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## RESEARCH ARTICLE

# Lyapunov Spectra Behavior for Linear Time-Varying Discrete Systems Under Small Perturbations

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**ABSTRACT** We investigate the behavior of the Lyapunov spectrum of a linear time-varying discrete system under the action of small perturbations in order to obtain some verifiable conditions for stability and openness of the Lyapunov spectrum. To this end we introduce the concepts of broken away solutions and splitted systems. The main results obtained are a necessary condition for stability and a sufficient condition for the openness of the Lyapunov spectrum, which is given in terms of the system itself. Finally, examples of using the obtained results are presented.

**INDEX TERMS** Linear time-varying discrete systems, Lyapunov spectrum, small perturbations, multiplicative perturbations.

## I. INTRODUCTION

### A. MATHEMATICAL GENESIS OF THE PROBLEM

One of the main questions in the theory of linear systems (both with continuous and discrete time) is the question about the behavior of solutions under an unbounded increasing argument. In the case of linear homogeneous systems with constant coefficients, the answer to this question depends on the signs of the real parts of the eigenvalues of the coefficient matrix of the system, and in the case of periodic linear homogeneous systems on the location of the multipliers of the system with respect to the unit circle on the complex plane. There are no such specific answers in the time-varying case, but there are fairly general results based on the method of characteristic exponents. This method studies the growth rate of solutions of a linear homogeneous system

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as compared to the exponentiation functions  $e^{\alpha t}$ . In the case of systems with constant coefficients, the Lyapunov indices coincide with the real parts of the eigenvalues of the system's coefficient matrix, and in the case of systems with periodic coefficients, with the logarithms of the system's multiplier modules divided by the period.

The concept of characteristic exponents of linear time-varying differential equations was introduced by A.M. Lyapunov in 1892 in his famous work [26]. Subsequently, the theory of Lyapunov characteristic exponents developed into a well-established asymptotic theory of linear systems [1], [7], [10], [22], [23]. The characteristic number  $\lambda$ , or *Lyapunov exponent*, as it is called nowadays, of a nonzero function  $\varphi: [t_0, +\infty) \rightarrow \mathbb{R}^s$  characterizes its growth as  $t$  tends to  $+\infty$  in the scale of exponents  $\alpha$  of exponential functions  $e^{\alpha t}$ , where  $\alpha \in \mathbb{R}$ , and it is defined to be

$$\lambda[\varphi] = \limsup_{t \rightarrow +\infty} t^{-1} \ln \|\varphi(t)\|.$$

A.M. Lyapunov showed that if the original differential system

$$\dot{x} = f(t, x), \quad t \geq t_0, \quad x \in \mathbb{R}^s, \quad (1)$$

has the trivial solution and Lyapunov exponents of all solutions of the linearized system

$$\dot{x} = A(t)x, \quad t \geq t_0, \quad x \in \mathbb{R}^s, \quad (2)$$

where  $A(t) = f'_x(t, 0)$ , are negative, then under certain conditions on function  $f(t, x)$ , the trivial solution of system (1) is asymptotically stable. This result was a basis of the so-called first Lyapunov method of studying the stability of solutions of differential systems. When studying Lyapunov exponents of linear systems, some unexpected effects were discovered, which Lyapunov himself probably did not suspect. In particular, it turned out that small perturbations of the coefficient matrix  $A(\cdot)$  of system (2) may lead to jumps of the Lyapunov exponents of its solutions. For example, the original linear system (2) may have all solutions with negative Lyapunov exponents (and for this reason to be exponentially stable), but an arbitrarily small additive perturbation  $Q(\cdot)$  of the matrix  $A(\cdot)$  may result in positive Lyapunov exponents of the perturbed system

$$\dot{x} = (A(t) + Q(t))x, \quad t \geq t_0, \quad x \in \mathbb{R}^s. \quad (3)$$

This instability of Lyapunov exponents was discovered by O. Perron [32] in 1930.

All of the above also applies to a linear discrete time-varying system

$$x(n+1) = A(n)x(n), \quad n \in \mathbb{N}, \quad x \in \mathbb{R}^s, \quad (4)$$

with the assumption that  $A(\cdot)$  is a Lyapunov sequence (see below, in the notation section, for the definition). It can be proved [4] that there are systems of the form (4) for which small perturbations of the sequence  $A(\cdot)$  lead to a significant change in the Lyapunov spectrum.

The phenomenon of instability of Lyapunov exponents naturally leads to the question for which systems it can occur or, alternatively, about the characterization of systems with stable exponents.

Necessary and sufficient conditions for stability of the Lyapunov spectrum of a continuous-time system of the form (2) were obtained by V.M. Millionshchikov [30] and at the same time by B.F. Bylov and N.A. Izobov [9] using the so-called Millionschikov rotation method (see [29]). They showed that in order for the stability of the Lyapunov spectrum of system (2) to hold, it is necessary and sufficient that this system can be reduced to a block-triangular form by some Lyapunov transformation, such that the blocks are integrally separated [11] and for each block, the upper and lower central exponents [9], [10], [30] coincide. Similar conditions hold for linear discrete-time systems [5].

It is important to notice that, in general these conditions are unverifiable, since for their application we must transform system (2) (or system (4)) into some special form by Lyapunov transformation, but the algorithms to construct this

transformation are unknown. Therefore, the question arises: is it possible to obtain any stability conditions for the Lyapunov spectrum that are, in some sense, verifiable? In this paper, we will present such conditions.

## B. MAJOR CONTRIBUTION

If the Lyapunov spectrum is stable, we are sure that sufficiently small perturbations of the system do not remove its Lyapunov spectrum from some small neighborhood of the original spectrum. The following question arises naturally in this connection: is it possible to move the Lyapunov spectrum to any prescribed position in a small vicinity of the original spectrum using appropriate small perturbations? This property can be called the openness of the Lyapunov spectrum of system (2) (or system (4)). Some results on the openness for continuous-time systems were obtained in [28] and for discrete-time systems in [5].

The foundations of the asymptotic theory of discrete-time systems including the issues close to the problems formulated above are presented in [13], [18], [19]. Questions on integral separateness and stability of Lyapunov spectrum of system (4) were considered in [4]–[6]. In this context, we also mention the paper of L. Barreira and C. Valls [8], where the problem of coincidence of Lyapunov spectra of perturbed and unperturbed systems is investigated. Note that these results do not allow us to achieve the goals of our article, which are to study the behavior of the Lyapunov spectrum of system (4) under small perturbations of the matrix  $A(\cdot)$  in order to obtain some criteria to detect stability or openness of this spectrum.

The main contribution of this article is to provide a necessary condition for stability of the Lyapunov spectrum of system (4), which do not require a reduction of this system to a special form, but is expressed in terms of the system itself. In addition, we obtain sufficient conditions for the openness of the spectrum of system (4). To solve these problems, we introduce and use the concept of splitness of system (4) based on the angular behavior of solutions of this system.

Note that the notion of splitness of system (2) was introduced in [27] in connection with control problems for the Lyapunov spectrum of a linear differential system.

## C. POTENTIAL FOR ENGINEERING APPLICATIONS

Although the presented work is theoretical in nature, obtained sufficient condition for the continuity of the Lyapunov spectrum have a great potential for applications in engineering practice for the following reasons.

The Lyapunov exponent technique is widely used in engineering practice for systems with variable coefficients as a natural counterpart to the pole placement method for systems with constant coefficients (see [34]). In applications, we usually know the values of the coefficients only to a certain approximation, and when numerically calculating the values of Lyapunov's exponents, the information about their continuity is crucial, because it allows to connect the inaccuracy of the coefficients of the system to the error in estimating the exponents [14]–[17]. Additionally, when using

some feedback design methods for systems with variable coefficients, it is required that the Lapunov exponents of the open system be stable (see Theorem 6.9 and Example 7.4 in [3]). Finally, let us note that the influence of small parametric disturbances on the dynamics of control systems is widely discussed in the literature [2], [24], [31], [33] and [12] and the presented results enable the application of techniques based on Lyapunov exponents to these problems.

**D. STRUCTURE OF THE ARTICLE**

This article is organized as follows. In Section II, the necessary notation is introduced. In Section III, the concept of the Lyapunov spectrum of system (4) is explained. In Section IV, the concept of the spectral set of this system under various perturbations of its coefficient matrix is considered. It is demonstrated that multiplicative perturbations are more adequate to the problem under consideration. A definition of the stability of the Lyapunov spectrum is also introduced. In Section V the concept of splitted systems is proposed and their properties are discussed. In Section VI, the main theorem on the property of splitted systems are shown. In Section VII, we prove several results that follow from the main theorem and demonstrate the importance of the introduced concept of splitted systems for studying the behavior of the Lyapunov spectrum under the action of small perturbations. Some examples are given in Section VII. The article is completed by Conclusions.

**II. NOTATION**

Let  $\mathbb{R}^s$  be an  $s$ -dimensional Euclidean space with a fixed orthonormal basis  $e_1, \dots, e_s$  and the standard norm  $\| \cdot \|$ . By  $\mathbb{R}^{s \times t}$  we shall denote the space of all real matrices of size  $s \times t$  with the spectral norm, i.e., with the operator norm generated in  $\mathbb{R}^{s \times t}$  by Euclidean norms in  $\mathbb{R}^s$  and  $\mathbb{R}^t$ , respectively. By  $[a_1, \dots, a_t] \in \mathbb{R}^{s \times t}$  we denote the matrix with the sequential columns  $a_1, \dots, a_t \in \mathbb{R}^s$ ;  $I \in \mathbb{R}^{s \times s}$  is the identity matrix. For any nonsingular matrix  $H \in \mathbb{R}^{s \times s}$  we denote by  $\kappa(H)$  the condition number of  $H$  with respect to spectral norm, i.e.,  $\kappa(H) = \|H\| \|H^{-1}\|$ . For any sequence  $F(\cdot) = (F(n))_{n \in \mathbb{N}}$ , where  $F(n) \in \mathbb{R}^{s \times t}$ ,  $n \in \mathbb{N}$ , we define  $\|F\|_\infty = \sup_{n \in \mathbb{N}} \|F(n)\|$ . Any bounded sequence  $L(\cdot)$  of invertible matrices  $L(n) \in \mathbb{R}^{s \times s}$ ,  $n \in \mathbb{N}$ , such that the sequence  $L^{-1}(\cdot)$  is bounded on  $\mathbb{N}$ , is called a *Lyapunov sequence*.

By  $\mathbb{R}^s_{\leq}$  (resp.  $\mathbb{R}^s_{>}$ ) we denote the set of all nondecreasing (resp. increasing) sequences of  $s$  real numbers. For a fixed sequence  $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{R}^s_{\leq}$  and any  $\delta > 0$  let us denote by  $\mathcal{O}_\delta(\mu)$  the set of all sequences  $\nu = (\nu_1, \dots, \nu_s) \in \mathbb{R}^s_{\leq}$  such that  $\max_{j=1, \dots, s} |\nu_j - \mu_j| < \delta$ . In other words,  $\mathcal{O}_\delta(\mu)$  is a  $\delta$ -neighborhood of the sequence  $\mu \in \mathbb{R}^s_{\leq}$  with respect to the metric generated by the vector  $l_\infty$  norm of the space  $\mathbb{R}^s$  on its subset  $\mathbb{R}^s_{\leq}$ .

By  $[\alpha]$  we shall denote the integer part of  $\alpha \in \mathbb{R}$ , that is,  $[\alpha]$  is the largest integer not exceeding  $\alpha$ .

Let us define the angle between a nonzero vector  $p \in \mathbb{R}^s$  and some non-trivial linear subspace  $V \subset \mathbb{R}^s$  by the equality

$$\angle(p; V) = \inf_{0 \neq q \in V} \angle(p, q),$$

where

$$\angle(p, q) = \arccos \frac{\langle p, q \rangle}{\|p\| \|q\|}$$

is the angle between the nonzero vectors  $p, q \in \mathbb{R}^s$ ,  $\langle p, q \rangle$  is the scalar product of the vectors  $p$  and  $q$ .

In our further considerations we shall use the following lemmas.

*Lemma 1 [27]:* Let  $V$  be a linear subspace of  $\mathbb{R}^s$ ,  $\dim V = s - 1$ ; and let  $p \in \mathbb{R}^s \setminus V$ . If  $X \in \mathbb{R}^{s \times s}$  is a nonsingular matrix, then

$$\angle(Xp; XV) \geq \frac{2}{\pi} \angle(p; V) (\kappa(X))^{1-s}.$$

*Lemma 2 [27]:* Let  $V$  be a linear subspace of  $\mathbb{R}^s$ ,  $\dim V = s - 1$ ; and let  $p \in \mathbb{R}^s \setminus V$  be an arbitrary nonzero vector. If a matrix  $H \in \mathbb{R}^{s \times s}$  satisfies the conditions  $Hp = p$  and  $Hx = 0$  for each  $x \in V$ , then

$$\|H\| = \frac{1}{\sin \angle(p; V)}.$$

*Lemma 3 [27]:* Let  $a: \mathbb{N} \rightarrow \mathbb{R}$  and  $b: \mathbb{N} \rightarrow \mathbb{R}$  be arbitrary bounded mappings, and let

$$\psi(\mu) = \limsup_{k \rightarrow \infty} (a(k) + \mu b(k)).$$

Then the following assertions are valid:

(1) The function  $\psi: \mathbb{R} \rightarrow \mathbb{R}$  is convex and satisfies the Lipschitz condition on  $\mathbb{R}$ .

(2) If there exists a strictly increasing sequence  $(k_j)_{j \in \mathbb{N}}$  of positive integers such that

$$\lim_{j \rightarrow \infty} a(k_j) = \psi(0), \quad \rho \doteq \lim_{j \rightarrow \infty} b(k_j) > 0,$$

then the function  $\psi(\cdot)$  is (strictly) monotone increasing on the interval  $[0, \infty)$  and the estimate

$$\psi(\mu) - \psi(0) \geq \rho \mu$$

is valid for all  $\mu \geq 0$ . Moreover for each  $t \geq 0$ , there exists a  $\mu_t$ ,  $0 \leq \mu_t \leq \rho^{-1}t$ , such that

$$\psi(\mu_t) = \psi(0) + t.$$

Consider a discrete linear time-varying system (4) with a Lyapunov sequence  $A: \mathbb{N} \rightarrow \mathbb{R}^{s \times s}$ . Put

$$a \doteq \max\{\|A\|_\infty, \|A^{-1}\|_\infty\} < \infty.$$

Note that

$$\begin{aligned} \|A\|_\infty + \|A^{-1}\|_\infty &\geq \|A(1)\| + \|A^{-1}(1)\| \\ &\geq \|A(1)\| + \|A(1)\|^{-1} \geq 2, \end{aligned}$$

hence  $a \geq 1$ .

We denote the transition matrix of system (4) by  $X_A(n, m)$ ,  $n, m \in \mathbb{N}$ . Then [19, p. 13–14] for each solution  $x(\cdot)$  of this system, we have the equality

$$x(n) = X_A(n, m)x(m) \quad \text{for all } n \in \mathbb{N}, m \in \mathbb{N},$$

and

$$X_A(n, m) = \begin{cases} \prod_{l=m}^{n-1} A(l) & \text{for } n > m, \\ I & \text{for } n = m, \\ X_A^{-1}(m, n) & \text{for } n < m. \end{cases}$$

Then for any  $n \in \mathbb{N}, m \in \mathbb{N}$  the following inequality

$$\|X_A(n, m)\| \leq a^{|n-m|} \tag{5}$$

is true.

Note that here and throughout the paper we put

$$\prod_{l=m}^{n-1} A(l) = A(n-1)A(n-2)\dots A(m),$$

i.e., the matrices are multiplied in descending order of the index.

Let  $\Phi(\cdot)$  be a fundamental system of solutions (FSS) of system (4), i.e., a set of  $s$  linearly independent solutions  $x_1(\cdot), \dots, x_s(\cdot)$  of system (4). We identify FSS  $\Phi(\cdot)$  with the matrix  $[x_1(\cdot), \dots, x_s(\cdot)]$ , which is called a fundamental matrix (FM) of system (4).

### III. LYAPUNOV SPECTRA

For any nontrivial solution  $x(\cdot)$  of system (4) the Lyapunov exponent  $\lambda[x]$  is defined as

$$\lambda[x] = \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \|x(n)\|.$$

It is well known [7] that if  $A(\cdot)$  is a Lyapunov sequence, then the set of Lyapunov exponents of all nontrivial solutions of system (4) are included in the interval  $[-\ln a, \ln a]$ , and contains at most  $s$  elements, say

$$\Lambda_1(A) < \Lambda_2(A) < \dots < \Lambda_r(A).$$

The Lyapunov exponent of the trivial solution of system (4) is set equal to  $-\infty$ .

*Remark 1:* The Lyapunov exponent of the trivial solution of system (4) is set equal to  $-\infty$  by definition in order to satisfy the property of monotonicity of the map  $x \mapsto \lambda[x]$ . Indeed, if  $0 < x_1(t) \leq x_2(t)$  for all  $t \geq 0$  we have  $\lambda[x_1] \leq \lambda[x_2]$ . Since  $\lambda[e^{\mu t}] = \mu$  for each  $\mu \in \mathbb{R}$  and  $e^{\mu t} > 0$ , then we are to have  $\lambda[0] < \mu$  for each  $\mu \in \mathbb{R}$ . Thus, the only variant for  $\lambda[0]$  is  $\lambda[0] = -\infty$ .

For each  $j \in \{1, \dots, r\}$  let us consider the set  $\mathcal{E}_j$  of all solutions of system (4), whose Lyapunov exponents do not exceed  $\Lambda_j$ . Moreover, by  $\mathcal{E}_0$  we denote the set that consists of the trivial solution of system (4).

Then [19, p. 54] each of the sets  $\mathcal{E}_j$  is a linear subspace, and the dimension of the subspace  $\mathcal{E}_j$  is equal to  $N_j$ , where  $N_j$  is the maximal number of linearly independent solutions

of system (4), which have Lyapunov exponents  $\Lambda_j$ . We put  $N_0 = \dim \mathcal{E}_0 \doteq 0$ . Since  $\mathcal{E}_j \subset \mathcal{E}_l$  for  $j < l$ , then  $N_0 < N_1 < \dots < N_r$ , and  $N_r = s$ .

Let  $\Phi(\cdot) = \{x_1(\cdot), \dots, x_s(\cdot)\}$  be an arbitrary FSS of system (4). For each  $j \in \{1, \dots, r\}$  consider the value  $s_j$  which is the number of solutions from the set  $\Phi(\cdot)$  with exponent equal to  $\Lambda_j$ . It is known [19, p. 54], that the following inequalities hold:

$$s_1 + \dots + s_j \leq N_j, \quad j = 1, \dots, r.$$

*Definition 1* [19, p. 53]: FSS  $\Phi(\cdot)$  is called *normal*, if the following equalities hold:

$$s_1 + \dots + s_j = N_j, \quad j = 1, \dots, r.$$

It is known [19, p. 55], that for each system (4) a normal FSS exists.

*Definition 2* [19, p. 55]: We say that the FSS  $\Phi(\cdot) = \{x_1(\cdot), \dots, x_s(\cdot)\}$  is *incompressible*, if for any nontrivial combination  $y(\cdot) = \sum_{j=1}^s c_j x_j(\cdot)$  the equality

$$\lambda[y] = \max\{\lambda[x_j]: c_j \neq 0\}$$

holds.

It is known (see [19, p. 55]), that a FSS  $\Phi(\cdot)$  is normal if and only if it is incompressible.

Definition 1 implies an important consequence: for each normal FSS of system (4), the number  $s_j$  of its solutions with the Lyapunov exponent  $\Lambda_j$  is the same and coincides with the value  $N_j - N_{j-1}$ ,  $j = 1, \dots, r$ . Thus, we can associate with each linear discrete time-varying system (4) a collection of  $s$  numbers  $\lambda_1, \lambda_2, \dots, \lambda_s$ , which are the Lyapunov exponents of the solutions included in any normal FSS of our system. This collection is called *the Lyapunov spectrum* of system (4) [19, p. 57]. Further we denote it by

$$\lambda(A) = (\lambda_1(A), \dots, \lambda_s(A)),$$

assuming that the inequalities  $\lambda_1(A) \leq \dots \leq \lambda_s(A)$  are satisfied, and therefore  $\lambda(A) \in \mathbb{R}_{\leq}^s$ .

### IV. PRELIMINARIES

Let us consider an additively perturbed system

$$x(n+1) = (A(n) + Q(n))x(n), \quad n \in \mathbb{N}, x \in \mathbb{R}^s, \tag{6}$$

with  $A(\cdot)$  being the Lyapunov sequence and  $Q(\cdot)$  being the additive perturbation.

*Definition 3* [5]: A sequence  $Q(\cdot)$  is said to be an *admissible additive perturbation* for system (4) if  $A(\cdot) + Q(\cdot)$  is a Lyapunov sequence.

Since  $A(\cdot)$  is a Lyapunov sequence, it is easy to see that the following lemma holds.

*Lemma 4* [5]: Sequence  $Q(\cdot)$  is an admissible additive perturbation for system (4) if and only if there exists a Lyapunov sequence  $R: \mathbb{N} \rightarrow \mathbb{R}^{s \times s}$  such that

$$Q(n) = A(n)R(n) - A(n), \quad n \in \mathbb{N}. \tag{7}$$



Equality (7) enables us to rewrite the perturbed system (6) in the following form

$$x(n + 1) = A(n)R(n)x(n), \quad n \in \mathbb{N}, \quad x \in \mathbb{R}^s. \quad (8)$$

Here the sequence  $R(\cdot)$  is called a *multiplicative perturbation* whereas system (8) is called a *multiplicatively perturbed system*. Since, according to our assumptions  $A(\cdot)$  is a Lyapunov sequence then we arrive at the following definition.

**Definition 4 [5]:** A multiplicative perturbation  $R(\cdot)$  is said to be an *admissible multiplicative perturbation* if  $R(\cdot)$  is a Lyapunov sequence.

**Remark 2:** Let us notice that for all systems (4) the set of admissible multiplicative perturbations is the same, whereas the set of admissible additive perturbations depends on the coefficient matrix  $A(\cdot)$  of system (4).

For a fixed sequence  $A(\cdot)$ , let  $\mathcal{Q}$  denote the set of all systems (6) corresponding to admissible additive perturbations  $Q(\cdot)$  for system (4) and similarly, let  $\mathcal{R}$  denote the set of all systems (8) corresponding to admissible multiplicative perturbations  $R(\cdot)$ . We write  $Q(\cdot) \in \mathcal{Q}$  identifying system (6) and the additive perturbation  $Q(\cdot)$  that defines this system. Similarly, we use the notation  $R(\cdot) \in \mathcal{R}$  for system (8) and the corresponding multiplicative perturbation  $R(\cdot)$ . From Lemma 4 and definitions of admissible perturbations we have

$$\mathcal{Q} = \mathcal{R}. \quad (9)$$

We also use some subsets of the sets  $\mathcal{Q}$  and  $\mathcal{R}$ . For any  $\delta > 0$  let us denote by  $\mathcal{Q}_\delta$  the subset of  $\mathcal{Q}$  corresponding to sequences  $Q(\cdot)$  satisfying

$$\|Q\|_\infty < \delta,$$

and by  $\mathcal{R}_\delta$  the subset of  $\mathcal{R}$  corresponding to sequences  $R(\cdot)$  satisfying

$$\|R - I\|_\infty < \delta.$$

**Lemma 5 [5]:** For any  $\delta > 0$  we have  $\mathcal{Q}_\delta \subset \mathcal{R}_{a\delta}$  and  $\mathcal{R}_\delta \subset \mathcal{Q}_{a\delta}$ , where  $a = \max\{\|A\|_\infty, \|A^{-1}\|_\infty\}$ .

*Proof:* Take any  $Q(\cdot) \in \mathcal{Q}_\delta$ . Put

$$R(n) = I + A^{-1}(n)Q(n), \quad n \in \mathbb{N}.$$

From Lemma 4 it follows that  $R(\cdot)$  is an admissible multiplicative perturbation for system (4). Since

$$R(n) = I + A^{-1}(n)Q(n), \quad n \in \mathbb{N},$$

we have  $R(\cdot) \in \mathcal{R}_{a\delta}$ . Hence  $\mathcal{Q}_\delta \subset \mathcal{R}_{a\delta}$ .

Now take any  $R(\cdot) \in \mathcal{R}_\delta$ . Put

$$Q(n) = A(n)R(n) - A(n), \quad n \in \mathbb{N}.$$

By Lemma 4 we see that  $Q(\cdot)$  is an admissible additive perturbation for system (4) and

$$\|Q\|_\infty \leq \|A\|_\infty \|R - I\|_\infty < a\delta.$$

Hence  $Q(\cdot) \in \mathcal{Q}_{a\delta}$  and, therefore,  $\mathcal{R}_\delta \subset \mathcal{Q}_{a\delta}$ . ■

If  $Q(\cdot) \in \mathcal{Q}$ , then we can define the Lyapunov spectrum  $\lambda(A + Q) \in \mathbb{R}^s_\leq$  of system (6). In a similar way we can

define the Lyapunov spectrum  $\lambda(AR) \in \mathbb{R}^s_\leq$  of system (8) for  $R(\cdot) \in \mathcal{R}$ .

The spectral set of system (4) corresponding to the class  $\mathcal{Q}$  is defined by the equality  $\lambda(\mathcal{Q}) \doteq \{\lambda(A + Q) : Q(\cdot) \in \mathcal{Q}\}$ . Similarly, we define the sets  $\lambda(\mathcal{Q}_\delta)$ ,  $\lambda(\mathcal{R})$  and  $\lambda(\mathcal{R}_\delta)$ .

From (9) and Lemma 5 we get the following statement.

**Corollary 1:** The equality  $\lambda(\mathcal{Q}) = \lambda(\mathcal{R})$  holds. Moreover, for any  $\delta > 0$  the inclusions

$$\lambda(\mathcal{Q}_\delta) \subset \lambda(\mathcal{R}_{a\delta}), \quad \lambda(\mathcal{R}_\delta) \subset \lambda(\mathcal{Q}_{a\delta})$$

hold.

**Definition 5 [5], [13]:** The Lyapunov spectrum of system (4) is called *stable* if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that the inclusion  $\lambda(\mathcal{Q}_\delta) \subset \mathcal{O}_\varepsilon(\lambda(A))$  is satisfied.

By Lemma 5 we obtain the following result.

**Theorem 1 [5]:** The Lyapunov spectrum of system (4) is stable if and only if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that the inclusion  $\lambda(\mathcal{R}_\delta) \subset \mathcal{O}_\varepsilon(\lambda(A))$  is satisfied.

## V. SPLITTED SYSTEMS

Let  $\{x_1(\cdot), \dots, x_s(\cdot)\}$  be some FSS of system (4). For any  $i \in \{1, \dots, s\}$  and  $n \in \mathbb{N}$ , by  $V_i(n)$  we denote the linear span of the vectors  $x_j(n), j \in \{1, \dots, s\} \setminus \{i\}$  and by  $\varphi_i(n) \doteq \angle(x_i(n); V_i(n))$  we denote the angle between the vector  $x_i(n)$  and the subspace  $V_i(n)$ . We take an arbitrary  $\sigma \in \mathbb{N}$ . For any  $\gamma \in (0, \frac{\pi}{2}]$ ,  $k \in \mathbb{N}$ , and  $i \in \{1, \dots, s\}$ , we set

$$\Gamma_i^\gamma(\sigma) \doteq \{j \in \mathbb{N} : \varphi_i(j\sigma) \geq \gamma\},$$

$$\Gamma_i^\gamma(k; \sigma) \doteq \Gamma_i^\gamma(\sigma) \cap \{1, \dots, k\}.$$

Let  $N_i^\gamma(k; \sigma)$  be the number of elements of the set  $\Gamma_i^\gamma(k; \sigma)$ . Let us also introduce the following notation

$$g_i^\gamma(k; \sigma) \doteq \frac{N_i^\gamma(k; \sigma)}{k}, \quad f_i(k; \sigma) \doteq \frac{\ln \|x_i(k\sigma)\|}{k\sigma}.$$

If the numbers  $\gamma$  and  $\sigma$  are given in advance, then the corresponding symbols in the notation introduced above will be omitted.

A sequence  $(n_k)_{k \in \mathbb{N}}$  of natural numbers strictly increasing to  $+\infty$  is referred to as a *realizing sequence* of a solution  $x(\cdot)$  of system (4) if

$$\lambda[x] = \lim_{k \rightarrow \infty} \frac{\ln \|x(n_k)\|}{n_k}.$$

**Definition 6:** We say that the solution  $x_j(\cdot)$  occurring in the FSS  $\Phi(\cdot) = \{x_1(\cdot), \dots, x_s(\cdot)\}$  is  $\sigma$ -*broken away* (from the remaining solutions of  $\Phi(\cdot)$ ) if for a given  $\sigma \in \mathbb{N}$ , there exists a  $\gamma \in (0, \frac{\pi}{2}]$  and a realizing sequence  $(k_n\sigma)_{n \in \mathbb{N}}$  of the solution  $x_j(\cdot)$ , where  $k_n \in \mathbb{N}$ , such that

$$\lim_{n \rightarrow \infty} g_j^\gamma(k_n; \sigma) > 0.$$

A FSS  $\Phi(\cdot)$  is said to be  $\sigma$ -*split* if each of the solutions of this FSS is  $\sigma$ -broken away.

Let us consider some basic properties of the notions introduced above.

**Theorem 2:** If the solution  $x_i(\cdot)$  occurring in the FSS  $\Phi(\cdot) = \{x_1(\cdot), \dots, x_s(\cdot)\}$  is  $\sigma_0$ -broken away for some  $\sigma_0 \in \mathbb{N}$ , then it is  $\sigma$ -broken away for any  $\sigma \in \mathbb{N}$ .

*Proof:* Let  $\gamma \in (0, \frac{\pi}{2}]$ , and let a strictly increasing sequence  $(k_n)_{n \in \mathbb{N}}$  be chosen so as to satisfy the conditions

$$\lim_{n \rightarrow \infty} f_i(k_n; \sigma_0) = \lambda[x_i]$$

and

$$\lim_{n \rightarrow \infty} g_i^\gamma(k_n; \sigma_0) > 0.$$

To each  $n \in \mathbb{N}$  we assign the nonnegative integer  $l_n$  such that

$$k_n \sigma_0 \in [l_n \sigma, (l_n + 1) \sigma), \quad (10)$$

i.e.,

$$k_n \frac{\sigma_0}{\sigma} - 1 < l_n \leq k_n \frac{\sigma_0}{\sigma}.$$

Since  $(k_n)_{n \in \mathbb{N}}$  is a strictly increasing sequence, it follows that  $(l_n)_{n \in \mathbb{N}}$  is nondecreasing. Moreover,

$$\lim_{n \rightarrow \infty} l_n = \infty.$$

Since

$$0 \leq k_n \sigma_0 - l_n \sigma < \sigma,$$

we have

$$\lim_{n \rightarrow \infty} \frac{k_n \sigma_0}{l_n \sigma} = 1 \quad (11)$$

and by (5) we get

$$\|X_A(l_n \sigma, k_n \sigma_0)\| \leq a^\sigma, \quad \|X_A(k_n \sigma_0, l_n \sigma)\| \leq a^\sigma. \quad (12)$$

This implies that

$$\begin{aligned} f_i(l_n; \sigma) &= \frac{\ln \|X_A(l_n \sigma, k_n \sigma_0)x_i(k_n \sigma_0)\|}{l_n \sigma} \\ &\leq \frac{\ln(\|X_A(l_n \sigma, k_n \sigma_0)\| \|x_i(k_n \sigma_0)\|)}{l_n \sigma} \\ &\leq \frac{1}{l_n \sigma} (\sigma \ln a + \ln \|x_i(k_n \sigma_0)\|) \end{aligned}$$

and, on the other hand,

$$\begin{aligned} f_i(l_n; \sigma) &\geq \frac{1}{l_n \sigma} \ln \left( \|X_A(k_n \sigma_0, l_n \sigma)\|^{-1} \|x_i(k_n \sigma_0)\| \right) \\ &\geq \frac{1}{l_n \sigma} (-\sigma \ln a + \ln \|x_i(k_n \sigma_0)\|). \end{aligned}$$

Now, by taking into account the above equalities and (11) we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{l_n \sigma} (\pm \sigma \ln a + \ln \|x_i(k_n \sigma_0)\|) \\ &= \lim_{n \rightarrow \infty} \frac{1}{l_n \sigma} \ln \|x_i(k_n \sigma_0)\| = \lim_{n \rightarrow \infty} \frac{1}{k_n \sigma_0} \ln \|x_i(k_n \sigma_0)\| \\ &= \lambda[x_i] \end{aligned}$$

and we obtain

$$\lim_{n \rightarrow \infty} f_i(l_n; \sigma) = \lambda[x_i].$$

For any  $n, m \in \mathbb{N}$  the linear subspace  $V_i(n)$  of vectors  $x_j(n)$ ,  $j \neq i$  can be represented in the form  $V_i(n) = X_A(n, m)V_i(m)$ . Let us denote  $\sigma_1 = \max\{\sigma_0, \sigma\}$  and consider  $n, m \in \mathbb{N}$  such that  $|n - m| \leq \sigma_1$ . Then by Lemma 1 we have

$$\begin{aligned} \varphi_i(n) &\doteq \angle(x_i(n); V_i(n)) \\ &= \angle(X_A(n, m)x_i(m); X_A(n, m)V_i(m)) \\ &\geq 2\angle(x_i(m); V_i(m))(\chi(X_A(n, m)))^{1-s} \frac{1}{\pi} \\ &= 2\varphi_i(m)(\chi(X_A(n, m)))^{1-s} \frac{1}{\pi} \geq c\varphi_i(m), \end{aligned}$$

where

$$c = \frac{2a^{2\sigma_1(1-s)}}{\pi}$$

by (12).

Two cases are possible.

*Case 1* ( $\sigma < \sigma_0$ ): Let  $p = \lceil \frac{\sigma_0}{\sigma} \rceil$ . We take a  $j \in \mathbb{N}$  such that  $k_j > 2$ . It follows from (10) that

$$l_j \leq k_j \frac{\sigma_0}{\sigma} < k_j \left( \left\lceil \frac{\sigma_0}{\sigma} \right\rceil + 1 \right) = k_j(p + 1),$$

i.e.,

$$\frac{k_j}{l_j} > \frac{1}{p + 1}.$$

Let  $m$  range over the set  $\Gamma_i^\gamma(k_j - 2; \sigma_0)$ . The interval  $\left[ \left(m - \frac{1}{2}\right)\sigma_0, \left(m + \frac{1}{2}\right)\sigma_0 \right)$  for each  $m$  contains at least  $p$  multiples of  $\sigma$ . At all of these points the angle  $\varphi_i$  is not less than  $c\gamma$ . In addition, all these points lie on the real line on the left of

$$\left(k_j - 2 + \frac{1}{2}\right)\sigma_0 < k_j \sigma_0 - \sigma_0 < l_j \sigma + \sigma - \sigma_0 < l_j \sigma,$$

i.e., to the left of  $l_j \sigma$ . Therefore all of them belong to the set  $\Gamma_i^{c\gamma}(l_j; \sigma)$ . Then for the total number of elements of the set  $\Gamma_i^{c\gamma}(l_j; \sigma)$  we have the estimate

$$N_i^{c\gamma}(l_j; \sigma) \geq pN_i^\gamma(k_j - 2; \sigma_0) \geq p(N_i^\gamma(k_j; \sigma_0) - 2)$$

and

$$\begin{aligned} g_i^{c\gamma}(l_j; \sigma) &= \frac{N_i^{c\gamma}(l_j; \sigma)}{l_j} \geq \frac{p(N_i^\gamma(k_j; \sigma_0) - 2)}{l_j} \\ &= \frac{pN_i^\gamma(k_j; \sigma_0)}{k_j} \frac{k_j}{l_j} - \frac{2p}{l_j} > \frac{pg_i^\gamma(k_j; \sigma_0)}{p + 1} - \frac{2p}{l_j}. \end{aligned}$$

Consequently

$$\begin{aligned} \limsup_{j \rightarrow \infty} g_i^{c\gamma}(l_j; \sigma) &\geq \lim_{j \rightarrow \infty} \left( \frac{pg_i^\gamma(k_j; \sigma_0)}{p + 1} - \frac{2p}{l_j} \right) \\ &= \frac{p}{p + 1} \lim_{j \rightarrow \infty} g_i^\gamma(k_j; \sigma_0) > 0. \end{aligned}$$

*Case 2* ( $\sigma \geq \sigma_0$ ): Denote  $q = \lceil \frac{\sigma}{\sigma_0} \rceil$ . Let  $j \in \mathbb{N}$  satisfy the condition  $l_j > 1$ . It follows from (10) that

$$\frac{k_j}{l_j} \geq \frac{\sigma}{\sigma_0} \geq \left\lceil \frac{\sigma}{\sigma_0} \right\rceil = q.$$

We choose some positive integer  $l < l_j$ . Then

$$\left(l + \frac{1}{2}\right)\sigma < (l + 1)\sigma \leq l_j\sigma \leq k_j\sigma_0.$$

The interval  $[(l - \frac{1}{2})\sigma, (l + \frac{1}{2})\sigma]$  contains at most  $q + 1$  multiples of  $\sigma_0$ . If at least one of them belongs to  $\Gamma_i^\gamma(k_j; \sigma_0)$ , then  $l \in \Gamma_i^{c\gamma}(l_j; \sigma)$ . Consequently

$$N_i^\gamma(k_j; \sigma_0) \leq (q + 1)N_i^{c\gamma}(l_j; \sigma)$$

and

$$\begin{aligned} g_i^{c\gamma}(l_j; \sigma) &= \frac{N_i^{c\gamma}(l_j; \sigma)}{l_j} \geq \frac{N_i^\gamma(k_j; \sigma_0)}{(q + 1)l_j} \\ &= \frac{N_i^\gamma(k_j; \sigma_0)}{k_j} \frac{k_j}{(q + 1)l_j} \geq \frac{g_i^\gamma(k_j; \sigma_0)q}{q + 1}. \end{aligned}$$

Therefore

$$\limsup_{j \rightarrow \infty} g_i^{c\gamma}(l_j; \sigma) \geq \frac{q}{q + 1} \lim_{j \rightarrow \infty} g_i^\gamma(k_j; \sigma_0) > 0.$$

Now from the sequence  $(l_j)_{j \in \mathbb{N}}$  we extract a strictly increasing subsequence  $(l_{j_m})_{m \in \mathbb{N}}$  on which the limit

$$\limsup_{j \rightarrow \infty} g_i^{c\gamma}(l_j; \sigma)$$

is realized. The subsequence satisfies the relations

$$\lim_{m \rightarrow \infty} f_i(l_{j_m}; \sigma) = \lambda[x_i], \quad \lim_{m \rightarrow \infty} g_i^{c\gamma}(l_{j_m}; \sigma) > 0,$$

it means that the solution  $x_i(\cdot)$  is  $\sigma$ -broken away. The proof is completed. ■

*Corollary 2:* If a FSS is  $\sigma_0$ -splitted for certain  $\sigma_0 \in \mathbb{N}$ , then it is  $\sigma$ -splitted for any  $\sigma \in \mathbb{N}$ .

Having in mind Theorem 2 and Corollary 2 we say that the solution  $x_i(\cdot)$  occurring in the FSS  $\{x_1(\cdot), \dots, x_s(\cdot)\}$  is *broken away* if it is  $\sigma$ -broken away for some  $\sigma \in \mathbb{N}$ . Accordingly, a FSS is called *splitted* if it is  $\sigma$ -splitted for some  $\sigma \in \mathbb{N}$ .

*Definition 7:* System (4) that has a splitted normal FSS is called a *splitted system*.

The next example shows that there are systems which are not splitted, in particular such that they have no splitted FSS.

*Example 1:* Consider system (4) with

$$A(n) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad n \in \mathbb{N}.$$

Obviously,

$$X_A(n, m) = \begin{pmatrix} 1 & n - m \\ 0 & 1 \end{pmatrix}.$$

Let  $x(\cdot)$  be any nontrivial solution of system (4) with given  $A(\cdot)$  and let  $x(1) = \text{col}(\alpha, \beta)$ . Then  $x(n) = X_A(n, 1)x(1) = \text{col}(\alpha + \beta n, \beta)$ . If  $\beta \neq 0$ , then  $\angle(x(n), e_1) \doteq \vartheta \rightarrow 0$  or  $\vartheta \rightarrow \pi$  as  $n \rightarrow \infty$ , since

$$\cos \vartheta = \frac{\langle x(n), e_1 \rangle}{\|e_1\| \|x(n)\|} = \frac{\alpha + \beta n}{\sqrt{\beta^2 + (\alpha + \beta n)^2}} \rightarrow \frac{\beta}{|\beta|}$$

as  $n \rightarrow \infty$ . On the other hand, if  $\beta = 0$ , then  $\vartheta = 0$  for all  $n \in \mathbb{N}$ . Hence the angle between any two solutions from any

FSS tends to 0 or  $\pi$ . By Definition 6 it means that any FSS of this system is not splitted.

*Definition 8* (see [20, p. 15], [19, p. 100]): Let  $L(\cdot)$  be a Lyapunov sequence. A linear transformation

$$y = L(n)x, \quad n \in \mathbb{N}, \tag{13}$$

where  $x, y \in \mathbb{R}^s$ , is called a *Lyapunov transformation*.

*Theorem 3:* A Lyapunov transformation preserves the property of a solution being broken away.

*Proof:* Suppose the solution  $x_i(\cdot)$  occurring in the FSS  $\{x_1(\cdot), \dots, x_s(\cdot)\}$  is broken away. Let us apply a Lyapunov transformation (13) to (4). We shall show that the solutions  $y_j(\cdot)$  from the FSS  $\{y_1(\cdot), \dots, y_s(\cdot)\}$ , where  $y_j(\cdot) \doteq L(\cdot)x_j(\cdot)$ ,  $j = 1, \dots, s$ , of the transformed system, is broken away.

We take an arbitrary  $\sigma \in \mathbb{N}$ , and by  $\psi_i(n)$  we denote the angle between  $y_i(n)$  and the linear span  $L(n)V_i(n)$  of the vectors  $y_k(n)$ ,  $k \neq i$ ,  $n \in \mathbb{N}$ . It follows from Lemma 1 that

$$\begin{aligned} \psi_i(n) &\geq \frac{2}{\pi} \varphi_i(n) \kappa^{1-s}(L(n)) \\ &= \frac{2}{\pi} \varphi_i(n) \|L(n)\|^{1-s} \|L^{-1}(n)\|^{1-s}. \end{aligned}$$

Since  $L(\cdot)$  is a Lyapunov sequence, it follows that there exists  $c > 0$  such that

$$\frac{2}{\pi} \|L(n)\|^{1-s} \|L^{-1}(n)\|^{1-s} > c$$

for all  $n \in \mathbb{N}$ . Consequently,

$$\psi_i(n) \geq c\varphi_i(n)$$

for all  $n \in \mathbb{N}$ . For  $\alpha \in (0, \frac{\pi}{2}]$ , we set

$$\begin{aligned} L\Gamma_i^\alpha &\doteq \{j \in \mathbb{N} : \psi_i(j\sigma) \geq \alpha\}, \\ L\Gamma_i^\alpha(k) &\doteq L\Gamma_i^\alpha \cap \{1, \dots, k\}, \\ Lg_i^\alpha(n) &= \frac{1}{k} \sum_{j \in L\Gamma_i^\alpha(k)} 1. \end{aligned}$$

If  $j \in \Gamma_i^\alpha$ , then

$$\psi_i(j\sigma) \geq c\varphi_i(j\sigma) \geq c\alpha,$$

i.e.,  $j \in L\Gamma_i^{c\alpha}$ . Consequently,

$$\Gamma_i^\alpha(k) \subset L\Gamma_i^{c\alpha}(k)$$

and

$$g_i^\alpha(k) \leq Lg_i^{c\alpha}(k)$$

for all  $k \in \mathbb{N}$ .

Let  $\gamma \in (0, \frac{\pi}{2}]$  and assume the sequence  $(k_j)_{j \in \mathbb{N}}$  satisfies the relations

$$\lim_{j \rightarrow \infty} \frac{\ln \|x_i(k_j\sigma)\|}{k_j\sigma} = \lambda[x_i]$$

and

$$\lim_{j \rightarrow \infty} g_i^\gamma(k_j) > 0.$$

Then

$$\lim_{j \rightarrow \infty} \frac{\ln \|y_i(k_j \sigma)\|}{k_j \sigma} = \lambda[y_i],$$

$$\limsup_{j \rightarrow \infty} Lg_i^{\sigma \gamma}(k_j) \geq \lim_{j \rightarrow \infty} g_i^\gamma(k_j) > 0.$$

We extract a subsequence  $(k_{j_m})_{m \in \mathbb{N}}$  of the sequence  $(k_j)_{j \in \mathbb{N}}$  on which the upper limit  $\limsup_{j \rightarrow \infty} Lg_i^{\sigma \gamma}(k_j)$  is realized. Then

$$\lim_{m \rightarrow \infty} \frac{\ln \|y_i(k_{j_m} \sigma)\|}{k_{j_m} \sigma} = \lambda[y_i]$$

and

$$\lim_{m \rightarrow \infty} Lg_i^{\sigma \gamma}(k_{j_m}) > 0.$$

It means that the solution  $y_i(\cdot)$  is  $\sigma$ -broken away. The proof of the theorem is complete. ■

*Corollary 3:* A Lyapunov transformation preserves the splitting property of a system. Moreover, a splitted FSS is transformed into a splitted FSS.

*Remark 3:* Each system (4) that can be reduced by a Lyapunov transformation to a diagonal form is splitted, since the solutions  $x_1(\cdot), \dots, x_s(\cdot)$  of a diagonal system with the initial conditions  $x_i(1) = e_i, i = 1, \dots, s$ , form a normal FSS of this system and preserve constant angles between themselves (equal to  $\frac{\pi}{2}$ ).

## VI. BASIC RESULT

In this section, we prove the main property of splitted systems.

*Theorem 4:* Suppose that system (4) has a splitted FSS  $x_1(\cdot), \dots, x_s(\cdot)$ . Then there exist  $\beta > 0$  and  $\delta > 0$  such that for any  $\xi_i \in [-\delta, \delta], i = 1, \dots, s$ , there exists an admissible multiplicative perturbation  $R(\cdot)$ , satisfying the estimate

$$\|R - I\|_\infty \leq \beta \max\{|\xi_i| : i = 1, \dots, s\} \quad (14)$$

and such that the solutions  $\bar{x}_i(\cdot), i = 1, \dots, s$ , of system (8) with the initial conditions  $\bar{x}_i(1) = x_i(1), i = 1, \dots, s$ , satisfy the relations

$$\lambda[\bar{x}_i] = \lambda[x_i] + \xi_i, \quad i = 1, \dots, s.$$

*Proof:* Fix  $\sigma = 1$ . Since the solutions  $x_i(\cdot), i = 1, \dots, s$ , are broken away, it follows that there exist a number  $\gamma \in (0, \frac{\pi}{2}]$  and realizing sequences  $(k_j(i))_{j \in \mathbb{N}} \subset \mathbb{N}$  for solutions  $x_i(\cdot), i = 1, \dots, s$ , such that

$$\rho_i = \lim_{j \rightarrow \infty} g_i^\gamma(k_j(i)) > 0$$

for any  $i = 1, \dots, s$ . Note that the inequality  $\rho_i \leq 1$  is always valid, since

$$\sup\{g_i^\gamma(k) : k \in \mathbb{N}, i = 1, \dots, s\} \leq 1.$$

This, together with Lemma 3, implies that each function

$$\Lambda_i^\gamma(\mu) \doteq \limsup_{k \rightarrow \infty} (f_i(k) + \mu g_i^\gamma(k))$$

satisfies the estimate

$$\lambda[x_i] + \mu \geq \Lambda_i^\gamma(\mu) \geq \lambda[x_i] + \rho_i \mu, \quad (15)$$

where

$$\rho = \min\{\rho_i : i = 1, \dots, s\},$$

and for each  $t \geq 0$ , there exists a  $\mu_t^i \in [0, \rho^{-1}t]$  such that

$$\Lambda_i^\gamma(\mu_t^i) = \lambda[x_i] + t. \quad (16)$$

Since  $\gamma$  is a fixed number throughout the proof, from now on we omit the superscript  $\gamma$ .

Let us fix  $r \in (0, 1)$  and  $\delta_1 \in (0, \ln(L_1 + 1))$ , where

$$L_1 = \frac{r \sin \gamma}{s}$$

and denote

$$L = \frac{L_1}{\delta_1}.$$

It is easy to verify that

$$|\exp(\delta_1) - 1| < L_1, \quad |\exp(-\delta_1) - 1| < L_1$$

and

$$|\exp(\tau) - 1| \leq L |\tau|$$

for all  $\tau \in (-\infty, \delta_1]$ . We set

$$\delta = \frac{\delta_1 \rho}{3}$$

and take an arbitrary  $\xi_i \in [-\delta, \delta], i = 1, \dots, s$ . Let  $\eta \doteq \min\{\xi_i : i = 1, \dots, s\}$  and  $\zeta_i \doteq \xi_i - \eta$ . Then  $|\eta| \leq \delta$  and  $0 = \xi_i - \xi_i \leq \xi_i - \eta = \zeta_i \leq \xi_i + |\eta| \leq 2\delta$ , that is,

$$0 \leq \zeta_i \leq 2\delta.$$

Let

$$\varepsilon \doteq \max\{|\xi_i| : i = 1, \dots, s\}.$$

Then

$$|\eta| \leq \varepsilon$$

and

$$\zeta_i \leq |\xi_i| + |\eta| \leq 2\varepsilon, \quad i = 1, \dots, s.$$

Let the quantities  $\mu_i, i = 1, \dots, s$ , be obtained from the conditions

$$\Lambda_i(\mu_i) = \lambda[x_i] + \zeta_i.$$

Since  $\zeta_i \geq 0$ , it follows that the numbers  $\mu_i$  are well defined by (16), and by (15) we have the estimates

$$\mu_i \geq \Lambda_i(\mu_i) - \lambda[x_i] = \zeta_i \geq \rho \mu_i \geq 0$$

for all  $i = 1, \dots, s$ .

For each  $n \in \mathbb{N}$ , we introduce a matrix  $R(n) \in \mathbb{R}^{s \times s}$  by the formulas

$$R(n)x_i(n) = x_i(n) \exp(s_i(n)), \quad i = 1, \dots, s, \quad (17)$$



where

$$s_i(n) = \begin{cases} \eta + \mu_i & \text{for } n \in \Gamma_i, \\ \eta & \text{for } n \notin \Gamma_i. \end{cases}$$

Then we have the estimates

$$\begin{aligned} |s_i(n)| &\leq |\eta| + \mu_i \leq |\eta| + \frac{\zeta_i}{\rho} \\ &\leq \delta + \frac{2\delta}{\rho} = \frac{\delta}{\rho}(\rho + 2) \leq \frac{3\delta}{\rho} = \delta_1 \end{aligned}$$

for  $i = 1, \dots, s$  and  $n \in \mathbb{N}$ . This, together with the definition of  $\delta_1$ , implies that

$$|\exp(s_i(n)) - 1| \leq L_1$$

and

$$\begin{aligned} |\exp(s_i(n)) - 1| &\leq L|s_i(n)| \leq L \left( |\eta| + \frac{\zeta_i}{\rho} \right) \\ &\leq L \left( |\eta| + \frac{2\varepsilon}{\rho} \right) \leq \left( 1 + \frac{2}{\rho} \right) L\varepsilon. \end{aligned}$$

By the definition of a FSS, the vectors  $x_1(n), \dots, x_s(n)$  are linearly independent for each  $n \in \mathbb{N}$  and, by (17), they are eigenvectors of the matrix  $R(n)$ . This implies that  $R(n)$  is a matrix of simple structure [25, p. 239, Proposition 2] and therefore it can be represented by the sum

$$R(n) = \sum_{i=1}^s P_n^i \exp(s_i(n)),$$

where the root projections  $P_n^i$  are given by the conditions

$$P_n^i x_i(n) = x_i(n)$$

and

$$P_n^i x_j(n) = 0$$

for  $j \neq i$ . Moreover

$$\sum_{i=1}^s P_n^i = I$$

for all  $n \in \mathbb{N}$ . By Lemma 2, we have the estimate

$$\|P_n^i\| \leq \frac{1}{\sin \gamma}$$

and hence the inequalities

$$\begin{aligned} \|R(n) - I\| &= \left\| \sum_{i=1}^s P_n^i (\exp(s_i(n)) - 1) \right\| \\ &\leq \sum_{i=1}^s \|P_n^i\| |\exp(s_i(n)) - 1| < s \frac{L_1}{\sin \gamma} = r. \end{aligned} \tag{18}$$

Moreover,

$$\|R(n) - I\| \leq s \frac{|\exp(s_i(n)) - 1|}{\sin \gamma} \leq \frac{s}{\sin \gamma} \left( 1 + \frac{2}{\rho} \right) L\varepsilon.$$

Hence, for the sequence  $R(\cdot) = (R(n))_{n \in \mathbb{N}}$  we have

$$\|R - I\|_\infty = \sup_{n \in \mathbb{N}} \|R(n) - I\| \leq \beta\varepsilon,$$

where

$$\beta = \frac{Ls \left( 1 + \frac{2}{\rho} \right)}{\sin \gamma}.$$

This proves (14).

Moreover, we have

$$X_{AR}(n + 1, n) = X_A(n + 1, n)R(n)$$

for all  $n \in \mathbb{N}$ . Since  $r \in (0, 1)$ , then the condition  $\|H - I\| < r$  implies that  $H$  is invertible and

$$\begin{aligned} \|H\| &\leq r + 1, \\ \|H^{-1}\| &\leq \frac{1}{1 - r} \end{aligned}$$

whatever  $H \in \mathbb{R}^{s \times s}$  is given [21, p. 301]. By (18) this in turn implies that the sequence  $R(\cdot)$  is an admissible multiplicative perturbation.

Consider the FSS  $\bar{x}_i(\cdot)$ ,  $i = 1, \dots, s$ , of (8) with such a perturbation with the initial conditions

$$\bar{x}_i(1) = x_i(1), \quad i = 1, \dots, s.$$

For every natural  $i \leq s$  and  $k \geq 2$  we have the equalities

$$\begin{aligned} \bar{x}_i(k) &= X_{AR}(k, 1)\bar{x}_i(1) = X_{AR}(k, 1)x_i(1) \\ &= \prod_{j=1}^{k-1} X_{AR}(j + 1, j)x_i(1) \\ &= \prod_{j=1}^{k-1} X_A(j + 1, j)R(j)x_i(1) \\ &= \prod_{j=1}^{k-1} X_A(j + 1, j) \exp(s_i(j))x_i(1) \\ &= \exp(s_i(1) + \dots + s_i(k - 1))X_A(k, 1)x_i(1) \\ &= \exp\left(\sum_{j=1}^{k-1} s_i(j)\right)x_i(k). \end{aligned}$$

It follows that the Lyapunov exponents of these solutions satisfy the relations

$$\begin{aligned} \lambda[\bar{x}_i] &= \limsup_{k \rightarrow \infty} \frac{1}{k} \ln \|\bar{x}_i(k)\| \\ &= \limsup_{k \rightarrow \infty} \frac{1}{k} \left( \ln \|x_i(k)\| + \sum_{j=1}^{k-1} s_i(j) \right) \\ &= \limsup_{k \rightarrow \infty} \left( f_i(k) + \frac{\mu_i N_i(k - 1)}{k} + \frac{\eta(k - 1)}{k} \right) \\ &= \limsup_{k \rightarrow \infty} \left( f_i(k) + \frac{\mu_i N_i(k)}{k} \right. \\ &\quad \left. + \frac{\mu_i(N_i(k - 1) - N_i(k))}{k} + \eta - \frac{\eta}{k} \right) \end{aligned}$$

$$\begin{aligned} &= \limsup_{k \rightarrow \infty} (f_i(k) + \mu_i g_i(k)) + \eta = \Lambda_i(\mu_i) + \eta \\ &= \lambda[x_i] + \zeta_i + \eta = \lambda[x_i] + \xi_i \end{aligned}$$

for  $i = 1, \dots, s$ . ■

### VII. APPLICATIONS

In this section, we prove several results that follow from Theorem 4 and demonstrate the importance of the introduced concept of splitted systems for studying the behavior of the Lyapunov spectrum under the action of small perturbations.

*Theorem 5:* If system (4) has a splitted FSS which is not normal, then the Lyapunov spectrum of system (4) is not stable.

*Proof:* Let  $\{x_1(\cdot), \dots, x_s(\cdot)\}$  be a splitted FSS of system (4) which is not normal. Denote

$$\mu_i = \lambda[x_i], \quad i = 1, \dots, s.$$

Without loss of generality, we assume that the sequence  $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{R}_{\leq}^s$ . Note that among the numbers  $\mu_1, \dots, \mu_s$ , there are equal ones, because otherwise the FSS  $\{x_1(\cdot), \dots, x_s(\cdot)\}$  would be incompressible, and therefore normal. If

$$\lambda(A) = (\lambda_1(A), \dots, \lambda_s(A)) \in \mathbb{R}_{\leq}^s$$

is the Lyapunov spectrum of (4), then  $\mu \neq \lambda(A)$ .

Choose a number  $\alpha > 0$  so small that the sets  $\mathcal{O}_\alpha(\mu)$  and  $\mathcal{O}_\alpha(\lambda(A))$  do not intersect. Take an arbitrary positive  $\varepsilon < \min\{\alpha, \delta\}$ , where  $\delta$  is from Theorem 4. With the selected  $\varepsilon$ , in the neighborhood of  $\mathcal{O}_\varepsilon(\mu)$  there is a sequence of numbers  $\mu' = (\mu'_1, \dots, \mu'_s)$ , all of whose elements are different, i.e.,  $\mu' \in \mathbb{R}_{<}^s$ . Take  $\xi_i \doteq \mu'_i - \mu_i, i = 1, \dots, s$ . Then  $|\xi_i| < \varepsilon < \delta$ , therefore, by Theorem 4, there exist an admissible multiplicative perturbation  $R(\cdot)$  satisfying the estimate  $\|R - I\|_\infty < \beta\varepsilon$  and such that the system (8) with such a perturbation has a FSS  $\{\bar{x}_1(\cdot), \dots, \bar{x}_s(\cdot)\}$  such that

$$\lambda[\bar{x}_i] = \lambda[x_i] + \xi_i = \mu_i + \xi_i = \mu'_i, \quad i = 1, \dots, s.$$

Since the numbers  $\mu'_i$  are pairwise distinct, this FSS is incompressible and therefore normal. Hence, the set  $\mu'$  is the Lyapunov spectrum of system (8). Obviously,  $\mu' \notin \mathcal{O}_\alpha(\lambda(A))$  for all such  $\varepsilon$ , which means that the Lyapunov spectrum of system (4) is not stable. ■

*Corollary 4:* If the Lyapunov spectrum of system (4) is stable, then each splitted FSS is normal.

It can be easily seen that stability of Lyapunov spectrum is equivalent to continuity of the map  $R(\cdot) \mapsto \lambda(AR)$  at the point  $R(n) \equiv I, n \in \mathbb{N}$ . Theorem 4 can be also used to study some other properties of this map.

*Definition 9:* The Lyapunov spectrum of system (4) is called *open*, if the mapping  $\lambda(AR): \mathcal{R} \rightarrow \mathbb{R}_{\leq}^s$  is open at the point  $R(n) \equiv I, n \in \mathbb{N}$ , that is, for any  $\varepsilon > 0$ , there exists  $\gamma = \gamma(\varepsilon) > 0$  such that the inclusion

$$\mathcal{O}_\gamma(\lambda(A)) \subset \lambda(\mathcal{R}_\varepsilon)$$

holds.

*Remark 4:* It was proved in [5, Theorem 3] that if the Lyapunov spectrum of system (4) is stable, then it is open. Here we obtain another sufficient condition for the Lyapunov spectrum to be open, expressed in terms of the splittness of system (4).

*Theorem 6:* If system (4) is splitted and has a non-multiple Lyapunov spectrum, i.e.  $\lambda(A) \in \mathbb{R}_{<}^s$ , then the Lyapunov spectrum of this system is open.

*Proof:* Let  $\lambda(A) = (\lambda_1, \dots, \lambda_s) \in \mathbb{R}_{<}^s$  and  $\{x_1(\cdot), \dots, x_s(\cdot)\}$  be a normal splitted FSS of system (4), such that  $\lambda[x_i] = \lambda_j, i = 1, \dots, s$ . Denote

$$\eta \doteq \frac{1}{3} \min\{\lambda_{i+1} - \lambda_i : i = 1, \dots, s - 1\}.$$

For arbitrary  $\varepsilon > 0$ , we put

$$\gamma = \gamma(\varepsilon) = \min\{\eta, \varepsilon/\beta, \delta/\beta, \delta\},$$

where  $\delta > 0$  and  $\beta > 0$  are the quantities from Theorem 4. Take any  $\mu = (\mu_1, \dots, \mu_s) \in \mathcal{O}_\gamma(\lambda(A))$  and prove that  $\mu \in \lambda(\mathcal{R}_\varepsilon)$ . Let  $\xi_i \doteq \mu_i - \lambda_i$ . Then  $|\xi_i| < \gamma \leq \delta$ . By theorem 4 there exists an admissible multiplicative perturbation  $R(\cdot)$  that ensures the equality

$$\lambda[\bar{x}_i] = \lambda[x_i] + \xi_i = \lambda_i + \xi_i = \mu_i$$

for the solution of system (8) with the initial condition  $\bar{x}_i(1) = x_i(1)$ , and such that the inequality

$$\|R - I\|_\infty < \beta \max\{|\xi_i| : i = 1, \dots, s\} < \beta\gamma \leq \varepsilon$$

holds, i.e.  $R(\cdot) \in \mathcal{R}_\varepsilon$ .

Consider the FSS  $\{\bar{x}_1(\cdot), \dots, \bar{x}_s(\cdot)\}$  of system (8). Let us note that

$$\begin{aligned} \lambda[\bar{x}_{i+1}] - \lambda[\bar{x}_i] &= \lambda_{i+1} - \lambda_i + \xi_{i+1} - \xi_i \\ &\geq 3\eta - |\xi_{i+1}| - |\xi_i| \geq 3\eta - 2\delta \geq 3\eta - 2\eta \\ &= \eta > 0 \end{aligned}$$

for  $i \in \{1, \dots, s-1\}$ . Hence the numbers  $\lambda[\bar{x}_1], \dots, \lambda[\bar{x}_s]$  are pairwise different, so the FSS  $\{\bar{x}_1(\cdot), \dots, \bar{x}_s(\cdot)\}$  of system (8) is normal and

$$\lambda(AR) = (\lambda[\bar{x}_1], \dots, \lambda[\bar{x}_s]) = \mu.$$

It means that  $\mu \in \lambda(\mathcal{R}_\varepsilon)$ . ■

The property of openness of the Lyapunov spectrum of system (4) can be interpreted as the property of local assignability of the Lyapunov spectrum of system (8) under the action of a multiplicative perturbation  $R(\cdot)$ , which in this context is considered as a matrix control.

*Definition 10:* The Lyapunov spectrum of system (8) is called *locally assignable* if for any  $\varepsilon > 0$  there exists such a  $\gamma = \gamma(\varepsilon) > 0$  that for any  $\mu \in \mathcal{O}_\gamma(\lambda(A))$  there is a matrix control  $R(\cdot) \in \mathcal{R}_\varepsilon$  such that  $\lambda(AR) = \mu$ .

It is clear that the property of openness of the Lyapunov spectrum of system (4) coincides with the property of local assignability of the Lyapunov spectrum of system (8); therefore, the following corollary holds.

Corollary 5: If system (4) is splitted and has a non-multiple Lyapunov spectrum, then the Lyapunov spectrum of system (8) is locally assignable.

VIII. EXAMPLES

In this section we shall present examples that show that there are both systems with broken away normal FSS and systems having broken away FSS which is not normal. We shall also show that the assumptions of the proved theorems can be effectively verified.

In the first example we present a system with splitted FSS which is not normal and therefore its Lyapunov spectrum is not stable. In this example we use the following result.

Lemma 6: We have

$$\limsup_{n \rightarrow \infty} \sin \ln n = 1, \tag{19}$$

$$\liminf_{n \rightarrow \infty} \sin \ln n = -1. \tag{20}$$

Proof: Consider the following sequences

$$t_k = \exp\left(2k + \frac{1}{2}\right)\pi, \quad n_k = [t_k], \quad k \in \mathbb{N}.$$

We have

$$\begin{aligned} 1 &\geq \limsup_{n \rightarrow \infty} \sin \ln n \geq \limsup_{k \rightarrow \infty} \sin \ln n_k \\ &= \limsup_{k \rightarrow \infty} \sin(\ln t_k + \ln(n_k/t_k)) \\ &= \limsup_{k \rightarrow \infty} \left( \sin(\ln t_k) \cos(\ln(n_k/t_k)) \right. \\ &\quad \left. + \cos(\ln t_k) \sin(\ln(n_k/t_k)) \right) \\ &= \limsup_{k \rightarrow \infty} \cos(\ln(n_k/t_k)) = \cos \ln\left(\lim_{k \rightarrow \infty} (n_k/t_k)\right) \\ &= \cos \ln 1 = 1. \end{aligned}$$

Considering sequences

$$t_k = \exp\left(2k + \frac{3}{2}\right)\pi, \quad n_k = [t_k], \quad k \in \mathbb{N},$$

we may prove in a similar way the equality (20). ■

Example 2: Let us consider system (4) with  $s = 2$  and a diagonal matrix

$$A(n) = \text{diag}(a_1(n), a_2(n)), \quad n \in \mathbb{N},$$

where

$$\begin{aligned} a_1(n) &= \exp(n \sin \ln n - (n + 1) \sin \ln(n + 1)), \\ a_2(n) &= \exp\left(2((n + 1) \sin \ln(n + 1) - n \sin \ln n)\right). \end{aligned}$$

The sequence  $A(\cdot)$  is a Lyapunov sequence since

$$|(n + 1) \sin \ln(n + 1) - n \sin \ln n| \leq \sqrt{2}, \quad n \in \mathbb{N}.$$

To obtain the last inequality it is enough to apply the Lagrange mean value theorem to the function  $f: [n, n + 1] \rightarrow \mathbb{R}, f(t) = t \sin \ln t$ . It is easy to see that the matrix

$$\Phi(n, 1) = \begin{pmatrix} \exp(-n \sin \ln n) & 0 \\ 0 & \exp(2n \sin \ln n) \end{pmatrix}, \quad n \in \mathbb{N},$$

is a fundamental matrix of this system. By Remark 3 the corresponding FSS is normal. By (19), (20) we have

$$\lambda(A) = (1, 2).$$

Consider now the FSS  $\{x_1(\cdot), x_2(\cdot)\}$ , where

$$\begin{aligned} x_1(n) &= \begin{pmatrix} \exp(-n \sin \ln n) \\ \exp(2n \sin \ln n) \end{pmatrix}, \\ x_2(n) &= \begin{pmatrix} 0 \\ \exp(2n \sin \ln n) \end{pmatrix}, \quad n \in \mathbb{N}. \end{aligned}$$

From (19) and (20) it follows that  $\lambda[x_1] = \lambda[x_2] = 2$  and that

$$n_k = [t_k] + 1, \quad k \in \mathbb{N},$$

is a realizing sequence for  $x_1(\cdot)$  and  $x_2(\cdot)$ , where  $t_k = \exp\left(2k + \frac{1}{2}\right)\pi$ . Therefore FSS  $\{x_1(\cdot), x_2(\cdot)\}$  is not normal. We shall show that it is splitted.

Denote by  $\varphi_1(n)$  the angle between  $x_1(n)$  and  $x_2(n)$ . After some simple calculations we have

$$\cos \varphi_1(n) = (1 + \exp(-6n \sin \ln n))^{-1/2}.$$

Let us fix a

$$c \in (\sqrt{2}/2, 1). \tag{21}$$

Let  $\gamma \doteq \arccos c$ , then  $\gamma \in (0, \pi/4)$ . Notice that  $n \in \Gamma_1^\gamma(M; 1)$  for some  $M \in \mathbb{N}$  if and only if  $n \in \mathbb{N}, n \leq M$  and

$$\cos^2 \varphi_1(n) \leq c^2,$$

i.e.,

$$1 + \exp(-6n \sin \ln n) \geq 1/c^2.$$

The last inequality is equivalent to the inequality

$$6n \sin \ln n \leq \ln\left(\frac{c^2}{1 - c^2}\right). \tag{22}$$

By the choice of  $c$  we know that  $\frac{c^2}{1 - c^2} > 1$  and therefore

$$\ln\left(\frac{c^2}{1 - c^2}\right) > 0.$$

The last inequality means that each  $n \in \mathbb{N}$  satisfying  $\sin \ln n \leq 0$  also satisfies inequality (22) and therefore

$$\{n \in \mathbb{N} : n \leq M, \sin \ln n \leq 0\} \subset \Gamma_1^\gamma(M; 1)$$

and

$$\text{card}\{n \in \mathbb{N} : n \leq M, \sin \ln n \leq 0\} \leq N_1^\gamma(M; 1),$$

where  $\text{card } B$  denotes the number of elements of the set  $B$ . Let us fix  $k \in \mathbb{N}, k > 1$ . Then for each

$$n \in [\exp((2k - 1)\pi), \exp(2k\pi)] \cap \mathbb{N}$$

we have  $\sin \ln n \leq 0$  and  $1 < n < n_k$ , therefore

$$[\exp((2k - 1)\pi), \exp(2k\pi)] \cap \mathbb{N} \subset \Gamma_1^\gamma(n_k; 1)$$

and

$$N_1^\gamma(n_k; 1) \geq \exp(2k\pi) - \exp((2k - 1)\pi) - 1.$$

Since  $n_k \leq \exp\left(\left(2k + \frac{1}{2}\right)\pi\right) + 1$ , we have

$$\limsup_{k \rightarrow \infty} \frac{N_1^\gamma(n_k; 1)}{n_k} \geq \frac{1 - \exp(-\pi)}{\exp(\pi/2)} > 0,$$

which completes the proof.

In the next example we present a system with a stable Lyapunov spectrum such that each of its FSS, which is not normal, is not splitted.

*Example 3:* Let us consider the system

$$x(n + 1) = \text{diag}(1, 2)x(n), \quad n \in \mathbb{N}, x \in \mathbb{R}^2. \quad (23)$$

Since the system is time-invariant, it follows from [5] that its Lyapunov spectrum is stable. Note that the normal FM of system (23) has the form

$$\Phi(n) = \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix},$$

and therefore the corresponding normal FSS of system (23) is splitted. Let us prove that every FSS of system (23), which is not normal, is not splitted. Let us divide the set of all non-trivial solutions of system (23) into two groups. We include in the first group all the solutions  $x(\cdot)$  of this system with the initial conditions  $x(1) = \alpha e_1$ , where  $\alpha \neq 0$ . Such solutions are constant and their Lyapunov exponents are equal to 0. In the second group we include all the solutions  $x(\cdot)$  of system (23) with initial conditions  $x(1) = \alpha e_1 + \beta e_2$ , where  $\beta \neq 0$ . They have the form  $x(n) = \text{col}(\alpha, 2^{n-1}\beta)$  and their Lyapunov exponents are equal to  $\ln 2$ . Each normal FSS contains solutions from both of these groups. If FSS is not normal, then it should contain two solutions only from the second group. We cannot construct a normal FSS from the solutions from the first group, since any two solutions from the first group are obtained one from the other by multiplying by some constant, i.e., they are linearly dependent.

Take any solution  $x(\cdot)$  from the second group and calculate the angle between  $x(\cdot)$  and the vector  $e_2$ . We have

$$\cos \angle(e_2, x(n)) = \frac{\langle e_2, x(n) \rangle}{\|e_2\| \|x(n)\|} = \frac{2^{n-1}\beta}{\sqrt{\alpha^2 + 4^{n-1}\beta^2}} \rightarrow \frac{\beta}{|\beta|} \text{ for } n \rightarrow \infty,$$

i.e.,  $\angle(e_2, x(n))$  tends to 0 or  $\pi$  when  $n \rightarrow \infty$ . It follows that the angle between any two vectors  $x_1(n)$  and  $x_2(n)$  corresponding to some solutions from the second group tends to 0 or  $\pi$  when  $n \rightarrow \infty$ . By Definition 6 it means that FSS  $\{x_1(\cdot), x_2(\cdot)\}$  which is not normal is also not splitted.

### IX. CONCLUSION

In this paper we study the stability and openness problems of Lyapunov spectra of discrete time-varying linear systems. To investigate this problems we introduce the concept of broken away solutions and the concept of splitted systems. We demonstrate some properties of these concepts and then

applied these properties to the stability and openness problems of Lyapunov spectra. One of the main results states that if the Lyapunov spectrum is stable then each splitted fundamental system of solutions is normal. It is worth mentioning that this condition does not require a reduction of the given system to any special form, but is rather expressed in terms of the system itself. Another important result is that if a given system is splitted and has a non-multiple Lyapunov spectrum, then it has an open Lyapunov spectrum. We expect that the proposed concepts of broken away solutions and splitted systems may be useful in the investigation of other problems in the theory of discrete time-varying linear systems, such as the problem of assignability of the Lyapunov spectrum. This will be a subject of our further research.

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