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RESEARCH ARTICLE

Energy-to-Peak Reduced Order Filtering for Continuous-Time Markov Jump Linear Systems With Partial Information on the Jump Parameter

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ABSTRACT This paper deals with the design of an energy-to-peak reduced-order filter (also called $L_2 - L_\infty$ filtering) for continuous-time Markov jump linear systems, assuming that the filter has only access to an estimate of the Markov parameter, coming from the output of a detector device. To model this situation we consider that the joint process formed by the Markov parameter and the detector information follows an exponential hidden Markov model. The result is given in terms of Linear Matrix Inequalities (LMI) so that the available numerical package tools can be readily implemented to solve the problem. The paper is concluded with some numerical simulations.

INDEX TERMS Reduced-order filtering, energy-to-peak performance, switched systems, hidden Markov models, linear matrix inequalities.

I. INTRODUCTION

The research on systems subject to sudden changes has been the focus of a great deal of attention in the last decades. Of special interest is the class of hybrid systems subject to switching rules modelled by random processes, in which Markov Jump Linear Systems (MJLS) arguably have been the focal point of study, see, for instance, [1]–[3]. By now, the use of MJLS as a tool for modeling and design has found space in a wide range of applications, such as Networked Control Systems (NCS) in [4], Active-Fault Tolerant Control Systems (AFTCS) in [5], among others.

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The main goal of this paper is on the so-called energy-to-peak reduced-order filtering problem, or $L_2 - L_\infty$ reduced-order filtering problem. The aim is to design a filter such that the ratio between the peak value of the estimation error and the energy of an L_2 external disturbance is less than a fixed positive value γ. As pointed out in [6], this class of filters is less conservative than the H_∞ filtering in the sense that the bound restriction is imposed for all t, instead of the L_2 norm of the estimation error, and has been applied in several types of applications such as electrical circuits, navigation systems, communication systems, and estimation in civil structures, see [7]. As an example of application, we can mention the estimates of the velocity or the sideslip angle of vehicles in which, as pointed out in [8], what usually matters is that the maximum value of these variables are limited to ensure a safe drive (see also [9]). Another application can be found in [6], which proposes the design of a reduced-order $L_2 - L_\infty$ filter for discrete-time systems subject to network-induced delays. The design problem of energy-to-peak filtering can be dated back as far as [10] and since then it has been an intensive area of research. This problem has been analyzed by several authors under different approaches as, for instance, the seminal paper [11] which provides necessary and sufficient conditions for the solution of the filter problem by using LMI and coupling nonconvex matrix rank constraints. The problem of robust energy-to-peak filtering for uncertain systems was considered in [7], [12]–[14]. The energy-to-peak filtering problem of Markov jump systems is investigated in [15], [16], while singular semi-Markov jump systems with unideal measurements was studied in [17]. For sampled nonlinear systems, the paper [18] focus on the L_2-L_∞ filtering problem by using Takagi–Sugeno (T–S) fuzzy systems.

A key point related to MJLS is the (lack of) availability of the Markov variable, denoted here by $\theta(t)$ for the continuous-time case. The case in which it is not possible to perfectly measure $\theta(t)$ was treated, within the context of AFTCS, in [5], [19]–[21]. According to [21], AFTCS consists, essentially, in automatically detecting and identifying the faulty components and then reconfiguring the control law on-line in response to this decision. In this context, the fault is modeled here by the Markov chain $\theta(t)$ while the fault detector, denoted by $\hat{\theta}(t)$, plays the role of the signal coming from a failure detection and identification (FDI) device. One important example comes from networked control systems (NCSs), which are inherently subject to packet dropouts and imperfect characteristics of the communication channel.

More recently, in [22], [23], a more general exponential hidden Markov model for the joint process $\widetilde{\theta}(t) = (\theta(t), \hat{\theta}(t))$ was proposed, allowing the study of the H_2 state-feedback as well as the H_∞ static output feedback control problems for MJLS under the assumption that the controllers can only have access to the unreliable information coming from the detector $\hat{\theta}(t)$, and thus $\theta(t)$ cannot be perfectly measured. Other approaches which consider the case in which $\theta(t)$ is not available are the asynchronous case see, for instance, [8], [24], and the ϵ-dependent rate of jump case, which was considered in [25].

Usually it is assumed in filtering problems that the dimension of the filter state variable is the same as the one of the original system (known as full-order filters). However, when the original system has a high dimension as, for instance, in meteorology and oceanography applications, full-order filters can be difficult to implement in real time (see, for instance, [26]). Due to that the study of reduced-order filters (that is, when the filter's dimension is smaller than that of the original system) has been receiving a great deal of attention. Regarding MJLS, we can mention the papers [6], which studies the design of robust reduced-order $L_2 - L_\infty$ filters for a class of discrete-time systems subject to network-induced delays governed by a Markov chain; [27], which addresses the reduced-order H_∞ filtering problem for continuous-time MJLS, where the jump parameters are modelled by a discrete-time Markov process; and [28], which deals with the robust H_2 and H_∞ reduced-order mode-dependent, partially mode-dependent, or mode-independent filters for discrete-time MJLS under a parameter-dependent LMI approach.

In this paper we focus on the design of $L_2 - L_\infty$ reduced-order filtering problem for continuous-time MJLS within the challenging context of the partial observation of the Markov parameter. The main contribution is to provide a design procedure, based on an LMI formulation, to obtain a reduced-order filter which relies only on the detector $\hat{\theta}(t)$, so that the $L_2 - L_\infty$ gain of the estimation error is smaller than an upper bound γ. On the one hand this set up makes the problem more realistic and interesting from the practical point of view but, on the other hand, it imposes new technical challenges in the deduction of the LMIs, requiring some special structures for the solution, as can be seen in the proof of Theorem 1 below. We believe that this convex approach for the reduced-order filter represents a challenging and important open problem which, as far as the authors are aware of, had not been previously analyzed in the literature under this exponential hidden Markov formulation. Next, we highlight the main novelties with respect to existing results:

1) The papers [15], [17] tackle the $L_2 - L_\infty$ filtering problem for continuous-time jump systems for the mode-dependent case, that is, the jump parameter is accessible to the filter. Differently from that, in the present paper we consider the more realistic case in which only an estimate of this parameter is available for the filter design.
2) When compared to [8], [24] it is important to stress that our formulation for modeling the detector is different from the one adopted in these papers, which is based on a conditional probability restriction that must hold for all time t (Eq. (4) in [24] and Eq. (6) in [8]), and also that, to avoid infinite jumps of $\hat{\theta}(t)$ in a finite time interval, there exists a minimum time interval for the execution of the detector (see [8]). On the other hand, under our formulation, the signal $\widetilde{\theta}(t)$ is modelled as an exponential hidden Markov process, so that the time evolution of the extended process $\widetilde{\theta}(t)$ is well defined as a continuous-time Markov chain with no extra conditions (see Section IV).
3) By using the hidden Markov formulation adopted in this paper, it is not necessary to consider the rate of jumps of $\hat{\theta}(t)$ as required in [25], which studied the H_∞ filtering problem, considering that the Markov chain $\theta(t)$ and the detector $\hat{\theta}^\epsilon(t)$ jointly follow an exponential hidden Markov chain model, with the parameter ϵ determining how fast $\hat{\theta}^\epsilon(t)$ will switch.
4) The formulation in the present paper encompasses the detector model used in the context of AFTCS as well

as the mode-dependent, mode-independent, and cluster cases, which are different approaches regarding the availability of $\theta(t)$ (see Remark 4 for further details).

5) The papers [29], [30] tackle the control problem of continuous-time MJLS under the aforementioned exponential hidden Markov, being the H_2, H_∞ and mixed H_2/H_∞ filtering problems also studied in [30], but not the $L_2 - L_\infty$ filtering problem, treated for the first time in this paper.
6) The robust filtering problem, considering polytopic uncertainties on the dynamic matrices of the system as well as the detector rates, can also be handled under our approach.
7) The developed procedure in this paper is illustrated by means of a numerical example based on the stable longitudinal dynamics of an unmanned aircraft derived in [31] subject to faulty sensor readings and noise.

The paper is organized as follows. In Section II we present the notation used in the paper. Section III introduces some preliminary results that will be required along the paper. The problem formulation is presented in Section IV. The main result of the paper is in Section V, which provides design conditions for obtaining a $\hat{\theta}(t)$-dependent filter such that the $L_2 - L_\infty$ gain of the estimation error is smaller than an upper bound γ. The perfect information and robust filtering problems are discussed in Subsections V-B and V-C respectively. Section VI illustrates the obtained results through some numerical examples. The paper is concluded in Section VII with some final comments.

II. NOTATION

The real n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. The linear (norm bounded) space of all $m \times n$ real matrices is represented by $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, and for simplicity, we set $\mathbb{B}(\mathbb{R}^n) = \mathbb{B}(\mathbb{R}^n, \mathbb{R}^n)$. The space of positive semi-definite $n \times n$ matrices is represented by $\mathbb{B}(\mathbb{R}^{n+})$. The superscript $'$ is the transpose of a matrix. The identity matrix of size $n \times n$ is represented by I_n (or just I) and the null matrix of size $m \times n$, by $0_{m\times n}$ (or just by 0). For $G \in \mathbb{B}(\mathbb{R}^n)$ we set $Her(R) \triangleq R + R'$. For positive integers N and M, the sets $\mathbb{N}$ and $\mathbb{M}$ are given by $\mathbb{N} \triangleq \{1, 2, 3\ldots, N\}$ and $\mathbb{M} \triangleq \{1, 2, 3, \ldots, M\}$, respectively.

On the probabilistic space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathcal{F}_t$, $\mathbf{E}(\cdot)$ is the expected value operator. The space of all continuous-time signals $\mathcal{F}_t$-adapted processes $w = \{w(t) \in \mathbb{R}^r,\ t \in \mathbb{R}^+\}$ such that $\|w\|_2^2 \triangleq \int_0^\infty \mathbf{E}(\|w(\tau)\|^2)d\tau < \infty$, is represented by $L_2^r(\Omega, \mathcal{F}, \mathbf{P})$, or just by L_2, for simplicity. Similarly, by changing the subscript 2 by ∞ we have that L_∞ represents the space of processes w such that $\|w\|_\infty^2 \triangleq \sup_{t\in\mathbb{R}^+} \mathbf{E}(\|w(t)\|^2) < \infty$.

III. PRELIMINARIES

We consider in this section the following MJLS defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$,

$$\widetilde{\mathcal{G}} : \begin{cases} \dot{x}(t) = A_{\widetilde{\theta}(t)}x(t) + J_{\widetilde{\theta}(t)}w(t) \\ z(t) = C_{\widetilde{\theta}(t)}x(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $z(t) \in \mathbb{R}^{n_z}$ is the estimated output, and $w(t) \in \mathbb{R}^{n_w}$ is the disturbance. $\widetilde{\theta}(t)$ is a homogeneous Markov chain taking values in a finite set $\widetilde{\mathcal{N}}$ with transition rate matrix $\widetilde{\Lambda} \triangleq [\widetilde{\lambda}_{sv}]$, and $\widetilde{\theta}(0) = \widetilde{\theta}_0$, where $\widetilde{\theta}_0$ is a random variable taking values in $\widetilde{\mathcal{N}}$.

We present next the stochastic stability definition used throughout this work.

Definition 1 (Stochastic stability, [32]): System (1) with $w(t) \equiv 0$ is said to be *stochastically stable* (SS) if $\|x\|_2^2 = \int_0^\infty \mathbf{E}(\|x(t)\|^2)dt < \infty$, for every finite second moment $x(0)$ and every $\widetilde{\theta}_0 \in \widetilde{\mathcal{N}}$.

The following Lyapunov equation result can be found in Chapter 3 of [2].

Proposition 1: If there exist $P_s > 0$, $s \in \widetilde{\mathcal{N}}$, such that

$$Her(P_sA_s) + \sum_{v\in\widetilde{\mathcal{N}}} \widetilde{\lambda}_{sv} \quad P_v < 0, \tag{2}$$

then system (1) is stochastically stable.

Proof 1: See Theorem 3.21 in [2].

Remark 1: It is worth recalling that, as shown in Chapter 4 of [2], if system (1) is stochastically stable then, for $x = \{x(t); t \in \mathbb{R}^+\}$ with $x(t)$ given by (1) we have that $x \in L_2$ whenever $w \in L_2$.

Before proceeding, we recall the Dynkin's formula for the MJLS (1), shown in Chapter 4 of [2]. Set the function $V(s, x) = x'P_sx$ for matrices $P_s > 0$, $s \in \widetilde{\mathcal{N}}$, $\chi_s \triangleq A_sx + J_sw$, and $\mathfrak{L}V(s, x, w)$ as:

$$\mathfrak{L}V(s, x, w) = x'P_s\chi_s + \chi_s'P_sx + x'\left(\sum_{v\in\widetilde{\mathcal{N}}} \lambda_{sv}P_v\right)x. \tag{3}$$

According to the Dynkin's formula (see equation (4.16) in [2]) we have the following equality:

$$\begin{aligned} &\mathbf{E}(x(t)'P_{\widetilde{\theta}(t)}x(t)) - \mathbf{E}(x(0)'P_{\widetilde{\theta}(0)}x(0)) \\ &\quad = \mathbf{E}\Big(\int_0^t \mathfrak{L}V(\widetilde{\theta}(\tau), x(\tau), w(\tau))d\tau\Big). \end{aligned} \tag{4}$$

In what follows, we consider that $x(0) = 0$. In the next proposition we provide sufficient conditions to assure that system (1) is stochastically stable and, for some $\gamma > 0$, we have, for any $t \in \mathbb{R}^+$, that $\mathbf{E}(\|z(t)\|^2) \leq \gamma^2\|w\|_2^2$. Then, it is clear that $\mathbf{E}(\|z(t)\|^2) \leq \|z\|_\infty^2 \leq \gamma^2\|w\|_2^2$.

Proposition 2: If there exist $P_s > 0$, $s \in \widetilde{\mathcal{N}}$, such that

$$\begin{bmatrix} Her(P_sA_s) + \sum_{v\in\widetilde{\mathcal{N}}} \widetilde{\lambda}_{sv}P_v & P_sJ_s \\ J_s'P_s & -\gamma^2 I \end{bmatrix} < 0, \tag{5}$$

$$\begin{bmatrix} P_s & C_s' \\ C_s & I \end{bmatrix} \geq 0, \tag{6}$$

then system (1) is SS and $\mathbf{E}(\|z(t)\|^2) \leq \gamma^2\|w\|_2^2$.

Proof 2: From (5) it is clear that (2) is satisfied and thus from Proposition 1 we have that system (1) is stochastically stable. Set

$$\Phi_s \triangleq \begin{bmatrix} Her(P_sA_s) + \sum_{v\in\widetilde{\mathcal{N}}} \widetilde{\lambda}_{sv}P_v & P_sJ_s \\ J_s'P_s & -\gamma^2 I \end{bmatrix}.$$

Consider any $w \in L_2$, $w \neq 0$. From (5) we get that

$$\begin{aligned}
&\begin{bmatrix} x(t) & w(t) \end{bmatrix} \Phi_{\widetilde{\theta}(t)} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \\
&= x(t)' P_{\widetilde{\theta}(t)} \Big(A_{\widetilde{\theta}(t)} x(t) \\
&\quad + J_{\widetilde{\theta}(t)} w(t) \Big) + \Big(A_{\widetilde{\theta}(t)} x(t) + J_{\widetilde{\theta}(t)} w(t) \Big)' P_{\widetilde{\theta}(t)} x(t) \\
&\quad + \sum_{v \in \widetilde{\mathcal{N}}} \widetilde{\lambda}_{\widetilde{\theta}(t) v} x(t)' P_v x(t) - \gamma^2 \|w(t)\|^2 < 0. \qquad (7)
\end{aligned}$$

From (3) we have that (7) can be re-written as

$$\mathfrak{L}V(\widetilde{\theta}(\tau), x(\tau), w(\tau)) - \gamma^2 \|w(\tau)\|^2 < 0. \qquad (8)$$

Recalling that $x(0) = 0$ we get, after integrating (8) and applying the Dynkin's formula (4), that

$$\begin{aligned}
\mathbf{E}\left(x(t)' P_{\widetilde{\theta}(t)} x(t) \right) &= \mathbf{E}\left(\int_0^t \mathfrak{L}V(\widetilde{\theta}(\tau), x(\tau), w(\tau)) d\tau \right) \\
&< \gamma^2 \int_0^t \mathbf{E}(\|w(\tau)\|^2) d\tau \leq \gamma^2 \|w\|_2^2. \qquad (9)
\end{aligned}$$

From (6) it follows that $P_s \geq C_s' C_s$ and thus $\mathbf{E}(x(t)' P_{\widetilde{\theta}(t)} x(t)) \geq \mathbf{E}(\|z(t)\|^2)$. By combining the previous equation and (9) we get that $\mathbf{E}(\|z(t)\|^2) \leq \gamma^2 \|w\|_2^2$.

IV. PROBLEM FORMULATION

We study the following MJLS on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$,

$$\mathcal{G}: \begin{cases} \dot{x}(t) = A_{\theta(t)} x(t) + J_{\theta(t)} w(t) \\ y(t) = L_{\theta(t)} x(t) + H_{\theta(t)} w(t) \\ z(t) = C_{\theta(t)} x(t), \end{cases} \qquad (10)$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^{n_y}$ is the measured output, $z(t) \in \mathbb{R}^{n_z}$ is the estimated output, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance in L_2, and $\theta(t)$ is a (homogeneous) Markov chain taking values in the finite set $\mathbb{N}$ with transition rate matrix $\Lambda \triangleq [\lambda_{ij}]$, and $\theta(0) = \theta_0$, where θ_0 is a random variable taking values in $\mathbb{N}$. We also assume that $x(0) = 0$.

We consider that the state of the Markov chain $\theta(t)$ is not observable, and that the only information available comes from some detector represented by $\hat{\theta}(t)$ taking values in $\mathbb{M}$. By defining the extended hidden process as $\widetilde{\theta}(t) \triangleq (\theta(t), \hat{\theta}(t))$, we consider that $\{\widetilde{\theta}(t)\}$ is a homogeneous Markov process with state space $\mathbb{N} \times \mathbb{M}$ and transition rate $\nu_{(i,k)(j,\ell)}$ satisfying:

$$\begin{aligned}
&\mathbf{P}\left(\widetilde{\theta}(t+h) = (j,\ell) \mid \widetilde{\theta}(t) = (i,k) \right) \\
&= \begin{cases} 1 + \nu_{(i,k)(i,k)} h + o(h), & (j,\ell) = (i,k) \\ \nu_{(i,k)(j,\ell)} h + o(h), & (j,\ell) \neq (i,k), \end{cases}
\end{aligned}$$

where, for (i,k) fixed, the transition rate parameters $\nu_{(i,k)(j,\ell)}$ are defined as:

$$\nu_{(i,k)(j,\ell)} = \begin{cases} \alpha_{j\ell}^k \lambda_{ij}, & j \neq i, \ell \in \mathbb{M}, \\ q_{k\ell}^i, & j = i, \ell \neq k, \\ \lambda_{ii} + q_{kk}^i, & j = i, \ell = k, \end{cases} \qquad (11)$$

with $\sum_{\ell \in \mathbb{M}} \alpha_{j\ell}^k = 1, \forall j \in \mathbb{N}, k \in \mathbb{M}$; $\lambda_{ij} \geq 0$ for all $i \neq j$; $q_{k\ell}^i \geq 0$, $\ell \neq k$, $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$, $q_{kk}^i = -\sum_{\ell \neq k} q_{k\ell}^i$. We define $\mathbb{V} \subseteq \mathbb{N} \times \mathbb{M}$, an invariant set for $\widetilde{\theta}(t)$ (that is, $\mathbf{P}(\widetilde{\theta}(t) \in \mathbb{V}) = 1$ whenever $\widetilde{\theta}(0) \in \mathbb{V}$).

Remark 2: Recalling that λ_{ij} represents the transition rate of $\theta(t)$, we get that $\alpha_{j\ell}^k$ and $q_{k\ell}^i$ models simultaneous and spontaneous jumps of $\hat{\theta}(t)$, that is, for small $h > 0$, $\mathbf{P}(\hat{\theta}(t+h) = \ell \mid \theta(t+h) = j, \widetilde{\theta}(t) = (ik)) = \alpha_{j\ell}^k + r(h)$ for some function such that $\lim_{h \to 0} r(h) = 0$, and $j \neq i$, $\ell \in \mathbb{M}$, and $\mathbf{P}(\hat{\theta}(t+h) = \ell \mid \theta(t+h) = i, \widetilde{\theta}(t) = (ik)) = q_{k\ell}^i h + o(h)$, for $\ell \neq k$, see [22] and the references therein for further details. As pointed out in [5], Chapter 4, depending on the values of the indexes (i, k, ℓ), different interpretations can be assigned to $q_{k\ell}^i$ such as detection delays, rate of false alarms, rate of errors in detection and identification, etc. Examples of that can be found in [5]. Moreover, as discussed in [5], Chapter 4, Monte Carlo simulations and prior information can be used to estimate the transition rates $q_{k\ell}^i$ and similarly for the probability rates $\alpha_{j\ell}^k$.

Remark 3: As mentioned in the introduction, the exponential hidden Markov model considered in this paper has a close connection with the AFTCS (see, e.g., [5] and the references therein). The main idea in AFTCS is that there is a fault detection and identification (FDI) device which provides an estimate $\hat{\theta}(t)$ of the occurrence of a fault, represented by $\theta(t)$, in a dynamic system. Several FDI approaches have been proposed in the literature, categorized into signal-based and model-based techniques. As described in [5], Chapter 2, signal-based methods detect faults by testing specific properties of measurement signals, with bandpass filters and spectral analysis being some examples of the employed techniques, while model-based methods are performed in two steps: the residual generation and the residual evaluation. As pointed out in [21], the FDI scheme can be interpreted as a stochastic hypothesis test, which can be implemented using single sample tests, moving window tests or sequential tests. As explained in [5], [21], in single sample tests, the information used for the FDI tests is gathered, processed, and discarded at each time sample. In such cases, and if the noise statistics on the information are white, then the FDI processing is memoryless, so that Markov models can be used to characterize the transition behaviour of the state of the FDI process conditioned on the fault status of the components. In our work we follow a similar approach and assume the existence of a detector that provides the estimate $\hat{\theta}(t)$ so that the joint model $\widetilde{\theta}(t)$ can be modeled through a hidden Markov process. Although a discussion on FDI devices and related algorithms is of great interest, this is a major problem on its own and falls outside the scope of this paper. We refer the interested reader to [5], Chapter 4, and the references therein for a deeper discussion on this subject.

The main goal is to design the following filter structure,

$$\mathcal{G}_f: \begin{cases} \dot{x}_f(t) = A_{f\hat{\theta}(t)} x_f(t) + B_{f\hat{\theta}(t)} y(t) \\ z_f(t) = C_{f\hat{\theta}(t)} x_f(t), \end{cases} \qquad (12)$$

that depends only on the observed variable $\hat{\theta}(t)$, such that, for a given positive scalar γ, we get that $\|e\|_\infty \leq \gamma\|w\|_2$, for any $w \in L_2$, $w \neq 0$, where $e(t) = z(t) - z_f(t)$. We assume that $x_f(0) = 0$ and that $x_f(t) \in \mathbb{R}^{n_f}$, $0 < n_f \leq n$, that is, a reduced-order filter (if $n_f < n$).

Remark 4: By properly defining the parameters of the detector we can retrieve several cases usually considered in the literature, as presented next (see also [23] for further details):

- *Mode-dependent case:* In this case we have $\hat{\theta} \equiv \theta$, which can be characterized in our model by taking $\mathbb{M} = \mathbb{N}$, $q^i_{k\ell} = 0$, $\alpha^k_{jj} = 1$, and $\alpha^k_{j\ell} = 0$ for $j \neq \ell$, with invariant set $\mathbb{V} = \{(i, i) \in \mathbb{N} \times \mathbb{N}\}$.
- *Only Mutual (simultaneous) Jumps*: This case is characterized by taking $q^i_{k\ell} = 0$ for all $i \in \mathbb{N}$, $k, \ell \in \mathbb{M}$, so that, whenever there is a transition of $\theta(t)$, there is a probability that $\hat{\theta}(t)$ will jump as well.
- *The Cluster Case*: In this case the states of the Markov chain can be written as the union of $M \leq N$ disjoint sets (clusters) $\mathbb{N}_i$ so that $\mathbb{N} = \cup_{i\in\mathbb{M}}\mathbb{N}_i$ and there is a function $g : \mathbb{N} \to \mathbb{M}$ such that $g(i) = j$ indicates that the state i is associated to the cluster j. Under this approach at each time t the filter would have only access to $g(\theta(t))$. This is equivalent, under our approach, to take $q^i_{k\ell} = 0$ and $\alpha^k_{ig(i)} = 1$, so that whenever $\theta(t)$ jumps to i, $\hat{\theta}(t)$ also jumps simultaneously to $g(i)$.
- *No Mutual Jumps*: This case is characterized by taking $\alpha^k_{jk} = 1$ and $\alpha^k_{j\ell} = 0$ for $k \neq \ell$ (as in the AFTCS model), and is useful for modelling detection delays and false alarms (see [20]).
- *Mode-independent case*: In this case, $\mathbb{M} = \{1\}$, $q^i_{k\ell} = 0$, and $\alpha^1_{j1} = 1$. Thus $\hat{\theta}(t)$ does not provide any information on $\theta(t)$.

Thus the results to be developed will hold for all these cases.

By combining (10) and (12) we get the extended system given as follows:

$$\mathcal{G}_c\colon \begin{cases} \dot{\tilde{x}}(t) = A_{\theta(t)\hat{\theta}(t)}\tilde{x}(t) + J_{\theta(t)\hat{\theta}(t)}w(t) \\ e(t) = C_{\theta(t)\hat{\theta}(t)}\tilde{x}(t), \end{cases} \tag{13}$$

where $\tilde{x}(t)' \triangleq \begin{bmatrix} x(t)' & x_f(t)' \end{bmatrix}$ and for $(i, k) \in \mathbb{V}$,

$$A_{ik} = \begin{bmatrix} A_i & 0 \\ B_{fk}L_i & A_{fk} \end{bmatrix}, \quad J_{ik} = \begin{bmatrix} J_i \\ B_{fk}H_i \end{bmatrix},$$
$$C_{ik} = \begin{bmatrix} C_i' \\ -C_{fk}' \end{bmatrix}'. \tag{14}$$

Thus, by setting $\tilde{\mathcal{N}} = \mathbb{V}$, $\tilde{\lambda}_{(i,k),(j\ell)} = \nu_{(i,k),(j\ell)}$, $A_s = A_{ik}$, $J_s = J_{ik}$, and $C_s = C_{ik}$ in (5)-(6), the main goal of this work can be written as follows:

Problem I: Find A_{fk}, B_{fk}, C_{fk} and $P_{ik} > 0$ such that (5)-(6) hold.

Solving Problem I amounts to finding the solution set of (5)-(6) involving the product of variables (P_{ik}, A_{fk}) and (P_{ik}, J_{fk}), which is non-convex. In the next section, we provide new convex design conditions formulated in an LMI set-up in order to provide a convex solution to Problem I.

V. MAIN RESULTS

A. DESIGN RESULTS FOR PROBLEM I

In this sub-section we provide design results for solving **Problem I**. Due to the possible mismatch between $\theta(t)$ and $\hat{\theta}(t)$, we need the time evolution of both state variables, $x(t)$ and $x_f(t)$, to obtain $e(t)$ (this kind of situation also occurs in other problems, like in robust filtering, etc). Since the filter does not affect the system's dynamic in (10) we must assume that system (10) is stochastically stable. Notice that the conservatism can be reduced for the the case in which $\theta(t)$ is perfectly known (that is, $\hat{\theta}(t) = \theta(t)$) and $n_f = n$. See sub-section V-B for further details of this case.

In order to get the desired (possibly reduced-order) filter, we need to define the following matrices:

$$V' \triangleq \begin{cases} \begin{bmatrix} I_{n_f} & 0_{n_f\times n-n_f} \end{bmatrix}, & \text{for } n > n_f \\ I_n, & \text{for } n = n_f. \end{cases} \tag{15}$$

In what follows we will consider matrices G_k with the following structure:

$$G_k = \begin{cases} \begin{bmatrix} G_{1k} & G_{2k} \\ 0_{n-n_f\times n_f} & G_{4k} \end{bmatrix}, & \text{for } n > n_f \\ G_{1k}, & \text{for } n = n_f \end{cases} \tag{16}$$

with $G_{1k} \in \mathbb{B}(\mathbb{R}^{n_f})$ for all $k \in \mathbb{M}$. We introduce the next inequalities,

$$P_{ik} > 0, \tag{17}$$
$$\mathcal{H}_{ik} + Her(\mathcal{Q}_{ik}\Phi_{ik}) + Her(\mathcal{E}_{ik}\Psi_{ik}) < 0, \tag{18}$$
$$\begin{bmatrix} P_{ik} & \cdot \\ \begin{bmatrix} C_i & -O_kV' \end{bmatrix} & I_{n_z\times n_z} \end{bmatrix} \geq 0, \tag{19}$$

for $(i, k) \in \mathbb{V}$, where

$$\mathcal{H}_{ik} \triangleq \begin{bmatrix} \sum_{(j,\ell)\in\mathbb{V}} \nu_{(i,k)(j,\ell)}P_{j\ell} & \cdot & \cdot \\ P_{ik} & 0_{2n\times 2n} & \cdot \\ 0_{n_w\times 2n} & 0_{n_w\times 2n} & -\upsilon I_{n_w} \end{bmatrix},$$
$$\mathcal{Q}_{ik}' \triangleq \begin{bmatrix} R_{1ik}' & Y_{1ik}' & R_{2ik}' & Y_{2ik}' & R_{3ik}' \end{bmatrix},$$
$$\Phi_{ik} \triangleq \begin{bmatrix} A_i & 0_{n\times n} & -I_n & 0_{n\times n} & J_i \end{bmatrix},$$
$$\mathcal{E}_{ik}' \triangleq \begin{bmatrix} I_n & I_n & I_n & I_n & 0_{n\times n_w} \end{bmatrix},$$
$$\Psi_{ik} \triangleq \begin{bmatrix} VF_kL_i & VX_kV' & 0_{n\times n} & -G_k & VF_kH_i \end{bmatrix}.$$

We have that the variables in (17)-(19) are: $\upsilon \in \mathbb{R}_+$, $P_{ik} > 0$, $P_{ik} \in \mathbb{B}(\mathbb{R}^{2n+})$, $(i, k) \in \mathbb{V}$, $F_k \in \mathbb{B}(\mathbb{R}^{n_y}, \mathbb{R}^{n_f})$, $X_k \in \mathbb{B}(\mathbb{R}^{n_f})$, $O_k \in \mathbb{B}(\mathbb{R}^{n_f}, \mathbb{R}^{n_z})$, $G_k \in \mathbb{B}(\mathbb{R}^n)$, $R_{1ik} \in \mathbb{B}(\mathbb{R}^n)$, $R_{2ik} \in \mathbb{B}(\mathbb{R}^n)$, $R_{3ik} \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^{n_w})$, $Y_{1ik} \in \mathbb{B}(\mathbb{R}^n)$, $Y_{2ik} \in \mathbb{B}(\mathbb{R}^n)$.

We present in the next theorem design conditions to obtain a filter as in (12), with order $0 < n_f \leq n$, such that $\|e\|_\infty \leq \gamma\|w\|_2$ is satisfied.

Theorem 1: If there exist υ, P_{ik}, X_k, F_k, O_k, G_k, R_{1ik}, R_{2ik}, R_{3ik}, Y_{1ik}, and Y_{2ik} such that (17)-(19) hold, with G_k as in (16), then, by setting $\gamma = \upsilon^{1/2}$, $A_{fk} = G_{1k}^{-1}X_k$, $B_{fk} = G_{1k}^{-1}F_k$, and $C_{fk} = O_k$, we get that $\|e\|_\infty \leq \gamma\|w\|_2$ holds.

Proof 3: The goal of this proof is to show that (17)-(19) imply that (5)-(6) hold so that, from Proposition 2, the result follows. From (18) it is easy to see that $-Her(G_k) < 0$, and therefore we have that G_k is non-singular so that G_{1k} is also non singular.

Since $X_k = G_{1k}A_{fk}$ and $F_k = G_{1k}B_{fk}$, we get that (18) can be re-written as

$$\mathcal{H}_{ik} + Her\left(\begin{bmatrix}\mathcal{Q}_{ik} & \mathcal{E}_{ik}G_k\end{bmatrix}\begin{bmatrix}\tilde{\Phi}_{ik}\\ \tilde{\Psi}_{ik}\end{bmatrix}\right) < 0, \quad (20)$$

and

$$\begin{bmatrix}\tilde{\Phi}_{ik}\\ \tilde{\Psi}_{ik}\end{bmatrix} = \begin{bmatrix}\tilde{A}_{ik} & -I_{2n\times 2n} & \tilde{J}_{ik}\end{bmatrix},$$

where

$$\tilde{A}_{ik} \triangleq \begin{bmatrix}A_i & 0_{n\times n}\\ VB_{fk}L_i & VA_{fk}V'\end{bmatrix}, \quad \tilde{J}_{ik} \triangleq \begin{bmatrix}J_i\\ VB_{fk}H_i\end{bmatrix},$$

$$\begin{bmatrix}\mathcal{Q}_{ik} & \mathcal{E}_{ik}G_k\end{bmatrix} = \begin{bmatrix}R_{1ik} & G_k\\ Y_{1ik} & G_k\\ R_{2ik} & G_k\\ Y_{2ik} & G_k\\ R_{3ik} & 0_{n_w\times n}\end{bmatrix} \triangleq \begin{bmatrix}G_{1ik}\\ G_{2ik}\\ G_{3ik}\end{bmatrix},$$

for $G_{1ik} \in \mathbb{B}(\mathbb{R}^{2n})$, $G_{2ik} \in \mathbb{B}(\mathbb{R}^{2n})$, and $G_{3ik} \in \mathbb{B}(\mathbb{R}^{2n}, \mathbb{R}^{n_w})$. Set

$$U \triangleq \begin{bmatrix}I_n & 0_{n\times n_f} & 0_{n\times(n-n_f)}\\ 0_{n_f\times n} & I_{n_f} & 0_{n_f\times(n-n_f)}\end{bmatrix}, \quad (21)$$

$$\bar{A}_{ik} \triangleq U'A_{ik}U, \quad \bar{J}_{ik} \triangleq U'J_{ik}, \quad (22)$$

where A_{ik} and J_{ik} are as in (14) and notice that

$$UU' = I_{n+n_f}.$$

We get that (20) can be re-written as follows

$$\begin{bmatrix}\sum_{(j,\ell)\in\mathbb{V}} \nu_{(i,k)(j,\ell)}P_{j\ell} & \cdot & \cdot\\ P_{ik} & 0 & \cdot\\ 0 & 0 & -\gamma^2 I_{n_w}\end{bmatrix} + Her\left(\begin{bmatrix}G_{1ik}\\ G_{2ik}\\ G_{3ik}\end{bmatrix}\begin{bmatrix}\bar{A}_{ik} & -I_{2n} & \bar{J}_{ik}\end{bmatrix}\right) < 0. \quad (23)$$

Define, for $\bar{A}_{ik}$ and $\bar{J}_{ik}$ as in (22), $T_\perp$ as

$$T_\perp \triangleq \begin{bmatrix}I_{2n} & \bar{A}'_{ik} & 0\\ 0 & \bar{J}'_{ik} & I_{n_w}\end{bmatrix}',$$

so that $\begin{bmatrix}\bar{A}_{ik} & -I_{2n} & \bar{J}_{ik}\end{bmatrix}T_\perp = 0$. Multiplying (23) to the left-hand side by $T'_\perp$ and to the right-hand side by its transpose we have that

$$T'_\perp\begin{bmatrix}\sum_{(j,\ell)\in\mathbb{V}} \nu_{(i,k)(j,\ell)}P_{j\ell} & \cdot & \cdot\\ P_{ik} & 0 & \cdot\\ 0 & 0 & -\gamma^2 I_{n_w}\end{bmatrix}T_\perp =$$

$$\begin{bmatrix}Her(P_{ik}U'A_{ik}U) + \sum_{(j,\ell)\in\mathbb{V}} \nu_{(i,k)(j,\ell)}P_{j\ell} & \cdot\\ (U'\bar{J}_{ik})'P_{ik} & -\gamma^2 I_{n_w}\end{bmatrix} < 0.$$

By multiplying the last inequality by $diag(U, I_{n_w})$ to the left-hand side, and $diag(U', I_{n_w})$ to the right-hand side, we get that (5) is satisfied, after taking $s = (i, k)$, $\nu = (j, \ell)$, $\widetilde{A}_s = A_{ik}$, $J_s = J_{ik}$, and $P_s = UP_{ik}U' > 0$. Notice now that (19) can be written as

$$\begin{bmatrix}P_{ik} & \cdot\\ C_{ik}U & I_{n_z}\end{bmatrix} \geq 0. \quad (24)$$

Multiplying (24) to the left-hand side by $diag(U, I_{n_z})$ and its transpose to the right-hand side we get (6), after setting $C_s = C_{ik} = \begin{bmatrix}C_i & -C_{fk}\end{bmatrix}$. From Proposition 2, we get the desired result.

From Theorem 1, the main goal in **Problem I** can be re-written as follows: **Problem II:** Given υ, find $\bar{\xi} \in \bar{\Xi}(\upsilon)$, where $\bar{\xi} = (P_{ik}, F_k, X_k, O_k, G_k, R_{1ik}, R_{2ik}, R_{3ik}, Y_{1ik}, Y_{2ik})$, and $\bar{\Xi}(\upsilon)$ is the solution set of (17)-(19) for a given $\upsilon > 0$.

From **Problem II** and the linear relation of υ in (17)-(19), we can write the following LMI optimization problem for obtaining the lowest upper bound value $\upsilon = \gamma^2$ which ensures that $\|e\|_\infty \leq \gamma\|w\|_2$: $\min_{\xi\in\Xi} \upsilon$,, where $\xi = (\upsilon, P_{ik}, F_k, X_k, O_k, G_k, R_{1ik}, R_{2ik}, R_{3ik}, Y_{1ik}, Y_{2ik})$ and Ξ is the solution set of (17)-(19) including υ as a decision variable.

Remark 5: Regarding the size of the optimization LMI problem presented above we have, considering $\mathbb{V} = \mathbb{N} \times \mathbb{M}$, that there are MN symmetric matrix variables, $M(5N+4)$ full matrix variables of different dimensions, one scalar variable, and $3MN$ LMI conditions.

B. PERFECT INFORMATION

As previously mentioned, less conservative results can be obtained for the case in which $\theta(t)$ is perfectly known (that is, $\hat{\theta}(t) = \theta(t)$) and $n_f = n$. In this case one could work with a filter in the observer-form as follows:

$$\dot{x}_f(t) = A_{\theta(t)}x_f(t) + B_{f\theta(t)}(y(t) - L_{\theta(t)}x_f(t)),$$
$$z_f(t) = C_{\theta(t)}x_f(t).$$

By defining $\tilde{x}(t) = x(t) - x_f(t)$ and $e(t) = z(t) - z_f(t)$, we end up with the equations

$$\dot{\tilde{x}}(t) = \bar{A}_{\theta(t)}\tilde{x}(t) + \bar{J}_{\theta(t)}w(t),$$
$$e(t) = C_{\theta(t)}\tilde{x}(t), \quad (25)$$

where $\bar{A}_i = A_i - B_{fi}L_i$, $\bar{J}_i = J_i - B_{fi}H_i$, with no need to consider the augmented system (13) neither that system (10) is stochastically stable. We have the following result:

Proposition 3: If there exist $P_i > 0$, V_i, $i \in \mathbb{N}$, such that

$$\begin{bmatrix}Her(P_iA_i - V_iL_i) + \sum_{j\in\mathbb{N}} \lambda_{ij}P_j & P_iJ_i - V_iH_i\\ J'_iP_i - H'_iV'_i & -\gamma^2 I\end{bmatrix} < 0,$$

$$\begin{bmatrix}P_i & C'_i\\ C_i & I\end{bmatrix} \geq 0,$$

then, by taking $B_{fi} = P_i^{-1}V_i$, we have that system (25) is SS and $\|e\| \leq \gamma\|w\|_2$.

Proof 4: The result follows from Proposition 2 after noticing that $P_i\bar{A}_i = P_iA_i - V_iL_i$ and that $P_i\bar{J}_i = P_iJ_i - V_iH_i$.

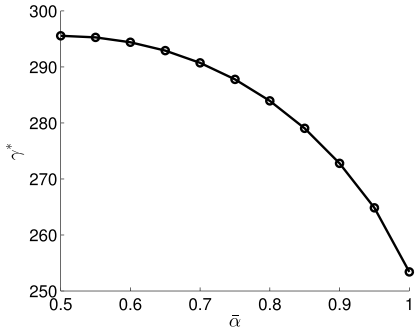

FIGURE 1. γ^* **against** $\bar{\alpha} \in [0.5, 1]$ **in** (27) **for** $n_f = 4$.

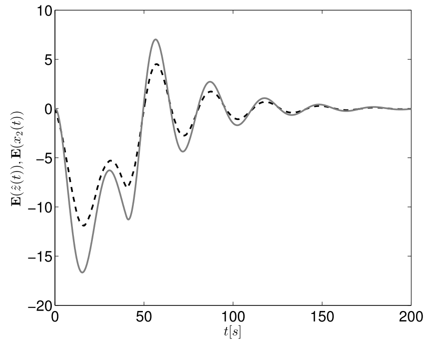

FIGURE 2. $\mathbf{E}(x_2(t))$ **(grey line) and** $\mathbf{E}(\hat{z}(t))$ **(black dashed line) against** t**, for** $\bar{\alpha} = 0.90$.

C. ROBUST FILTERING

In this section we study the robust filtering case, considering polytopic uncertainties on A_i, J_i, L_i, H_i, C_i as well as on the detector rates $\alpha^k_{j\ell}$ and $q^i_{k\ell}$. First of all, note that the summation in $\nu_{(i,k)(j,\ell)}$ in (18), considering (11), can be rewritten as

$$\sum_{(j,\ell)\in\mathbb{V}} \nu_{(i,k)(j,\ell)} P_{j\ell} = (\lambda_{ii} + q^i_{kk}) P_{ik} + \sum_{(j,\ell)\in\mathbb{V},\, \ell\neq k} q^i_{k\ell} P_{i\ell} + \sum_{(j,\ell)\in\mathbb{V},\, j\neq i} \lambda_{ij}\alpha^k_{j\ell} P_{j\ell} \tag{26}$$

for all $(i, k) \in \mathbb{V}$. Note that the rates $\alpha^k_{j\ell}$ and $q^i_{k\ell}$ are affine in (26). Set $\Psi_i = (A_i, J_i, L_i, H_i, C_i)$ and

$$\underline{q}^i_k \triangleq \begin{bmatrix} q^i_{k1} & \ldots & q^i_{kM} \end{bmatrix}' \in \mathbb{R}^M, \quad \Upsilon_k \triangleq [\alpha^k_{j\ell}].$$

We assume that there are $\Psi_i(s)$, $\alpha^k_{j\ell}(s)$ and $q^i_{k\ell}(s)$, for $s = 1, \ldots, \sigma$, such that

$$(\underline{q}^i_k, \Upsilon_k) = \sum_{s=1}^{\sigma} \eta_s (\underline{q}^i_k(s), \Upsilon_k(s)),$$

$$\Psi_i = \sum_{s=1}^{\sigma} \eta_s \Psi_i(s),$$

for some $\eta_s \geq 0$, $s = 1, \ldots, \sigma$, $\sum_{s=1}^{\sigma} \eta_s = 1$ Then, since (18) is affine in Ψ_i, $\alpha^k_{j\ell}$ and $q^i_{k\ell}$, by setting $\Psi_i = \Psi_i(s)$, $\alpha^k_{j\ell} = \alpha^k_{j\ell}(s)$ and $q^i_{k\ell} = q^i_{k\ell}(s)$ and solving (17)-(19) for each vertex $s \in \{1, \ldots, \sigma\}$, we also get that $\|e\|_\infty \leq \gamma \|w\|_2$ with the filter matrices given as in Theorem 1.

Remark 6: Notice that from (26) we could have the polytopic uncertainties in λ_{ij} instead of $\alpha^k_{j\ell}$.

VI. ILLUSTRATIVE EXAMPLE

We consider an adapted version of the linearized model of the stable longitudinal dynamics of an unmanned aircraft discussed in [31]. The original nonlinear model is obtained by classical (Newtonian) mechanics, by considering a rigid-body motion and assuming that the Earth is an inertial (Galilean) frame so that the Coriolis acceleration is ignored. The components of the state vector are the variations on the pitch rate, airspeed, angle of attack, and pitch angle. After linearization, the nominal matrices are given by

$$\bar{A} = \begin{bmatrix} -4.7796 & 0 & -4.5420 & 0 \\ 0 & -0.0830 & -0.8660 & -9.8100 \\ 1.0000 & -0.0215 & -3.6573 & 0 \\ 1.0000 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{J}' = \begin{bmatrix} 27.4128 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We assume that we can measure the angle of attack and pitch angle so that

$$\bar{L} = \begin{bmatrix} 0_{2\times 2} & I_2 \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} 0_{2\times 1} & I_2 \end{bmatrix}$$

but, however, the sensors are subject to faults. In this case, the process $\theta(t)$ is a Markov chain with three possible states, so that $\mathbb{N} = \{1, 2, 3\}$, with:

- State 1 representing the nominal case, so that $L_1 = \bar{L}$ and $H_1 = \bar{H}$;
- State 2 representing a faulty mode of operation whereas there is an attenuation in the measurements of the angle of attack and pitch angle so that, in this case, $L_2 = 0.5\bar{L}$ and $H_2 = \bar{H}$;
- State 3 also representing a faulty mode in which the measurements are completely lost, that is, $L_3 = 0$, and the noise level increases so that $H_3 = 1.2\bar{H}$.

For all modes of operation, we set $A_i = \bar{A}$, $J_i = \bar{J}$, $i \in \mathbb{N}$. The transition rates are given by

$$[\lambda_{ij}] = \begin{bmatrix} -0.4 & 0.4 & 0 \\ 0.7 & -1.0 & 0.3 \\ 0 & 0.5 & -0.5 \end{bmatrix}.$$

We consider that it is not possible to measure the fault process perfectly but, instead, there is a detector $\hat{\theta}(t)$ which provides some information regarding $\theta(t)$, so that $\mathbb{M} = \mathbb{N}$. The goal is to design the matrices $A_{f\ell}$, $B_{f\ell}$, $C_{f\ell}$ in (12) to obtain an estimation of the variations on airspeed, that is, $C_i = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$, $i \in \mathbb{N}$. We notice that by combining the matrices A_i, J_i, L_i, H_i, C_i of system (10) introduced above with the filter equations $A_{f\ell}$, $B_{f\ell}$, $C_{f\ell}$ of the system (12) we get the extended system given by (13).

In this example, we analyse five different cases (see Remark 4 for an explanation of these cases):

(1) The mode-dependent case;
(2) The case in which we only have *mutual (simultaneous) jumps* of $\theta(t)$ and $\hat{\theta}(t)$, that is, $q^i_{k\ell} = 0$;
(3) The cluster case;
(4) The case of no mutual jumps;
(5) The mode-independent case,

We recall that, as pointed out in Remark 4, case (2) is useful for modelling the situation of simultaneous jumps of $\theta(t)$ and $\hat{\theta}(t)$, but with a possible mismatch between them, while case (4) is useful for modelling detection delays and false alarms. Case (1) represents the situation of perfect information of the mode of operation $\theta(t)$ while cases (4) and (5) represent the situation of partial or no information at all of the the mode of operation $\theta(t)$. For the mode-dependent, simultaneous jumps, and cluster cases, and considering Remark 4, we set the rates of spontaneous jumps $q^i_{k\ell}$ to zero. For the rate of simultaneous jumps, we set

$$[\alpha^k_{j\ell}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \bar{\alpha} & 1-\bar{\alpha} \\ 0 & 1-\bar{\alpha} & \bar{\alpha} \end{bmatrix} \quad (27)$$

where $0 \leq \bar{\alpha} \leq 1.0$. That is, whenever there is a transition of $\theta(t)$ to "1", $\hat{\theta}(t)$ will also go to "1", meaning a perfect detection for the nominal mode of operation. On the other hand, if $\theta(t)$ jumps to "2" (or "3"), there is a probability $\bar{\alpha}$ of $\hat{\theta}(t)$ going to "2" (or "3"), that is, $\bar{\alpha}$ is the probability of correct detection for this two modes of operation. Considering Remark 4, we get the mode-dependent case (1) by setting $\bar{\alpha} = 1$ and $\mathbb{V} = \{(1, 1), (2, 2), (3, 3)\}$ and the simultaneous jump case (2) by setting a chosen value of $\bar{\alpha}$ within the interval [0, 1]. As for the cluster case (3) (see Remark 4), we set the clusters $\mathbb{N}_1 = \{1\}$ and $\mathbb{N}_2 = \{2, 3\}$, that is, the faulty modes are indistinguishable. Regarding the no mutual jumps case (4) (see Remark 4), we will consider the situation of detection delays with expected delay time of $\frac{1}{2.5}$ s, which corresponds to taking $\alpha^k_{jk} = 1$ and $\alpha^k_{j\ell} = 0$ for $k \neq \ell$, along with $q^1_{21} = q^1_{31} = q^2_{12} = q^2_{32} = q^3_{13} = q^3_{23} = 2.5$, and $q^i_{k\ell} = 0$ for the other cases. Finally, case (5) is the mode-independent situation, in which we have only one filter for all modes of operation. By minimizing υ in (17)-(19), we get $\gamma^* = \sqrt{\upsilon^*}$ for all cases of Remark 4: (1) $\bar{\alpha} = 1$ (the mode-dependent case), (2) $\bar{\alpha} = 0.9$ (a case of simultaneous jumps); (3) the cluster case; (4) the no mutual jump case; and (5) the mode-independent case. The costs γ^* obtained through (17)-(19), are shown in Table 1 for $n_f \in \{1, \ldots, 4\}$. From Table 1, we note that, as we decrease n_f, the costs will degrade for all cases, which is expected since we are using a smaller number of filter states for the estimation. More specifically, we also note that,

- For any chosen case, the performance of the filters is similar for the cases $n_f = 3$ and $n_f = 2$, but considerably worse than the performance of the case $n_f = 4$.

TABLE 1. γ^* **for** $n_f \in \{1, \ldots, 4\}$ **for (1) the mode-dependent case; (2) a simultaneous jump case with** $\bar{\alpha} = 0.9$**; (3) the cluster case; (4) the case of no mutual jumps; and (5) the mode-independent case.**

n_f	γ^* (1)	γ^* (2)	γ^* (3)	γ^* (4)	γ^*(5)
1	600.4	600.6	600.7	601.2	602.3
2	434.4	445.7	457.5	451.1	514.7
3	425.8	440.1	455.5	446.0	507.9
4	253.4	272.8	295.6	273.5	350.9

- As expected the worst performance is obtained with $n_f = 1$, which corresponds to the filter with only one state variable. This is not a too realistic situation, since one is trying to use a scalar filter to estimate an output from a forth-order system.

The conclusion, in terms of the order n_f of the filter, is that the designer must ponder if it is acceptable to increase the costs in this proportion whilst using less computational resources with a reduced order filter. Comparing now the cases, the best costs, as expected, are obtained for the mode-dependent case (1), corresponding to $\bar{\alpha} = 1.0$, since in this case there is no mismatch between the switching process of the filter and of the plant. Compared to case (1), there is a slight degradation of the value of the costs for case (2), with the cluster case (3) presenting some intermediary values in this example. Case (4), for our choice of rates $q^i_{k\ell}$, presents a similar performance as the one for case (2), with both cases outperforming the cluster case (3). The worst case scenario occurs, in terms of costs, for the mode-independent case, since the filter is the same for all modes of operation. Finally, the filter matrices for $n_f = 3$ and the (3) cluster case are given as follows,

$$A_{f1} = \begin{bmatrix} -6.6115 & 0.0434 & 16.5391 \\ 22.6821 & -0.1192 & 74.3876 \\ 1.1395 & -0.0184 & -5.7569 \end{bmatrix},$$

$$B_{f1} = \begin{bmatrix} 14.9512 & -0.8588 \\ 55.0244 & 15.1125 \\ -2.9764 & 0.0476 \end{bmatrix},$$

$$C_{f1} = \begin{bmatrix} -13.5725 & -0.9362 & -69.9869 \end{bmatrix}$$

and, for $\ell \in \{2, 3\}$,

$$A_{f\ell} = \begin{bmatrix} -2.0317 & 0.0424 & 6.7956 \\ 41.6391 & -1.3833 & -234.2737 \\ 0.5071 & -0.0114 & -2.0871 \end{bmatrix},$$

$$B_{f\ell} = \begin{bmatrix} -0.0501 & -0.1901 \\ -7.9312 & 26.6796 \\ -0.0873 & -0.0829 \end{bmatrix}$$

$$C_{f\ell} = \begin{bmatrix} -11.6305 & -0.5710 & 2.7278 \end{bmatrix}.$$

We can have a closer look at the costs for $n_f = 4$ by varying $\bar{\alpha} \in [0.5, 1.0]$ while minimizing υ in order to get the values of γ^* against $\bar{\alpha}$, as shown in Figure 1.

From Figure 1, we can see that the smallest cost is obtained for $\bar{\alpha} = 1.0$, which corresponds to the mode-dependent case (see Remark 4). As we decrease $\bar{\alpha}$, we increase the mismatch frequency between θ and $\hat{\theta}$, consequently

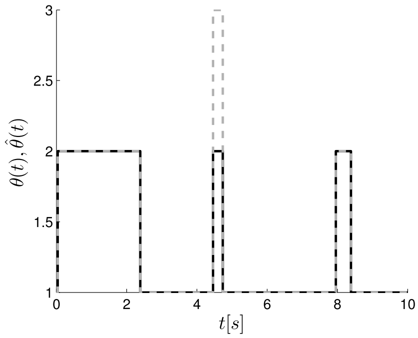

FIGURE 3. One trajectory of $\theta(t)$ (full black line) and $\hat{\theta}(t)$ (dashed gray line) sampled from the Monte Carlo simulation of 200 rounds.

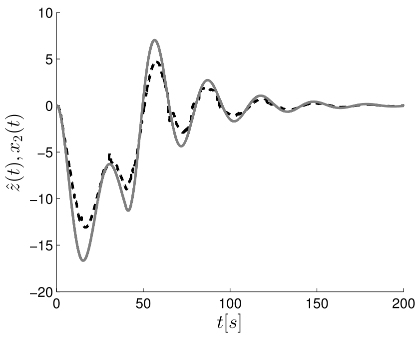

FIGURE 4. One trajectory of $x_2(t)$ and its estimation $z_f(t)$ obtained with the sample path of θ and $\hat{\theta}$ shown in Figure 3 ($n_f = 3, \bar{\alpha} = 0.9$).

increasing the costs γ, with the biggest value of γ given for $\bar{\alpha} = 0.5$, that is, when it is equally likely to obtain the correct and the incorrect information from the detector. For this situation the filter matrices are the same for $\mathbb{N}_1 = \{1\}$ and $\mathbb{N}_2 = \{2, 3\}$, which corresponds to the cluster case. We now run a Monte Carlo simulation of 200 rounds for the filter calculated for the case $n_f = 3$ (a reduced order filter) and $\bar{\alpha} = 0.9$. We consider $\theta(0) = \hat{\theta}(0) = 1$ and the exogenous input as $w(t) = 0.01 \left[1\ 0\ 0\right]', 0 \leq t < 40\ s$, and $w(t) = 0_{3\times 1},\ t \geq 40$. We get that $\|z\|_\infty / \|w\|_2 \approx 85.3 < 440.1$, as expected. Figure 2 shows the actual state $\mathbf{E}(x_2(t))$ and its estimation $\mathbf{E}(\hat{z}(t))$ against the time). We note that, after the initial transient, the reduced order filter is able to successfully track the desired state. Besides, in order to illustrate the behavior of $\theta(t)$ and $\hat{\theta}(t)$, we plot one trajectory of both curves against the time obtained from the Monte Carlo simulation used to plot Figure 3. We note that the Markov chain, which represents the fault process, transitions between two of the three possible states. Whenever there is a transition of $\theta(t)$, there is a 90% of chance that $\hat{\theta}(t)$ will go to the same state, provided $\theta(t) = 2$ or $\theta(t) = 3$. Thus there is a 10% of chance of a mismatch between $\theta(t)$ and $\hat{\theta}(t)$ after a jump from states 2 and 3. For instance, from Figure 3 we notice that, at around 4 s, the Markov chain goes to "2", but the detector goes to "3". We plot in Figure 4 the actual state $x_2(t)$ and its estimation $z_f(t)$ associated with the sample path of $\theta(t)$ and $\hat{\theta}(t)$ shown in Figure 3, obtained through the Monte Carlo simulation. We note that the filter is able to satisfactorily track the state $x_2(t)$, even with the reduced order structure and the mismatch effect.

VII. FINAL COMMENTS

In this paper we investigated the energy-to-peak reduced-order filtering problem for continuous-time Markov jump linear systems considering that the Markov state cannot be directly measured. To cope with this partial information set-up, it is assumed that there exists another stochastic process, called a detector, which provides the only information regarding the Markov chain, so that the joint process follows the so-called exponential hidden Markov model. A sufficient design condition, written in terms of LMIs, is given for obtaining the filter matrices that depend only on the detector process and such that the ratio of the expected value of the Euclidean norm of the estimation error by the energy of the exogenous input signal is bounded for all time. Finally, we present an illustrative example in the context of an unmanned aircraft subject to faulty sensor readings.

As a future work, the design of energy-to-peak output dynamic controllers with partial information on the state variable as well as the jump variable is an interesting line of research which, as far as the authors are aware of, was not tackled in the literature yet. Another possible promising field of research would be to apply the obtained results in filtering issues related to communication protocols or cyber-attacks, as considered, for instance, in [33], [34].

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