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RESEARCH ARTICLE

Fault-Tolerant Partition Resolvability in Mesh Related Networks and Applications

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ABSTRACT Fault-tolerance of a system measures its working capability in the presence of faulty components in the system. The fault-tolerant partition dimension of a network computes the least number of subcomponents of network required to distinctively identify each node in the presence of faults, having promising applications in telecommunication, robot navigation and geographical routing protocols. In this paper, certain triangular mesh networks including, triangular ladder (Tl_s), triangular mesh (T_s), reflection triangular mesh ($rl(T_s)$), tower triangular mesh (Tr_s) and reflection tower triangular mesh ($rl(Tr_s)$) networks are discussed for their partition and fault-tolerant partition resolvability. In this regard, it is shown that the partition dimension of these networks is 3, whereas their fault-tolerant partition dimension is 4.

INDEX TERMS Triangular ladder, triangular mesh, metric dimension, partition dimension, fault-tolerant partition dimension.

I. INTRODUCTION AND BASIC TERMINOLOGIES

Networks are the foundation for gathering and sending data, which should be used to inform and evolve strategies and efficiencies. These networks have been recognized as versatile interconnection networks having topologies that reflect the communication pattern of a wide variety of natural problems. Mesh related networks have been around for some time, but are growing in popularity as the prevalence of internet of things (IoT) is increasing in several areas like, industrial automation, agriculture, emergency services and smart cities. Mesh networks are quite reliable as they contain reasonably high degree of redundancy so that alternative routes can be made available to detour faulty nodes. Due to reliability of mesh networks, military organizations, emergency services like police and fire services often use mesh topologies in order to avoid breakdowns in communication. Triangular meshes are a widely adopted standard for representing surfaces where the geometry of an object is defined by the vertex coordinates of triangles, while the edges and triangular faces encode the topology, providing a light and concise representation of graphical data [11]. Adar and Epstein [1] and Nazeer *et al.* [27] computed metric dimension of extended meshes and triangular mesh related graphs respectively. Akhtar *et al.* obtained the largest subgraphs for the enhanced mesh networks [2].

Due to the applicability of metric dimension in identification of nodes in networks, metric dimension and parameters related to it are very common among the researchers nowadays. One of the unique and important parameter is partition dimension of graph which is generalization of metric dimension of graph. The partition dimension is described in several real world applications such as process of identifying network discovery and verification [9], coding and decoding of games and other strategies of games [10], the popular Djokovic-Winkler relationship [17] and the piloting or guidance of a robot [20].

The notion of partition resolvability in graphs and partition dimension was proposed as a generalized version of metric resolvability in graphs and metric dimension by Chartrand *et al.* [13]. Consider $\Im$ be a graph of order s with $V(\Im)$ as vertex and $E(\Im)$ as edge set. If two vertices

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$\delta, \xi \in V(\Im)$, then distance $d(\delta, \xi)$ between these vertices is the least number of edges in $\delta - \xi$ path. The distance between a vertex ξ and $U \subseteq V(\Im)$ is defined as $\min\{d(\xi, y)|y \in U\}$ and is denoted by $d(\xi, U)$. For a vertex $\xi \in V(\Im)$, $N(\xi)$ represents collection of open neighbours of ξ in Υ, *i.e.* $N(\xi) = \{b \in V(\Im) : b$ is adjacent to $\xi\}$ and the collection of closed neighbours of ξ will be represented by $N[\xi]$, *i.e.* $N[\xi] = N(\xi) \cup \{\xi\}$ [24]. Consider $\nu = \{\zeta_1, \zeta_2, \ldots, \zeta_f\}$ as an ordered collection of vertices in $\Im$. The notation $r(\xi|\nu)$ is the representation of ξ in association with ν and is defined by an f-ordered distance vector $(d(\xi, \zeta_1), d(\xi, \zeta_2), \ldots, d(\xi, \zeta_f))$. If each vertex of $\Im$ has unique representation associated to ν, then the subset ν is called a resolving set of $\Im$. The metric dimension (MD) of $\Im$ denoted by $\beta(\Im)$ is defined as $\min\{|\nu| : \nu$ is resolving set of $\Im\}$. Bousquet *et al.* studied the MD on sparse graphs and its applications to zero forcing sets [8]. Tomáš related MD to directed and undirected circulant graphs [32]. In 2021, MD of ideal-intersection graph of the ring $\mathbb{Z}_n$ was discussed by Saha *et al.* [28].

The fault-tolerant idea of the definition of resolving set was initiated by Hernando *et al.* in 2008 [18]. If for every pair of distinct vertices $\rho, \xi \in V(\Im)$, there exists at least two vertices $\alpha_1, \alpha_2 \in \nu$ such that $d(\rho, \alpha_m) \neq d(\xi, \alpha_m)$ for $m \in \{1, 2\}$, then, the set ν of $V(\Im)$ is called fault-tolerant resolving set. The set ν of such kind with least size is referred as fault-tolerant metric basis and its respective size as the fault-tolerant metric dimension (FTMD) of $\Im$, represented by $\beta'(\Im)$. The FTMD of generalized wheels and convex polytopes are discussed by Zheng *et al.* [35]. The FTMD of inter connection networks are discussed by Hayat *et al.* [16]. Sharma and Bhat discussed the FTMD of zero-divisor graphs of commutative rings [31]. Mehmood *et al.* discussed energy-efficient and cooperative fault-tolerant communication approach for wireless body area network, and reliable communication scheme for remote patient monitoring in wireless body area networks [21], [22].

Consider, $\mathbb{Y} = \{\xi_1, \xi_2, \ldots, \xi_f\}$ be a set of $V(\Im)$ of connected graph $\Im$ with f partition classes. The representation $r(\lambda|\mathbb{Y})$ of vertex λ in association with partition set $\mathbb{Y}$ is f-vector $(d(\lambda, \xi_1), d(\lambda, \xi_2), \ldots, d(\lambda, \xi_f))$. If representation of all the vertices in $\Im$ are distinct, then the partition $\mathbb{Y}$ is called resolving partition (RP) of $\Im$. The partition dimension (PD) of graph $\Im$ is defined as, $\min\{|\mathbb{Y}| : \mathbb{Y}$ is RP of $\Im\}$, and is notated by $\text{pd}(\Im)$. The graphs having PD 2 or n were characterised by Chartrand *et al.* [13]. Computational complexity for metric dimension of general graphs relates to non-deterministic polynomial time-hardness [12], [15], [20]. As partition dimension is generalized variant of metric dimension, so computing PD for general graphs is also NP hard. Due to this limitation, researchers are captivated to work on partition dimension problems. The concept of $\text{pd}(\Im)$ for numerous classes of graphs has discussed extensively in literature. For instance, the PD of starphene graph was evaluated by Ahmad *et al.* [3], bridge graph by Amrullah *et al.* [4] and of tri-hexagonal α-boron nanotube by Shabbir and Azeem [30]. The bounds on PD of generalized Mobius ladder were provided by Hussain *et al.* [19] and PD for certain classes of series parallel graph was studied by Monica *et al.* [23]. Chu *et al.* [14], Wei *et al.* [33] and Yero *et al.* [34] discussed the PD problem for convex polytopes, cycle related graphs and strong product graphs and cartesian product graphs respectively.

The furtherance in the avenue of research of PD as fault-tolerant partition dimension (FTPD) of graph was revealed by Salman *et al.* [29]. Consider, $\mathbb{Y} = \{\xi_1, \xi_2, \ldots, \xi_f\}$ be a set of $V(\Im)$ with f partition classes. The partition $\mathbb{Y}$ is known to be fault-tolerant resolving partition (FTRP) of $\Im$ if for every pair of distinct vertices $\rho, \xi \in V(\Im)$, $r(\rho|\mathbb{Y})$ and $r(\xi|\mathbb{Y})$ have at least two different coordinates. The FTPD of $\Im$ denoted by $\mathcal{F}(\Im)$ is defined as the least number of subsets in set $\mathbb{Y}$. Azhar *et al.* provided the exact value of FTPD of homogeneous caterpillar [5], cyclic networks [6], tadpole and necklace graph [7]. Nadeem *et al.* discussed FTPD of toeplitz networks and circulant graphs having $\{1, 2\}$ connection set [25], [26]. In this paper we investigate the FTPD for two triangular mesh architectures derived from the standard triangular ladder.

Chartrand *et al.* explored subsequent conclusions on $\text{pd}(\Im)$.

Proposition 1 [13]: Let $\Im$ be a graph, then;

(a) $\text{pd}(\Im) \leq \beta(\Im) + 1$.

(b) $\text{pd}(\Im) = 2$ *iff* $\Im = P_s$ *where* P_s *is a path.*

Salman *et al.* explored following conclusions regarding $\mathcal{F}(\Im)$.

Proposition 2 [29]: (a) $\mathcal{F}(\Im) \leq \beta'(\Im) + 1$, *whenever,* $s \geq 2$.

(b) $3 \leq \mathcal{F}(\Im) \leq s$, *whenever,* $s \geq 3$.

A. MAJOR CONTRIBUTIONS

The investigations conducted in this work lead to the following results:

For every $s \geq 2$.

(1) $\text{pd}(Tl_s) = 3$ and $\mathcal{F}(Tl_s) = 4$.

For every $s \geq 3$.

(2) $\text{pd}(T_s) = 3$ and $\mathcal{F}(T_s) = 4$.

(3) $\text{pd}(rl(T_s)) = 3$ and $\mathcal{F}(rl(T_s)) = 4$.

(4) $\text{pd}(Tr_s) = 3$ and $\mathcal{F}(Tr_s) = 4$.

(5) $\text{pd}(rl(Tr_s)) = 3$ and $\mathcal{F}(rl(Tr_s)) = 4$.

The rest of the article is organized in following manner: The Section 2, contains the calculation of exact values of FTPD of triangular ladder, and Section 3, for triangular mesh related graphs. Section 4, contains an application of the work done in context of sensors deployment in smart city. Finally, the article is concluded in fifth section by mentioning limitations and providing future research direction.

II. FAULT-TOLERANT PARTITION DIMENSION OF TRIANGULAR LADDER

We compute FTPD of triangular ladder in this section. The number of triangles in a graph represent the length of the triangular ladder. The triangular ladder of length s is denoted by Tl_s, and its order is $2s + 1$. Triangular ladder Tl_s is shown

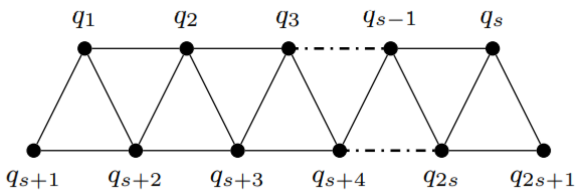

FIGURE 1. **Triangular ladder Tl_s.**

in Figure 1. We compute pd(Tl_s) and $\mathcal{F}(Tl_s)$ in subsequent theorems.

Theorem 1: For every $s \geq 2$, pd$(Tl_s) = 3$.

Proof: Let $\mathbb{Y} = \{\xi_1, \xi_2, \xi_3\}$ be a partition set of $V(Tl_s)$ for $s \geq 2$. The $r(q|\mathbb{Y})$ associated to $\xi_1 = \{q_i : 1 \leq i \leq s\} \cup \{q_i : s+2 \leq i \leq 2s\}$, $\xi_2 = \{q_{s+1}\}$ and $\xi_3 = \{q_{2s+1}\}$ are provided below:

$$r(q_\varrho|\mathbb{Y}) = \begin{cases} (0, \varrho, s-\varrho+1) & 1 \leq \varrho \leq s; \\ (1, 0, s) & \varrho = s+1; \\ (0, \varrho-s-1, 2s-\varrho+1) & s+2 \leq \varrho \leq 2s; \\ (1, s, 0) & \varrho = 2s+1. \end{cases}$$

Above distinct identifications justify that $\mathbb{Y}$ is resolving partition of Tl_s, so, pd$(Tl_s) \leq 3$. Now as Tl_s is not a path graph and Proposition 1(b), indicates that partition dimension of Tl_s cannot be 2, hence, pd$(Tl_s) \geq 3$. So from both acquired inequalities, pd$(Tl_s) = 3$, completes the proof. □

Theorem 2: For every $s \geq 2$, $\mathcal{F}(Tl_s) = 4$.

Proof: Let $\mathbb{Y} = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ be a partition set of $V(Tl_s)$ with 4 partition classes. The $r(q|\mathbb{Y})$ considering $\xi_1 = \{q_i : 1 \leq i \leq s\}$, $\xi_2 = \{q_i : s+1 \leq i \leq 2s-1\}$, $\xi_3 = \{q_{2s}\}$ and $\xi_4 = \{q_{2s+1}\}$ are provided below:

$$r(q_\varrho|\mathbb{Y}) = \begin{cases} (0, 1, s-\varrho, s-\varrho+1) & 1 \leq \varrho \leq s-1; \\ (0, 2, 1, 1) & \varrho = s; \\ (1, 0, 2s-\varrho, 2s-\varrho+1) & s+1 \leq \varrho \leq 2s-1; \\ (1, 1, 0, 1) & \varrho = 2s; \\ (1, 2, 1, 0) & \varrho = 2s+1. \end{cases}$$

It can be verified that the representations stated above differ from each other in atleast two positions and therefore $\mathbb{Y}$ forms a FTRP of Tl_s. Consequently, $\mathcal{F}(Tl_s) \leq 4$.

Now in order to show that $\mathcal{F}(Tl_s) \geq 4$, we proceed by establishing that $\mathcal{F}(Tl_s) \neq 3$. Suppose on contrary that $\mathbb{Y} = \{\xi_1, \xi_2, \xi_3\}$ be a fault-tolerant partition basis of Tl_s. Any of the subsets ξ_1, ξ_2 or ξ_3 of $V(Tl_s)$ contains one degree 3 vertex of at the minimum. As q_1 and q_s are only vertices of degree 3, so without losing generality we surmise that q_1 is contained in ξ_1, and $N(q_1) = \{q_2, q_{s+1}, q_{s+2}\}$. Suppose that $|\xi_1| = 1$, and $N(q_1) \subseteq \xi_2 \cup \xi_3$, $|N(q_1) \cap \xi_2| \geq 2$ or $|N(q_1) \cap \xi_3| \geq 2$. Without loss of generality we surmise that at least two vertices $f_1, f_2 \in N(q_1) \cap \xi_3$. As $r(f_1|\mathbb{Y}) = (1, k_1, 0)$ and $r(f_2|\mathbb{Y}) = (1, k_2, 0)$ has same first and third coordinates, thus, a contradiction. Now in support of the contradiction, we discuss the ensuing cases when $|\xi_1| \geq 2$ and $q_1 \in \xi_1$.

Case 1: If $N(q_1) \cap \xi_1 = \{q_2, q_{s+1}, q_{s+2}\}$, then, $r(q_1|\mathbb{Y}) = (0, b_0, c_0)$, $r(q_2|\mathbb{Y}) = (0, b_1, c_1)$, $r(q_{s+1}|\mathbb{Y}) = (0, b_2, c_2)$ and $r(q_{s+2}|\mathbb{Y}) = (0, b_3, c_3)$. As $b_0 - 1 \leq b_1, b_2, b_3 \leq b_0 + 1$, so two vertices will have same distance from vertex set ξ_2, thus a contradiction.

Case 2: If ξ_1 contains two neighbours of q_1, and ξ_3 contains its one neighbour, then we address these cases:

Case 2(a) If $N(q_1) \cap \xi_1 = \{q_2, q_{s+1}\}$, and vertex $q_{s+2} \in \xi_3$, then, $r(q_1|\mathbb{Y}) = (0, c_0, 1)$ and $r(q_2|\mathbb{Y}) = (0, c_1, 1)$.

Case 2(b) If $N(q_1) \cap \xi_1 = \{q_2, q_{s+2}\}$, and one vertex $q_{s+1} \in \xi_3$, then, $r(q_1|\mathbb{Y}) = (0, c_0, 1)$, $r(q_2|\mathbb{Y}) = (0, c_1, b_1)$ and $r(q_{s+2}|\mathbb{Y}) = (0, c_2, 1)$. It is clear from these cases that two vertices have same distance from vertex set ξ_3, thus a contradiction.

Case 3: If $N(q_1) \cap \xi_1 = \{q_2\}$ and two vertices $q_{s+1}, q_{s+2} \in \xi_3$, then, $r(q_1|\mathbb{Y}) = (0, c_0, 1)$, $r(q_2|\mathbb{Y}) = (0, c_1, b_1)$, $r(q_{s+1}|\mathbb{Y}) = (1, c_2, 0)$ and $r(q_{s+2}|\mathbb{Y}) = (1, c_3, 0)$. Since, q_{s+1} and q_{s+2} have same distance from vertex set ξ_1, so a contradiction.

Case 4: If $N(q_1) \cap \xi_1 = \{q_2\}$, $q_{s+1} \in \xi_2$ and $q_{s+2} \in \xi_3$, then, $r(q_1|\mathbb{Y}) = (0, 1, 1)$ and $r(q_2|\mathbb{Y}) = (0, c_4, 1)$. Since, q_1 and q_2 have same distance from vertex set ξ_3, this leads to a contradiction.

Case 5: Consider, $N(q_1) \cap \xi_1 = \emptyset$, and at least two neighbours of q_1 belong to vertex set ξ_3. Without loss of generality, we assume that $q_2, q_{s+1} \in \xi_3$, then $r(q_1|\mathbb{Y}) = (0, c_0, 1)$, $r(q_2|\mathbb{Y}) = (1, c_1, 0)$ and $r(q_{s+1}|\mathbb{Y}) = (1, c_2, 0)$. Again q_2 and q_{s+1} have same distance from vertex set ξ_3, so a contradiction.

All the above cases illustrate that $\mathcal{F}(Tl_s) \geq 4$. We end up the conclusion by relating both inequalities, so, $\mathcal{F}(Tl_s) = 4$. □

III. FAULT-TOLERANT PARTITION DIMENSION OF TRIANGULAR MESH GRAPH

In the current section, the FTPD of the triangular mesh graph T_s will be computed. Triangular mesh graph is formed by joining the layers of triangular ladder. Order of the triangular mesh is $\frac{s^2+s}{2}$. Rows of the mesh are numbered from top to bottom as $v_1^1, v_1^2, \ldots, v_1^s$. The triangular mesh graph T_6 is shown in Figure 2.

Nazeer *et al.* computed the metric dimension of triangular mesh graph.

Lemma 1 ([27]): The metric dimension of triangular mesh graph is 2.

Lemma 2: Let $\Im$ be a triangular mesh related graph with s number of rows where, $s \geq 3$, and $v \in V(\Im)$ such that deg$(v) \neq 3$, then, $\mathcal{F}(\Im) \geq 4$.

Proof: In order to verify that $\mathcal{F}(\Im) \geq 4$, we get $\mathcal{F}(\Im) \neq 3$ by a contradiction. Suppose $\mathbb{Y} = \{\xi_1, \xi_2, \xi_3\}$ be a fault-tolerant partition resolving set of $\Im$. At least one vertex of degree 4 will be contained in one of the subset ξ_1, ξ_2 or ξ_3 of $V(\Im)$. With no loss of generality, we surmise that ξ_1 contains q that is a vertex of degree 4, and $N(q) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Assume that $|\xi_1| = 1$, and $N(q) \subseteq \xi_2 \cup \xi_3$, $|N(q) \cap \xi_2| \geq 2$ or $|N(q) \cap \xi_3| \geq 2$. With no loss of generality, it can be assumed that at least two vertices $e, \alpha \in N(q) \cap \xi_3$, thus, $r(e|\mathbb{Y}) = (1, c_1, 0)$ and $r(\alpha|\mathbb{Y}) = (1, c_2, 0)$. As vertices

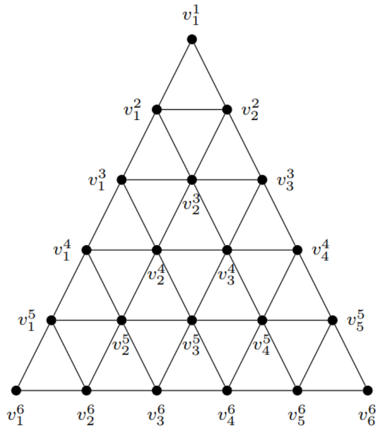

FIGURE 2. Triangular mesh graph T_6.

e and α have identical distance from set ξ_1, so a contradiction. Now in support of the contradiction, we discuss the ensuing cases when $|\xi_1| \geq 2$ and $q \in \xi_1$.

Case 1: If $N(q) \cap \xi_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, then, $r(q|\mathbb{Y}) = (0, j_0, k_0)$, $r(\alpha_1|\mathbb{Y}) = (0, j_1, k_1)$, $r(\alpha_2|\mathbb{Y}) = (0, j_2, k_2)$, $r(\alpha_3|\mathbb{Y}) = (0, j_3, k_3)$ and $r(\alpha_4|\mathbb{Y}) = (0, j_4, k_4)$. As $j_0 - 1 \leq j_1, j_2, j_3, j_4 \leq j_0 + 1$, so two vertices will have same distance from vertex set ξ_2, thus a contradiction.

Case 2: If $N(q) \cap \xi_1 = \{\alpha_1, \alpha_2, \alpha_3\}$, and $\alpha_4 \in \xi_3$, then, $r(q|\mathbb{Y}) = (0, k_0, 1)$, $r(\alpha_1|\mathbb{Y}) = (0, k_1, j_1)$, $r(\alpha_2|\mathbb{Y}) = (0, k_2, j_2)$, $r(\alpha_3|\mathbb{Y}) = (0, k_3, j_3)$ and $r(\alpha_4|\mathbb{Y}) = (1, k_4, 0)$. Since $1 \leq j_1, j_2, j_3 \leq 2$, so two vertices from α_1, α_2 and α_3 have same distance from ξ_3, thus a contradiction.

Case 3: If $N(q) \cap \xi_1 = \{\alpha_1, \alpha_2\}$ and any two vertices $\alpha_3, \alpha_4 \in \xi_3$, then, $r(q|\mathbb{Y}) = (0, k_0, 1)$, $r(\alpha_1|\mathbb{Y}) = (0, k_1, j_1)$, $r(\alpha_2|\mathbb{Y}) = (0, k_2, j_2)$, $r(\alpha_3|\mathbb{Y}) = (1, k_3, 0)$ and $r(\alpha_4|\mathbb{Y}) = (1, k_4, 0)$. As α_3 and α_4 have same distance from ξ_1, which is a contradiction.

Case 4: Consider, $N(q) \cap \xi_1 = \{\alpha_1\}$, and ξ_3 contains at least two vertices from $N(q)$. If without losing generality, $\alpha_2, \alpha_3 \in \xi_3$, then $r(q|\mathbb{Y}) = (0, k_0, 1)$, $r(\alpha_1|\mathbb{Y}) = (0, k_1, j_1)$, $r(\alpha_2|\mathbb{Y}) = (1, k_2, 0)$ and $r(\alpha_3|\mathbb{Y}) = (1, k_3, 0)$. Since distance of α_2 and α_3 from ξ_1 are identical, this leads to a contradiction.

Case 5: Consider, $N(q) \cap \xi_1 = \emptyset$, and ξ_3 contains at minimum two vertices from $N(q)$. If without losing generality, $\alpha_1, \alpha_2 \in \xi_3$, then $r(q|\mathbb{Y}) = (0, s_0, 1)$, $r(\alpha_1|\mathbb{Y}) = (1, k_1, 0)$ and $r(\alpha_2|\mathbb{Y}) = (1, k_2, 0)$. Since distance of α_2 and α_3 from ξ_1 are identical, thus a contradiction.

Case 6: If $N(q) \cap \xi_1 = \{\alpha_1, \alpha_2\}$, $\alpha_3 \in \xi_2$ and $\alpha_4 \in \xi_3$, then, $r(q|\mathbb{Y}) = (0, 1, 1)$, $r(\alpha_1|\mathbb{Y}) = (0, j_1, k_1)$ and $r(\alpha_2|\mathbb{Y}) = (0, j_2, k_2)$. Since $1 \leq j_1, j_2 \leq 2$, so distance of α_1 and α_2 will be same from ξ_2, which is a contradiction.

All the above cases illustrate that $\mathcal{F}(\Im) \geq 4$. □

We compute $\mathrm{pd}(T_s)$ and $\mathcal{F}(T_s)$ in the ensuing theorems.

Theorem 3: For the triangular mesh graph T_s for $s \geq 3$, the partition dimension is 3.

Proof: According to Lemma 1, $\beta(T_s) = 2$, for $s \geq 3$, so, by Proposition 1(a), $\mathrm{pd}(T_s) \leq 3$. Now as T_s is not a path graph and Proposition 1(b), indicates that 2 partition dimension of T_s is not possible, hence, $\mathrm{pd}(T_s) \geq 3$. So from both acquired inequalities, $\mathrm{pd}(T_s) = 3$, concludes the proof. □

Theorem 4: The FTPD of triangular mesh graph T_s for $s \geq 3$, is 4.

Proof: We use the basic method of double inequality to prove that $\mathcal{F}(T_s) = 4$. In this regard, first we show that $\mathcal{F}(T_s) \leq 4$. Consider $\mathbb{Y} = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ be a partition set of $V(T_s)$ for $s \geq 3$. When $s = 3$, consider, $\xi_1 = \{v_1^1, v_1^2\}$, $\xi_2 = \{v_1^3\}$, $\xi_3 = \{v_2^3, v_3^3\}$ and $\xi_4 = \{v_2^2\}$. It is easy to verify that $\mathbb{Y}$ is FTRP of T_s for $s = 3$. Now for $s \geq 4$, the $r(v_l^m|\mathbb{Y})$, where $\xi_1 = \{v_1^i : 1 \leq i \leq s\}$, $\xi_2 = \{v_i^s : 2 \leq i \leq s\}$, $\xi_3 = \{v_i^i : 2 \leq i \leq s-1\}$ and $\xi_4 = V(T_s) \setminus \{\xi_1, \xi_2, \xi_3\}$ are provided as follows:

$$r(v_w^t|\mathbb{Y}) = \begin{cases} (0, s-1, 1, 2) & \text{for } t = w = 1; \\ (0, s-\varrho, \varrho-1, 1) & \text{for } t = \varrho, w = 1, \\ & \quad 2 \leq \varrho \leq s-1; \\ (0, 1, s-1, 2) & \text{for } t = s, w = 1; \\ (\varrho-1, 0, s-\varrho, 1) & \text{for } t = s, w = \varrho, \\ & \quad 2 \leq \varrho \leq s-1; \\ (\varrho-1, s-\varrho, 0, 1) & \text{for } t = \varrho, w = \varrho, \\ & \quad 2 \leq \varrho \leq s-1; \\ (s-1, 0, 1, 2) & \text{for } t = s, w = s; \\ (q-1, s-\varrho, \varrho-q, 0) & \text{for } t = \varrho, w = q, \\ & \quad q+1 \leq \varrho \leq s-1, \\ & \quad q = 2, 3, \ldots, (s-2). \end{cases}$$

Above distinct representations verify that $\mathbb{Y}$ is FTRP of T_s, so, $\mathcal{F}(T_s) \leq 4$. Also by Lemma 2, $\mathcal{F}(\Im) \geq 4$, hence we end up the conclusion by relating both inequalities, so, $\mathcal{F}(T_s) = 4$. □

A. FAULT-TOLERANT PARTITION DIMENSION OF REFLECTION TRIANGULAR MESH GRAPH

In this section, we compute the $\mathcal{F}(rl(T_s))$, where $rl(T_s)$ is reflection triangular mesh graph. Mirror image of T_s is named as reflection triangular mesh graph. We number the rows of the mesh from top to bottom as $v_1^{(1)1}, v_1^{(1)2}, \ldots, v_1^{(2)1}$. The order of reflection triangular mesh graph is s^2. The reflection triangular mesh graph $rl(T_6)$ is shown in Figure 3.

Nazeer *et al.* computed the metric dimension of reflection triangular mesh graph.

Lemma 3 ([27]): The metric dimension of reflection triangular mesh graph for $s \geq 3$ is 2.

We compute partition dimension and FTPD of reflection triangular mesh graph in the ensuing theorems.

Theorem 5: The partition dimension of reflection triangular mesh graph $\mathrm{pd}(rl(T_s))$ for $s \geq 3$, is 3.

Proof: According to Lemma 3, $\beta(rl(T_s)) = 2$ for $s \geq 3$, so, by Proposition 1(a), $\mathrm{pd}(rl(T_s)) \leq 3$. Now as $rl(T_s)$ is not a path graph and Proposition 1(b), indicates that partition dimension of $rl(T_s)$ to be 2 is not possible, hence, $\mathrm{pd}(rl(T_s)) \geq 3$. So from both acquired inequalities, $\mathrm{pd}(rl(T_s)) = 3$, concludes the proof. □

Theorem 6: The FTPD of reflection triangular mesh graph $rl(T_s)$ for $s \geq 3$, is 4.

Proof: We use the basic method of double inequality to prove that $\mathcal{F}(rl(T_s)) = 4$. In this regard, first we show that $\mathcal{F}(rl(T_s)) \leq 4$. Consider $\mathbb{Y} = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ be a partition set of $V(rl(T_s))$ for $s \geq 3$. The $r(v_l^m|\mathbb{Y})$, where $\xi_1 = V(rl(T_s)) \setminus \{\xi_2, \xi_3, \xi_4\}$, $\xi_2 = \{v_i^{(1)i} : 2 \leq i \leq s\}$, $\xi_3 = \{v_i^{(2)i} : 2 \leq i \leq s-1\}$ and $\xi_4 = \{v_1^{(2)1}\}$ are provided as follows:

$$r(v_w^{(1)t}|\mathbb{Y}) = \begin{cases} (0, 1, s, 2s-2) & \text{for } t = w = 1; \\ (0, \varrho-1, s-1, 2s-\varrho-1) & \text{for } t = \varrho, w = 1, \\ & 2 \leq \varrho \leq s; \\ (0, s-\varrho, s-\varrho, s-1) & \text{for } t = s, w = \varrho, \\ & 2 \leq \varrho \leq s-1; \\ (1, 0, s-\varrho+1, 2s-\varrho-1) & \text{for } t = \varrho, w = \varrho, \\ & 2 \leq \varrho \leq s; \\ (0, \varrho-q, s-q, 2s-\varrho-1) & \text{for } t = \varrho, w = q, \\ & q+1 \leq \varrho \leq s-1, \\ & q = 2, 3, \ldots, (s-2). \end{cases}$$

$$r(v_w^{(2)t}|\mathbb{Y}) = \begin{cases} (1, s-1, 1, 0) & \text{for } t = w = 1; \\ (0, s-1, \varrho-1, \varrho-1) & \text{for } t = \varrho, w = 1, \\ ; & 2 \leq \varrho \leq s-1; \\ (1, s-\varrho, 0, \varrho-1) & \text{for } t = \varrho, w = \varrho, \\ & 2 \leq \varrho \leq s-1; \\ (0, s-q, \varrho-q, \varrho) & \text{for } t = \varrho, w = q, \\ & q+1 \leq \varrho \leq s-1, \\ & q = 2, 3, \ldots, (s-2). \end{cases}$$

Now for verifying that $\mathcal{F}(rl(T_s)) \geq 4$, we get $\mathcal{F}(rl(T_s)) \neq 3$, by a contradiction. Suppose $\mathbb{Y} = \{\xi_1, \xi_2, \xi_3\}$ be a fault-tolerant partition resolving set of $rl(T_s)$. Only $v_1^{(1)s}$ and $v_s^{(1)s}$ are vertices of degree 3, and at least one of them will be contained in one of the subsets ξ_1, ξ_2 or ξ_3 of $V(rl(T_s))$. Without loss of generality, we surmise that ξ_1 contains $v_1^{(1)s}$, and $N(v_1^{(1)s}) = \{v_1^{(1)s-1}, v_1^{(2)s-1}, v_2^{(1)s}\}$. Assume that $|\xi_1| = 1$, and $N(v_1^{(1)s}) \subseteq \xi_2 \cup \xi_3$, $|N(v_1^{(1)s}) \cap \xi_2| \geq 2$ or $|N(v_1^{(1)s}) \cap \xi_3| \geq 2$. Without loss of generality, we assume that at least two vertices $v_1^{(1)s-1}, v_1^{(2)s-1} \in N(v_1^{(1)s}) \cap \xi_3$, thus, $r(v_1^{(1)s-1}|\mathbb{Y}) = (1, c_1, 0)$ and $r(v_1^{(2)s-1}|\mathbb{Y}) = (1, c_2, 0)$. As vertices $v_1^{(1)s-1}$ and $v_1^{(2)s-1}$ have identical distance from set ξ_1, so a contradiction. Now in support of the contradiction, we discuss the ensuing cases when $|\xi_1| \geq 2$ and $v_1^{(1)s} \in \xi_1$.

Case 1: If $N(v_1^{(1)s}) \cap \xi_1 = \{v_1^{(1)s-1}, v_1^{(2)s-1}, v_2^{(1)s}\}$, then, $r(v_1^{(1)s}|\mathbb{Y}) = (0, j_0, k_0)$, $r(v_1^{(1)s-1}|\mathbb{Y}) = (0, j_1, k_1)$, $r(v_1^{(2)s-1}|\mathbb{Y}) = (0, j_2, k_2)$, and $r(v_2^{(1)s}|\mathbb{Y}) = (0, j_3, k_3)$. As $j_0 - 1 \leq j_1, j_2, j_3 \leq j_0 + 1$, so two vertices will have same distance from vertex set ξ_2, thus a contradiction.

Case 2: If ξ_1 contains two neighbours of $v_1^{(1)s}$, and ξ_3 contains its one neighbour, then we address these cases:

Case 2(a) If $N(v_1^{(1)s}) \cap \xi_1 = \{v_1^{(1)s-1}, v_1^{(2)s-1}\}$, and $v_2^{(1)s} \in \xi_3$, then, $r(v_1^{(1)s}|\mathbb{Y}) = (0, k_0, 1)$ and $r(v_1^{(1)s-1}|\mathbb{Y}) = (0, k_1, 1)$.

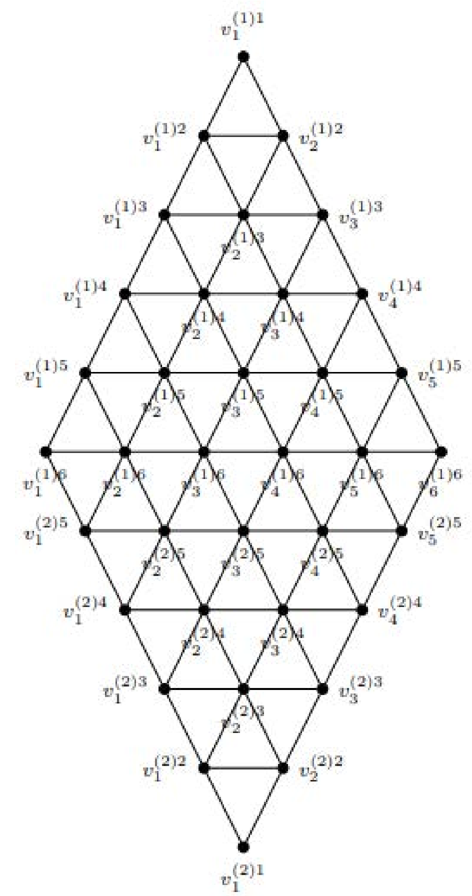

FIGURE 3. Reflection triangular mesh graph $rl(T_6)$.

Since $v_1^{(1)s}$ and $v_1^{(1)s-1}$ have same distance from ξ_3, thus a contradiction.

Case 2(b) If $N(v_1^{(1)s}) \cap \xi_1 = \{v_1^{(1)s-1}, v_2^{(1)s}\}$ and $v_1^{(2)s-1} \in \xi_3$, then, $r(v_1^{(1)s}|\mathbb{Y}) = (0, k_0, 1)$, $r(v_1^{(1)s-1}|\mathbb{Y}) = (0, k_1, j_1)$ and $r(v_2^{(1)s}|\mathbb{Y}) = (0, k_2, 1)$. Since $v_1^{(1)s}$ and $v_2^{(1)s}$ have same distance from ξ_3, thus a contradiction.

Case 3: Consider, $N(v_1^{(1)s}) \cap \xi_1 = \{v_1^{(1)s-1}\}$, and $v_1^{(2)s-1}, v_2^{(1)s} \in \xi_3$, then $r(v_1^{(1)s}|\mathbb{Y}) = (0, k_0, 1)$, $r(v_1^{(1)s-1}|\mathbb{Y}) = (0, k_1, j_1)$, $r(v_1^{(2)s-1}|\mathbb{Y}) = (1, k_2, 0)$ and $r(v_2^{(1)s}|\mathbb{Y}) = (1, k_3, 0)$. Since distance of $v_1^{(2)s-1}$ and $v_2^{(1)s}$ from ξ_1 are identical,this leads to a contradiction.

Case 4: Consider, $N(v_1^{(1)s}) \cap \xi_1 = \emptyset$, and ξ_3 contains at least two vertices from $N(v_1^{(1)s})$. If without losing generality, $v_1^{(1)s-1}, v_1^{(2)s-1} \in \xi_3$, then $r(v_1^{(1)s}|\mathbb{Y}) = (0, s_0, 1)$, $r(v_1^{(1)s-1}|\mathbb{Y}) = (1, k_1, 0)$ and $r(v_1^{(2)s-1}|\mathbb{Y}) = (1, k_2, 0)$. Since distance of $v_1^{(1)s-1}$ and $v_1^{(2)s-1}$ from ξ_1 are identical, thus a contradiction.

Case 5: If each of ξ_1, ξ_2 and ξ_3 contains one neighbour of $v_1^{(1)s}$, then we address these cases:

Case 5(a) If $N(v_1^{(1)s}) \cap \xi_1 = \{v_1^{(1)s-1}\}$, $v_1^{(2)s-1} \in \xi_2$ and $v_2^{(1)s} \in \xi_3$, then, $r(v_1^{(1)s}|\mathbb{Y}) = (0, 1, 1)$ and $r(v_1^{(1)s-1}|\mathbb{Y}) = (0, j_1, 1)$. Since distance of $v_1^{(1)s}$ and $v_1^{(1)s-1}$ from ξ_3 are identical, thus a contradiction.

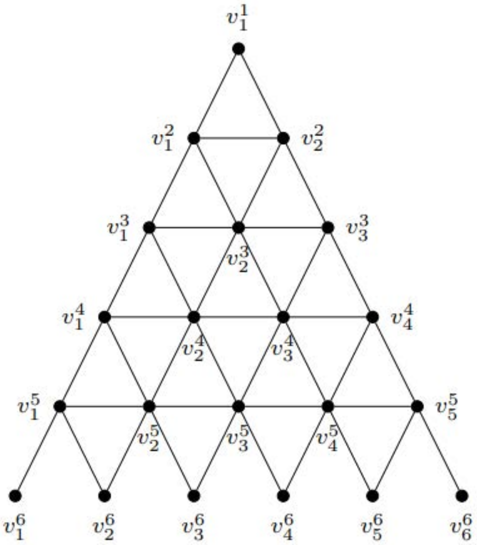

FIGURE 4. Tower triangular mesh graph Tr_6.

Case 5(b) If $N(v_1^{(1)s}) \cap \xi_1 = \{v_1^{(2)s-1}\}$, $v_1^{(1)s-1} \in \xi_2$ and $v_2^{(1)s} \in \xi_3$, then, $r(v_1^{(1)s}|\mathbb{Y}) = (0, 1, 1)$ and $r(v_1^{(2)s-1}|\mathbb{Y}) = (0, j_1, 1)$. Since distance of $v_1^{(1)s}$ and $v_1^{(2)s-1}$ from ξ_3 are identical, thus a contradiction.

Case 5(c) If $N(v_1^{(1)s}) \cap \xi_1 = \{v_2^{(1)s}\}$, $v_1^{(1)s-1} \in \xi_2$ and $v_1^{(2)s-1} \in \xi_3$, then, $r(v_1^{(1)s}|\mathbb{Y}) = (0, 1, 1)$ and $r(v_2^{(1)s}|\mathbb{Y}) = (0, 1, 1)$. Since $r(v_1^{(1)s}|\mathbb{Y})$ and $r(v_2^{(1)s}|\mathbb{Y})$ are identical, thus a contradiction. All the above cases illustrate that $\mathcal{F}(rl(T_s)) \geq 4$. We end up the conclusion by relating both inequalities, so, $\mathcal{F}(rl(T_s)) = 4$. □

B. FAULT-TOLERANT PARTITION DIMENSION OF TOWER TRIANGULAR MESH GRAPH

In this section, we compute the $\mathcal{F}(Tr_s)$, where Tr_s is tower triangular mesh graph. The tower triangular mesh graph is formed by joining the layers of triangular ladder such that there is no path among the vertices $v_1^s, v_2^s, \ldots, v_s^s$. Order of the triangular mesh is $\frac{s^2+s}{2}$, and rows of the mesh are numbered from top to bottom as $v_1^1, v_1^2, \ldots, v_1^s$. The tower triangular mesh graph Tr_6 is shown in Figure 4.

Nazeer *et al.* computed the metric dimension of tower triangular mesh graph.

Lemma 4 ([27]): The metric dimension of tower triangular mesh graph is 2*, for* $s \geq 3$.

We compute partition dimension and FTPD of tower triangular mesh graph in the ensuing theorems.

Theorem 7: The partition dimension of tower triangular mesh graph Tr_s *for* $s \geq 3$*, is* 3.

Proof: According to Lemma 4, $\beta(Tr_s) = 2$ for $s \geq 3$, so, by Proposition 1(a), $\mathrm{pd}(Tr_s) \leq 3$. Now as Tr_s is not a path graph and Proposition 1(b), indicates that PD of Tr_s to be 2 is not possible, hence, $\mathrm{pd}(Tr_s) \geq 3$. So from both acquired inequalities, $\mathrm{pd}(Tr_s) = 3$, concludes the proof. □

Theorem 8: The FTPD of tower triangular mesh graph Tr_s *for* $s \geq 3$*, is* 4.

Proof: We use the basic method of double inequality to prove that $\mathcal{F}(Tr_s) = 4$. In this regard, first we show that $\mathcal{F}(Tr_s) \leq 4$. Consider $\mathbb{Y} = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ be a partition set of $V(Tr_s)$ for $s \geq 3$. When $s = 3$, consider, $\xi_1 = \{v_1^1, v_1^2\}$, $\xi_2 = \{v_1^3\}$, $\xi_3 = \{v_2^3, v_3^3\}$ and $\xi_4 = \{v_2^2\}$. It is easy to verify that $\mathbb{Y}$ is FTRP of Tr_s for $s = 3$. Now for $s \geq 4$, the $r(v_l^m|\mathbb{Y})$, where $\xi_1 = \{v_1^i : 1 \leq i \leq s\}$, $\xi_2 = \{v_i^s : 2 \leq i \leq s\}$, $\xi_3 = \{v_i^i : 2 \leq i \leq s-1\}$ and $\xi_4 = V(Tr_s) \setminus \{\xi_1, \xi_2, \xi_3\}$ are provided as follows:

$$r(v_w^t|\mathbb{Y}) = \begin{cases} (0, s-1, 1, 2) & \text{for } t = w = 1; \\ (0, s-\varrho, \varrho-1, 1) & \text{for } t = \varrho, w = 1, \\ & 2 \leq \varrho \leq s-1; \\ (0, 2, s-1, 2) & \text{for } t = s, w = 1; \\ (\varrho-1, 0, s-\varrho, 1) & \text{for } t = s, w = \varrho, \\ & 2 \leq \varrho \leq s-1; \\ (\varrho-1, s-\varrho, 0, 1) & \text{for } t = \varrho, w = \varrho' \\ & 2 \leq \varrho \leq s-1; \\ (s, 0, 1, 2) & \text{for } t = s, w = s; \\ (q-1, s-\varrho, \varrho-q, 0) & \text{for } t = \varrho, w = q, \\ & q+1 \leq \varrho \leq s-1, \\ & q = 2, 3, \ldots, (s-2). \end{cases}$$

Above distinct representations verify that $\mathbb{Y}$ is FTRP of Tr_s, so, $\mathcal{F}(Tr_s) \leq 4$. Also by Lemma 2, $\mathcal{F}(\Im) \geq 4$, hence we end up the conclusion by relating both inequalities, so, $\mathcal{F}(Tr_s) = 4$. □

C. FAULT-TOLERANT PARTITION DIMENSION OF REFLECTION TOWER TRIANGULAR MESH GRAPH

In this section, we compute the $\mathcal{F}(rl(Tr_s))$, where $rl(Tr_s)$ is reflection tower triangular mesh graph. Reflection tower triangular mesh graph is mirror image of tower triangular mesh graph and its order is s^2. The reflection tower triangular mesh graph $rl(Tr_6)$ is shown in Figure 5.

Nazeer *et al.* computed the metric dimension of reflection tower triangular mesh graph.

Lemma 5 ([27]): The metric dimension of reflection tower triangular mesh graph is 2 *for* $s \geq 3$.

We compute partition dimension and FTPD of reflection tower triangular mesh graph in the ensuing theorems.

Theorem 9: The partition dimension of reflection tower triangular mesh graph $rl(Tr_s)$ *for* $s \geq 3$*, is* 3.

Proof: According to Lemma 5, $\beta(rl(Tr_s)) = 2$ for $s \geq 3$, so, by Proposition 1(a), $\mathrm{pd}(rl(Tr_s)) \leq 3$. Now as $rl(Tr_s)$ is not a path graph and Proposition 1(b), indicates that PD of $rl(Tr_s)$ cannot be 2, hence, $\mathrm{pd}(rl(Tr_s)) \geq 3$. So from both acquired inequalities, $\mathrm{pd}(rl(Tr_s)) = 3$, concludes the proof. □

Theorem 10: The FTPD of reflection tower triangular mesh graph $rl(Tr_s)$*, for* $s \geq 3$*, is* 4.

Proof: We use the basic method of double inequality to prove that $\mathcal{F}(rl(Tr_s)) = 4$. In this regard, first we show that $\mathcal{F}(rl(Tr_s)) \leq 4$. Consider $\mathbb{Y} = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ be a partition set of $V(rl(Tr_s))$ for $s \geq 2$. The $r(v_l^m|\mathbb{Y})$, where $\xi_1 = V(rl(T_s)) \setminus \{\xi_2, \xi_3, \xi_4\}$, $\xi_2 = \{v_i^{(1)i} : 2 \leq i \leq s\}$,

$\xi_3 = \{v_i^{(2)i} : 2 \le i \le s-1\}$ and $\xi_4 = \{v_1^{(2)1}\}$ are provided as follows:

$$r(v_w^{(1)t}|\mathbb{Y}) = \begin{cases} (0, 1, s, 2s-2) & \text{for } t = w = 1; \\ (0, \varrho-1, s-1, 2s-\varrho-1) & \text{for } t = \varrho, w = 1, \\ & \quad 2 \le \varrho \le s; \\ (0, s-\varrho, s-\varrho, s-1) & \text{for } t = s, w = \varrho' \\ & \quad 2 \le \varrho \le s-1; \\ (1, 0, s-\varrho+1, 2s-\varrho-1) & \text{for } t = \varrho, w = \varrho, \\ & \quad 2 \le \varrho \le s-1; \\ (2, 0, 1, s-1) & \text{for } t = w = s; \\ (0, \varrho-q, s-q, 2s-\varrho-1) & \text{for } t = \varrho, w = q, \\ & \quad q+1 \le \varrho \le s-1, \\ & \quad q = 2, 3, \ldots, (s-2). \end{cases}$$

$$r(v_w^{(2)t}|\mathbb{Y}) = \begin{cases} (1, s-1, 1, 0) & \text{for } t = w = 1; \\ (0, s-1, \varrho-1, \varrho-1) & \text{for } t = \varrho, w = 1, \\ & \quad 2 \le \varrho \le s-1; \\ (1, s-\varrho, 0, \varrho-1) & \text{for } t = \varrho, w = \varrho, \\ & \quad 2 \le \varrho \le s-1; \\ (0, s-q, \varrho-q, \varrho) & \text{for } t = \varrho, w = q, \\ & \quad q+1 \le \varrho \le s-1, \\ & \quad q = 2, 3, \ldots, (s-2). \end{cases}$$

From above, it can be concluded that all the representations are distinct, so $\mathbb{Y}$ is FTRP of $rl(Tr_s)$, so, $\mathcal{F}(rl(Tr_s)) \le 4$. Also by Lemma 2, $\mathcal{F}(\Im) \ge 4$, hence we end up the conclusion by relating both inequalities, so, $\mathcal{F}(rl(Tr_s)) = 4$. □

The following algorithms are produced in the support of above findings which can be used in aid of Matlab or other simulation tools to compute PD and FTPD.

Algorithm 1: Consider a finite connected network $\Im$ of order s with vertex set $V(\Im)$. Then following steps will compute the PD of $\Im$.

I: *Insert the input adjacency matrix $A = [a_{ij}]$ of the network $\Im$ such that $a_{ij} = 1$, if v_i and v_j are adjacent nodes in $\Im$, otherwise and $a_{ij} = 0$;*

II: *Compute the matrix D of distances;*

III: *For $p = 3$, compute all partition sets $\mathbb{Y}_{\alpha p} = \{\xi_1, \xi_2, \ldots, \xi_p\}$.*

IV: *For $\alpha = 1$, $\xi_i \in \mathbb{Y}_{\alpha p}$ and $v \in V(\Im)$, compute,*

$$d(v, \xi_i) = \begin{cases} 0 & \text{if } v \in \xi_i; \\ \min(d(v, v_j)) & \text{where } v_j \in \xi_i. \end{cases}$$

V: *If for $p = 3$, and if for $\alpha = 1$, $r(v|\mathbb{Y}_{\alpha p}) = (d(v, \xi_1), d(v, \xi_2), \ldots, d(v, \xi_p))$ are distinct for all $v \in V(\Im)$, then STOP; otherwise, go to Step IV for $\alpha = \alpha+1$, otherwise, go to Step III for $p = p+1$.*

VI: *Return, $\mathbb{Y}_{\alpha p}$ is the partition basis of $\Im$ and $\mathrm{pd}(\Im) = p$.*

Remark 1: In order to compute FTPD of $\Im$, Steps V and VI in Algorithm 1 are replaced by following steps:

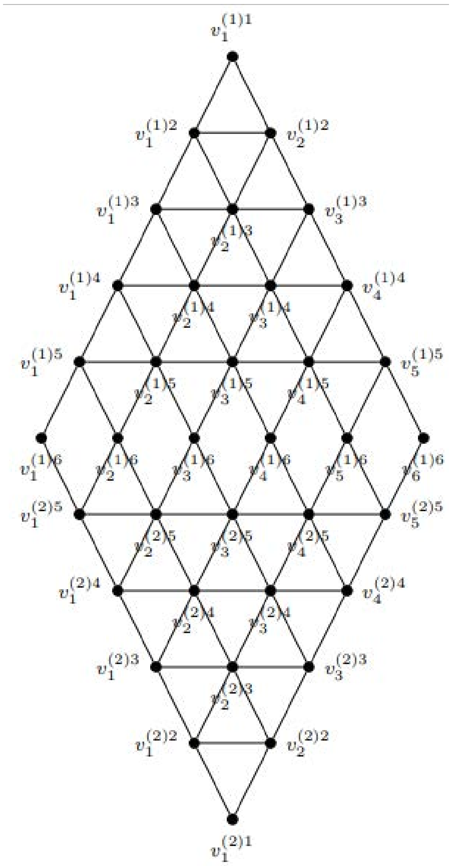

FIGURE 5. Reflection tower triangular mesh graph $rl(Tr_6)$.

V*: *If for $p = 3$, and if for $\alpha = 1$, $r(v|\mathbb{Y}_{\alpha p}) = (d(v, \xi_1), d(v, \xi_2), \ldots, d(v, \xi_p))$ are distinct at two places for all $v \in V(\Im)$, then STOP; otherwise, go to Step IV for $\alpha = \alpha + 1$, otherwise, go to Step III for $p = p + 1$.*

VI*: *Return, $\mathbb{Y}_{\alpha p}$ is the fault-tolerant partition basis of $\Im$ and $\mathcal{F}(\Im) = p$.*

IV. APPLICATION

Some recent applications of partition dimension and fault-tolerant partition dimension can be seen in areas of routing optimization problem [5] and supply chain optimization problem [7]. In particular, we consider an application of sensors deployed in homes located in smart cities. Wireless sensor networks are critical components of smart cities, where a group of IOT, like, broadcasting units collect and process data received from sensors and then send them to a remote administrative center that helps in providing different services. As broadcasting unit has to receive signals from each partition set, so, this propagation of data will be degraded due to following problems:

1. Receiving and synchronized time for the broadcasting unit will increase as the number of the partitioning sets increases.
2. More transmitting power will utilize as the distance between sensor and broadcasting unit increases.

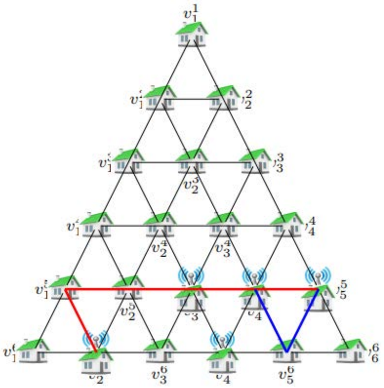

FIGURE 6. Partition dimension.

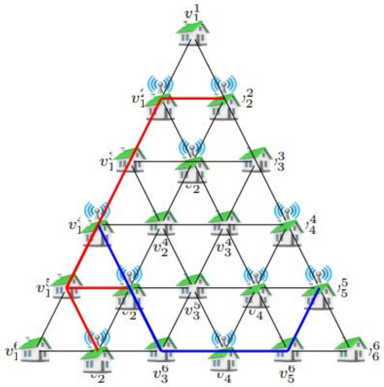

FIGURE 7. Fault-tolerant partition dimension.

The said problem can be rephrased in terms of graph theory as follows:

Consider, sensors (nodes) and broadcasting units (nodes) installed at homes in smart cities are arranged in the form of triangular mesh network T_6. If sensors are deployed in all the homes, while, broadcasting units are installed on few homes, then, what are the fewest number of broadcasting units required in minimum number of blocks (partition sets), such that each node has unique representation depending upon minimum distance from node to partition set.

By Theorem 3, arrange nodes in 3 groups by taking partition sets $\xi_1 = \{v_1^1, v_1^2, v_2^2, v_1^3, v_2^3, v_3^3, v_1^4, v_2^4, v_3^4, v_4^4, v_1^5, v_2^5, v_3^5, v_4^5, v_1^6\}$, $\xi_2 = \{v_2^6, v_3^6, v_4^6, v_5^6, v_6^6\}$ and $\xi_3 = \{v_5^5\}$. It is clear from Figure 6, that data propagation will be optimal by placing broadcasting units at nodes v_3^5, v_4^5, v_5^5, v_2^6 and v_4^6. This scenario explains concept of partition dimension of graph. Now, if concurrent data is transmitted from sensors to broadcasting node, then there will be delay in data propagation to remote administrative center. Group the nodes according to 4 partition sets given in Theorem 4. This problem of delay in data propagation will be handled by placing broadcasting units at $v_1^2, v_2^2, v_2^3, v_1^4, v_4^4, v_2^5, v_4^5, v_5^5, v_2^6$ and v_4^6. It is clear from Figure 7, if one of the partition set is not accessible for sensor node, then sensor will find some alternate partition set with nearest idle/low data broadcasting unit, such that broadcasting unit in alternate partition set tolerates the delay and optimize the data propagation. This explains applicability of FTPD in optimal data transfer problems.

V. CONCLUSION

The aim of this work is to compute PD and FTPD of triangular mesh related networks. We conclude that PD of triangular ladder and triangular mesh related networks for $s \geq 3$, is 3 whereas, their FTPD is 4, and both parameters are free from the order of graph. An algorithm to compute PD and FTPD of a network is produced which can be used in aid of Matlab or any other simulation tool. Further, an application of the work done in context of deploying broadcasting units in smart cities is also furnished.

Computation of PD and FTPD for a network is an NP hard problem [15] which can also be seen from the Algorithm 1. This states the limitations in the computation of these parameters as well as the significance of the computed results. In future, it will be interesting to compute FTPD of some other classes of mesh networks.

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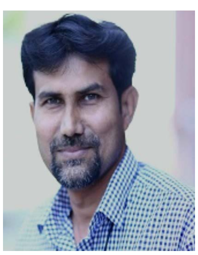

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