IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received 10 June 2022, accepted 8 July 2022, date of publication 13 July 2022, date of current version 19 July 2022.

Digital Object Identifier 10.1109/ACCESS.2022.3190312

RESEARCH ARTICLE

Exponential Stability Analysis of Stochastic Semi-Linear Systems With Lèvy Noise

K. MATHIYALAGAN[1], R. RAGUL[1], G. SANGEETHA[2], AND YONG-KI MA[3]

[1]Department of Mathematics, Bharathiar University, Coimbatore 641046, India
[2]Department of Mathematics, KGISL Institute of Technology, Coimbatore 641035, India
[3]Department of Applied Mathematics, Kongju National University, Chungcheongnam-do 32588, South Korea

Corresponding authors: K. Mathiyalagan (kmathimath@gmail.com) and Yong-Ki Ma (ykma@kongju.ac.kr)

The work of K. Mathiyalagan was supported by the University Grants Commission (UGC), New Delhi, India through the BSR Start-up Research Grant under Grant F.30-410/2018(BSR). The work of Yong-Ki Ma was supported by the National Research Foundation of Korea (NRF) Grant through the Korea Government (MSIT) under Grant 2021R1F1A1048937.

ABSTRACT The exponential stability of semi-linear stochastic partial differential equations (SPDEs) involving Lèvy type noise is investigated in this paper. By constructing an appropriate Lyapunov function, a new set of sufficient conditions are established in terms of linear matrix inequalities (LMIs) which ensures the mean-square exponentially stability (MSES) of given system with Neumann boundary conditions. Then the H_∞ performance index is introduced to eliminate the disturbance which occurs in the considered system. The boundary control gain is obtained by solving the LMI conditions using the standard MATLAB software. Finally, a numerical example is provided to demonstrate the usefulness of the proposed methods.

INDEX TERMS Parabolic system, Boundary control, Lèvy noise, H_∞ performance, Lyapunov stability.

I. INTRODUCTION

Partial differential equations (PDEs) are equations which can be used to model the complex phenomenons in nature. Mainly, semi-linear reaction-diffusion PDEs are commonly used to model a variety of real-world phenomenon such as population dynamics and chemical reactions etc., [31], [40]. Many researchers have focused their attention on reaction-diffusion equations due to their wide range of applications, see [1], [11], [12], [17], [25], [33]. Random noise in dynamical systems is caused by external disruptions, measurement errors, and a lack of knowledge of specific parameters. Deterministic systems were extended to stochastic systems in order to express this type of dynamical system. SPDEs helps to describe the dynamics of chemical engineering, ecology, neurophysiology, statistical physics, biology and martial science [3], [10], [16]. In recent years research on SPDEs is an active area with many results and developments, see [19], [20], [37] and references therein.

The associate editor coordinating the review of this manuscript and approving it for publication was Xiaojie Su.

On the other side, H_∞ performance has received huge exposure for its variety of applications and the ability to reject noise disturbances and model parameters. Ju *et al.* [13] addressed the stochastic H_∞ performance for Markov jump systems along with stochastic Lyapunov functional for obtaining the conservativeness. In [32] the authors investigated the H_∞ control for diffusion systems and explained the results using Brownian motion and the Poisson process. Ding and Zhu in [7] were presented the H_∞ based synchronization results for uncertain stochastic systems with time delays. Pan *et al.* [23] investigated the H_∞ boundary control approach for reaction-diffusion equations. Yan *et al.* [39] addressed the reaction-diffusion Brusselator system with NBC.

In practice, external disturbances in modeling may have an effect on the system's performance. As a result, designing an appropriate controller or performance indexes that can accept the effects of disturbances while maintaining stability is a difficult task. Disturbance signals are unexpected inputs that alter the output of control systems and lead to an increase in system error. Also, as a consequence, on establishing the stability conditions, noise or external disturbances must be addressed more effectively. In this connection, there are numerous results addressed for disturbances in the literature. For example, Dus [8] studied the exponential stability of general systems of discretized scalar conservation laws using boundary feedback laws. Wei *et al.* [34] introduced a multiple disturbances for stochastic systems and subsequently designed an observer to estimate the disturbance. Luo and Zhang [21] addressed the exponential stability results for PDEs with uncertainties and used the Lyapunov technique to achieve the sufficient conditions. Wu *et al.* [36] studied MSES analysis for linear stochastic PDEs with Brownian motion and proposed a boundary controller. Exponential stability of stochastic functional differential equations were presented in [22]. Many results were found for SPDE with stochastic process like Brownian motion, White noise, Gaussian motion and so on.

Moreover, most of the stability results are presented for SPDEs induced by Brownian motion. Brownian motion is a stochastic process with continuous paths, hence it could not be used to model stochastic disturbances in real-world systems like neurobiological systems, genetic regulatory models, singular systems and financial systems etc., [6], [14], [30], [41]. These systems are complicated in nature and have discontinuous paths, as a result the Brownian motion stochastic differential equation is falling to deal these issues. In this instance, Lèvy noise is used to cope with the small and large fluctuations occurs in the system, as well as it has the combination of Brownian motion and Poisson process [2], [18], [38]. It is one of the stochastic processes with fixed and independent increments. Stochastic outcomes with Lèvy noise were analyzed for different kind of systems in [9], [26]. Brzezniak *et al.* [5] has been studied the concept of Lèvy noise to derive the strong solutions of stochastic PDEs. Song*et al.* [28] addressed the robust stability analysis for stochastic systems with random jumps. The reaction-diffusion equations with Lèvy noise addressed for stochastic systems in [24] and sufficient conditions are established to analyze the exponential stability results. Recently, the authors in [15] investigated the results for continuous-time stochastic systems with Lèvy noise. Applications of Lèvy noise can be explained clearly through Chua's circuit in [42]. The effects of large fluctuations on reaction-diffusion equations were examined in [4].

Stability analysis of SPDE driven by Lèvy noise still not yet investigated. With this motivation this article attempted to reach the MSES for SPDEs driven by Lèvy noise using Lyapunov stability theory. The main objective of proposed work is to derive the sufficient conditions for the considered reaction-diffusion equations to guarantees the MSES. We presented boundary control for semi-linear SPDEs driven by Lèvy noise. The main contributions of present work are:

- Boundary control is proposed to study the MSES of the considered SPDEs.
- Lèvy process which includes both the Wiener and Poisson jump processes is first time addressed for the considered parabolic system.

To ensure the MSES of the system under investigation, a new set of sufficient LMI conditions is presented. Following that, a H_∞ performance is introduced to deal the disturbance which occurs in the system. Finally, a numerical example is presented to validate the applicability of the proposed model.

II. PROBLEM DESCRIPTION

Consider the following semi-linear SPDE:

$$\begin{aligned} d\wp(\varkappa,t) &= \left\{\frac{\partial^2\wp(\varkappa,t)}{\partial\varkappa^2} + A\wp(\varkappa,t) + g(t,\wp(\varkappa,t))\right\}dt \\ &\quad +\sigma(t,\wp(\varkappa,t))dW(\varkappa,t) + \int_Z \phi(t,\wp(\varkappa,t),z) \\ &\quad \times\tilde{N}(dt,dz), \\ \wp(\varkappa,0) &= \wp_0(\varkappa). \end{aligned} \tag{1}$$

where $t \geq 0$, $\varkappa \in (0,1)$ denote time and space variables, respectively. $\wp(\varkappa,t)$ is the system state, A is known constant matrix, $W(\varkappa,t)$ is a Brownian motion defined on a complete probability space $(\Sigma,\mathcal{F},P)$ adapted to a right continuous filtration $\mathcal{F}_{t\geq 0}$ and $\mathbb{E}\Big[dW(\varkappa,t)\Big] = 0$. Let $z \in (Z,\mathcal{B}(Z))$ be a measurable space. Then $\tilde{N}(dt,dz) = N(dt,dz) - \upsilon(dz)dt$ denoted as the compensated Poisson measure in which $N(dt,dz)$ represents the Poisson random measure with intensity measure $\upsilon(dz)$ for $z \in \mathcal{B}(Z)$. Assume that $W(\cdot)$ and $N(\cdot)$ are independent and the function $\phi(\wp(\varkappa,t),z)$ satisfies $\int_Z\int_0^1 \phi^T(\wp(\varkappa,t),z)\phi(\wp(\varkappa,t),z)d\varkappa\upsilon(dz) < \infty$.

The system (1) is investigated with the NBC

$$\frac{\partial\wp(\varkappa,t)}{\partial\varkappa}\Big|_{\varkappa=0} = 0, \quad \frac{\partial\wp(\varkappa,t)}{\partial\varkappa}\Big|_{\varkappa=1} = u(t), \tag{2}$$

$u(t)$ is the boundary control input given as $u(t) = K\wp(1,t)$, where K is control gain to be designed.

The assumptions listed below are important in getting our main results.

(A1) There exists a constant $m > 0$,

$$\int_0^1 g^T(t,\wp(\varkappa,t))g(t,\wp(\varkappa,t))d\varkappa \leq \int_0^1 m\wp^T(\varkappa,t)\wp(\varkappa,t)d\varkappa. \tag{3}$$

(A2) There exists a constant $c > 0$,

$$\mathrm{tr}(\sigma^T(t,\wp(\varkappa,t))\sigma(t,\wp(\varkappa,t))) \leq c\wp^T(\varkappa,t)\wp(\varkappa,t). \tag{4}$$

(A3) There exists a constant $q > 0$,

$$\int_Z\left(\int_0^1 \phi^T(t,\wp(\varkappa,t),z)\phi(t,\wp(\varkappa,t),z)d\varkappa\right)\upsilon(dz) \leq q\int_0^1 \wp^T(\varkappa,t)\wp(\varkappa,t)d\varkappa. \tag{5}$$

To discuss the existence of solution for system (1) by semigroup theory rewrite the system as time dependent ordinary differential equation. For the purpose following functions are considered as only time dependent, $\wp_t = \wp(\cdot,t)$, $G_t(\wp_t) = g(t,\wp(\cdot,t))$, $\Sigma(\wp_t) = \sigma(t,\wp(\cdot,t))$,

$dW_t = W(\cdot, t)$, $\Phi_t(\wp_t) = \phi(t, \wp(\cdot, t), z)$, then we rewrite the equation (1) as,

$$d\wp_t = [\mathcal{C}\wp_t + A\wp_t + G_t(\wp_t)]dt + \Sigma(\wp_t)dW_t + \int_Z \Phi_t(\wp_t, z)\tilde{N}(dt, dz) \tag{6}$$

where $\mathcal{C} = \frac{\partial^2}{\partial \varkappa^2}$ is linear operator from H^1 into H^{-1} with domain $D(\mathcal{C}) = H_0^1 \cap H^2$. If "$\mathcal{C}$" satisfies the coercivity condition, G_t, Σ_t, satisfy the Lipschitz continuity and boundedness conditions. Then the system (6) has a unique global strong solution $\wp \in L_2([0, T]; H^{-1}) \cap D([0, T]; H)$. For detailed proof see Theorem 3.2 in [27].

Remark 1: It should be emphasized that, Lèvy processes are stochastic processes with independent and stationary increments. Brownian motion, the Poisson process, as well as stable and self-decomposable processes, are all instances of Lèvy processes. Brownian motion is a well-known stochastic process that happens continuously. On the other hand, many practical systems may be influenced by random jump type unexpected interference, such as dramatic stock market fluctuations caused by the global financial crisis, and so on. In such cases, Brownian motion-based systems are unable to match the demands of reality. In order to develop more acceptable findings, Lèvy noise has been introduced into stochastic systems. So the problem addressed in this paper will be more useful in practice.

A. PRELIMINARIES

Here we introduce the some basic definitions and lemmas that are essential for getting the most important results.

Definition 1: System (1) is said to be MSES if there exist a positive constants $\beta > 0$ and $\delta > 0$ such that

$$\mathbb{E}\|\wp(\varkappa, t)\|^2 \leq \rho e^{-\delta t}\mathbb{E}\|\wp_0(\varkappa)\|^2, t \geq 0.$$

for all $\wp_0(\varkappa) \in L_2(0, 1)$.

Lemma 1: [35] Let $\wp \in W^{1,2}([0, L]; \mathbb{R}^n)$ be a vector function with $\wp(0) = 0$ or $\wp(L) = 0$. Then, for matrix $R > 0$, we have the following integral inequality:

$$\int_0^L \wp^T(s)R\wp(s)ds \leq \frac{4L^2}{\pi^2}\int_0^L \left(\frac{d\wp(s)}{ds}\right)^T R\left(\frac{d\wp(s)}{ds}\right).$$

Here, an Itö operator $\mathcal{L}$ defined for further use(see, [43]).

$$\begin{aligned}\mathcal{L}V(\cdot) = &\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \wp}g(t, \wp(\varkappa, t)) + \frac{1}{2}tr\big(\sigma^T(t, \wp(\varkappa, t)) \\ &\times \frac{\partial^2 V}{\partial \wp^2}\sigma(t, \wp(\varkappa, t))\big) + \frac{\partial V}{\partial \wp}\sigma(t, \wp(\varkappa, t))dW(\varkappa, t) \\ &+ \int_Z \Big[V(\wp(\varkappa, t) + \phi(t, \wp(\varkappa, t), z)) - V(\wp(\varkappa, t)) \\ &-\phi(t, \wp(\varkappa, t), z)\frac{\partial V}{\partial \wp}\Big]\upsilon(dz). \end{aligned} \tag{7}$$

The notations that were used in this paper are described below. $\mathbb{R}^n$ is n- dimensional Euclidean space. The superscript 'T' stands for matrix transposition. I stands for identity matrix. The symmetric elements are denoted by asterisk $'*'$. Let $\|\cdot\|$ denotes L_2 norm. $W_\wp^{p,q}$ denotes the Sobolev space of absolutely continuous integrable functions defined over $\wp$. $'\mathbb{E}'$ denotes mathematical expectation.

III. MAIN RESULTS

In this section, first a boundary controller is designed to achieve MSES for the considered semi-linear stochastic parabolic system. Moreover, we use a controller to achieve MSES for a parabolic system and derive a sufficient conditions for ensuring H_∞ performance. The following theorem is presented to ensure that stochastic parabolic system (1) achieves MSES under the boundary controller $u(t)$.

Theorem 1: Consider that the assumptions (3)-(5) holds. For given scalars m, c, q, known constant matrix K, and decay rate $\delta > 0$, there exist a symmetric positive definite matrix P and positive scalars $\bar{\rho}$, $\underline{\rho}$ and ρ, such that the following LMIs hold:

$$\Xi = \begin{bmatrix} \Xi_{11} & \frac{\pi^2}{2}P & P \\ * & \Xi_{22} & 0 \\ * & * & -I \end{bmatrix} < 0, \tag{8}$$

$$\underline{\rho}I \leq P \leq \bar{\rho}I, \tag{9}$$

where $\Xi_{11} = PA + A^TP^T + mI + \bar{\rho}cI + \bar{\rho}qI - \frac{\pi^2}{2}P + \delta P + P^T\delta^T$, $\Xi_{22} = PK + K^TP^T - \frac{\pi^2}{2}P$. Further, for any given initial condition, the decay rate satisfies

$$\mathbb{E}\|\wp(\varkappa, t)\|^2 \leq \rho e^{-\delta t}\mathbb{E}\|\wp_0(\varkappa)\|^2. \tag{10}$$

Then, the system (1) is MSES.

Proof: Consider the Lyapunov functional

$$V(\wp(\varkappa, t)) = \int_0^1 \wp^T(\varkappa, t)P\wp(\varkappa, t)d\varkappa.$$

Now with help of Itö formula, we get that

$$\begin{aligned}\mathcal{L}V = &\int_0^1 \Big(2\wp^T(\varkappa, t)P\Big[\frac{\partial^2\wp(\varkappa, t)}{\partial \varkappa^2} + A\wp(\varkappa, t) + g(t, \wp(\varkappa, t))\Big] \\ &+ \mathrm{tr}(\sigma^T(t, \wp(\varkappa, t))P\sigma(t, \wp(\varkappa, t)))\Big)d\varkappa \\ &+2\int_0^1 \wp^T(\varkappa, t)P\sigma(t, \wp(\varkappa, t))d\varkappa dW(\varkappa, t) \\ &+ \int_Z\int_0^1 \phi^T(t, \wp(\varkappa, t), z)P\phi(t, \wp(\varkappa, t), z)d\varkappa\upsilon(dz).\end{aligned}$$

Using integration by parts with the boundary conditions of system (1) and by taking $\bar{\wp}(\varkappa, t) = \wp(\varkappa, t) - \wp(1, t)$, obviously, we have $\bar{\wp}(1, t) = 0$, and $\frac{\partial \bar{\wp}}{\partial \varkappa} = \frac{\partial \wp}{\partial \varkappa}$. With the help of Lemma 1, we obtain

$$\begin{aligned}\int_0^1 \wp^T(\cdot)P\frac{\partial^2\wp(\cdot)}{\partial \varkappa^2}d\varkappa &= \wp^T(1, t)PK\wp(1, t) - \int_0^1 \left(\frac{\partial \wp}{\partial \varkappa}\right)^T P\left(\frac{\partial \wp}{\partial \varkappa}\right)d\varkappa \\ &= \wp^T(1, t)PK\wp(1, t) - \int_0^1 \left(\frac{\partial \bar{\wp}}{\partial \varkappa}\right)^T P\left(\frac{\partial \bar{\wp}}{\partial \varkappa}\right)d\varkappa\end{aligned}$$

$$
\begin{aligned}
&\leq \wp^T(1,t)PK\wp(1,t) \\
&\quad -\frac{\pi^2}{4}\int_0^1 \bar{\wp}(\varkappa,t)P\bar{\wp}(\varkappa,t)d\varkappa \\
&\leq \wp^T(1,t)PK\wp(1,t) \\
&\quad -\frac{\pi^2}{4}\int_0^1 [\wp(\varkappa,t)-\wp(1,t)]^T \\
&\quad \times P[\wp(\varkappa,t)-\wp(1,t)]d\varkappa. \quad (11)
\end{aligned}
$$

From assumptions (3), (4) and (5), we get that

$$
\int_0^1 \Big[m\wp^T(\varkappa,t)\wp(\varkappa,t) - g^T(\wp(\varkappa,t))g(\wp(\varkappa,t))\Big]d\varkappa \geq 0, \quad (12)
$$

$$
\mathrm{tr}\Big[\sigma^T(t,\wp(\varkappa,t))P\sigma(t,\wp(\varkappa,t))\Big] \leq \bar{\rho}c\wp^T(\varkappa,t)\wp(\varkappa,t), \quad (13)
$$

$$
\begin{aligned}
&\int_Z\int_0^1 \phi^T(t,\wp(\varkappa,t),z)P\phi(t,\wp(\varkappa,t),z)d\varkappa\upsilon(dz) \\
&\qquad \leq \bar{\rho}q\int_0^1 \wp^T(\varkappa,t)\wp(\varkappa,t)d\varkappa. \quad (14)
\end{aligned}
$$

By arranging equations from (11) to (14), we have

$$
\begin{aligned}
\mathcal{L}V \leq &\int_0^1 \wp^T(\varkappa,t)\Big[PA+A^TP^T+mI+\rho cI+\rho qI\Big] \\
&\times\wp(\varkappa,t)d\varkappa + \wp^T(1,t)(PK+K^TP^T)\wp(1,t) \\
&-\frac{\pi^2}{2}\int_0^1 \Big[\wp(\varkappa,t)-\wp(1,t)\Big]^T P\Big[\wp(\varkappa,t) \\
&-\wp(1,t)\Big]d\varkappa - \int_0^1 g^T(t,\wp(\cdot))g(t,\wp(\cdot))d\varkappa \\
&+2\int_0^1 \wp^T(\varkappa,t)P\sigma(t,\wp(\varkappa,t))d\varkappa dW(\varkappa,t) \\
&+\int_0^1 2\wp^T(\varkappa,t)Pg(t,\wp(\varkappa,t)d\varkappa.
\end{aligned}
$$

Then by setting $\chi(\cdot) = \big[\wp^T(\varkappa,t)\ \wp^T(1,t)\ g^T(\wp(\varkappa,t))\big]^T$ and for decay rate δ, we have

$$
\begin{aligned}
\mathcal{L}V(\cdot)+2\delta V(\cdot) \leq &\int_0^1 \chi^T(\cdot)\Xi\chi(\cdot)d\varkappa \\
&+2\int_0^1 \wp^T(\varkappa,t)P\sigma(t,\wp(\varkappa,t))d\varkappa dW(\varkappa,t)
\end{aligned}
$$

Taking the expectations on both sides according to the properties of Itö integral [43], we arrive at

$$
\mathbb{E}\Big[\mathcal{L}V(\cdot)+2\delta V(\cdot)\Big] \leq \mathbb{E}\Big[\int_0^1 \chi^T(\cdot)\Xi\chi(\cdot)d\varkappa\Big].
$$

If LMI (8) holds, we get that

$$
\mathbb{E}\Big[\mathcal{L}V(\cdot)+2\delta V(\cdot)\Big] \leq 0.
$$

Next by using comparison principle, for any arbitrary initial condition $\wp(\varkappa,0)=\wp_0(\varkappa)$, the following inequality holds

$$
\begin{aligned}
\mathbb{E}\Big[V(t,\wp(\varkappa,t))\Big] &\leq e^{-2\delta t}\mathbb{E}\Big[V(0,\wp(\varkappa,0))\Big], \\
&\Rightarrow \underline{\rho}\mathbb{E}\left\|\wp(\varkappa,t)\right\|^2 \leq \bar{\rho}e^{-\delta t}\mathbb{E}\left\|\wp_0(\varkappa)\right\|^2, \\
&\Rightarrow \mathbb{E}\left\|\wp(\varkappa,t)\right\|^2 \leq \rho e^{-\delta t}\mathbb{E}\left\|\wp_0(\varkappa)\right\|^2 \quad (15)
\end{aligned}
$$

where, $\rho = \frac{\bar{\rho}}{\underline{\rho}}$. According to Definition 1 system (1) is MSES. This completes the proof.

Suppose, in Theorem 1, if control gain K is unknown then which affects the linearity of (8). In this case, by letting $\mathcal{K} = PK$, one can preserve the linearity and the corresponding exponential stability results can be presented as follows.

Corollary 1: Consider that the assumptions (3)-(5) holds. For given scalars m, c, q, and decay rate $\delta > 0$, there exist a symmetric positive definite matrix P and positive scalars $\bar{\rho}$, $\underline{\rho}$ and ρ, and and appreciate matrix $\mathcal{K}$, such that the following LMI together with conditions (9), (10) hold:

$$
\Xi_1 = \begin{bmatrix} \Xi_{11} & \frac{\pi^2}{2}P & P \\ * & \mathcal{K}+\mathcal{K}^T-\frac{\pi^2}{2}P & 0 \\ * & * & -I \end{bmatrix} < 0. \quad (16)
$$

Then, the system (1) is MSES. Moreover, the control gain matrix is given by $K = P^{-1}\mathcal{K}$.

A. H_∞ ANALYSIS FOR PARABOLIC SYSTEM

External noise is inevitable in general owing to the environmental disturbances. These external disturbances have an impact on the system's stability. We are interested to investigate the system (1) in the presence of external disturbances. Consider the following semilinear stochastic parabolic system with external disturbance expressed in the Itö differential form:

$$
\begin{aligned}
d\wp(\varkappa,t) = &\Big\{\tfrac{\partial^2\wp(\varkappa,t)}{\partial\varkappa^2} + A\wp(\varkappa,t) + g(t,\wp(\varkappa,t)) \\
&+Cv(\varkappa,t)\Big\}dt + \sigma(t,\wp(\varkappa,t))dW(\varkappa,t) \\
&+\textstyle\int_Z \phi(t,\wp(\varkappa,t),z)\tilde{N}(dt,dz), \quad (17)
\end{aligned}
$$

where $v(\varkappa,t)$ is external disturbance which satisfies

$$
\int_0^\infty\int_0^1 v^T(\varkappa,t)v(\varkappa,t)d\varkappa < \infty. \quad (18)
$$

Definition 2: System (17) is said to be MSES with given disturbance attenuation level $\gamma > 0$, if it is robustly stable and the state vector $\wp(\varkappa,t)$ under zero initial condition satisfies

$$
\mathbb{E}||\wp(\varkappa,t)||^2 \leq \gamma^2\mathbb{E}||v(\varkappa,t)||^2,
$$

for every non-zero $v(\varkappa,t)$ satisfies (18).

The H_∞ performance is a useful one for determining the robustness of disturbed system. In order to discuss the H_∞ performance of system (1), we consider the following performance index,

$$
J(\cdot) = \textstyle\int_0^1(\wp^T(\varkappa,t)\wp(x,t) - \gamma^2v^T(\varkappa,t)v(\varkappa,t))d\varkappa. \quad (19)
$$

Theorem 2: Consider that the assumptions (3)-(5) holds. For given scalars m, c, q and decay rate $\delta > 0$, there exist a symmetric positive definite matrix P, positive scalars $\bar{\rho}$, $\underline{\rho}$ and appreciate matrix $\mathcal{K}$, such that the following LMI together with conditions (9), (10) hold:

$$\bar{\Xi} = \begin{bmatrix} \bar{\Xi}_{11} & \frac{\pi^2}{2}P & P & PC \\ * & \bar{\Xi}_{22} & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \qquad (20)$$

where $\bar{\Xi}_{11} = \Xi_{11} + I$, $\bar{\Xi}_{22} = \mathcal{K} + \mathcal{K}^T - \frac{\pi^2}{2}P$. Then, the system (17) is MSES with H_∞ performance attenuation level $\gamma > 0$. Moreover, the control gain is given by $K = P^{-1}\mathcal{K}$.

Proof: The desired results were obtained using similar techniques to those used in the proof of Theorem 1, and the detailed proof is omitted for clarity.

By setting $\bar{\chi}(\cdot) = \left[\wp^T(\varkappa,t) \quad \wp^T(\varkappa,t) \quad g^T(t,\wp(\varkappa,t)) \quad v^T(\varkappa,t)\right]^T$, and by letting $\mathcal{K} = PK$, we have

$$\begin{aligned} &\mathcal{L}V(\cdot) + 2\delta V(\cdot) + J(\cdot) \\ &\quad \le \int_0^1 \bar{\chi}^T(\cdot)\bar{\Xi}\bar{\chi}(\cdot)d\varkappa + 2\int_0^1 \wp^T(\varkappa,t)P \\ &\quad \times\sigma(t,\wp(\varkappa,t))d\varkappa dW(\varkappa,t) \end{aligned}$$

By taking the expectations on both sides, we get

$$\begin{aligned} \mathbb{E}\Big[\mathcal{L}V(t,\wp(\varkappa,t)) + 2\delta V(t,\wp(\varkappa,t)) + J(t,\wp(\varkappa,t),v(\varkappa,t))\Big] \\ \le \mathbb{E}\Big[\textstyle\int_0^1 \bar{\chi}^T(\cdot)\bar{\Xi}\bar{\chi}(\cdot)d\varkappa\Big]. \end{aligned}$$

If LMI (20) holds, it easy to see that

$$\mathbb{E}\Big[\mathcal{L}V(\cdot) + 2\delta V(\cdot) + J(\cdot)\Big] \le 0.$$

Integrating the above inequality over $[0,t]$, which gives that

$$\begin{aligned} \mathbb{E}\big[V(\wp(t,\varkappa,t))\big] &\le e^{-2\delta t}\mathbb{E}\big[V(0,\wp(\varkappa,0))\big] \\ &+\mathbb{E}\textstyle\int_0^t e^{-2\delta(t-s)}J(t,\wp(\varkappa,s),v(\varkappa,s))ds. \end{aligned}$$

With zero initial condition, we have,

$$\mathbb{E}\int_0^t e^{-2\delta(t-s)}J(s,\wp(\varkappa,s),v(\varkappa,s))ds \le 0.$$

Integrating over $[0,\infty)$, obtained as

$$\mathbb{E}\int_0^\infty\int_0^t e^{-2\delta(t-s)}J(t,\wp(\varkappa,s),v(\varkappa,s))dsdt \le 0.$$

Changing the order of integration which yields that

$$\begin{aligned} &\mathbb{E}\int_0^\infty J(t,\wp(\varkappa,s),v(\varkappa,s))\left(\int_s^\infty e^{-2\delta(t-s)}dt\right)ds \\ &\quad = \frac{1}{2\delta}\mathbb{E}\int_0^\infty J(t,\wp(\varkappa,s),v(\varkappa,s))ds \le 0. \end{aligned}$$

It is easy to verify that

$$\mathbb{E}\left\|\wp(\varkappa,t)\right\|^2 \le \gamma^2\mathbb{E}\left\|v(\varkappa,t)\right\|^2.$$

We can concluded that from Definition 2 that the system (17) is MSES with H_∞ performance. This completes the proof.

Remark 2: If the system (1) considered with different boundary conditions like Dirchlet boundary conditions, Robin type boundary conditions the the above results are still valid with minor changes in finding $\mathcal{L}V$.

IV. NUMERICAL EXAMPLES

Example 1: This section presents a numerical example that demonstrate how the obtained results can be applied for FitzHugh-Nagumo equation. Consider the stochastic PDEs with Lèvy noise in the form of FitzHugh-Nagumo equation [29].

$$\begin{aligned} d\wp_1(\varkappa,t) = \Big\{&\frac{\partial^2\wp_1(\varkappa,t)}{\partial\varkappa^2} - \lambda\wp_2(\varkappa,t) + \eta\wp_1(\varkappa,t) \\ &\times(\wp_1(\varkappa,t)-\alpha)(1-\wp_1(\varkappa,t))\Big\}dt + \mathcal{L}_1, \end{aligned} \qquad (21)$$

$$\begin{aligned} &d\wp_2(\varkappa,t) = \Big\{\frac{\partial^2\wp_1(\varkappa,t)}{\partial\varkappa^2} + [\kappa\wp_1(\varkappa,t) - \beta\wp_2(\varkappa,t)]\Big\}dt \\ &\quad +\mathcal{L}_2, \\ &\frac{\partial\wp_1(0,t)}{\partial\varkappa} = \frac{\partial\wp_2(0,t)}{\partial\varkappa} = 0, \\ &\frac{\partial\wp_1(1,t)}{\partial\varkappa} = u_1(t), \ \frac{\partial\wp_2(1,t)}{\partial\varkappa} = u_2(t), \\ &\wp_1(x,0) = \wp_1^0(\varkappa), \ \wp_2(x,0) = \wp_2^0(\varkappa) \end{aligned} \qquad (22)$$

where, $\wp_1(x,t)$ voltage variable, $\wp_2(x,t)$ recovery variable and the semi-linear term $\eta\wp_1(\wp_1-\alpha)(1-\wp_1)$ represents the voltage threshold. The electrical potential on the molecular membrane can be affected by several forms of random terms due to the surrounding medium. Since this Levy approach allows for jumps, which gives more realistic to real world models, the representations of L_1 and L_2 are given by

$$\begin{aligned} \mathcal{L}_1 &= \mu_1\wp_1^2(\varkappa,t)dW_1(\varkappa,t) + \zeta_1\int_0^\infty z\wp_1^2(\varkappa,t)\tilde{N}_1(dt,dz), \\ \mathcal{L}_2 &= \mu_2\wp_2^2(\varkappa,t)dW_2(\varkappa,t) + \zeta_2\int_0^\infty z\wp_2^2(\varkappa,t)\tilde{N}_2(dt,dz). \end{aligned}$$

By setting

$$\begin{aligned} A &= \begin{bmatrix} -\eta\alpha & -\lambda \\ \kappa & -\beta \end{bmatrix}, \\ \sigma(t,\wp(\varkappa,t),z) &= \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}\begin{bmatrix} \wp_1^2(\varkappa,t) \\ \wp_2^2(\varkappa,t) \end{bmatrix}, \\ g(t,\wp(\varkappa,t)) &= \begin{bmatrix} \eta[(1+\alpha)\wp_1^2(\varkappa,t) - \wp_1^3(\varkappa,t)] \\ 0 \end{bmatrix}, \\ \phi(t,\wp(\varkappa,t),z) &= \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix}\begin{bmatrix} \wp_1^2(\varkappa,t) \\ \wp_2^2(\varkappa,t) \end{bmatrix}. \end{aligned}$$

From Equation (21) and (22) one can reach system (1).

Let the constant parameters $\alpha = 0.5$, $\lambda = 0.1$, $\kappa = 0.8$, $\beta = 0.54$, $\eta = 0.1$, $\mu_1 = \mu_2 = 0.01$, $\zeta_1 = \zeta_2 = 0.02$ and $m = 2.25$, $q = 3.5$, $c = 1$. Now solving LMIs in Corollary 1 using the LMI toolbox in MATLAB, the feasible solutions to guarantee the exponential stability of the considered system can be obtained with decay rate $\delta = 0.25$. Furthermore, the controller gain value is found as

$$K = \begin{bmatrix} 8.7184 & 0.5882 \\ 0.5882 & 9.1231 \end{bmatrix}.$$

The validity of the developed feedback controller is demonstrated in Figs 1 - 4. The state trajectories do not meet the

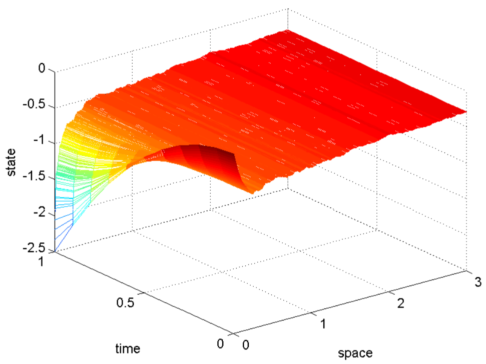

FIGURE 1. State $\wp_2(\varkappa, t)$ of (22) without control.

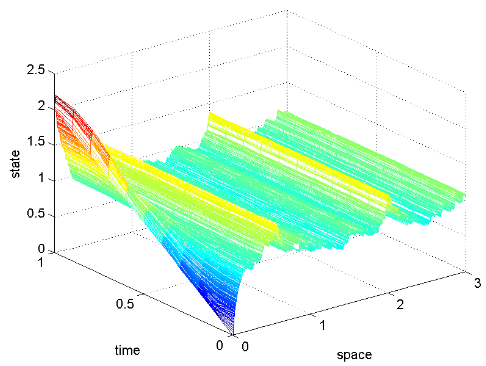

FIGURE 2. State $\wp_2(\varkappa, t)$ of (22) without control.

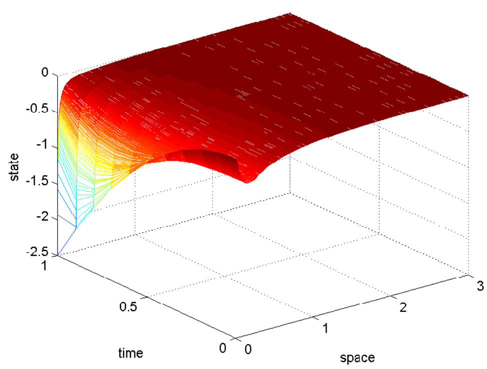

FIGURE 3. State $\wp_1(\varkappa, t)$ of (21) with control.

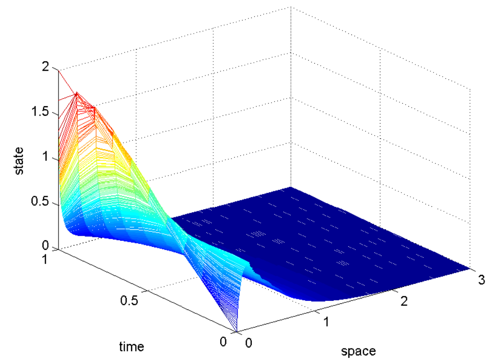

FIGURE 4. State $\wp_2(\varkappa, t)$ of (22) with control.

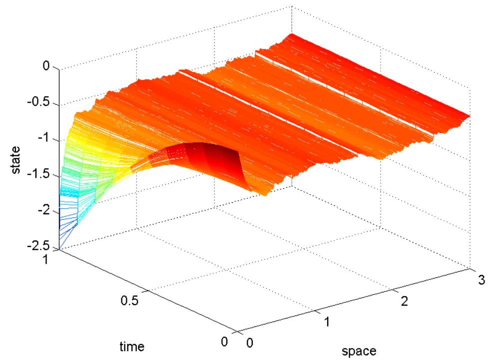

FIGURE 5. State $\wp_1(\varkappa, t)$ of (23) without control.

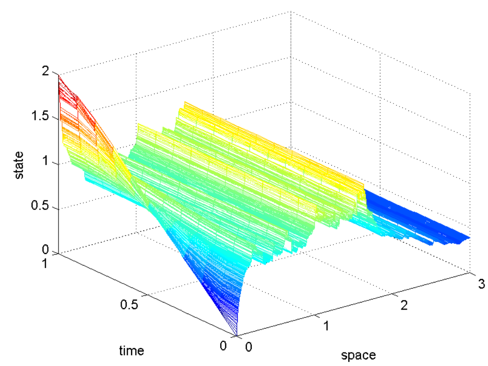

FIGURE 6. State $\wp_2(\varkappa, t)$ of (24) without control.

equilibrium point when the control input $u(t) = 0$ which are shown in Figures 1 & 2. Then we see that in the presence of the controller system state achieves the convergence as shown in Figures 3 & 4. This demonstrates the designed boundary controls effectiveness.

Example 2: Next, we consider (21) and (22) with external disturbance $v(\varkappa, t)$

$$d\wp_1(\varkappa, t) = \Big\{ \frac{\partial^2 \wp_1(\varkappa,t)}{\partial \varkappa^2} - \lambda \wp_2(\varkappa, t) + \eta \wp_1(\varkappa, t)(\wp_1(\varkappa, t) - \alpha)(1 - \wp_1(\varkappa, t)) + 0.01 v(\varkappa, t) \Big\} + \mathcal{L}_1, \tag{23}$$

$$d\wp_2(\varkappa, t) = \Big\{ \frac{\partial^2 \wp_1(\varkappa,t)}{\partial \varkappa^2} + [\kappa \wp_1(\varkappa, t) - \beta \wp_2(\varkappa, t)] + 0.02 v(\varkappa, t) \Big\} + \mathcal{L}_2, \tag{24}$$

with the system parameters are chosen as $\alpha = 0.9$, $\lambda = 0.75$, $\kappa = 0.4$, $\beta = 0.9$, $\eta = 0.3$, $\mu_1 = \mu_2 = 0.05$, $\zeta_1 = \zeta_2 = 0.02$, $m = 3$, $q = 1$, $c = 3$ and $v(\varkappa, t) = 1.5\varkappa + \cos(t)$. Where, $\mathcal{L}_1$ and $\mathcal{L}_2$ taken same as in example 1.

For $C = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.2 \end{bmatrix}$, by solving the LMI conditions in Theorem 2, we get a feasible solutions to guarantee the exponential stability of the considered system can be obtained with the control gain

$$K = \begin{bmatrix} 1.5295 & 0.0503 \\ 0.0503 & 1.2725 \end{bmatrix}.$$

and we can obtain the optimized H_∞ performance level $\gamma_{\min} = 0.1124$.

Simulation results for the considered system with and without the boundary feedback control are shown in

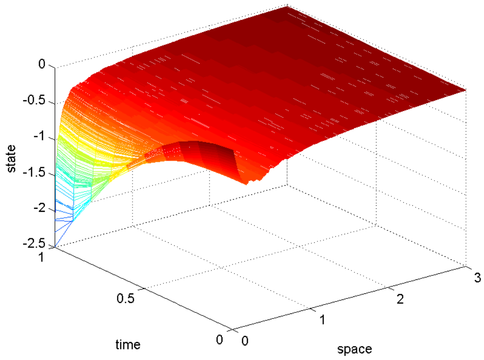

FIGURE 7. State $\wp_1(x, t)$ of (23) with control.

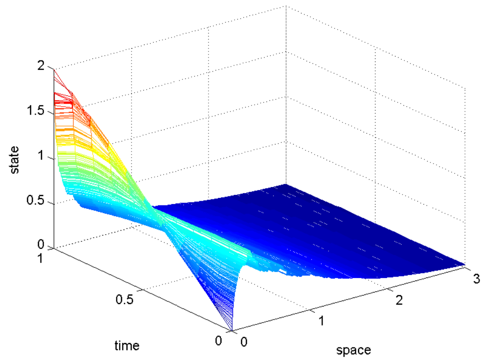

FIGURE 8. State $\wp_2(x, t)$ of (24) with control.

Figures 5 - 8. It is evident that the state responses have good transient response with the help of designed the boundary controller and achieve the exponential stability in the presence of disturbance.

V. CONCLUSION

In this paper, the problem of MSES of semi-linear stochastic PDE driven by Lèvy noise using boundary feedback control has been investigated. The system's mean-square exponential stability was guaranteed using Lyapunov theory and LMI technique. The disturbances that exist in the system has been dealt with the aid of the H_∞ performance. Finally, numerical example establishes the effectiveness of the proposed techniques. In future, due to the importance of SPDE results can be extended to investigate the observer based control with varies form of performance indexes such as passivity, dissipativity, extended dissipativity and so on.

REFERENCES

[1] V. Anaya, M. Bendahmane, D. Mora, and M. Sepulveda, "A virtual element method for a nonlocal FitzHugh–Nagumo model of cardiac electrophysiology," *IMA J. Numer. Anal.*, vol. 40, no. 2, pp. 1544–1576, Apr. 2020.

[2] D. Applebaum, *Levy Processes and Stochastic Calculus*. Cambridge, U.K.: Cambridge Univ. Press, 2009.

[3] M. Bachar, J. J. Batzel, and S. Ditlevsen, *Stochastic Bio-Mathematical Models With Applications to Neuronal Modeling*. Berlin, Germany: Springer-Verlag, 2013.

[4] J. Bao and C. Yuan, "Blow-up for stochastic reaction-diffusion equations with jumps," *J. Theor. Probab.*, vol. 29, pp. 617–631, Jun. 2016.

[5] Z. Brzezniak, W. Liu, and J. Zhuc, "Strong solutions for stochastic partial differential equations with locally monotone coefficients driven by Lévy noise," *Nonlinear Anal., Real World Appl.*, vol. 17, pp. 283–310, Jun. 2014.

[6] X. Dong, W. Si, and C. Wang, "Global identification of FitzHugh–Nagumo equation via deterministic learning and interpolation," *IEEE Access*, vol. 7, pp. 107334–107345, 2019.

[7] K. Ding and Q. Zhu, "H_∞ synchronization of uncertain stochastic time-varying delay systems with exogenous disturbance via intermittent control," *Chaos Solitons Fractals*, vol. 127, pp. 244–256, Oct. 2019.

[8] M. Dus, "Exponential stability of a general slope limiter scheme for scalar conservation laws subject to a dissipative boundary condition," *Math. Control, Signals, Syst.*, vol. 34, pp. 37–65, Mar. 2022.

[9] K. R. Ghusinga and A. Singh, "Analysis of stochastic timing of intracellular events with gene switching," in *Proc. 6th Indian Control Conf. (ICC)*, Dec. 2019, pp. 280–285, doi: 10.1109/ICC47138.2019.9123214.

[10] A. Grover and B. Lall, "A Stochastic model for characterizing fluctuations in chemical sensing," *IEEE Trans. Signal Process.*, vol. 69, pp. 1938–1951, 2021.

[11] J. Zheng and G. Zhu, "A weak maximum principle-based approach for input-to-state stability analysis of nonlinear parabolic PDEs with boundary disturbances," *Math. Control, Signals, Syst.*, vol. 32, no. 2, pp. 157–176, Apr. 2020.

[12] Y. C. Ji and F. H. Fenton, "Numerical solutions of reaction-diffusion equations: Application to neural and cardiac models," *Amer. J. Phys.*, vol. 84, no. 8, pp. 626–638, May 2016.

[13] Y. Ju, G. Cheng, and Z. Ding, "Stochastic H_∞ finite-time control for linear neutral semi-Markovian jumping systems under event triggering scheme," *J. Franklin Inst.*, vol. 358, pp. 1529–1552, Dec. 2021.

[14] J. Keener and J. Sneyd, *Mathematical Physiology I: Cellular Physiology*. New York, NY, USA: Springer-Verlag, 2009.

[15] G. Li and Q. Yang, "Stabilization of hybrid stochastic systems with Levy noise by discrete-time feedback control," *Int. J. Control*, vol. 95, no. 1, pp. 1–16, Jun. 2020.

[16] S.-H. Li and Q.-X. Zhu, "Stochastic impact in Fitzhugh–Nagumo neural system with time delays driven by colored noises," *Chin. J. Phys.*, vol. 56, no. 1, pp. 346–354, Feb. 2018.

[17] T. Lv, Q. Gan, and F. Xiao, "Stability for a class of generalized reaction–diffusion uncertain stochastic neural networks with mixed delays," *Int. J. Mach. Learn. Cybern.*, vol. 10, no. 5, pp. 967–978, May 2019.

[18] W. Ma, X. Luo, and Q. Zhu, "Practical exponential stability of stochastic age-dependent capital system with Lévy noise," *Syst. Control Lett.*, vol. 144, Oct. 2020, Art. no. 104759.

[19] J. D. Murray, *Mathematical Biology I. An Introduction*. New York, NY, USA: Springer-Verlag, 2002.

[20] J. D. Murray, *Mathematical Biology I. Spatial Models and Biomedical Applications*. New York, NY, USA: Springer-Verlag, 2003.

[21] Q. Luo and Y.-T. Zhang, "Exponential stability in mean square of neutral stochastic partial differential equations with parameter uncertainties," in *Proc. Int. Conf. Mach. Learn. Cybern.*, 2006, pp. 585–589, doi: 10.1109/ICMLC.2006.258381.

[22] P. H. A. Ngoc, "A new approach to mean square exponential stability of stochastic functional differential equations," *IEEE Control Syst. Lett.*, vol. 5, no. 5, pp. 1645–1650, Nov. 2021.

[23] P. Pan, J. Wang, and K. Wu, "Boundary stabilization and H_∞ control for stochastic reaction-diffusion Systems," in *Proc. Chin. Control Decis. Conf. (CCDC)*, May 2016, pp. 2279–2283.

[24] J. Peng and Z. Liu, "Stability analysis of stochastic reaction-diffusion delayed neural networks with Levy noise," *Neural Comput. Appl.*, vol. 20, no. 4, pp. 535–541, Feb. 2011.

[25] B. Priya, M. S. Ali, G. K. Thakur, S. Sanober, and B. Dhupia, "p^{th} moment exponential stability of memristor Cohen–Grossberg BAM neural networks with time-varying delays and reaction–diffusion," *Chin. J. Phys.*, vol. 74, pp. 184–194, Dec. 2021.

[26] S. Pushpak, K. Ebrahimi, and U. Vaidya, "Distributed optimization via primal-dual gradient dynamics with stochastic interactions," in *Proc. 4th Indian Control Conf. (ICC)*, 2018, pp. 18–23, doi: 10.1109/INDIANCC.2018.8307947.

[27] M. Röckner and T. Zhang, "Stochastic evolution equations of jump type: Existence, uniqueness and large deviation principles," *Potential Anal.*, vol. 26, pp. 255–279, Feb. 2007.

[28] B. Song, H. Zhou, P. Xu, and X. X. Ma, "Robust stability and stabilization of uncertain stochastic systems involving jump random noise," in *Proc. 39th Chinese Control Conf.*, 2020, pp. 1007–1010.

[29] M. E. Schonbek, "Boundary value problems for the FitzHugh-Nagumo equations," *J. Differ. Equ.*, vol. 30, no. 1, pp. 119–147, Oct. 1978.
[30] A. N. Vargas, C. F. Caruntu, J. Y. Ishihara, and H. Bouzahir, "Stochastic stability of switching linear systems with application to an automotive powertrain model," *Math. Comput. Simul.*, vol. 191, pp. 278–287, Jan. 2022.
[31] M. F. B. Veron, M. G. Huidobro, and C. Yarur, "On a semilinear parabolic system of reaction-diffusion with absorption," *Asymptotic Anal.*, vol. 36, pp. 241–283, Dec. 2003.
[32] M. Wang, Q. Meng, and Y. Shen, "H_2/H_∞ control for stochastic jump-diffusion systems with Markovian switching," *J. Syst. Sci. Complex.*, vol. 34, pp. 924–954, Nov. 2021.
[33] Y. Wang, X. C. Luo, and H. X. Wang, "GPU-based model predictive control of nonlinear parabolic partial differential equations system and its application in continuous casting," *IEEE Access*, vol. 7, pp. 79337–79353, 2019.
[34] X. J. Wei, Z. J. Wu, and H. R. Harimi, "Disturbance observer based disturbance attenuation control for a class of stochastic systems," *Automatica*, vol. 63, pp. 21–25, Jan. 2016.
[35] H.-N. Wu, J.-W. Wang, and H.-X. Li, "Fuzzy boundary control design for a class of nonlinear parabolic distributed parameter systems," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 3, pp. 642–652, Jun. 2014.
[36] K.-N. Wu, X.-Z. Liu, P. Shi, and C.-C. Lim, "Boundary control of linear stochastic reaction-diffusion systems," *Int. J. Robust Nonlinear Control*, vol. 29, no. 1, pp. 268–282, Oct. 2019.
[37] H. N. Wu and X. M. Zhang, "Boundary static output feedback control for nonlinear stochastic parabolic partial differential systems via fuzzy-model-based approach," *IEEE Trans. Fuzzy Syst.*, vol. 28, no. 10, pp. 2581–2591, Oct. 2020.
[38] Y. Xu, Z. He, and P. Wang, "P^{th} moment asymptotic stability for neutral stochastic functional differential equations with Lévy processes," *Appl. Math. Comput.*, vol. 269, pp. 594–605, Oct. 2015.
[39] X. P. Yan, P. Zhang, and C. H. Zhang, "Turing instability and spatially homogeneous Hopf bifurcation in a diffusive Brusselator system," *Nonlinear Anal., Model. Control*, vol. 25, no. 4, pp. 638–657, 2020.
[40] S. Yoshizawa, "Population growth process described by a semilinear parabolic equation," *Math. Bio-Sci.*, vol. 7, pp. 291–303, Apr. 1970.
[41] F. Zhao, X. Chen, J. Cao, M. Guo, and J. Qiu, "Finite-time stochastic input-to-state stability and observer-based controller design for singular nonlinear systems," *Nonlinear Anal., Model. Control*, vol. 25, no. 6, pp. 980–996, Nov. 2020.
[42] H. Zhou, Y. Zhang, and W. Li, "Synchronization of stochastic Levy noise and its applications of Chua's circuits," *IEEE Trans. Circuits Syst.*, vol. 66, no. 7, pp. 2709–2722, Jul. 2019.
[43] W. Zhou, J. Yang, L. Zhou, and D. Tong, *Stability and Synchronization Control of Stochastic Neural Networks*. Cham, Switzerland: Springer, 2016.

K. MATHIYALAGAN received the B.Sc., M.Sc., and M.Phil. degrees in mathematics from Bharathiar University, Coimbatore, India, in 2005, 2007, and 2008, respectively, and the Ph.D. degree in mathematics from Anna University, Chennai, India, in 2012.

He was a Senior Engineering Research Fellow and a CSIR Senior Research Fellow at the Department of Mathematics, Anna University of Technology, Coimbatore, from June 2009 to December 2012. He was a Postdoctoral Research Fellow at the Institute of Cyber-System and Control, Zhejiang University, China, in 2013. He was a Postdoctoral Research Associate and a Research Professor at the Department of Electrical Engineering, Yeungnam University, South Korea. Then he joined as a Dr.D.S.Kothari Postdoctoral Fellow of University Grants Commission, India at the Department of Mathematics, Bharathiar University, where he is currently working as an Assistant Professor with the Department of Mathematics. His current research interests include control theory and its applications for ordinary and partial differential equations, and time-delay systems. He was a recipient of the Highly Cited Researcher Award by Clarivate Analytics (formerly, Thomson Reuters) in 2019. He is an Associate Editor of *International Journal of Control, Automation and Systems* and *Mathematical Problems in Engineering*.

R. RAGUL received the B.Sc. degree in mathematics from Bharathiar University, Coimbatore, India, in 2014, the B.Ed. degree in mathematics from Tamilnadu Teachers Education University, Chennai, India, in 2015, and the M.Sc., and M.Phil. degrees from Bharathiar University, in 2017 and 2019, respectively. He is currently pursuing the Ph.D. degree with Bharathiar University. His current research interests include stability, robust control for partial differential equations, and time-delay systems.

G. SANGEETHA received the B.Sc. degree in mathematics from Periyar University, in 2007, and the M.Sc., M.Phil., and Ph.D. degrees in mathematics from Bharathiar University, Coimbatore, India, in 2009, 2010, and 2021, respectively. She is currently working as an Assistant Professor with the Department of Mathematics, KGISL Institute of Technology, Coimbatore. Her current research interests include control theory and its applications for different kind of dynamical systems.

YONG-KI MA received the M.S. and Ph.D. degrees in mathematics from Yonsei University, South Korea, in 2007 and 2011, respectively.

From 2011 to 2012, he was a Postdoctoral Fellow at the Department of Statistics, Seoul National University, South Korea. He is currently a Professor of applied mathematics with Kongju National University, South Korea. His research interests include control theory and stochastic modeling.