathcal{P}\delta}(F) = & \{s_3\}; \\ U_1{}^{\mathcal{P}\delta}(F) = & \{s_2, s_6\}; \\ Bnd_1{}^{\mathcal{P}\delta}(F) = & \{s_2, s_6\}; \\ Acc_1{}^{\mathcal{P}\delta}(F) = & \frac{1}{4}; \\ Rough_1{}^{\mathcal{P}\delta}(F) = & \frac{3}{4}. \end{cases} \quad (55)$$

The proposed approach is compared with its counterpart given in [19] by calculating the right neighborhood N_r, then the maximal right neighborhood M_r and finally the maximal union neighborhood M_u for each $s \in S$.

$$N_r(s_1) = \{s_3\} \quad N_r(s_4) = \{s_2, s_3, s_6\}$$
$$N_r(s_2) = \{s_1, s_4, s_5\} \quad N_r(s_5) = \{s_3, s_6\}$$
$$N_r(s_3) = \phi \quad N_r(s_6) = \{s_4\}$$
$$M_r(s_1) = M_r(s_4) = M_r(s_5) = \{s_1, s_4, s_5\}$$
$$M_r(s_2) = M_r(s_3) = M_r(s_6) = \{s_2, s_3, s_6\}$$
$$M_u(s_1) = M_u(s_4) = M_u(s_5) = \{s_1, s_4, s_5\}$$
$$M_u(s_2) = M_u(s_3) = M_u(s_6) = \{s_2, s_3, s_6\}$$

According to the approach given in [19] (see, Definition 10) we find that

$$\begin{cases} Low_1{}^{\mathcal{P}\delta}(F) = & \phi; \\ Upp_1{}^{\mathcal{P}\delta}(F) = & \{s_2, s_3, s_6\}; \\ Boundary_1{}^{\mathcal{P}\delta}(F) = & \{s_2, s_3, s_6\}; \\ Accuracy_1{}^{\mathcal{P}\delta}(F) = & 0; \\ Roughness_1{}^{\mathcal{P}\delta}(F) = & 1. \end{cases} \quad (56)$$

It follows from the above calculations that the boundary regions of the set F generated by approaches given in [4] and [19] are $\{s_1, s_2, s_4, s_5, s_6\}$ and $\{s_2, s_3, s_6\}$, respectively. Whereas, the boundary region of the set F generated by the suggested approach introduced in Definition 14 is $\{s_2, s_6\}$, which implies that the uncertainty/vagueness area is minimized by the proposed approach more than approaches displayed in [4], [19]. Hence, a decision made according to the calculations of the present approach is more accurate.

According to the above discussion, it can be seen that there are various methods or approaches used to approximate the subsets. The current technique " maximal left neighborhoods and ideals" is a vital tool to eliminate the ambiguity of the data in the real-life issues and produces more accurate decisions since it decreases the boundary region by enlarging the lower approximations and dwindling the upper approximations, and hence, increases the value of accuracy compared to the other types such those were discussed in [4], [19].

VI. CONCLUSION

One of the recent successfully tool to handel uncertainty problems is rough set. It was proposed with the goal of the induction of approximations of concepts; it offers mathematical tools to discover patterns hidden in data. This manuscript had been written to contribute to this field by introducing some novel kinds of approximation spaces generated by "maximal left neighborhoods and ideals" which generalize the old concepts and get preferable results by reducing the boundary regions.

First, we scrutinized their main properties and provided some illustrative counterexample to elucidate the obtained results. By the way, it was proved that the current approach preserved main characterizations of Pawlak's model and kept the property of monotonicity. Then, we compared between the proposed methods and discussed their advantages compared to the previous methods in terms of improvement the approximation operators and accuracy measures. Finally, a numerical example was given and demonstrated how the current methods expanded the knowledge obtained from the information systems.

As it is well-known that the interior and closure topological operators behave similarly to the lower and upper approximations; so, in forthcoming works, we plan to study the counterparts of these models via topological structures. In addition, we will benefit from the hybridization of rough set theory with some approaches such as soft sets and fuzzy sets [30], [34] to introduce these approximation spaces via these hybridized frames and show their role in efficiently dealing with uncertain knowledge.

CONFLICTS OF INTEREST

The authors declare that they have no competing interests.

REFERENCES

[1] H. M. Abu-Donia, "Comparison between different kinds of approximations by using a family of binary relations," *Knowl.-Based Syst.*, vol. 21, no. 8, pp. 911–919, Dec. 2008.
[2] A. A. Allam, M. Y. Bakeir, and E. A. Abo-Tabl, "New approach for basic rough set concepts," in *Proc. Int. Workshop Rough Sets, Fuzzy Sets, Data Mining, Granular-Soft Comput.* (Lecture Notes in Artificial Intelligence), vol. 3641. Regina, SK, Canada: Springer, 2005, pp. 64–73.

[3] T. M. Al-shami, ''An improvement of rough sets' accuracy measure using containment neighborhoods with a medical application,'' *Inf. Sci.*, vol. 569, pp. 110–124, Aug. 2021.

[4] T. M. Al-shami, ''Maximal rough neighborhoods with a medical application,'' *J. Ambient Intell. Humanized Comput.*, vol. 2022, pp. 1–12, May 2022, doi: 10.1007/S12652-022-03858-1.

[5] T. M. Al-shami, ''Topological approach to generate new rough set models,'' *Complex Intell. Syst.*, vol. 2022, pp. 1–13, Mar. 2022, doi: 10.1007/S40747-022-00704-X.

[6] T. M. Al-shami and D. Ciucci, ''Subset neighborhood rough sets,'' *Knowl.-Based Syst.*, vol. 237, Feb. 2022, Art. no. 107868, doi: 10.1016/j.knosys.2021.107868.

[7] T. M. Al-Shami, H. Işik, A. S. Nawar, and R. A. Hosny, ''Some topological approaches for generalized rough sets via ideals,'' *Math. Problems Eng.*, vol. 2021, pp. 1–11, Aug. 2021, doi: 10.1155/2021/5642982.

[8] M. Atef, A. M. Khalil, S. G. Li, A. Azzam, and A. E. Atik, ''Comparison of six types of rough approximations based on j-neighborhood space and j-adhesion neighborhood space,'' *J. Intell. Fuzzy Syst.*, vol. 39, no. 3, pp. 4515–4531, 2020.

[9] J. Dai, S. Gao, and G. Zheng, ''Generalized rough set models determined by multiple neighborhoods generated from a similarity relation,'' *Soft Comput.*, vol. 22, no. 7, pp. 2081–2094, Apr. 2018.

[10] M. K. El-Bably and T. M. Al-Shami, ''Different kinds of generalized rough sets based on neighborhoods with a medical application,'' *Int. J. Biomath.*, vol. 14, no. 8, Nov. 2021, Art. no. 2150086, doi: 10.1142/S1793524521500868.

[11] M. K. El-Bably, T. M. Al-shami, A. Nawar, and A. Mhemdi, ''Corrigendum to 'comparison of six types of rough approximations based on j-neighborhood space and j-adhesion neighborhood space,''' *J. Intell. Fuzzy Syst.*, vol. 41, no. 6, pp. 7353–7361, 2021.

[12] A. Ç. Güler, E. D. Yildirim, and O. Özbakir, ''Rough approximations based on different topologies via ideals,'' *Turkish J. Math.*, vol. 46, no. 4, pp. 1177–1192, Jan. 2022.

[13] M. Hosny, ''Idealization of j-approximation spaces,'' *Filomat*, vol. 34, no. 2, pp. 287–301, 2020.

[14] M. Hosny, ''Topological approach for rough sets by using J-nearly concepts via ideals,'' *Filomat*, vol. 34, no. 2, pp. 273–286, 2020.

[15] M. Hosny, ''Topologies generated by two ideals and the corresponding j-approximations spaces with applications,'' *J. Math.*, vol. 2021, pp. 1–13, Oct. 2021, doi: 10.1155/2021/6391266.

[16] M. Hosny, ''Rough sets theory via new topological notions based on ideals and applications,'' *AIMS Math.*, vol. 7, no. 1, pp. 869–902, 2021.

[17] M. Hosny, ''Generalization of rough sets using maximal right neighborhood systems and ideals with medical applications,'' *AIMS Math.*, vol. 7, no. 7, pp. 13104–13138, 2022.

[18] R. A. Hosny, T. M. Al-shami, A. A. Azzam, and A. S. Nawar, ''Knowledge based on rough approximations and ideals,'' *Math. Problems Eng.*, vol. 2022, pp. 1–12, May 2022.

[19] M. Hosny, T. M. Al-shami, and A. Mhemdi, ''Rough approximation spaces via maximal union neighborhoods and ideals with a medical application,'' 2022.

[20] R. A. Hosny, B. A. Asaad, A. A. Azzam, and T. M. Al-Shami, ''Various topologies generated from E_j-neighbourhoods via ideals,'' *Complexity*, vol. 2021, pp. 1–11, Jun. 2021.

[21] D. Jankovic and T. R. Hamlet, ''New topologies from old via ideals,'' *Amer. Math. Monthly*, vol. 97, pp. 295–310, Apr. 1990.

[22] A. Kandil, S. A. El-Sheikh, M. Hosny, and M. Raafat, ''Bi-ideal approximation spaces and their applications,'' *Soft Comput.*, vol. 24, no. 17, pp. 12989–13001, Sep. 2020.

[23] A. Kandil, M. M. Yakout, and A. Zakaria, ''Generalized rough sets via ideals,'' *Ann. Fuzzy Math. Inform.*, vol. 5, no. 3, pp. 525–532, 2013.

[24] M. Kondo, ''On the structure of generalized rough sets,'' *Inf. Sci.*, vol. 176, no. 5, pp. 589–600, Mar. 2006.

[25] A. M. Kozae, S. A. El-Sheikh, and M. Hosny, ''On generalized rough sets and closure spaces,'' *Int. J. Appl. Math.*, vol. 23, no. 6, pp. 997–1023, 2010.

[26] E. F. Lashin, A. M. Kozae, A. A. A. Khadra, and T. Medhat, ''Rough set theory for topological spaces,'' *Int. J. Approx. Reasoning*, vol. 40, pp. 35–43, Jul. 2005.

[27] R. Mareay, ''Generalized rough sets based on neighborhood systems and topological spaces,'' *J. Egyptian Math. Soc.*, vol. 24, no. 4, pp. 603–608, Oct. 2016.

[28] Z. Pawlak, ''Rough sets,'' *Int. J. Comput. Inf. Sci.*, vol. 11, no. 5, pp. 341–356, Oct. 1982.

[29] Z. Pawlak, ''Rough concept analysis,'' *Bull. Polish Acad. Sci. Math.*, vol. 33, pp. 9–10, Jan. 1985.

[30] J. Ping, M. Atef, A. M. Khalil, M. Riaz, and N. Hassan, ''Soft rough q-rung orthopair m-polar fuzzy sets and q-rung orthopair m-polar fuzzy soft rough sets and their applications,'' *IEEE Access*, vol. 9, pp. 139186–139200, 2021, doi: 10.1109/ACCESS.2021.3118055.

[31] K. Qin, J. Yang, and Z. Pei, ''Generalized rough sets based on reflexive and transitive relations,'' *Inf. Sci.*, vol. 178, no. 21, pp. 4138–4141, Nov. 2008.

[32] R. Slowinski and D. Vanderpooten, ''A generalized definition of rough approximations based on similarity,'' *IEEE Trans. Knowl. Data Eng.*, vol. 12, no. 2, pp. 331–336, Mar. 2000.

[33] O. A. E. Tantawy and H. I. Mustafa, ''On rough approximations via ideal,'' *Inf. Sci.*, vol. 251, pp. 114–125, Dec. 2013.

[34] B. Yang, ''Fuzzy covering-based rough set on two different universes and its application,'' *Artif. Intell. Rev.*, vol. 55, no. 6, pp. 4717–4753, Aug. 2022, doi: 10.1007/S10462-021-10115-Y.

[35] Y. Y. Yao, ''Two views of the theory of rough sets in finite universes,'' *Int. J. Approx. Reasoning*, vol. 15, no. 4, pp. 291–317, Nov. 1996.

[36] Y. Y. Yao, ''Relational interpretations of neighborhood operators and rough set approximation operators,'' *Inf. Sci.*, vol. 111, nos. 1–4, pp. 239–259, Nov. 1998.

[37] Z. Zhao, ''On some types of covering rough sets from topological points of view,'' *Int. J. Approx. Reson.*, vol. 68, pp. 1–14, Jan. 2016.

TAREQ M. AL-SHAMI received the M.S. and Ph.D. degrees in pure mathematics from the Department of Mathematics, Faculty of Science, Mansour University, Egypt.

He is currently an Assistant Professor with the Department of Mathematics, Sana'a University, Sana'a, Yemen. He has published more than 118 research articles in international peer-reviewed SCIE and ESCI journals, such as *Information Sciences*, *Knowledge-Based Systems*, *Journal of Ambient Intelligence and Humanized Computing*, *Applied and Computational Mathematics*, *Computational and Applied Mathematics*, and *Soft Computing* journal among others. His research interests include pure mathematics, topology and its extensions, ordered topology, soft set theory, rough set theory and fuzzy set theory with applications in decision-making, medical diagnosis, artificial intelligence, computational intelligence, information measures, and information aggregation.

Dr. Al-Shami received Obada-Prize for postgraduate students, in February 2019. Also, he obtained the first rank at Sana'a University according to Scopus data of the region of Yemen for the period, from 2018 to 2121.

MONA HOSNY received the M.S. and Ph.D. degrees in topology from Ain Shams University, Cairo, Egypt.

She is currently an Assistant Professor in pure mathematics with the Department of Mathematics, Faculty of Science for Girls, King Khalid University, Abha, Saudi Arabia. She is also an Assistant Professor in pure mathematics with the Department of Mathematic, Faculty of Education, Ain Shams University. She has published many research articles in reputed international journals. She was a supervisor of many Ph.D. and M.Sc. Thesis. Her research interests include general topology, rough set theory and their applications, multiset theory, hesitant fuzzy soft multisets and their applications, nano topology, and ordered topology.

Dr. Hosny awarded Dr. Bahaa Eldeen Helmy Esmaeel's Award for the Best Ph.D. thesis, in 2016.