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RESEARCH ARTICLE

Impulse Moments Influence on Single-Frequency Torsional Oscillations of Nonlinear Flexible Bodies

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ABSTRACT The method of analytical study of the influence of impulse moments on nonlinear torsional oscillations of a homogeneous constant cross-section of a body under classical boundary conditions of the first, second, and third types have been developed. When the flexible material properties meet the body close to the power law of flexibility, mathematical models of the process have been obtained. It is the boundary value problem for an equation of hyperbolic type with a small parameter at the discrete right-hand side. The latter expresses the effect of pulse momentum on the oscillatory process. Under the effect of periodic pulse momentum on a flexible body, resonant and non-resonant processes are possible. Resonant processes occur when the amplitude of natural oscillations approaches a fixed value. The peculiarities of resonant oscillations are established. The amplitude of passing through the primary resonance is significant for larger values of the nonlinearity parameter and, in the case of the action of pulse momentum, closer to the middle of the body. If the initial perturbation amplitude is less than the amplitude at which the resonance occurs in the presence of only internal forces of viscous friction. So, the external periodic impulse moments of resonance processes have occurred. The extreme case confirms the reliability of the results obtained related to the dynamics of the respective objects under the continuous action of autonomous perturbation.

INDEX TERMS Impulse perturbations, nonlinear flexible bodies, Ateb-function, oscillations, mathematical model, resonance, amplitude, frequency.

I. INTRODUCTION

The creation and use in various branches of mechanical engineering and construction industry of structural elements from new materials, the physical and mechanical properties of which differ from the classical correlations arising from the linear law of flexibility, require consideration of new mathematical models for evaluations of their flexibility efficiency. We are talking primarily about dynamic tension caused by deformations due to various external factors. Suppose the dynamic deformations under continuous action on such elements for the case of simple stress state (torsion, tension, pure bending) obtained based on the mathematical models (torsional, longitudinal, bending) of oscillations [1]–[7] for quasilinear [8] linear or partial strongly nonlinear [9]–[11] laws of flexibility, for discrete action. In that case, the question for many cases remains open.

The problem is related to constructing analytical solutions to the corresponding boundary (mixed) problems for non-autonomous linear (quasilinear) equations of hyperbolic type. Numerical simulations of these mathematical models [2], [4], [8] cannot provide generalized answers to many practical problems related to the discrete action of external load. First, these are the resonance conditions and peculiarities of its passage under the periodic impulse of external perturbation. Such problems use for the case of the flexible elements of structures under the condition of essentially (strongly) nonlinear law of its flexible properties of the consideration

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subject of work, whence the urgency of their consideration follows.

The subject of the work is the application of the asymptotic method of nonlinear mechanics in combination with the application of special periodic Ateb-functions for new classes of dynamical systems. The dynamical systems considered in the work concern the oscillatory processes of flexible bodies, the material of which satisfies the strongly nonlinear law of flexibility with different ways of fixing their ends.

Mathematical models [10], [12], [13] of the process are boundary value problems for nonlinear differential equations with partial derivatives and discontinuous (irregular) right-hand sides. Using the properties of Eigen functions that describe the forms of nonlinear oscillations and the properties of delta functions, a method of partial regularization was developed, and we managed to obtain equations in standard form for non-resonant and resonant cases. These equations describe the laws of change of the basic parameters of nonlinear flexible body oscillations. The significant difference of the latter case in comparison with quasilinear resonant processes of flexible bodies is shown. By the way, quasilinear processes in flexible bodies are described as a partial case of the work where $\nu = 0$.

On the one hand, the urgency of solving such problems is associated with creating new materials whose flexible properties are described by nonlinear relations. On the other is the inefficiency of using numerical simulation for their analysis. The latter is manifested in studying such important practical phenomena as resonances. Due to the non-isochronous nature of the undisturbed process, the conditions of their existence and peculiarities of passing through numerical simulation are challenging to describe.

The results obtained in this work can be used to solve problems, not only analysis of dynamic processes in strongly nonlinear systems with distributed parameters. Synthesis problems are the choice of parameters that prevent resonant processes, such as longitudinal oscillations, flexible bodies under the periodic action of impulse forces in turbine rotors, and elements of protective structures under shock loads. The most basic idea of the work can be generalized in case of disturbed boundary conditions.

In summary, the paper makes the following contributions:

1) Relative torsional oscillations of a nonlinear flexible body that rotates around the axis with a constant portable angular velocity consider the periodic action of pulse momentum acting in a fixed cross-section.
2) The amplitude of passing through the primary resonance is more significant for larger values of the nonlinearity parameter and, in the case of the action of pulse momentum, closer to the middle of the body.
3) If the initial perturbation amplitude is less than the amplitude at which the resonance occurs in the presence of only internal forces of viscous friction, the external periodic impulse moments of resonance processes have occurred.

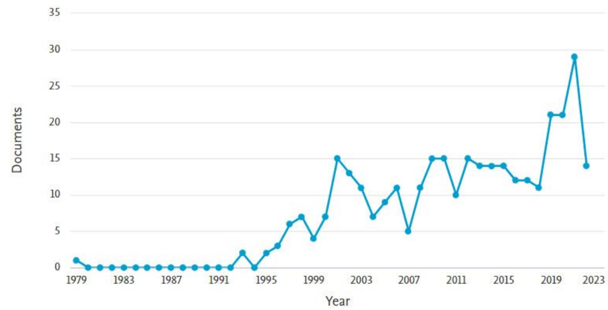

FIGURE 1. Statistics of the Scopus documents by selected year range 1979 – 2022 using the search quivery "nonlinear flexible bodies."

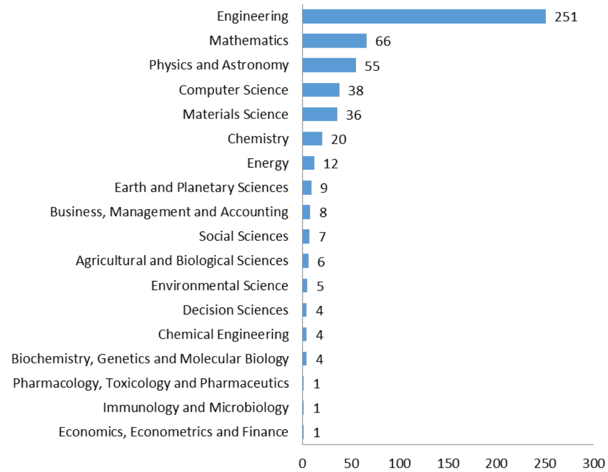

FIGURE 2. The number of articles published by subject areas.

II. REVIEW OF RELATED WORK

The authors of this paper carried out a comprehensive review of present studies about the investigation of nonlinear flexible bodies. The authors made the selection procedure of the Scopus documents in the fields of the nonlinear flexible bodies. The 331 documents of the Scopus database are selected using a searching query:

[ALL (*nonlinear* AND *flexible* AND *bodies*) AND (LIMIT-TO (AFFILCOUNTRY, *"Ukraine"*))]

The number of articles by subject areas published in Scopus of Ukrainian scientists about nonlinear flexible bodies is shown in Figure 2.

The section of the review of literature sources introduces the authors' research framework. The process of filtering relevant papers in the PRISMA diagram in Figure 3 is presented.

For a thorough review of scientific papers in the field of nonlinear flexible bodies, we used the PRISMA method. Authors collected the data using a searching query after conducting the search process by searching for journal articles through four databases, which were:

- Web of Science (1,195)
- Scopus (42,172)
- Google Scholar (18,600)
- EEE Xplore Digital Library (220)

The specific search in each database using titles with keywords as in the following: ALL (*nonlinear* AND *flexible*

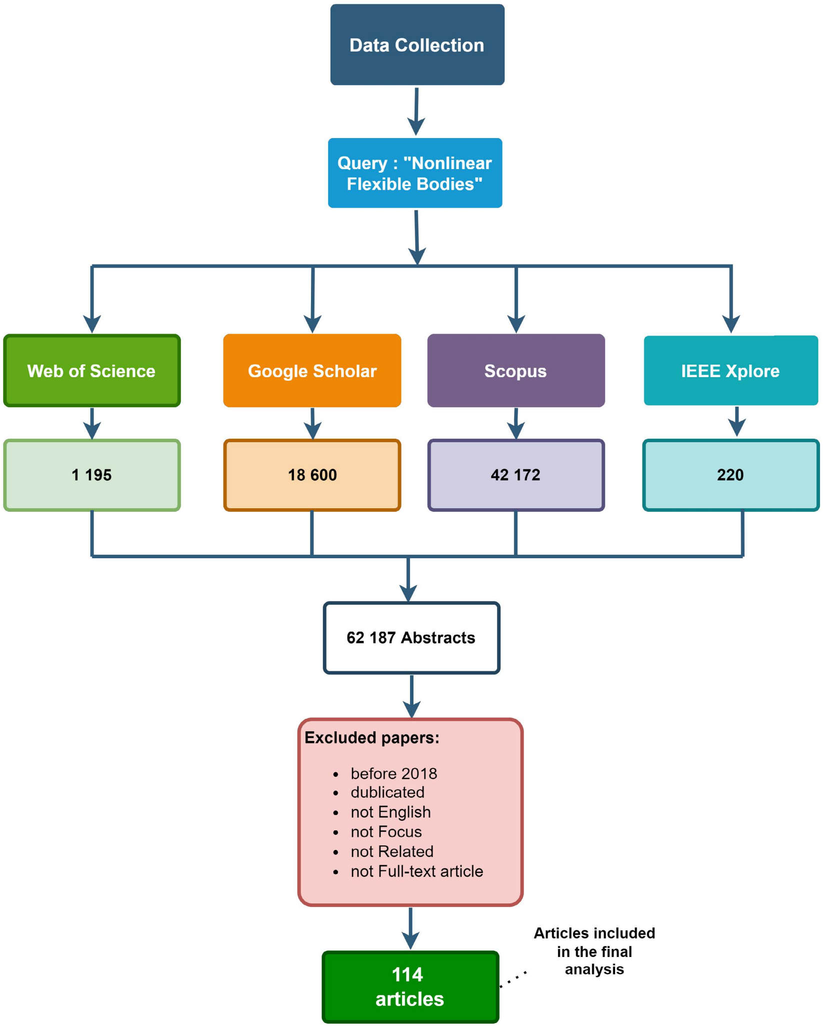

FIGURE 3. **PRISMA flow diagram for the study selection procedure.**

AND *bodies*). After the overview analysis, we found 62,187 unique abstracts. Then we excluded papers written not in English, published before 2018, not full-paper articles, and we removed duplicate records, not related and not focused.

The 144 document results we selected by the year range to analyze 2018-2022 years, 2022 (19 articles), 2021 (46), 2020 (25), 2019 (27), and 2018 (27).

Figure 4 gives the TOP-15 the most active countries in the field of study of nonlinear flexible bodies. They are TOP-15 because only 15 countries published more than 3 studies.

This result of this analysis of the review of literature sources of the nonlinear flexible bodies in the Scopus database is investigated based on the first authors' country of institutions.

The result based on Figure 4 identified 45 active countries. These countries are the following countries China (45 studies), the United States (33), Germany (16), the United Kingdom (13), France (10), Canada (9), Italy (8), Japan (7), Switzerland (7), Australia (6), Netherlands (5), South Korea (5), Austria (4), Belgium (4), Finland (4),

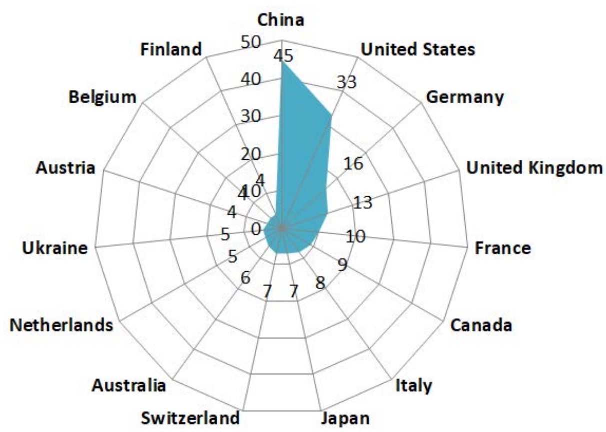

FIGURE 4. **The TOP-15 countries published in the field of study of the nonlinear flexible bodies.**

Brazil (3), India (3), Singapore (3), 2 articles from each of the following countries: Denmark, Hong Kong, Norway, Pakistan, Spain, Sweden, and 1 article from each of the following countries: Chile, Colombia, Czech Republic, Georgia, Greece, Indonesia, Iraq, Ireland, Israel, Jordan, Macao, Malaysia, Malta, Mexico, Portugal, Serbia, Slovakia, United Arab Emirates, Viet Nam, respectively.

The most effective methods of investigation of oscillating processes in nonlinear systems [14]–[17] with concentrated masses and distributed parameters are those based on introductory provisions of perturbation methods [18]–[23], combined with the principle of simultaneity of oscillations.

A significant development was found when the set of single-frequency solutions could be built for undisturbed analogs of the interconnected systems with the help of trigonometric and periodic Ateb-functions [24], [24], [26]–[29].

The latter describes the dynamics of a wide range of systems with power nonlinearity [30]–[33] and has the following several features that are not typical for a linear system:

1) Firstly, the absence of the principle of superposition;
2) Secondly, the oscillatory processes of these systems are characterized by such fundamentally different from linear systems features as the dependence of the period of the dynamic process on the amplitude;
3) Thirdly, general analytical approaches to studying perturbed analogs of systems are absent.

All the above significantly complicates constructing an analytical solution of the corresponding mathematical models of the dynamic process under the action of impulse [21], [34] (discrete) load and obtaining on its basis correlations that would be the basis for engineering calculations and other mathematical methods [35]–[38]. Thus, this work aims to develop a mathematical apparatus for studying the torsional oscillations [39]–[41] of a flexible body whose flexible properties are described as close to the power law of flexibility [36] ($\sigma = k\varepsilon^{\nu+1}$), where σ is a tension in the body, $\varepsilon$is a relative deformation, $\nu + 1$ is a nonlinearity index, $\nu > -1$ and which is exposed to external impulse action. In strongly nonlinear oscillating systems, the natural frequency depends on the amplitude. External periodic perturbation causes a resonant process in the case when its period is close to the period of natural oscillations. This is the essential difference between the forced oscillations of quasilinear and strongly nonlinear systems. If this resonance phenomenon occurs in the first approximation, provided that the periods of natural oscillations and the forcing force are the same, then for strongly nonlinear periods (natural frequency) for which the resonant process takes place exists only when the amplitude of natural oscillations approaches the resonance amplitude as $\frac{2\Pi}{\omega(a^*)} = \tau$, τ is a period of impulse perturbation.

III. THE MATHEMATICAL MODEL OF THE DYNAMIC OF A NONLINEAR FLEXIBLE BODY UNDER THE ACTION OF IMPULSE MOMENTS

Impulse perturbations are the moment of representation of short-term action on the system of external moments. Resonance in strongly nonlinear systems differs significantly from the resonance process in quasilinear systems and occurs when the frequency of light oscillations approaches the frequency of the forcing force: the natural frequency depends on the amplitude, so resonance in strongly nonlinear systems occurs when the amplitude approaches a fixed value. If the system's oscillations amplitude is less than the amplitude and there are only dissipative forces, then the resonance phenomenon will not be observed. If you bring the system into the resonance region, the dissipative forces reduce the amplitude of oscillations to a value close to the resonance amplitude. The resonant process is the amplitude increases, so the process continues.

It obtains without difficulty based on the dynamic equilibrium of the conditionally selected element of the body. Its mass is considered evenly distributed along the length l, and the cross-section is constant.

The deformation of such a body is unambiguously determined by its torsion $\phi\,(t, x)$ (torsion angle of the cross-section with the coordinate x at any time). The latter is determined by torque in the specified cross-sections and the physical and mechanical properties of the material $\phi\,(t, x) = g\,(M_k)$. For the case considered in this work $g\,(M_k)$ becomes $g\,(M_k) = GJ_P\left(\frac{\partial\phi}{\partial x}\right)^{\nu+1}$, G is a modulus of flexibility of the second kind, J_P a moment of inertia of cross-section about the axis that passes through its geometric center. In this case, the ratio stems from dynamic equilibrium for the flexible body element placed between cross-sections x, and $x + dx$ we have $\rho J_p \phi_{tt} = \frac{\partial}{\partial x}\left(GJ_P\frac{\partial\phi}{\partial x}\right)^{\nu+1}$, ρ is a density of body material. Should be we take into account the small value of the resistance force $\varepsilon f\,(\phi_t, \phi_x, \phi_{xx})$, the above ratio is transformed into the form of

$$\phi_{tt} - \alpha^2\,(\phi_x)^{\nu}\,\phi_{xx} = \frac{\mu}{\rho} f\,(\phi_t, \phi_x, \phi_{xx}) \tag{1}$$

In ration (1) $\alpha^2 = (\nu + 1)\,\frac{G}{\rho}$, μ is the small parameter. This parameter shows that the function's maximum value of the right-hand side equation (1) is a small value compared to the maximum value. In the case of the action on the flexible body at different times,$t_s$the momentum value, which generally depends on $\phi_t, \phi_x, \phi_{xx}$ equation (1), is transformed into the form of

$$\begin{aligned}\phi_{tt} - \alpha^2\,(\phi_x)^{\nu}\,\phi_{xx} &\\ = \frac{\mu}{\rho}&\left\{\begin{array}{l} f\,(\phi_t, \phi_x, \phi_{xx}) + \\ \sum\limits_{s=1} \hbar_s\,(\phi, \phi_t, \phi_x, \phi_{xx})\delta\,(t - t_s)\,\delta\,(x - x_s)\end{array}\right\}\end{aligned} \tag{2}$$

In equation (2) $\varepsilon\hbar_s\,(\phi\,(t_s, x_s)\,, \phi_t\,(t_s, x_s)\,, \phi_x\,(t_s, x_s)\,,$ $\phi_{xx}\,(t_s, x_s))$, the magnitude of the momentum acting at time t_s in the cross-section of the body with coordinate x_s $(0 < x_s < l\,)$ $\delta\,(\ldots)$ is a delta function of the corresponding argument [15].

As to the boundary conditions, they are classic: 1st, 2nd, and 3rd kind

$$\phi(x,t)_{|x=0} = 0, \quad \phi(x,t)_{|x=l} = 0 \tag{3}$$

$$\phi(x,t)_{|x=0} = 0, \quad \phi_x(x,t)_{|x=l} = 0 \tag{4}$$

$$\phi_x(x,t)_{|x=0} = 0, \quad \phi_x(x,t)_{|x=l} = 0 \tag{5}$$

and correspond, respectively, to the absence of angles of torsion of the end of the flexible body – (3); free, unloaded ends – (4); the absence of angle of torsion at the beginning of the body and free another end of the flexible body – (5).

It should be noted that boundary value problems formulated above also describe relative torsional oscillations of a flexible body that rotated around its axis of symmetry with a constant angular velocity Ω and $\phi(x,t)$, in such a case, is the relative angle of torsion. From the properties $\delta(\ldots)$ of the function, it is shown that the accuracy of the right-hand side of the differential equation (2), more precisely additive components which describe impulse action on the flexible body, will not change if presented as

$$\sum_{s=1} \hbar_s(\phi,\phi_t,\phi_x,\phi_{xx})\delta(t-t_s)\,\delta(x-x_s)$$
$$= \sum_{s=1} \hbar_s(\phi,\phi_t,\phi_x,\phi_{xx})cos\theta_s\delta\left(\frac{\theta-\bar{\theta}_s}{\Theta_s}\right)\delta(x-x_s),$$
$$\theta_s = \Theta_s t, \quad \bar{\theta}_s = \Theta_s t_s, \quad \Theta_s = \frac{2\pi}{t_{s+1}-t_s}$$

This allows us to consider differential equations formally

$$\varphi_{tt} - \alpha^2(\varphi_x)^{\nu}\varphi_{xx} = \frac{\mu}{\rho}\left\{\begin{array}{l} f(\varphi_t,\varphi_x,\varphi_{xx}) + \\ \sum_{s=1}\hbar_s(\varphi,\varphi_t,\varphi_x,\varphi_{xx})\cos\theta_s\delta\left(\frac{\theta_s-\bar{\theta}_s}{\Theta_s}\right)\delta(x-x_s)\end{array}\right\} \tag{6}$$

non-autonomous, and the non-autonomous part is proportional to the small parameter.

Many works [21], [34] show that in a non-resonant case, a small periodic perturbation of the non-autonomous type for the first approximation does not affect the basic parameters of the dynamic process.

Therefore, only the case of periodic action of impulse perturbation, i.e.,$t_{s+1} - t_s = \tau = \text{const}, \ \ s = 1, 2, \ldots$ will be considered below. In this case, the differential equations (6) can be written as

$$\varphi_{tt} - \alpha^2(\varphi_x)^{\nu}\varphi_{xx} = \frac{\mu}{\rho}\left\{\begin{array}{l} f(\varphi_t,\varphi_x,\varphi_{xx}) + \\ +\sum_{s=1}\hbar_s(\varphi,\varphi_t,\varphi_x,\varphi_{xx})\cos\theta\delta\left(\frac{\theta-2\pi(s-1)}{\tau}\right) \\ \times\delta(x-x_s)\end{array}\right\} \tag{7}$$

IV. METHODOLOGY OF ANALYTICAL STUDY OF THE INFLUENCE OF IMPULSE MOMENTS ON TORSIONAL OSCILLATIONS OF THE FLEXIBLE BODIES

A. NONRESONANT CASE

Using the general ideas of asymptotic methods of nonlinear mechanics [42], [43] adapted for boundary value problems for equation (1) in [43], the first approximation of the asymptotic solution of equation (7) in the form close to k-form of dynamic equilibrium is presented as

$$f(x,t) = a_k(t)F_k(x)\,ca(n+1,1,y_k) + mU_{1k}(a_k,x,y_k,\theta), \ \ y_k(t) = w_k(a_k)t + J_k \tag{8}$$

where $a_k(t)$, $\psi_k(t)$ respectively, are the amplitude and phase of a flexible body's single-frequency dynamic process, a system of functions that describes the forms of oscillations of undisturbed motion and its frequency. They are expressed for boundary conditions (3) - (5) through periodic Ateb-functions as

$$\{\Phi_k(x)\} = \left\{\begin{array}{l} sa\left(1,\frac{1}{\nu+1},\frac{k}{l}\Pi_x x\right), \\ ca\left(1,\frac{1}{\nu+1}\frac{k}{l}\Pi_x x\right), \\ sa\left(1,\frac{1}{\nu+1},\frac{2k+1}{2l}\Pi_x x\right)\end{array}\right\} \tag{9}$$

$$\omega_k(a_k) = \left\{\begin{array}{l} \alpha\left(\frac{k\Pi_x}{l}\right)^{1+\frac{\nu}{2}} a_k^{\frac{\nu}{2}}, \\ \alpha\left(\frac{k\Pi_x}{l}\right)^{1+\frac{\nu}{2}} a_k^{\frac{\nu}{2}}, \\ \alpha\left(\frac{(2k+1)\Pi_x}{2l}\right)^{1+\frac{\nu}{2}} a_k^{\frac{\nu}{2}},\end{array}\right. \tag{10}$$

$\Pi_x = \sqrt{\pi}\Gamma\left(\frac{\nu+1}{\nu+2}\right)\left(\Gamma\left(\frac{1}{2}+\frac{\nu+1}{\nu+2}\right)\right)^{-1}$ is a half-period of used Ateb-functions, which describes the forms of oscillations.

As for the function $U_{1k}(a_k,x,\psi_k,\theta)$, which considers the influence of small impulse forces and dissipative forces, it must be periodic by arguments ψ_k and θ, and by an argument ψ_k with a period, which is equal $2\Pi = 2\sqrt{\pi}\Gamma\left(\frac{1}{\nu+2}\right)\left(\Gamma\left(\frac{1}{2}+\frac{1}{\nu+2}\right)\right)^{-1}$, and by arguments $\theta\_2\pi$ is periodic. In addition, it should not contain k modes of natural oscillations of undisturbed motion. The later is equivalent to the following

$$\int_0^{2\Pi_\psi} U_{1k}(a_k,x,\psi_k,\theta)\left\{\begin{array}{l} ca(\nu+1,1,\psi_k) \\ sa(1,\nu+1,\psi_k)\end{array}\right\}d\psi_k = 0 \tag{11}$$

The physical interpretation imposed on the function is the following: the amplitude of $k$is a mode of the dynamic process is accepted as the amplitude of single-frequency torsional oscillations of the flexible body.

In addition, this function must satisfy the boundary conditions that are consistent with (3) - (5), i.e.

$$\begin{aligned} U_{1k}(a_k, x, \psi_k, \theta)_{|x=0} &= 0, \\ U_{1k}(a_k, x, \psi_k, \theta)_{|x=l} &= 0 \end{aligned} \tag{12}$$

$$\begin{aligned} \left.\frac{U_{1k}(a_k, x, \psi_k, \theta)}{\partial x}\right|_{x=0} &= 0, \\ \left.\frac{U_{1k}(a_k, x, \psi_k, \theta)}{\partial x}\right|_{x=0\,|x=l} &= 0 \end{aligned} \tag{13}$$

$$\begin{aligned} U_{1k}(a_k, x, \psi_k, \theta)_{|x=0} &= 0, \\ \left.\frac{U_{1k}(a_k, x, \psi_k, \theta)}{\partial x}\right|_{x=0\,|x=l} &= 0 \end{aligned} \tag{14}$$

In figures 5a) and 5b) are presented, respectively, dependencies (10) for the boundary conditions (3) or (4) change of the natural frequency of the oscillations due to amplitude and nonlinearity parameter, and in figures 5c) and 5d) are the natural frequency dependence of oscillations on the parameter α and the amplitude of oscillations at fixed values of the nonlinearity parameter. As for the law of change of the frequency of natural nonlinear oscillations under boundary conditions (5), the qualitative picture of their change does not change. The quantitative one takes on slightly different values due to a different parameter value at the coefficient α.

The presented graphical dependences show:

- for more significant values of nonlinearity parameter ν (for all other invariant characteristics of flexible body), the natural frequency of torsional oscillations is lower;
- the rate of decrease of the amplitude depending on the parameter ν is more significant for its smaller values;
- for greater values of the amplitude of the oscillations in case $-1 < \nu < 0$ the natural frequency decreases, and in the case $\nu > 0$ is an increase;
- with an increased coefficient of stiffness α at $\nu > 0$ natural frequency increases (for all other constant parameters of the flexible body), it decreases.

If we substitute the equation (7), in place of the unknown function $\phi(x, t)$ and its derivatives, the correlations arising from the representation of the asymptotic solution in the form (8), then after equalizing the coefficients in the right and left parts of the obtained expression at small parameter μ we obtain linear differential equation which connects the desired functions $a_k(t)$ and $\psi_k(t)$, more precisely their time derivatives $\frac{da_k}{dt}$, $\frac{d\vartheta_k}{dt}$ function $U_{1k}(a_k, x, \psi_k, \theta)$ and operation moments as

$$\begin{aligned} &\frac{\partial^2 U_{1k}}{\partial \psi_k^2}\omega_k^2(a_k) + \frac{\partial^2 U_{1k}}{\partial \theta^2}\mu^2 + 2\frac{\partial^2 U_{1k}}{\partial\theta\partial\psi_k}\mu\omega_k(a_k) \\ &\quad - \alpha^2\left(a_k ca(\nu+1, 1, \psi_k)\right)^{\nu} \\ &\quad \times \left\{ \begin{array}{l} \left(\frac{d\Phi_k(x)}{dx}\right)^{\nu}\frac{\partial^2 U_{1k}}{\partial x^2} + \\ \nu\left(\frac{d\Phi_k(x)}{dx}\right)^{\nu-1}\frac{d^2\Phi_k(x)}{dx^2}\frac{\partial U_{1k}}{\partial x} \end{array} \right\} \\ &= \frac{2}{(\nu+2)}\left[\frac{da_k}{dt}\left(\omega_k(a_k) + a_k\frac{d\omega_k}{da_k}\right) sa\,(1, \nu+1, \psi_k)\right. \\ &\quad \left. + a_k\omega_k(a_k)\frac{d\vartheta}{dt}ca^{\nu+1}(\nu+1, 1, \psi_k)\right]\Phi_k(x) \\ &\quad + \rho^{-1}F_{1k}(a_k, x, \psi_k) \\ &\quad + \rho^{-1}\sum_{s=1} G_{1s}(a_k, x, \psi_k)\cos\theta\,\delta\left(\frac{\theta}{\tau} - \frac{2(s-1)\pi}{\tau}\right) \\ &\quad \times \delta(x - x_s) \end{aligned} \tag{15}$$

where $F_{1k}(a_k, x, \psi_k)$ and $G_{1s}(a_k, x, \psi_k)$ correspond to the values of functions $f(\phi, \phi_t, \phi_x, \phi_{xx})$and $g_s(\phi, \phi_t, \phi_x, \phi_{xx})$ provided that they $\phi(x, t)$ and their derivatives are determined according to the principal value of a specified function $\phi(x, t)$, i.e.$\phi(x, t) = a_k\Phi_k(x)\, ca(\nu+1, 1, \psi_k)$.

Thus, the problem for the first asymptotic approximation is to find such correlations for the unknown functions $a_k(t)$ and $\psi_k(t)$. The functions $a_k(t)$ and $\psi_k(t)$ with (8) and (15) satisfy the initial equation (7) with the considered accuracy. Solve this problem to some extent allows the properties of completeness and orthonormality of the system of functions that describe the shape of oscillations, i.e.

$$\int_0^l \Phi_p(x)\,\Phi_q(x)\,dx = \begin{cases} 0 & where\ \mathrm{p} \neq \mathrm{q} \\ \Delta & where\ p = q \end{cases} \tag{16}$$

and $\Delta = \frac{\nu+2}{3\nu+4}l$ for the boundary conditions (12) and (13) and $\Delta = \frac{\nu+2}{3\nu+4}\frac{l}{2}$ is a for boundary conditions (14). The specified property of the system of functions $\{\Phi(x)\}$ allows the delta function on the linear variable to be presented in the form

$$\delta(x - x_s) = \frac{1}{\Delta}\sum_{j=1}\Phi_j(x_s)\,\Phi_j(x) \tag{17}$$

that is, to avoid sampling the basic equation by a linear variable; and unknown functions $U_{1k}(a_k, x, \psi_k, \theta)$ in the asymptotic representation of a single-frequency solution are presented as

$$U_{1k}(a_k, x, \psi_k, \theta) = \sum_{\substack{m=1 \\ m\neq k}}\Phi_m(x)\,U_{1km}(a_k, \psi_k, \theta) \tag{18}$$

the boundary conditions, in this case, will be fulfilled automatically.

Its derivatives have the same property if the function $U_{1k}(a_k, x, \psi_k, \theta)$ is 2Π periodic by argument ψ_k and does not contain the first harmonic schedule ψ_k.

Therefore, by substituting the correlations equations (17) and (18) into the differential equation (15), at the same time using the properties of the system of functions $\{\Phi(x)\}$, simple differential equations which describe the change in the amplitude of time and phase of the single-frequency process of torsional oscillations of the flexible body can be found from the obtained expression, as well as equations in partial derivatives which connect unknown coefficients $U_{1km}(a_k, \psi_k, \theta)$ of the decomposition of the function

$U_{1k}(a_k, x, \psi_k, \theta)$ as

$$\frac{da_k}{dt} = -\frac{\mu sa\,(1, \nu+1,\ \psi_k)}{2\omega_k(a_k)\,\rho\Delta} \times \int_0^l \left\{ \begin{array}{l} F_{1k}(a_k, x, \psi_k) + \\ +\sum\limits_{s=1} \begin{array}{c} G_{1s}(a_k, x, \psi_k) \\ \cos\theta\delta(\frac{\theta}{\tau} - \frac{2(s-1)\pi}{\tau}) \end{array} \\ \sum\limits_{j=1} \Phi_j(x_s)\,\Phi_j(x) \end{array} \right\} \Phi_k(x)dx$$

$$\frac{d\psi_k}{dt} = \omega_k(a_k) - \frac{\mu(\nu+2)\,ca^{\nu+1}\,(\nu+1,\ 1,\ \psi_k)}{2a_k\omega_k(a_k)\,\rho\Delta} \times \int_0^l \left\{ \begin{array}{l} F_{1k}(a, x, \psi) + \\ +\frac{1}{\Delta}\sum\limits_{s=1} \begin{array}{c} G_{1s}(a_k, x, \psi_k) \\ \cos\theta\delta(\frac{\theta}{\tau} - \frac{2(s-1)\pi}{\tau}) \end{array} \\ \sum\limits_{j=1} \Phi_j(x_s)\,\Phi_j(x) \end{array} \right\} \Phi_k(x) \times dx \tag{19}$$

$$\frac{\partial^2 U_{1km}(a_k, \psi_k, \theta)}{\partial\psi_k^2}\omega_k^2(a_{1k}) + \frac{\partial^2 U_{1km(a_k,\psi_k,\theta)}}{\partial\theta^2}\mu^2 + 2\frac{\partial^2 U_{1km}(a_k, \psi_k, \theta)}{\partial\theta\partial\psi_k}\mu\omega_k(a_k) - \bar{\alpha}_{km}^2(a_k, \psi_k)\,U_{1km}(a_k, \psi_k, \theta)$$
$$= \rho^{-1}F_{1km}(a_k, \psi_k) + (\rho\Delta)^{-1}\sum_{s=1} G_{1sm}(a_k, \psi_k)\cos\theta \times \delta(\frac{\theta}{\tau} - \frac{2(s-1)\pi}{\tau})\sum_{j=1}\Phi_j(x_s)\,\Phi_j(x) \tag{20}$$

where $F_{1km}(a_k, \psi_k) = \frac{1}{\Delta}\int\limits_0^l F_{1k}(a_k, x, \psi_k)\,\Phi_m(x)\,dx$

$G_{1sm}(a_k, \psi_k) = \frac{1}{\Delta}\int\limits_0^l \left\{ G_{1s}(a_k, x, \psi_k) \sum\limits_{j=1}\Phi_j(x_s)\,\Phi_j(x) \right\}$ $\Phi_m(x)\,dx$ $\bar{\alpha}_{km}^2(a_k, \psi_k)$ is a known function.

Below we consider only differential equations which describe the laws of change of amplitude and frequency of torsional oscillations of a flexible body, i.e., the system of differential equation (19). Its right-hand sides are 2Π periodic $\psi_k = \omega_k(a_k)\,t + \vartheta$ and $2\Pi$are periodic by the argument $\theta = \frac{2\pi}{\tau}t$. Thus, it is necessary to consider two cases for the considered system: $p\frac{2\Pi}{\omega(a)} \neq q\tau$ (p, q – mutually prime numbers). The first case is called resonant. It occurs when there is a connection between the period of natural $\frac{2\Pi}{\omega(a)}$and pulsed external perturbation in the form of rational correlation. In non-resonant cases, the phases of natural oscillations and pulsed external perturbation are not connected by rational correlation.

As follows from the first equation of the system (19), the amplitude of oscillations is proportional to the small parameter μ. This means that during natural or forced oscillations, the amplitude changes by the value of the order μ. This is the basis for using the averaging apparatus for differential equations (19) for the phases of natural oscillations and reduced pulse perturbation. Thus, in the non-resonant case, the main parameters of torsional oscillations of a nonlinear flexible body are described by the correlations

$$\frac{da_k}{dt} = -\frac{\mu}{8\pi\Pi\omega_k(a_k)\,\rho\Delta} \times \int_0^l\int_0^{2\Pi}\int_0^{2\pi} \left\{ \begin{array}{l} F_{1k}(a_k, x, \psi_k) + \\ +\sum\limits_{s=1} \begin{array}{c} G_{1s}(a_k, x, \psi_k) \\ \cos\theta\delta(\frac{\theta}{\tau} - \frac{2(s-1)\pi}{\tau}) \end{array} \\ \sum\limits_{j=1} \Phi_j(x_s)\,\Phi_j(x) \end{array} \right\} \Phi_k(x)\,sa\,(1, \nu+1,\ \psi_k) \times dxd\psi_k d\theta,$$

$$\frac{d\psi_k}{dt} = \omega_k(a_k) - \frac{\mu(\nu+2)}{16\pi\Pi a_k\omega_k(a_k)\,\rho\Delta} \times \int_0^l\int_0^{2\Pi}\int_0^{2\pi} \left\{ \begin{array}{l} F_{1k}(a_k, x, \psi_k) + \\ +\sum\limits_{s=1} \begin{array}{c} G_{1s}(a_k, x, \psi_k) \\ \cos\theta\delta(\frac{\theta}{\tau} - \frac{2(s-1)\pi}{\tau}) \end{array} \\ \sum\limits_{j=1} \Phi_j(x_s)\,\Phi_j(x) \end{array} \right\} \Phi_k(x)\,ca^{\nu+1}\,(\nu+1,\ 1,\ \psi_k) \times dxd\psi_k d\theta \tag{21}$$

If the properties of $\delta$is used functions, the system of differential equations (21) can be somewhat simplified as

$$\frac{da_k}{dt} = -\frac{\mu}{4\pi\Pi\omega_k(a_k)\,\rho\Delta} \times \int_0^l\int_0^{2\Pi} \left\{ \begin{array}{l} F_{1k}(a_k, x, \psi_k) + \\ +\sum\limits_{s=1} \begin{array}{c} G_{1s}(a_k, x, \psi_k) \\ \cos(2(s-1)\pi) \end{array} \\ \sum\limits_{j=1} \Phi_j(x_s)\,\Phi_j(x) \end{array} \right\} \times \Phi_k(x)\,sa\,(1, \nu+1,\ \psi_k)\,dxd\psi_k$$

$$\frac{d\psi_k}{dt} = \omega_k(a_k) - \frac{\mu(\nu+2)}{8\pi\Pi a_k\omega_k(a_k)\,\rho\Delta} \times \int_0^l\int_0^{2\Pi} \left\{ \begin{array}{l} F_{1k}(a_k, x, \psi_k) + \\ +\sum\limits_{s=1} \begin{array}{c} G_{1s}(a_k, x, \psi_k) \\ \cos(2(s-1)\pi) \end{array} \\ \sum\limits_{j=1} \Phi_j(x_s)\,\Phi_j(x) \end{array} \right\} \times \Phi_k(x)\,ca^{\nu+1}\,(\nu+1,\ 1,\ \psi_k)\,dxd\psi_k \tag{22}$$

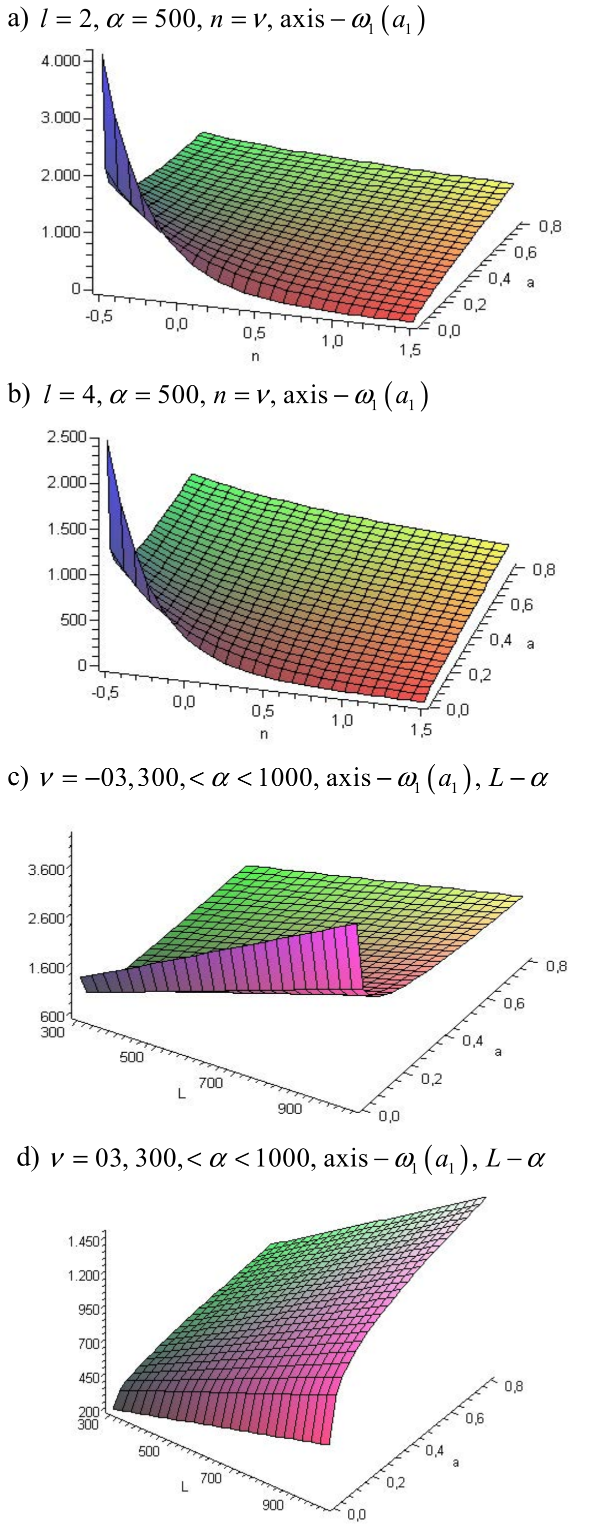

FIGURE 5. Dependence of natural frequency of torsional oscillations of the nonlinear flexible body on the amplitude of oscillations and other characteristics of the body.

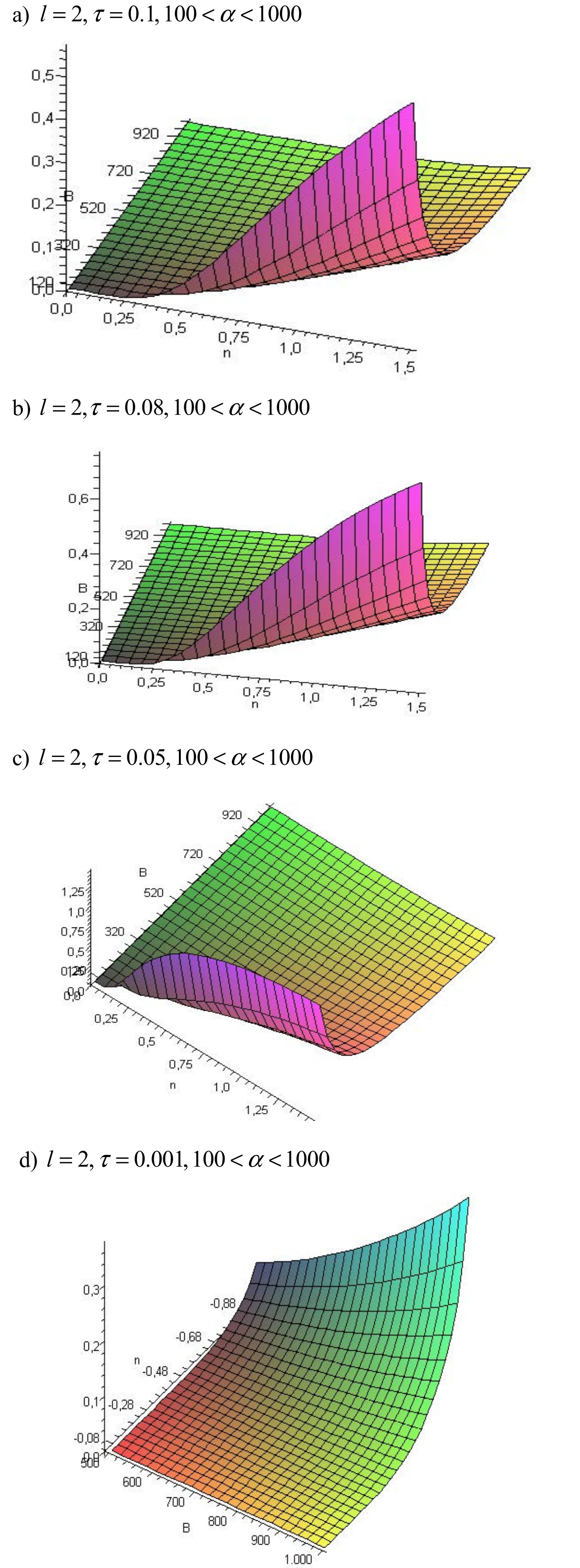

FIGURE 6. The dependence of the amplitude at which there is a resonance a^* on the parameters of the system. α is a vert axis a^*.

B. THE METHODS OF RESEARCH OF NONLINEAR RESONANT TORSIONAL OSCILLATIONS

The resonant case is much more challenging to study and, at the same time, more important from the practical side. It differs significantly from the resonant case of quasilinear oscillations of flexible bodies. The frequency of natural

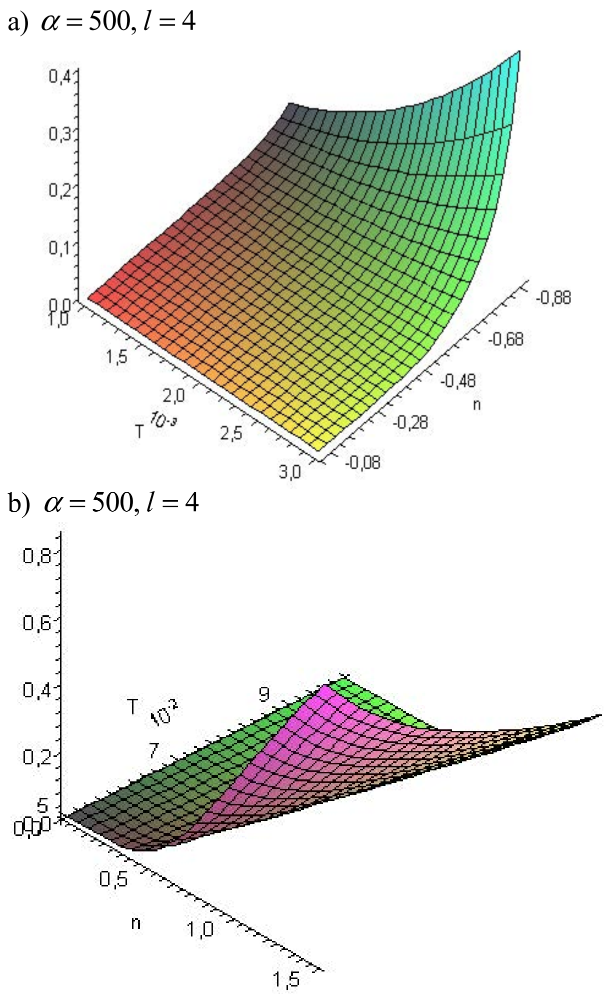

FIGURE 7. Dependence of the resonant amplitude on the nonlinearity parameter and period of pulsed perturbation (n -ν, T – τ vert axisa^*).

torsional oscillations of a nonlinear flexible body depends on the amplitude of oscillations. Thus, the resonance in the considered flexible body takes place under the condition that the amplitude of the single-frequency process approaches the value

$$a^* = \left(\frac{2p\Pi}{\alpha q\tau}\right)^{\frac{2}{\nu}} \left(\frac{l}{\lambda \Pi_x}\right)^{1+\frac{2}{\nu}} \tag{23}$$

Below Fig. 6 and Fig. 7 show the amplitude dependence at which is a resonance for some parameters of the flexible body.

The above graphical dependences show that in the case when the value of the nonlinearity parameter $\nu > 0$ for smaller values of the parameter of stiffness, α the resonance amplitude takes larger values, and the rate of increase of the resonance amplitude is more significant for more extensive parameters ν, while for larger values of the pulse perturbation period resonance amplitude is more minor. For this case $-1 < \nu < 0$, a more considerable parameter value α responds to a more significant resonance amplitude value, and the resonance amplitude's growth rate increases significantly when the parameter νapproaches -1. In this case, the immense value of the pulse perturbation period responds to larger resonance amplitude. As to the passing of resonance of the proposed flexible body, it, as for all nonlinear systems, depends significantly on the phase difference of natural oscillations and periodic perturbation is $\gamma = q\frac{\pi}{\Pi}\psi - p\theta \to \psi = \frac{\Pi}{q\pi}(p\theta + \gamma)$ is entered the specified parameter into the system of differential equations (21) and used the procedure of averaging by the phase of external pulse perturbation. We obtain for the case of main resonance $p = q = 1$.

$$\begin{aligned}
\frac{da_k}{dt} &= -\frac{\mu}{4\pi\Pi\omega_k(a_k)\rho\Delta} \\
&\quad \times \left\{ \int_0^l \int_0^{2\Pi} \begin{bmatrix} F_{1k}(a_k, x, \psi_k) \\ \Phi_k(x)\, sa(1, \nu+1, \ \psi_k)\, dx d\psi_k \end{bmatrix} \right. \\
&\qquad \frac{1}{\Delta} \int_0^l \int_0^{2\pi} \sum_{s=1} G_{1s}\left(a_k, x, \frac{\Pi}{\pi}(\theta+\gamma)\right) \\
&\quad - \times \cos\theta\, \delta\left(\frac{\theta}{\mu} - \frac{2(s-1)\pi}{\mu}\right. \\
&\qquad \sum_{j=1} \Phi_j(x_s)\, \Phi_j(x)\, sa\left(1, \nu+1, \frac{\Pi}{\pi}(\theta+\gamma)\right) \\
&\qquad \left. \times\, dx d\theta \vphantom{\int_0^l} \right\} \\
\frac{d\gamma}{dt} &= \frac{\Pi}{\pi}\omega_k(a_k) - \frac{2\pi}{\tau} \\
&\quad - \frac{\mu(\nu+2)}{4\pi\Pi a_k \omega_k(a_k)\rho\Delta} \\
&\quad \times \left\{ \int_0^l \int_0^{2\Pi} \begin{bmatrix} F_{1k}(a_k, x, \psi_k) \\ \Phi_k(x)\, ca^{\nu+1}(\nu+1, \ 1\psi_k)\, dx d\psi_k \end{bmatrix} \right. \\
&\qquad \int_0^l \int_0^{2\pi} \sum_{s=1} G_{1s}\left(a_k, x, \frac{\Pi}{\pi}(\theta+\gamma)\right) \\
&\quad + \frac{1}{\Delta} \times \cos\theta\, \delta\left(\frac{\theta}{\mu} - \frac{2(s-1)\pi}{\mu}\right. \\
&\qquad \sum_{j=1} \Phi_j(x_s)\, \Phi_j(x)\, ca^{\nu+1} \\
&\qquad \left. \times \left(1, \nu+1, \frac{\Pi}{\pi}(\theta+\gamma)\right) dx d\theta \right\}
\end{aligned} \tag{24}$$

Since the resonance phenomenon occurs when the amplitude of oscillations approaches the value a_k^* during the numerical simulation of the obtained system of differential equations (24), the following approaches are considered:

- for the resonance case, the value $\frac{\Pi}{\pi}\omega_k(a_k) - \frac{2\pi}{\tau}$ in the second formula of dependences (24) can be replaced by the following: $\frac{d\omega_k(a_k)}{da_k}\big|_{a_k=a_k^*}\left(a_k - a_k^*\right)$ as it should be small;
- as for some reasons, it is known, for example, that the frequency (period) of pulse perturbation is a slowly varying function of time ($\frac{2\pi}{\tau} = \frac{\Pi}{\pi}\omega_k\left(a_k^*\right) + \varepsilon\eta(t)$, $\eta(t)$ is a known function). The value $\frac{\Pi}{\pi}\omega_k(a_k) - \frac{2\pi}{\tau}$in the

second equation of correspondence (24) can be replasesd by $\varepsilon\eta(t)$.

Notes:

1) The paper considers that the studied body has a limited size, so the amplitude and frequency change only with time. The long systems [43] can be the subject of separate studies.
2) As for the initial conditions, they must ensure the existence of oscillating motion in an undisturbed system.
3) The initial value of the amplitude for a system of differential equations (24) in both of the above cases should be close to a_k^*.
4) The boundary value problems considered in this paper also describe the relative torsional oscillations of a flexible body under the action of a system of impulse moments, provided that the portable angular velocity of rotation of the body is a constant value.

C. RESONANCE RELATIVE TORSIONAL OSCILLATIONS OF THE NONLINEAR FLEXIBLE BODY ROTATE AT A CONSTANT ANGULAR VELOCITY UNDER THE ACTION OF A PERIODIC SYSTEM OF IMPULSE MOMENTS

A crucial case of the dynamics of the flexible body specified in the above section of the notes is when the pulsing momentum acts on the flexible body at a fixed point and is repeated during the complete rotation of the body N times at regular intervals.

If the following conditions are satisfied:

- the force of resistance is proportional to the speed of the relative torsion angle;
- flexible properties are described by dependence $\sigma = k\varepsilon^{\nu+1} + \mu\beta\varepsilon^3$

then the differential equation of relative flexible torsional vibrations of such a body is represented as follows

$$\phi_{tt} - \alpha^2 (\phi_x)^\nu \phi_{xx} = -\frac{\mu}{\rho}\left\{\begin{array}{l}\beta_1\phi_t + \beta_2(\phi_x)^2\phi_{xx} \\ -M\delta(x-x_0)\sum_{s=1}\delta\left(t - s\frac{2\pi}{N\Omega}\right)\end{array}\right\} \tag{25}$$

where $\bar{\beta}_1$, $\bar{\beta}_2$ are the known constants, Ω is an angular velocity of rotation of the body, $\frac{2\pi}{N\Omega}$ is the period of action of pulse momentum, the value of which is equal μM, x_0 is the point of application of the moment. In such case, non-resonant correlations (22) for the case of boundary conditions (2) can be represented as

$$\frac{da_k}{dt} = -\mu\bar{\beta}_1 a_k + \frac{\mu M}{4\pi\Pi\omega_k(a_k)\rho\Delta^2} \times \sum_{s=1}\int_0^l\int_0^{2\Pi}\int_0^{2\pi}\left\{\begin{array}{l}\cos\theta\left(\delta\left(\frac{N\theta\Omega}{2\pi} - s\frac{2\pi}{N\Omega}\right)\right) \\ \sum_{j=1}\Phi_j(x_0)\Phi_j(x)\, sa \\ \times(1,\nu+1,\ \psi_k)\, dxd\,\psi_k d\theta\end{array}\right\}$$

$$\frac{d\psi_k}{dt} = \omega_k(a_k) + \bar{\beta}_2 a_k^{2-\frac{\nu}{2}} + \frac{\mu(\nu+2)M}{8\pi\Pi a_k\omega_k(a_k)\rho\Delta^2} \times \sum_{s=1}\int_0^l\int_0^{2\Pi}\int_0^{2\pi}\left\{\begin{array}{l}\cos\theta\left(\delta\left(\frac{N\theta\Omega}{2\pi} - s\frac{2\pi}{N\Omega}\right)\right) \\ \sum_{j=1}\Phi_j(x_0)\Phi_j(x)\, ca^{\nu+1} \\ \times(\nu+1,\ 1,\ \psi_k)\, dxd\,\psi_k d\theta\end{array}\right\} \tag{26}$$

where $\bar{\beta}_1$, $\bar{\beta}_2$ are the known constants.

From the properties of periodic Ateb-functions [24]–[29], the independence of the phases of natural and forced (pulse) oscillations, it follows that the integrals in the right parts of the dependences (26) are zero because $\int_0^{2\Pi} sa\,(1,\nu+1,\ \psi_k)\,d\psi_k = \int_0^{2\Pi} ca^{\nu+1}\,(1,\nu+1,\ \psi_k)\,d\psi_k = 0$, and therefore non-resonant torsional oscillations, in this case, are described by differential equations

$$\frac{da_k}{dt} = -\mu\bar{\beta}_1 a_k,\quad \frac{d\psi_k}{dt} = \omega_k(a_k) + \bar{\beta}_2 a,^{2-\frac{\nu}{2}} \tag{27}$$

Thus, relation (27) confirms that periodic impulse perturbation, which does not coincide with the period of natural torsional oscillations, determines the dynamic process only by dissipative and nonlinear flexible forces. As for the resonant oscillations of the above case of torsional oscillations of a flexible body, the differential equations that describe the principal resonance in the first mode of oscillations under boundary conditions (3) take the form

$$\frac{da}{dt} = -\mu\bar{\beta}_1 a_k - \frac{\mu M}{4\pi\Pi\omega(a)\rho\Delta^2} sa\left(1,\nu+1,\frac{\Pi}{\pi}\gamma\right) \times \sum_{j=1} sa\left(1,\frac{1}{\nu+1},\frac{j}{l}\Pi_x x_0\right),$$

$$\frac{d\gamma}{dt} = \frac{d\omega(a)}{da}_{|a=a*}(a-a^*) + \bar{\beta}_2 a,^{2-\frac{\nu}{2}} + \frac{\mu(\nu+2)M}{8\pi\Pi a\omega(a)\rho\Delta^2} ca^{\nu+1}\left(\nu+1,1,\frac{\Pi}{\pi}\gamma\right) \times \sum_{j=1} sa\left(1,\frac{1}{\nu+1},\frac{j}{l}\Pi_x x_0\right) \tag{28}$$

Below, in Fig. 8, following the differential equations (28) presented for the resonance case, the laws of change of the amplitude of oscillations during the transition through the main resonance $l = 1$, $\nu = 0.22$green;$\nu = 0.44$ *red*; $\nu = -0.22$ *blue* are for all below

where

a) $x_0 = 1,\ l = 2, \alpha = 1 \cdot 10^3,$ $-curve1, -curve2, -curve3;$

b) $x_0 = 0.8,\ l = 2, \alpha = 1 \cdot 10^3,\ \nu = 0.1,$ $-curve1, -curve2, -curve3;$

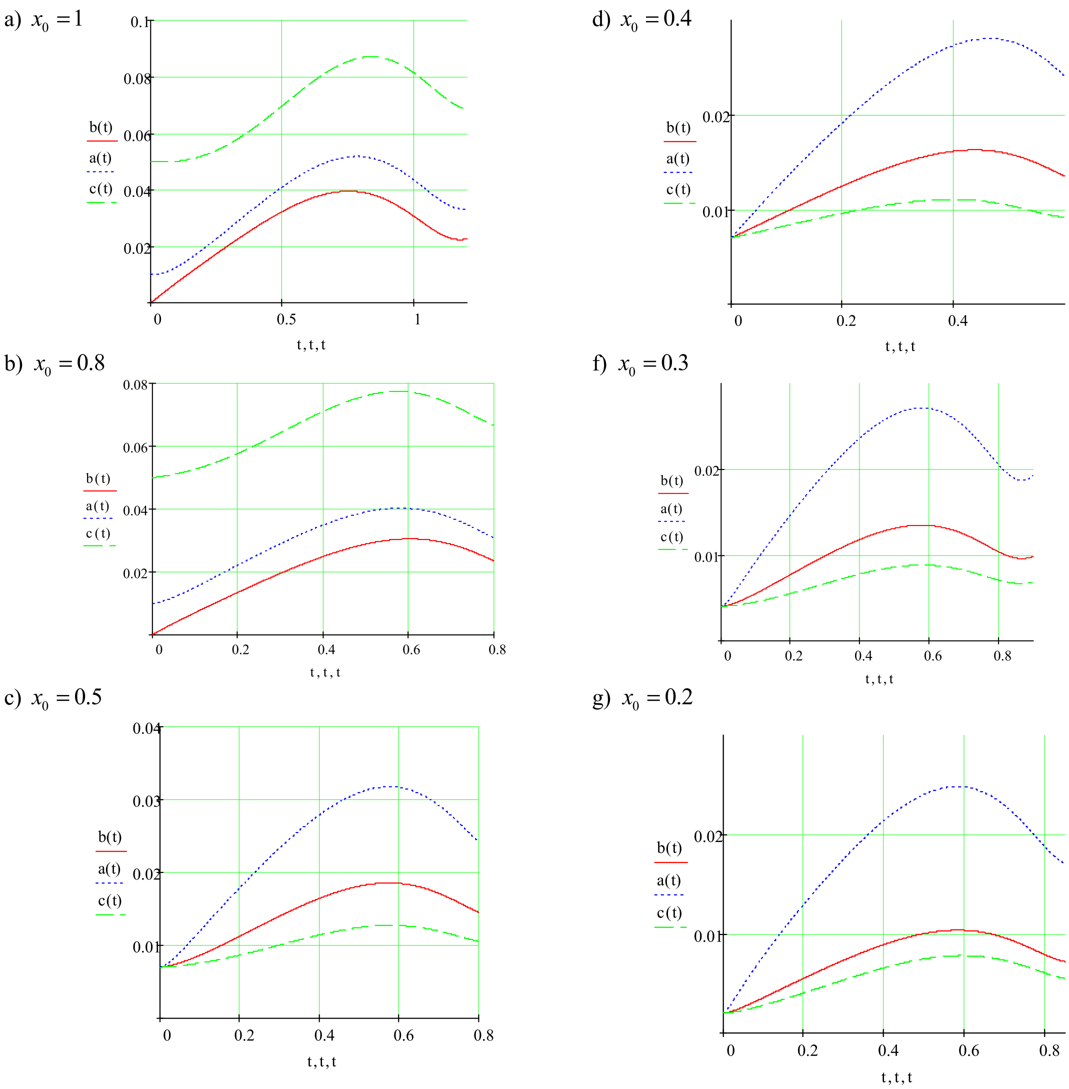

FIGURE 8. **The laws of the change of the amplitude of torsional oscillations at transition through the primary resonance at various points of application of pulse perturbation.**

c) $x_0 = 0.5,\ l = 1, \alpha = 1 \cdot 10^3,\ \nu = 0.22,$
$-curve1, -curve2, -curve3;$

d) $x_0 = 0.4,\ l = 1, \alpha = 1 \cdot 10^3,\ \nu = 0.22,$
$-curve1, -curve2, -curve3;$

f) $x_0 = 0.3,\ l = 1, \alpha = 1 \cdot 10^3,\ \nu = 0.22,$
$-curve1, -curve2,\ \nu = -0.22,\ -curve3;$

g) $x_0 = 0.2,\ l = 1, \alpha = 1 \cdot 10^3,\ \nu = 0.22,$
$-curve1,\ \nu = 0.44, -curve2,\ \nu = -0.22,\ -curve3.$

Theoretical results and graphical dependences based on their resonant torsional oscillations of a nonlinear flexible body under the action of a periodic system of pulse momentum applied to one geometric cross-section of the body under boundary conditions (3) are shown in particular that the amplitude of resonance transition is larger:

- for larger values of nonlinearity parameter ν (at all other invariant basic body parameters and $\nu > 0$);
- in the action case, the momentum closer to the middle of the body for the boundary conditions (2), (3), and for the boundary conditions (4) is a, on the contrary, less.

V. CONCLUSION

The research method of the influence of impulse moments on torsional oscillations of strongly nonlinear flexible bodies is developed in work. Analysis of the obtained theoretical and constructed graphical results shows that their periodic actions on the body of impulse moments in the body occur non-resonant and resonant oscillations. Resonant oscillations occur when the amplitude of torsional oscillations approaches a specific value is the amplitude of resonance. The amplitude of resonance in a case when the parameter of nonlinearity is $\nu > 0$ for:

- smaller values of the parameter of stiffness α take the more considerable value, and its growth rate is more significant for larger values of the parameter
- for larger values of the period of pulsed perturbation, the resonance amplitude is more minor.

In this case, $-1 < \nu < 0$ a more considerable value α corresponds to the immense amplitude resonance value, and the amplitude's growth rate increases significantly while approaching ν to -1. To avoid simultaneous resonance oscillations, we suppose the nonlinear flexible body is only under the periodic system of impulse moments forces and dissipative forces. In that case, the initial perturbation amplitude must be slightly smaller than the resonance amplitude. Otherwise, a phenomenon similar to beating in linear systems will occur. As for the amplitude of the passage through the resonance under the action on the body of the periodic system of pulse moments of the same magnitude in the same cross-section of the body, the amplitude of the transition through the resonance is greater:

- for larger values of the nonlinearity parameter ν (for all other invariant basic body parameters and $\nu > 0$);
- in case of impulse moment closer to the middle of the body and for boundary conditions (3).

The reliability of the work results is confirmed by the fact that in the extreme case of them ($\nu = 0$, $\hbar_s(\phi, \phi_t, \phi_x, \phi_{xx}) = 0$) we have known from the literature [21], [44]. Depending on the system's initial conditions, the question of the flow and the conditions of a single-frequency process can be the subject of separate studies.

Practical application of the obtained results. The results obtained in this work can be the basis for the choice of operating parameters of machine elements of mechanisms that are exposed to instantaneous moments to avoid resonant phenomena, in particular, rotors of multistage compressors of gas turbine engines [44]. Research dynamic processes in machines with flexible high-speed rotors to justify the choice of the most stable frequency of its rotation.

This article's main result presents a method related to creating new materials. Nonlinear relations describe the flexible properties of these materials. The authors of the article found that the use of numerical simulation for the analysis of these materials is ineffective. This is the study of resonances, which are essential phenomena in practice. It is not easy to describe the conditions for the existence of resonances and their peculiarities with the help of numerical simulation due to the non-isochronous nature of the undisturbed process.

The results obtained in this paper are in demand for solving the problems of analysis of dynamic processes in strongly nonlinear systems with distributed parameters and for the synthesis problems, i.e., the choice of their parameters that make resonant processes impossible. For example, in the case of longitudinal oscillations of flexible bodies under the periodic action of pulse forces, these are turbine rotors and elements of protective structures under the action of shock loads. Also, the main idea of the work can be generalized and used to solve problems in case of disturbed boundary conditions.

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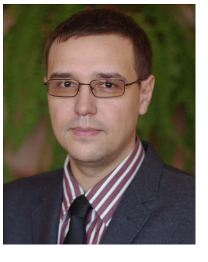

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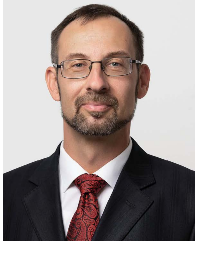

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