IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 5 June 2022, accepted 20 June 2022, date of publication 29 June 2022, date of current version 5 July 2022.

Digital Object Identifier 10.1109/ACCESS.2022.3187104

RESEARCH ARTICLE

Distributed Output-Feedback Tracking for Stochastic Nonlinear Multiagent Systems With Time-Varying Delays

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This work was supported in part by the Shandong Provincial Natural Science Foundation for Distinguished Young Scholars, China, under Grant ZR2019JQ22; and in part by the Shandong Province Higher Educational Excellent Youth Innovation Team, China, under Grant 2019KJN017.

ABSTRACT In this paper, we investigate the distributed output-feedback tracking control for stochastic nonlinear multi-agent systems (MASs) with time-varying delays. We propose a new distributed stochastic homogeneous domination method. Specifically, we first design distributed output-feedback controllers for the corresponding nominal systems. Then, by selecting the gains of controllers and observers, we solve the distributed tracking problem for stochastic MASs. After that, based on the coordinates transformation, with a proper Lyapunov-Krasoviskii (L-K) functional, it can be shown that the tracking error can be adjusted to arbitrarily small and all the states of the closed-loop system are bounded in probability. Finally, we give a simulation example to demonstrate the effectiveness of the control scheme.

INDEX TERMS Distributed stochastic homogeneous domination, stochastic nonlinear multi-agent systems, output-feedback tracking, time-varying delays.

I. INTRODUCTION

Due to the ubiquity of stochastic noise in applications, the study on stochastic systems has attracted attention in the fields of biology and environmental science. Since stochastic stabilization theory was proposed by [1], the research on its design and stabilization problem has been made great progress [2]–[6]. Zhang *et al.* [7] studied the finite-time stabilization of the feedforward systems by adding a power integrator and sign function. In addition, stability of time-delay systems is also an important topic [8]–[10]. Based on matrix inequality, the H-index with Markov jump systems and its application in H-fault detection filter (FDF) were studied in [11].

Multi-agent systems (MASs) exist widely in control engineering, such as power and traffic systems [12], [13]. It has received widespread concern. For nonlinear MASs, by inequality technique, a leader-following consensus problem was considered in [14]. Draw support from neural network (NN) approximation, [15] solved the unknown dynamics problem. Wang *et al.* [16] adopted a two-layer distributed hierarchical control strategy to deal with systems with unknown and inconsistent control direction. For stochastic nonlinear MASs, [17] investigated consensus of partially mixed impulse time-delay systems by comparison principle. In particular, distributed output tracking is becoming more and more popular. Li *et al.* [18], [19] solved the stochastic distributed output tracking problems by developing a new distributed integrator backstepping method. Xing *et al.* [20] proposed a distributed hybrid event-triggered mechanism to save communication resources. But the schemes in [18]–[20] required the all the states of agents are available.

From practical perspectives, the agents' states may not always be known or measurable. For this reason, it is necessary to study output-feedback tracking schemes. For single-agent systems, [21], [22] first designed homogeneous observers to estimate unmeasurable states, and then the output-feedback controller was designed to solve the tracking problem. For multi-agent systems, [23], [24] solved distributed output-feedback tracking problems for nonlinear

The associate editor coordinating the review of this manuscript and approving it for publication was Feiqi Deng.

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systems. However, there are rare results on distributed output-feedback tracking control for stochastic systems with time-varying delay.

Inspired by the previous discussion, this paper studies the distributed tracking problem of stochastic nonlinear time-delay MASs by output-feedback. The main contributions include:

1) The systems under investigation is more general than that in [18] and [19]. Li *et al.* [18], Li *et al.* [19] did not consider time-delay, and required that all the states are measurable. The existence of time-varying delay makes it difficult to select a proper Lyapunov-Krasoviskii (L-K) functional. The unmeasurable states makes the distributed controllers design more challenging.

2) Due to the influence of the Hessian term and time-varying delay, the distributed homogeneous domination approach developed in [23] is invalid. A new design scheme is developed in this paper.

The remainder of this paper is organized as follows. Section II is for preliminaries and problem formulation. In Section III states the main results. Section IV gives a simulation example. Section V is the conclusion.

II. PROBLEM FORMULATION AND PRELIMINARIES

A. PRELIMINARIES

Consider the stochastic time-delay system

$$dx(t) = f(t, x(t), x(t-d(t)))dt + g^T(t, x(t), x(t-d(t)))dw, \quad \forall t \geq 0, \qquad (1)$$

with initial data $\{x(\theta) : -d \leq \theta \leq 0\} = \xi \in \mathcal{C}^b_{\mathcal{F}_0}([-d, 0]; \mathbb{R}^n)$, where $d(t) : \mathbb{R}_+ \to [0, d]$ is time-varying delay; ω is an m-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$; $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz with $f(t, 0, 0) \equiv 0$, $g(t, 0, 0) \equiv 0$.

Definition 1 [10]: For any given $V(x(t), t) \in \mathcal{C}^{2,1}$ associated with system (1), the differential operator $\mathcal{L}$ is defined as

$$\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f + \frac{1}{2}\mathrm{Tr}\{g\frac{\partial^2 V}{\partial x^2}g^T\}.$$

Lemma 1 [22]: For $(x, y) \in \mathbb{R}^2$ the inequality holds:

$$xy \leq \frac{v^p}{p}|x|^p + \frac{1}{qv^q}|y|^q,$$

where $v > 0$, the constants $p > 1$ and $q > 1$ satisfy $(p-1)(q-1) = 1$.

Lemma 2 [10]: Let $x_1, x_2, \cdots, x_n, p$ be positive real numbers, then

$$(x_1 + x_2 + \cdots + x_n)^p \leq \max\{n^{p-1}, 1\}(x_1^p + x_2^p + \cdots + x_n^p).$$

In this paper, we consider a network $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ including N followers and one leader (labeled by 0). Define the matrix $B = diag(b_1, b_2, \cdots, b_{\mathbb{N}})$, where $b_i > 0$ if the leader can directly send information to the ith follower, and $b_i = 0$, otherwise. Let the followers' digraph be $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$. The Laplacian of $\mathcal{G}$ is set as $L = diag(\sum_{j\in\mathcal{N}_1} a_{1j}, \sum_{j\in\mathcal{N}_2} a_{2j}, \cdots, \sum_{j\in\mathcal{N}_{\mathbb{N}}} a_{\mathbb{N}j}) - (a_{ij})_{\mathbb{N}\times\mathbb{N}}$. Define $H = B + L$. More notations about graph theory can be found in [28].

B. PROBLEM FORMULATION

The followers' dynamics are described by

$$\begin{aligned} dx_{i1}(t) &= x_{i2}(t)dt + f_{i1}(t, x_{i1}(t), x_{i1}(t-d_{i1}(t)))dt \\ &\quad + g_{i1}^T(t, x_{i1}(t), x_{i1}(t-d_{i1}(t)))dw, \\ dx_{i2}(t) &= u_i(t)dt + f_{i2}(t, \bar{x}_{i2}(t), \bar{x}_{i2}(t-d_{i2}(t)))dt \\ &\quad + g_{i2}^T(t, \bar{x}_{i2}(t), \bar{x}_{i2}(t-d_{i2}(t)))dw, \\ y_i(t) &= \sum_{j\in\mathcal{N}_i} a_{ij}(x_{i1}(t) - x_{j1}(t)) + b_i(x_{i1}(t) - y_0(t)), \end{aligned} \qquad (2)$$

where $\bar{x}_{i2}(t) = (x_{i1}(t), x_{i2}(t))^T \in \mathbb{R}^2$, $u_i \in \mathbb{R}$, and $y_i \in \mathbb{R}$ are the state, input, output of the ith follower, respectively. $d_{ij}(t) : \mathbb{R}_+ \to [0, d], j = 1, 2$, are time-varying delay. The unknown functions f_{ij} and g_{ij} are $\mathcal{C}^1$ functions with $f_{ij}(t, 0, 0) = 0$, $g_{ij}(t, 0, 0) = 0$, $i = 1, 2, \cdots, N, j = 1, 2$. $y_0(t) \in \mathbb{R}$ is the leader's output.

Assumption 1: There are constants $\mu_i > 0, \bar{\mu}_i > 0$, and $\tau_i > 0$ such that

$$\begin{aligned} &|f_{i1}(t, x_{i1}(t), x_{i1}(t-d_{i1}(t)))| \\ &\quad \leq \mu_i\Big(|x_{i1}(t)| + |x_{i1}(t-d_{i1}(t))|\Big) + \tau_i, \\ &|f_{i2}(t, \bar{x}_{i2}(t), \bar{x}_{i2}(t-d_{i2}(t)))| \\ &\quad \leq \mu_i\Big(\sum_{k=1}^{2}|x_{ik}(t)| + \sum_{k=1}^{2}|x_{ik}(t-d_{ik}(t))|\Big) + \tau_i, \\ &|g_{i1}(t, x_{i1}(t), x_{i1}(t-d_{i1}(t)))| \\ &\quad \leq \bar{\mu}_i\Big(|x_{i1}(t)| + |x_{i1}(t-d_{i1}(t))|\Big) + \tau_i, \\ &|g_{i2}(t, \bar{x}_{i2}(t), \bar{x}_{i2}(t-d_{i2}(t)))| \\ &\quad \leq \bar{\mu}_i\Big(\sum_{k=1}^{2}|x_{ik}(t)| + \sum_{k=1}^{2}|x_{ik}(t-d_{ik}(t))|\Big) + \tau_i. \end{aligned}$$

Assumption 2: $y_0(t) \in \mathbb{R}$ and $\dot{y}_0(t)$ are bounded and only apply to those followers satisfying $0 \in \mathcal{N}_i, i = 1, 2, \cdots, N$.

Assumption 3: The leader is the root of a spanning tree in $\bar{\mathcal{G}}$.

Assumption 4: Time-varying delay $d_{ij}(t)$ satisfy $d \geq \max\{d_{ij}(t), i = 1, 2, \cdots, N, j = 1, 2\}$, and $\dot{d}_{ij}(t) \leq \gamma < 1$, where $\gamma < 1$ is a constant.

Remark 1: When $\tau_i = 0$, Assumption 1 reduces to that in [25]. In fact, τ_i admits constant disturbances, while μ_i and $\bar{\mu}_i$ allow diminishing disturbances. τ_i is independent of μ_i and $\bar{\mu}_i$.

III. MAIN RESULTS

A. NOMINAL MASs ANALYSIS

Consider the following nominal MASs:

$$\begin{aligned} \dot{z}_{i1} &= d_i z_{i2} - \sum_{j\in\mathcal{N}_i} a_{ij} z_{j2}, \\ \dot{z}_{i2} &= v_i, \\ y_i &= z_{i1}, \quad i = 1, 2, \cdots, N, \end{aligned} \tag{3}$$

where $d_i = b_i + \sum_{j\in\mathcal{N}_i} a_{ij}$.

From [18, Lemma 1] and Assumption 3, we can conclude H is invertible. Define

$$H^{-1} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1N} \\ h_{21} & h_{22} & \cdots & h_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N1} & h_{N2} & \cdots & h_{NN} \end{bmatrix}. \tag{4}$$

According to [23], we construct distributed reduced-order observers for (3) as

$$\dot{w}_{i2} = -d_i w_{i2} + \sum_{j=1}^{N} a_{ij} w_{j2} - d_i z_{i1} + \sum_{j=1}^{N} a_{ij} z_{j1} + u_i \tag{5}$$

and the distributed output-feedback controller as

$$v_i = -\alpha_i \left(\hat{z}_{i2} + \sum_{j=1}^{N} h_{ij} c_{j1} z_{j1} \right), \tag{6}$$

where $\hat{z}_{i2} = z_{i1} + w_{i2}$, α_i and $c_{j1} > 0$ are constants.

For system (3), construct the Lyapunov function as

$$\begin{aligned} V(\bar{\xi}_{i2}(t), e(t)) &= U(\bar{\xi}_{i2}(t)) + W(e(t)), \\ U(\bar{\xi}_{i2}(t)) &= \frac{1}{4}\sum_{i=1}^{N}(\xi_{i1}^4(t) + \xi_{i2}^4(t)), \\ W(e(t)) &= (e(t)e^T(t))^T P (e(t)e^T(t)), \end{aligned} \tag{7}$$

where $\xi_{i1} = z_{i1}$, $\xi_{i2} = z_{i2} - z_{i2}^*$, $z_{i2}^* = -\sum_{j=1}^{N} h_{ij} c_{j1} z_{j1}$, $z_{i3}^* = -\alpha_i \xi_{i2}$, $e_{i2} = z_{i2} - \hat{z}_{i2}$, $e = (e_{12}, \cdots, e_{N2})$, $i = 1, 2, \cdots, N$.

Defining

$$\mathcal{Z} = (z_{11}, z_{12}, \cdots, z_{N1}, z_{N2}, w_{12}, w_{22}, \cdots, w_{N2})^T,$$

from (3), (5), and (6), we get

$$\begin{aligned} \dot{\mathcal{Z}} &= E(\mathcal{Z}) \\ &= (d_1 z_{12} - \sum_{j=1}^{N} a_{1j} z_{j2}, v_1, \cdots, d_N z_{N2} - \sum_{j=1}^{N} a_{Nj} z_{j2}, \\ &\qquad v_N, f_{2N+1}, \cdots, f_{3N})^T, \end{aligned} \tag{8}$$

where $f_{2N+i} = -d_i w_{i2} + \sum_{j=1}^{N} a_{ij} w_{j2} - d_i z_{i1} + \sum_{j=1}^{N} a_{ij} z_{j1} + u_i$, $i = 1, 2, \cdots, N$.

Lemma 3: 1) $E(\mathcal{Z})$ and $V(\mathcal{Z})$ are homogeneous of degree 1 and 3 respectively, with the dilation weight:

$$\triangle = (\underbrace{1,\ 1,\ \cdots,\ 1,\ 1}_{\text{for } z_{11}, z_{12}, \cdots, z_{N1}, z_{N2}}, \underbrace{1,\ 1,\ \cdots,\ 1}_{\text{for } w_{12}, \cdots, w_{N2}}). \tag{9}$$

2) the derivative of $V(\mathcal{Z})$ satisfies

$$\dot{V}(\mathcal{Z}) = \frac{\partial V}{\partial \mathcal{Z}} E(\mathcal{Z}) \le -\underline{c}_0 \|\mathcal{Z}\|_\triangle^4, \tag{10}$$

where constant $\underline{c}_0 > 0$ and $\|\mathcal{Z}\|_\triangle = (\sum_{i=1}^{n} |z_{i1}|^2 + \sum_{i=1}^{n} |z_{i2}|^2 + \sum_{i=1}^{n} |w_{i2}|^2)^{1/2}$.

Remark 2: In the corresponding results on deterministic systems [23], it uses the Lyapunov function of the form

$$V = \frac{1}{2}\sum_{i=1}^{N}(\xi_{i1}^2 + \xi_{i2}^2) + e^T P e. \tag{11}$$

Due to the existence of $\frac{1}{2}\mathrm{Tr}\{g \frac{\partial^2 V}{\partial x^2} g^T\}$ in stochastic differential, the Lyapunov function (11) is invalid for stochastic system.

Let's consider a simple example

$$\begin{aligned} dx &= u dt + dw, \\ y &= x - y_r(t), \end{aligned}$$

where $y_r(t) = \sin t$.

Define $y = z$, $v = u/L$, where $L > 1$ is a design parameter. We have

$$\begin{aligned} dz &= (Lv - \cos t)dt + dw, \\ y &= z, \end{aligned}$$

where $f = -\cos t$, $g = 1$. Clearly, Assumptions 1 is satisfied with $\mu_1 = \bar{\mu}_1 = 1$, $\tau_1 = 1$. When we choose $V = \frac{1}{2}z^2$, we get

$$\begin{aligned} \frac{1}{2}\mathrm{Tr}\left\{ g\frac{\partial^2 V}{\partial z^2} g^T \right\} &\le \frac{1}{2} r \left| g\frac{\partial^2 V}{\partial z^2} g^T \right|_\infty \\ &\le \frac{1}{2} r\sqrt{r} \left| \frac{\partial^2 V}{\partial z^2} g^T g \right| \\ &\le \frac{1}{2}. \end{aligned} \tag{12}$$

Obviously, by adjusting the gain L, $\frac{1}{2}\mathrm{Tr}\left\{ g\frac{\partial^2 V}{\partial z^2} g^T \right\}$ cannot be made arbitrarily small. In this paper, we employ quartic Lyapunov functions.

B. DISTRIBUTED OUTPUT-FEEDBACK CONTROLLER DESIGN

We introduce the following coordinates transformations

$$\begin{aligned} z_{i1} &= \sum_{j=1}^{N} a_{ij}(x_{i1} - x_{j1}) + b_i(x_{i1} - y_0), \\ z_{i2} &= \frac{x_{i2}}{L}, \quad v_i = \frac{u_i}{L^2}, \quad i = 1, 2, \cdots, N, \end{aligned} \tag{13}$$

where $L > 1$ is a design constant.

Using (13), systems (2) becomes

$$\begin{aligned} dz_{i1}(t) = {} & L(d_i z_{i2}(t) - \sum_{j=1}^{N} a_{ij} z_{j2}(t))dt \\ & + \tilde{f}_{i1}(t, z_{i1}(t), z_{i1}(t - d_{i1}(t)))dt \\ & + \tilde{g}_{i1}^T(t, z_{i1}(t), z_{i1}(t - d_{i1}(t)))dw, \end{aligned}$$

$$
\begin{aligned}
dz_{i2}(t) &= Lv_i(t)dt + \tilde{f}_{i2}(t, \bar{z}_{i2}(t), \bar{z}_{i2}(t-d_{i2}(t)))dt \\
&\quad + \tilde{g}_{i2}^T(t, \bar{z}_{i2}(t), \bar{z}_{i2}(t-d_{i2}(t)))dw, \\
y_i(t) &= z_{i1}(t), \quad i = 1, 2, \cdots, N,
\end{aligned} \tag{14}
$$

where

$$
\begin{aligned}
&\tilde{f}_{i1}(t, z_{i1}(t), z_{i1}(t-d_{i1}(t))) \\
&\quad = -\sum_{j=1}^{N} a_{ij} f_{j1}(t, x_{j1}(t), x_{j1}(t-d_{j1}(t))) - b_i\dot{y}_0(t) \\
&\qquad + d_i f_{i1}(t, x_{i1}(t), x_{i1}(t-d_{i1}(t))), \\
&\tilde{f}_{i2}(t, \bar{z}_{i2}(t), \bar{z}_{i2}(t-d_{i2}(t))) \\
&\quad = \frac{1}{L} f_{i2}(t, \bar{x}_{i2}(t), \bar{x}_{i2}(t-d_{i2}(t))), \\
&\tilde{g}_{i1}(t, z_{i1}(t), z_{i1}(t-d_{i1}(t))) \\
&\quad = -\sum_{j=1}^{N} a_{ij} g_{j1}(t, x_{j1}(t), x_{j1}(t-d_{j1}(t))) \\
&\qquad + d_i g_{i1}(t, x_{i1}(t), x_{i1}(t-d_{i1}(t))), \\
&\tilde{g}_{i2}(t, \bar{z}_{i2}(t), \bar{z}_{i2}(t-d_{i2}(t))) \\
&\quad = \frac{1}{L} g_{i2}(t, \bar{x}_{i2}(t), \bar{x}_{i2}(t-d_{i2}(t))).
\end{aligned}
$$

We design the observers as

$$
\dot{w}_{i2} = L\Big(-d_i w_{i2} + \sum_{j=1}^{N} a_{ij} w_{j2} - d_i z_{i1} + \sum_{j=1}^{N} a_{ij} z_{j1} + u_i\Big), \tag{15}
$$

and distributed output-feedback controllers as

$$
v_i = -\alpha_i\Big(\hat{z}_{i2} + \sum_{j=1}^{N} h_{ij} c_{j1} z_{j1}\Big). \tag{16}
$$

Then from (14)–(16) we have

$$
d\mathcal{Z} = (LE(\mathcal{Z}) + F(\mathcal{Z}))dt + G^T(\mathcal{Z})dw, \tag{17}
$$

where

$$
\begin{aligned}
F(\mathcal{Z}) &= (\tilde{f}_{11}, \tilde{f}_{12}, \cdots, \tilde{f}_{N1}, \tilde{f}_{N2}, 0, \cdots, 0)^T, \\
G(\mathcal{Z}) &= (\tilde{g}_{11}, \tilde{g}_{12}, \cdots, \tilde{g}_{N1}, \tilde{g}_{N2}, 0, \cdots, 0)^T.
\end{aligned}
$$

Lemma 4: If Assumptions 1–4 hold, we get

$$
\begin{aligned}
\left|\frac{\partial V}{\partial \mathcal{Z}} F(\mathcal{Z})\right| &\le (\bar{c}_{01} + \bar{c}_{05}L^{1/2} + \bar{c}_{06}L^{-1})\|\mathcal{Z}(t)\|_\triangle^4 \\
&\quad + \bar{c}_{07}\sum_{j=1}^{N}\sum_{k=1}^{2}\|\mathcal{Z}(t-d_{jk}(t))\|_\triangle^4 \\
&\quad + \frac{1}{4}(L^{-3/2} + L^{-1}), \\
\frac{1}{2}\mathrm{Tr}\left\{G\frac{\partial^2 V}{\partial \mathcal{Z}^2}G^T\right\} &\le (\tilde{c}_{20} + \tilde{c}_{31}L^{1/2} + \tilde{c}_{32}L^{-1} \\
&\quad + \tilde{c}_{33}L^{-2})\|\mathcal{Z}(t)\|_\triangle^4 \\
&\quad + \tilde{c}_{34}\sum_{i=1}^{N}\sum_{j=1}^{2}\|\mathcal{Z}(t-d_{ij}(t))\|_\triangle^4 \\
&\quad + \frac{1}{2}(L^{-1/2} + L^{-1} + L^{-2}),
\end{aligned}
$$

where $\tilde{c}_{20}, \bar{c}_{0i}$, and $\tilde{c}_{3j}, i = 1, 5, 6, 7, j = 1, \cdots, 4$, are positive constants.

Proof. From (13) and [18, Lemma 1], we have

$$
\begin{bmatrix} z_{11} \\ \vdots \\ z_{N1} \end{bmatrix} = H \begin{bmatrix} x_{11} - y_0 \\ \vdots \\ x_{N1} - y_0 \end{bmatrix}. \tag{18}
$$

From (4) and (18), we then have

$$
x_{i1} = y_0 + \sum_{j=1}^{N} h_{ij} z_{j1}, \quad i = 1, 2, \cdots, N. \tag{19}
$$

According to Assumption 2, we get

$$
|y_0| + |\dot{y}_0| \le M, \tag{20}
$$

where $M > 0$ is a constant.

From (19)–(20), Assumption 1, and [22, Lemma 2.2], we obtain

$$
\begin{aligned}
|\tilde{f}_{i1}| &= \left|-\sum_{j=1}^{N} a_{ij} f_{j1}(t, x_{j1}(t), x_{j1}(t-d_{j1}(t))) - b_i\dot{y}_0(t)\right| \\
&\quad + \left|d_i f_{i1}(t, x_{i1}(t), x_{i1}(t-d_{i1}(t)))\right| \\
&\le \sum_{j=1}^{N} a_{ij}\mu_j(|x_{j1}(t)| + |x_{j1}(t-d_{j1}(t))|) + \sum_{j=1}^{N} a_{ij}\tau_j \\
&\quad + d_i\mu_i(|x_{i1}(t)| + |x_{i1}(t-d_{i1}(t))|) + d_i\tau_i + b_i M \\
&\le d_i\mu_i\Big(\sum_{j=1}^{N}|h_{ij}||z_{j1}(t)| + \sum_{j=1}^{N}|h_{ij}||z_{j1}(t-d_{j1}(t))|\Big) \\
&\quad + 2d_i\mu_i M + \sum_{j=1}^{N} a_{ij}\mu_j\Big(\sum_{s=1}^{N}|h_{js}||z_{s1}(t-d_{s1}(t))| \\
&\quad + \sum_{s=1}^{N}|h_{js}||z_{s1}(t)|\Big) + 2\sum_{j=1}^{N} a_{ij}\mu_j M + \sum_{j=1}^{N} a_{ij}\tau_j \\
&\quad + d_i\tau_i + b_i M \\
&\le d_i\mu_i\sum_{j=1}^{N}|h_{ij}|(|z_{j1}(t)| + |z_{j1}(t-d_{j1}(t))|) + b_i M \\
&\quad + \sum_{j=1}^{N}\sum_{s=1}^{N} a_{ij}\mu_j|h_{js}|(|z_{s1}(t)| + |z_{s1}(t-d_{s1}(t))|) \\
&\quad + 2\sum_{j=1}^{N} a_{ij}\mu_j M + d_i\tau_i + \sum_{j=1}^{N} a_{ij}\tau_j + 2d_i\mu_i M \\
&\le \sum_{j=1}^{N}\sigma_{i1j}(|z_{j1}(t)| + |z_{j1}(t-d_{j1}(t))|) + \beta_{i1} \\
&\le \tilde{c}_{01}(\|\mathcal{Z}(t)\|_\triangle + \sum_{j=1}^{N}\|\mathcal{Z}(t-d_{j1}(t))\|_\triangle) + \beta_{i1},
\end{aligned} \tag{21}
$$

where $\tilde{c}_{01} > 0$ is a constant, $\sigma_{i1j} = d_i\mu_i|h_{ij}| + \sum_{s=1}^{N} a_{is}\mu_s|h_{sj}|$, $\beta_{i1} = 2d_i\mu_i M + 2\sum_{j=1}^{N} a_{ij}\mu_j M + d_i\tau_i + \sum_{j=1}^{N} a_{ij}\tau_j + b_i M$.

Similarly, by (19)–(20), Assumption 1, and [22, Lemma 2.2], we get

$$
\begin{aligned}
|\tilde{f}_{i2}| &= \left|\frac{1}{L} f_{i2}(t, \bar{x}_{i2}(t), \bar{x}_{i2}(t - d_{i2}(t)))\right| \\
&\le \frac{1}{L}\mu_i\left(\sum_{k=1}^{2}|x_{ik}(t)| + \sum_{k=1}^{2}|x_{ik}(t - d_{ik}(t))|\right) + \frac{1}{L}\tau_i \\
&\le \frac{1}{L}\mu_i\left(\sum_{j=1}^{N}|h_{ij}||z_{j1}(t - d_{j1}(t))| + \sum_{j=1}^{N}|h_{ij}||z_{j1}(t)| \right.\\
&\quad \left. + L|z_{i2}(t - d_{i2}(t))| + 2|y_0(t)| + L|z_{i2}(t)|\right) + \frac{1}{L}\tau_i \\
&\le \frac{1}{L}\mu_i\sum_{j=1}^{N}|h_{ij}|(|z_{j1}(t)| + |z_{j1}(t - d_{j1}(t))|) + \frac{1}{L}\tau_i \\
&\quad + \mu_i(|z_{i2}(t)| + |z_{i2}(t - d_{i2}(t))|) + \frac{1}{L}2\mu_i M \\
&\le (L^{-1}\tilde{c}_{02} + \tilde{c}_{03})\left(\sum_{j=1}^{N}\sum_{k=1}^{2}\|\mathcal{Z}(t - d_{jk}(t))\|_\triangle \right.\\
&\quad \left. + \|\mathcal{Z}(t)\|_\triangle\right) + L^{-1}(2\mu_i M + \tau_i), \qquad (22)
\end{aligned}
$$

where $\tilde{c}_{02} > 0$ and $\tilde{c}_{03} > 0$ are constants.

By (7) and [22, Lemma 2.2], we obtain $\frac{\partial V}{\partial \mathcal{Z}_i}$ is homogeneous of degree 3. From (21)–(22), using [22, Lemmas 2.1 and 2.2], we have

$$
\begin{aligned}
\left|\frac{\partial V}{\partial \mathcal{Z}}F(\mathcal{Z})\right| &\le \sum_{i=1}^{N}\left(\left|\frac{\partial V}{\partial z_{i1}}\right||\tilde{f}_{i1}| + \left|\frac{\partial V}{\partial z_{i2}}\right||\tilde{f}_{i2}|\right) \\
&\le (\bar{c}_{01} + \bar{c}_{02}L^{-1})\left(\|\mathcal{Z}(t)\|_\triangle^4 + \sum_{j=1}^{N}\sum_{k=1}^{2}\|\mathcal{Z}(t - d_{jk}(t))\|_\triangle^4\right) \\
&\quad + \sum_{i=1}^{N}\left(\left|\frac{\partial V}{\partial z_{i1}}\right|\beta_{i1} + \left|\frac{\partial V}{\partial z_{i2}}\right|L^{-1}(2\mu_i M + \tau_i)\right) \\
&\le (\bar{c}_{01} + \bar{c}_{02}L^{-1})(\|\mathcal{Z}(t)\|_\triangle^4 + \sum_{j=1}^{N}\sum_{k=1}^{2}\|\mathcal{Z}(t - d_{jk}(t))\|_\triangle^4) \\
&\quad + (\bar{c}_{03} + \bar{c}_{04}L^{-1})\|\mathcal{Z}(t)\|_\triangle^3 \\
&\le (\bar{c}_{01} + \bar{c}_{02}L^{-1})(\|\mathcal{Z}(t)\|_\triangle^4 + \sum_{j=1}^{N}\sum_{k=1}^{2}\|\mathcal{Z}(t - d_{jk}(t))\|_\triangle^4) \\
&\quad + \left(\frac{3}{4}\bar{c}_{03}^{4/3}L^{1/2} + \frac{3}{4}\bar{c}_{04}^{4/3}L^{-1}\right)\|\mathcal{Z}(t)\|_\triangle^4 + \frac{L^{-3/2} + L^{-1}}{4} \\
&\le \left(\bar{c}_{01} + \frac{3}{4}\bar{c}_{03}^{4/3}L^{1/2} + (\bar{c}_{02} + \frac{3}{4}\bar{c}_{04}^{4/3})L^{-1}\right)\|\mathcal{Z}(t)\|_\triangle^4 \\
&\quad + (\bar{c}_{01} + \bar{c}_{02}L^{-1})\sum_{i=1}^{N}\sum_{j=1}^{2}\|\mathcal{Z}(t - d_{ij}(t))\|_\triangle^4 \\
&\quad + \frac{1}{4}(L^{-3/2} + L^{-1}), \qquad (23)
\end{aligned}
$$

where constants $\bar{c}_{0i} > 0$, $i = 1, 2, 3, 4,$.

Since $L > 1$, we get

$$
\begin{aligned}
\left|\frac{\partial V}{\partial \mathcal{Z}}F(\mathcal{Z})\right| &\le \left(\bar{c}_{01} + \frac{3}{4}\bar{c}_{03}^{4/3}L^{1/2} + (\bar{c}_{02} + \frac{3}{4}\bar{c}_{04}^{4/3})L^{-1}\right) \\
&\quad \cdot \|\mathcal{Z}(t)\|_\triangle^4 + \frac{1}{4}(L^{-3/2} + L^{-1}) + (\bar{c}_{01} \\
&\quad + \bar{c}_{02})\sum_{i=1}^{N}\sum_{j=1}^{2}\|\mathcal{Z}(t - d_{ij}(t))\|_\triangle^4 \\
&\le (\bar{c}_{01} + \bar{c}_{05}L^{1/2} + \bar{c}_{06}L^{-1})\|\mathcal{Z}(t)\|_\triangle^4 \\
&\quad + \bar{c}_{07}\sum_{i=1}^{N}\sum_{j=1}^{2}\|\mathcal{Z}(t - d_{ij}(t))\|_\triangle^4 \\
&\quad + \frac{1}{4}(L^{-3/2} + L^{-1}),
\end{aligned}
$$

where $\bar{c}_{0i}$, $i = 5, 6, 7$, are positive constants.

Similarly to (21)–(22), by (19)–(20), Assumption 1, and [22, Lemma 2.2], we obtain

$$
\begin{aligned}
|\tilde{g}_{i1}| &\le \sum_{j=1}^{N} a_{ij}\bar{\mu}_j|x_{j1}(t) + x_{j1}(t - d_{j1}(t))| + \sum_{j=1}^{N} a_{ij}\tau_j \\
&\quad + d_i\bar{\mu}_i|x_{i1}(t) + x_{i1}(t - d_{i1}(t))| + d_i\tau_i \\
&\le \sum_{j=1}^{N}\sum_{s=1}^{N} a_{is}\bar{\mu}_s|h_{sj}|(|z_{j1}(t)| + |z_{j1}(t - d_{j1}(t))|) \\
&\quad + d_i\bar{\mu}_i\sum_{j=1}^{N}|h_{ij}|(|z_{j1}(t)| + |z_{j1}(t - d_{j1}(t))|) \\
&\quad + \sum_{j=1}^{N} a_{ij}\tau_j + 2\sum_{j=1}^{N} a_{ij}\bar{\mu}_j M + 2d_i\bar{\mu}_i M + d_i\tau_i \\
&\le \sum_{j=1}^{N}\bar{\sigma}_{i1j}(|z_{j1}(t)| + |z_{j1}(t - d_{j1}(t))|) + \bar{\beta}_{i1} \\
&\le c_1(\|\mathcal{Z}(t)\|_\triangle + \sum_{j=1}^{N}\|\mathcal{Z}(t - d_{j1}(t))\|_\triangle) + \bar{\beta}_{i1}, \qquad (24)
\end{aligned}
$$

where $c_1 > 0$ is a constant and

$$
\begin{aligned}
\bar{\sigma}_{i1j} &= d_i\bar{\mu}_i|h_{ij}| + \sum_{s=1}^{N} a_{is}\bar{\mu}_s|h_{sj}|, \\
\bar{\beta}_{i1} &= 2d_i\bar{\mu}_i M + 2\sum_{j=1}^{N} a_{ij}\bar{\mu}_j M + d_i\tau_i + \sum_{j=1}^{N} a_{ij}\tau_j.
\end{aligned}
$$

By (19)–(20), Assumption 1, and [22, Lemma 2.2], we get

$$
\begin{aligned}
|\tilde{g}_{i2}| &\le \frac{1}{L}\bar{\mu}_i\left(\sum_{k=1}^{2}|x_{ik}(t)| + \sum_{k=1}^{2}|x_{ik}(t - d_{ik}(t))|\right) + \frac{1}{L}\tau_i \\
&\le \frac{1}{L}\bar{\mu}_i\sum_{j=1}^{N}|h_{ij}|(|z_{j1}(t)| + |z_{j1}(t - d_{j1}(t))|) + \frac{1}{L}\tau_i \\
&\quad + \bar{\mu}_i(|z_{i2}(t)| + |z_{i2}(t - d_{i2}(t))|) + \frac{1}{L}2\bar{\mu}_i M
\end{aligned}
$$

$$\leq (L^{-1}c_{11}+c_{12})\Big(\sum_{j=1}^{N}\sum_{k=1}^{2}\|\mathcal{Z}(t-d_{jk}(t))\|_{\triangle} + \|\mathcal{Z}(t)\|_{\triangle}\Big) + L^{-1}(2\bar{\mu}_i M+\tau_i), \quad (25)$$

where c_{11} and c_{12} are positive constants.

From (24), Lemmas 1–2, we have

$$\begin{aligned}|\tilde{g}_{i1}^T||\tilde{g}_{i1}| &\leq c_1^2(\|\mathcal{Z}(t)\|_{\triangle}+\sum_{j=1}^{N}\|\mathcal{Z}(t-d_{j1}(t))\|_{\triangle})^2+\bar{\beta}_{i1}^2 \\ &\quad +2\bar{\beta}_{i1}c_1(\|\mathcal{Z}(t)\|_{\triangle}+\sum_{j=1}^{N}\|\mathcal{Z}(t-d_{j1}(t))\|_{\triangle}) \\ &\leq (2c_1^2N+\bar{\beta}_{i1}c_1)\Big(\sum_{j=1}^{N}\|\mathcal{Z}(t-d_{j1}(t))\|_{\triangle}^2 + \|\mathcal{Z}(t)\|_{\triangle}^2\Big)+2\bar{\beta}_{i1}c_1+\bar{\beta}_{i1}^2 \\ &\leq \tilde{c}_{10}(\|\mathcal{Z}(t)\|_{\triangle}^2+\sum_{j=1}^{N}\|\mathcal{Z}(t-d_{j1}(t))\|_{\triangle}^2)+\tilde{c}_{11},\end{aligned} \quad (26)$$

where constants $\tilde{c}_{10}>0$ and $\tilde{c}_{11}>0$.

By (24)–(25), using [22, Lemma 2.1] and Lemmas 1–2, we obtain

$$\begin{aligned}|\tilde{g}_{i2}^T||\tilde{g}_{i2}| &\leq (\tilde{c}_{12}L^{-1}+\tilde{c}_{13}L^{-2}+\tilde{c}_{110})\Big(\|\mathcal{Z}(t)\|_{\triangle}^2 + \sum_{j=1}^{N}\sum_{k=1}^{2}\|\mathcal{Z}(t-d_{jk}(t))\|_{\triangle}^2\Big)+\tilde{c}_{15}L^{-2} \\ &\quad +\tilde{c}_{14}L^{-1}, \\ |\tilde{g}_{i1}^T||\tilde{g}_{i2}| &\leq (\tilde{c}_{16}+\tilde{c}_{17}L^{-1})\Big(\sum_{j=1}^{N}\sum_{k=1}^{2}\|\mathcal{Z}(t-d_{jk}(t))\|_{\triangle}^2 + \|\mathcal{Z}(t)\|_{\triangle}^2\Big)+(\tilde{c}_{18}+\tilde{c}_{19}L^{-1}),\end{aligned} \quad (27)$$

where $\tilde{c}_{1i}, i=2,3,\cdots,9,10$ are positive constants.

From (26)–(27), using [22, Lemmas 2.1 and 2.2], and Lemma 1, we have

$$\begin{aligned}&\frac{1}{2}\mathrm{Tr}\left\{G(\mathcal{Z})\frac{\partial^2 V}{\partial\mathcal{Z}^2}G^T(\mathcal{Z})\right\} \\ &\leq \frac{1}{2}r\left|G(\mathcal{Z})\frac{\partial^2 V}{\partial\mathcal{Z}^2}G^T(\mathcal{Z})\right|_{\infty} \\ &\leq \frac{1}{2}r\sqrt{r}\left|G(\mathcal{Z})\frac{\partial^2 V}{\partial\mathcal{Z}^2}G^T(\mathcal{Z})\right| \\ &\leq \frac{1}{2}r\sqrt{r}\sum_{i=1}^{N}\sum_{m=1}^{2}\sum_{s=1}^{2}\left(\left|\frac{\partial^2 V}{\partial z_{im}\partial z_{is}}\right|\left|\tilde{g}_{im}^T\right||\tilde{g}_{is}|\right) \\ &\leq (\tilde{c}_{20}+\tilde{c}_{21}L^{-1}+\tilde{c}_{22}L^{-2})\|\mathcal{Z}(t)\|_{\triangle}^4+(\tilde{c}_{24}L^{-1} \\ &\quad +\tilde{c}_{23}+\tilde{c}_{25}L^{-2})\sum_{j=1}^{N}\sum_{k=1}^{2}\|\mathcal{Z}(t-d_{jk}(t))\|_{\triangle}^4 \\ &\quad +(\tilde{c}_{26}+\tilde{c}_{27}L^{-1}+\tilde{c}_{28}L^{-2})\|\mathcal{Z}(t)\|_{\triangle}^2,\end{aligned} \quad (28)$$

where $\tilde{c}_{2i}, i=0,1,\cdots,8$ are positive constants.

By using Lemma 1, we obtain

$$\begin{aligned}\tilde{c}_{26}\|\mathcal{Z}(t)\|_{\triangle}^2 &\leq \frac{1}{2}\tilde{c}_{26}^2L^{1/2}\|\mathcal{Z}(t)\|_{\triangle}^4+\frac{1}{2}L^{-1/2}, \\ \tilde{c}_{27}L^{-1}\|\mathcal{Z}(t)\|_{\triangle}^2 &\leq \frac{1}{2}\tilde{c}_{27}^2L^{-1}\|\mathcal{Z}(t)\|_{\triangle}^4+\frac{1}{2}L^{-1}, \\ \tilde{c}_{28}L^{-2}\|\mathcal{Z}(t)\|_{\triangle}^2 &\leq \frac{1}{2}\tilde{c}_{28}^2L^{-2}\|\mathcal{Z}(t)\|_{\triangle}^4+\frac{1}{2}L^{-2}.\end{aligned} \quad (29)$$

Noting $L>1$ and substituting (29) into (28), we get

$$\begin{aligned}&\frac{1}{2}\mathrm{Tr}\left\{G(\mathcal{Z})\frac{\partial^2 V}{\partial\mathcal{Z}^2}G^T(\mathcal{Z})\right\} \\ &\leq \Big(\tilde{c}_{20}+\frac{1}{2}\tilde{c}_{26}^2L^{1/2}+(\tilde{c}_{21}+\frac{1}{2}\tilde{c}_{27}^2)L^{-1}+(\tilde{c}_{22} \\ &\quad +\frac{1}{2}\tilde{c}_{28}^2)L^{-2}\Big)\|\mathcal{Z}(t)\|_{\triangle}^4+(\tilde{c}_{23}+\tilde{c}_{24}L^{-1} \\ &\quad +\tilde{c}_{25}L^{-2})\sum_{j=1}^{N}\sum_{k=1}^{2}\|\mathcal{Z}(t-d_{jk}(t))\|_{\triangle}^4 \\ &\quad +\frac{1}{2}(L^{-1/2}+L^{-1}+L^{-2}) \\ &\leq (\tilde{c}_{20}+\tilde{c}_{31}L^{1/2}+\tilde{c}_{32}L^{-1}+\tilde{c}_{33}L^{-2})\|\mathcal{Z}(t)\|_{\triangle}^4 \\ &\quad +(\tilde{c}_{26}+\tilde{c}_{27}+\tilde{c}_{28})\sum_{j=1}^{N}\sum_{k=1}^{2}\|\mathcal{Z}(t-d_{jk}(t))\|_{\triangle}^4 \\ &\quad +\frac{1}{2}(L^{-1/2}+L^{-1}+L^{-2}) \\ &\leq (\tilde{c}_{20}+\tilde{c}_{31}L^{1/2}+\tilde{c}_{32}L^{-1}+\tilde{c}_{33}L^{-2})\|\mathcal{Z}(t)\|_{\triangle}^4 \\ &\quad +\tilde{c}_{34}\sum_{i=1}^{N}\sum_{j=1}^{2}\|\mathcal{Z}(t-d_{ij}(t))\|_{\triangle}^4+\frac{1}{2}(L^{-1/2} \\ &\quad +L^{-1}+L^{-2}),\end{aligned}$$

where constants $\tilde{c}_{3i}>0, i=1,\cdots,4$.

C. MAIN RESULTS

Theorem 1: If Assumptions 1–4 hold, under the distributed observer (15) and controller (16), the distributed output-feedback tracking problem of the system (2) is solvable. Specifically,

1) For any given ε and initial value $x(t_0)$, there is a finite-time $T(x(t_0),\varepsilon)$ such that

$$E|x_{i1}(t)-y_0(t)|^4<\varepsilon, \quad \forall t>T(x(t_0),\varepsilon), i=1,2,\cdots,N.$$

2) All the states of the closed-loop system are bounded in probability.

Proof. Step 1. We construct a L-K functional

$$\begin{aligned}T(\mathcal{Z}(t)) &= \sum_{i=1}^{N}\sum_{j=1}^{2}\frac{\bar{c}_{07}+\tilde{c}_{34}}{(1-\gamma)e^{-d}}\int_{t-d_{ij}(t)}^{t}e^{\sigma-t}\|\mathcal{Z}(\sigma)\|_{\triangle}^4 d\sigma \\ &\quad +V(\mathcal{Z}(t)) \\ &= Y(\mathcal{Z}(t))+V(\mathcal{Z}(t)),\end{aligned} \quad (30)$$

where $\bar{c}_{07}$ and $\tilde{c}_{34}$ are positive parameters and

$$Y(\mathcal{Z}(t))=\sum_{i=1}^{N}\sum_{j=1}^{2}\frac{\bar{c}_{07}+\tilde{c}_{34}}{(1-\gamma)e^{-d}}\int_{t-d_{ij}(t)}^{t}e^{\sigma-t}\|\mathcal{Z}(\sigma)\|_{\triangle}^{4}d\sigma.$$

We can verify that $T(\mathcal{Z}(t))$ is $\mathcal{C}^2$ on $\mathcal{Z}(t)$. By [26, Lemma 4.3], there exist two class $\mathcal{K}_\infty$ functions δ_1 and δ_2, such that

$$\delta_1(|\mathcal{Z}(t)|)\le V(\mathcal{Z}(t))\le\delta_2(|\mathcal{Z}(t)|). \tag{31}$$

From $d_{ij}(t):\mathbb{R}_+\to[0,d]$, (31), and [22, Lemma 2.2], there exist constants $\tilde{c}>0$, $c>0$ and a class $\mathcal{K}_\infty$ functions δ_3, $\bar{\delta}_3$, such that

$$\begin{aligned}&\frac{\bar{c}_{07}+\tilde{c}_{34}}{(1-\gamma)e^{-d}}\int_{t-d_{ij}(t)}^{t}e^{\sigma-t}\|\mathcal{Z}(\sigma)\|_{\triangle}^{4}d\sigma\\&\le\tilde{c}\int_{t-d_{ij}(t)}^{t}\delta_3|\mathcal{Z}(\sigma)|d\sigma\\&\overset{\sigma=s+t}{\le}\tilde{c}\int_{-d_{ij}(t)}^{0}\delta_3|\mathcal{Z}(s+t)|d(s+t)\\&\le c\sup_{-d\le s\le 0}(\delta_3|\mathcal{Z}(s+t)|)\\&\le\bar{\delta}_3\Big(\sup_{-d\le s\le 0}|\mathcal{Z}(s+t)|\Big).\end{aligned}\tag{32}$$

Since $|\mathcal{Z}(s+t)|\le\sup_{-d\le s\le 0}|\mathcal{Z}(s+t)|$, so $\delta_2(|\mathcal{Z}(s+t)|)\le\delta_2(\sup_{-d\le s\le 0}|\mathcal{Z}(s+t)|)$.

Defining $\delta_4=\delta_2+\bar{\delta}_3$, from (30)–(32), we get

$$\delta_1(|\mathcal{Z}(t)|)\le T(\mathcal{Z}(t))\le\delta_4(\sup_{-d\le s\le 0}|\mathcal{Z}(s+t)|). \tag{33}$$

Step 2. From (30) and Lemmas 3–4, we have

$$\begin{aligned}&\mathcal{L}T(\mathcal{Z}(t))\\&\le L\frac{\partial V}{\partial\mathcal{Z}}E(\mathcal{Z})+\frac{\partial V}{\partial\mathcal{Z}}F(\mathcal{Z})-Y(\mathcal{Z}(t))\\&\quad+\frac{1}{2}Tr\Big\{G(\mathcal{Z})\frac{\partial^2V(\mathcal{Z}(t))}{\partial\mathcal{Z}^2(t)}G^T(\mathcal{Z})\Big\}+2N\frac{\bar{c}_{07}+\tilde{c}_{34}}{(1-\gamma)e^{-d}}\\&\quad\cdot\|\mathcal{Z}(t)\|_{\triangle}^{4}-\sum_{i=1}^{N}\sum_{j=1}^{2}(\bar{c}_{07}+\tilde{c}_{34})\|\mathcal{Z}(t-d_{ij}(t))\|_{\triangle}^{4}\\&\le -L\Big(\underline{c}_0-(\bar{c}_{05}+\tilde{c}_{31})L^{-1/2}+(\bar{c}_{06}+\tilde{c}_{32})L^{-2}\\&\quad\times(\bar{c}_{01}+\tilde{c}_{20}+2N\frac{\bar{c}_{07}+\tilde{c}_{34}}{(1-\gamma)e^{-d}})L^{-1}+\tilde{c}_{33}L^{-3}\Big)\|\mathcal{Z}(t)\|_{\triangle}^{4}\\&\quad+\frac{1}{4}(L^{-3/2}+L^{-1})+\frac{1}{2}(L^{-1/2}+L^{-2}+L^{-1})-Y(\mathcal{Z}(t))\\&\le -L(\underline{c}_0-(\bar{c}_{05}+\tilde{c}_{31}+\bar{c}_{01}+\tilde{c}_{20}+2N\frac{\bar{c}_{07}+\tilde{c}_{34}}{(1-\gamma)e^{-d}}\\&\quad+\bar{c}_{06}+\tilde{c}_{32}+\tilde{c}_{33})L^{-1/2})\|\mathcal{Z}(t)\|_{\triangle}^{4}-Y(\mathcal{Z}(t))+\beta^*,\end{aligned}\tag{34}$$

where

$$\beta^*=\frac{1}{4}L^{-3/2}+\frac{1}{2}L^{-1/2}+\frac{3}{4}L^{-1}+\frac{1}{2}L^{-2}.$$

Since $\underline{c}_0$ is independent of $\bar{c}_{20}$, $\bar{c}_{0i}$, and $\tilde{c}_{3j}$, $i=1,5,6,7$, $j=1,\cdots,4$, we can choose

$$\begin{aligned}L\ge\max\Big\{&\frac{1}{\underline{c}_0}(\bar{c}_{05}+\tilde{c}_{31}+\bar{c}_{01}+\tilde{c}_{20}+\bar{c}_{06}\\&+2N\frac{\bar{c}_{07}+\tilde{c}_{34}}{(1-\gamma)e^{-d}}+\tilde{c}_{32}+\tilde{c}_{33})^2,1\Big\}.\end{aligned}\tag{35}$$

Then for any constant $\tilde{c}_0>0$, (34) can be rewritten as

$$\mathcal{L}T(\mathcal{Z}(t))\le-\tilde{c}_0\|\mathcal{Z}\|_{\triangle}^{4}-Y(\mathcal{Z}(t))+\beta^*. \tag{36}$$

By [22, Lemma 2.2], there are constants $\check{c}>0$, $\hat{c}>0$ such that

$$\check{c}\|\mathcal{Z}\|_{\triangle}^{4}\le V(\mathcal{Z}(t))\le\hat{c}\|\mathcal{Z}\|_{\triangle}^{4}. \tag{37}$$

By (36) and (37) we get

$$\begin{aligned}\mathcal{L}T(\mathcal{Z}(t))&\le-\tilde{c}_0\hat{c}^{-1}V(\mathcal{Z}(t))-Y(\mathcal{Z}(t))+\beta^*\\&\le-c_0^*T(\mathcal{Z}(t))+\beta^*,\end{aligned}\tag{38}$$

where $c_0^*=\min\{\frac{\tilde{c}_0}{\hat{c}},1\}$.

By (38) and [27, Th.1] system (2) and (14) have an almost surely unique solution on $[0,\infty]$.

Let

$$\eta_l=\inf\{t:t\ge t_0\,|\mathcal{Z}(t)|\ge l\},\quad\forall l>0,$$

and $t_l=\min\{\eta_l,t\}$ for any $t\ge t_0$. Since $|\mathcal{Z}(\cdot)|$ is bounded on $[t_0,t_l]$ a.s., $T(\mathcal{Z})$ is bounded on $[t_0,t_l]$ a.s. It then follows from (38) that $\mathcal{L}T$ is bounded on $[t_0,t_l]$ a.s. By Dynkin formul, we have

$$\begin{aligned}E(e^{c_0^*t_l}T(\mathcal{Z}(t_l)))&\le E\int_{t_0}^{t_l}e^{c_0^*s}\mathcal{L}T(\mathcal{Z}(s))ds+ET(\mathcal{Z}(t_0))\\&\quad\cdot e^{c_0^*t_0}+c_0^*E\int_{t_0}^{t_l}e^{c_0^*s}T(\mathcal{Z}(s))ds.\end{aligned}\tag{39}$$

Note that $\lim_{l\to\infty}\eta_l=\infty$. Then, letting $l\to\infty$, we get

$$\begin{aligned}e^{c_0^*t}E(T(\mathcal{Z}(t))&\le E\int_{t_0}^{t}e^{c_0^*s}\mathcal{L}T(\mathcal{Z}(s))ds+ET(\mathcal{Z}(t_0))\\&\quad\cdot e^{c_0^*t_0}+c_0^*E\int_{t_0}^{t}e^{c_0^*s}T(\mathcal{Z}(s))ds.\end{aligned}\tag{40}$$

From (38), we get

$$e^{c_0^*t}ET(\mathcal{Z}(t))\le e^{c_0^*t_0}ET(\mathcal{Z}(t_0))+\frac{\beta^*}{c_0^*}e^{c_0^*t}-\frac{\beta^*}{c_0^*}e^{c_0^*t_0},$$

or equivalently,

$$ET(\mathcal{Z}(t))\le e^{-c_0^*(t-t_0)}ET(\mathcal{Z}(t_0))+\frac{\beta^*}{c_0^*}(1-e^{-c_0^*(t-t_0)}). \tag{41}$$

Step 3. Denoting $z_1(t)=(z_{11}(t),\cdots,z_{N1}(t))^T$, by (41) we can obtain

$$\begin{aligned}E\,|z_1(t)|^4&=E(z_{11}^2(t)+z_{21}^2(t)+\cdots+z_{N1}^2(t))^2\\&\le NE(z_{11}^4(t)+z_{21}^4(t)+\cdots+z_{N1}^4(t))\\&\le 4NET(\mathcal{Z}(t))\\&\le 4N\Big(e^{-c_0^*(t-t_0)}ET(\mathcal{Z}(t_0))+\frac{\beta^*}{c_0^*}\\&\quad-\frac{\beta^*}{c_0^*}e^{-c_0^*(t-t_0)}\Big).\end{aligned}\tag{42}$$

From (13), we get

$$\begin{aligned} z_1(t) &= \Bigg(\sum_{s=1}^{N} a_{1s}(x_{11}(t) - x_{s1}(t)) + b_1(x_{11}(t) - y_0(t)), \\ &\qquad \cdots, \sum_{s=1}^{N} a_{Ns}(x_{N1}(t) - x_{s1}(t)) + b_N(x_{N1}(t) \\ &\qquad - y_0(t))\Bigg)^T \\ &= \Bigg(\sum_{s=1}^{N} a_{1s}(x_{11}(t) - y_0(t)) - \sum_{s=1}^{N} a_{1s}(x_{s1}(t) \\ &\qquad - y_0(t)) + b_1(x_{11}(t) - y_0(t)), \\ &\qquad \cdots, \sum_{s=1}^{N} a_{Ns}(x_{N1}(t) - y_0(t)) - \sum_{s=1}^{N} a_{Ns}(x_{N1}(t) \\ &\qquad - y_0(t)) + b_N(x_{N1}(t) - y_0(t))\Bigg)^T \\ &= H(x_1(t) - \mathbf{1}_N y_0), \end{aligned} \tag{43}$$

where $x_1(t) = (x_{11}(t), \cdots, x_{N1}(t))^T$ and $\mathbf{1}_N = (\underbrace{1, 1, \cdots, 1}_{N})^T$.

By (42)–(43) we obtain

$$E|x_1(t) - \mathbf{1}_N y_0|^4 \leq 4N|H^{-1}|^4\Big(e^{-c_0^*(t-t_0)}ET(\mathcal{Z}(t_0)) + \frac{\beta^*}{c_0^*}(1 - e^{-c_0^*(t-t_0)})\Big). \tag{44}$$

Since $\beta^* = \frac{1}{4}L^{-3/2} + \frac{1}{2}L^{-1/2} + \frac{3}{4}L^{-1} + \frac{1}{2}L^{-2}$, by tuning the gain L, β^* can be made arbitrarily small. From (44), we obtain that for any $\varepsilon > 0$ and initial value $x(t_0)$, there is a finite-time $T(x(t_0), \varepsilon)$, such that

$$E|x_{i1}(t) - y_0(t)|^4 \leq \varepsilon, \quad \forall t > T(x(t_0), \varepsilon), i = 1, 2, \cdots, N, \tag{45}$$

which means that conclusion 1) holds.

Step 4. From (41) we obtain

$$ET(\mathcal{Z}(t)) \leq ET(\mathcal{Z}(t_0)) + \frac{\beta^*}{c_0^*}. \tag{46}$$

Let $\xi = \mathcal{Z}(t)$ and

$$ET(\xi) \geq \int_{|\xi|>c} T(\xi)P(dw) \geq \inf_{|\xi|>c} T(\xi)P(|\xi| > c). \tag{47}$$

From (46)–(47) we have

$$P(|\xi| > c) \leq \frac{ET(\mathcal{Z}(t_0)) + \frac{\beta^*}{c_0^*}}{\inf_{|\xi|>c} T(\xi)}, \tag{48}$$

together with (41) we get

$$\lim_{c\to\infty} \sup_{t>t_0} P(|\xi| > c) \leq \lim_{c\to\infty} \sup_{t>t_0} \frac{ET(\mathcal{Z}(t_0)) + \frac{\beta^*}{c_0^*}}{\inf_{|\xi|>c} T(\xi)}. \tag{49}$$

By (49) and [19, Definition 1], ξ is bounded in probability. Using (43) and Assumption 2, we get $y_i(t) = z_{i1}(t)$ is bounded in probability, and

$$\xi_{i2} = z_{i2} + \sum_{j=1}^{N} h_{ij}c_{j1}z_{j1}, \quad i = 1, \cdots, N. \tag{50}$$

Notice that $\xi_{i1}(t)$, $\xi_{i2}(t)$ and $z_{i1}(t)$ are bounded in probability, by (50), we get $z_{i2}(t)$ is bounded in probability, $i = 1, 2, \cdots, N$. By (13), we can conclude $x_{i1}(t)$ and $x_{i2}(t)$ are bounded in probability, $i = 1, 2, \cdots, N$. Thence, we are able to conclude that all the states of the closed-loop system are bounded in probability.

The proof is completed.

Remark 3: The parameter ε is an arbitrary positive constant, which is pre-given according to the control objective. In the control scheme, We first set ε, then we design the control (depends on ε). Usually, the smaller requires larger control.

Remark 4: The nonlinear drift terms and nonlinear diffusions terms in (2) make all the existing distributed control methods invalid. To overcome this difficulty, we develop a new distributed stochastic homogeneous domination method. Specifically, we first focus on the nominal MASs analysis to produce negative terms, then we use these negative terms to dominate nonlinear terms appeared in the distributed output-feedback control design. The novelty of this approach is that no precise information of the nonlinearities is needed. Thus, this approach provided a new perspective to deal with the distributed output-feedback control problem.

IV. A SIMULATION EXAMPLE

Consider the following MASs. Fig. 1 shows the topology.

$$\begin{aligned} dx_{i1}(t) &= x_{i2}(t)dt + f_{i1}(t, x_{i1}(t), x_{i1}(t - d_{i1}(t)))dt \\ &\quad + g_{i1}^T(t, x_{i1}(t), x_{i1}(t - d_{i1}(t)))dw, \\ dx_{i2}(t) &= u_i(t)dt + f_{i2}(t, \bar{x}_{i2}(t), \bar{x}_{i2}(t - d_{i2}(t)))dt \\ &\quad + g_{i2}^T(t, \bar{x}_{i2}(t), \bar{x}_{i2}(t - d_{i2}(t)))dw, \\ y_i(t) &= \sum_{j=1}^{N} a_{ij}(x_{i1}(t) - x_{j1}(t)) + b_i(x_{i1}(t) - y_0(t)), \end{aligned} \tag{51}$$

where $f_{11} = x_{11}(t - d_{11}(t)), g_{11} = 0, f_{12} = 0, g_{12} = \frac{1}{2}x_{12}(t - d_{12}(t)), f_{21} = x_{21}(t - d_{21}(t)), g_{21} = \frac{1}{2}x_{21}(t), f_{22} = \frac{1}{2}x_{22}(t), g_{22} = \frac{1}{5}x_{22}(t - d_{22}(t)), f_{31} = x_{31}(t - d_{31}(t)), g_{31} = x_{31}(t), f_{32} = x_{32}(t), g_{32} = x_{32}(t - d_{32}(t)), d_{11}(t) = \frac{1}{4}\sin t, d_{12}(t) = \frac{1}{10}\cos t, d_{21}(t) = \frac{1}{5}\sin t, d_{22}(t) = \frac{1}{4}\cos t, d_{31}(t) = \frac{1}{4}\sin t, d_{32}(t) = \frac{1}{10}\sin t, y_0(t) = \frac{1}{1+t}$. Assumptions 1 and 4 are satisfied with $\mu_i = \bar{\mu}_i = 1, \tau_i = 1, d_{ij}(t) \leq 1$ and $\dot{d}_{ij}(t) \leq 1, i = 1, 2, 3, j = 1, 2$.

Following the design scheme in Section III, choosing $c_{11} = \frac{5}{4}, c_{12} = 1, c_{21} = \frac{9}{4}, c_{22} = 14.8, c_{31} = \frac{5}{4}, c_{32} = \frac{1}{4}$ and $L = 38$, we can get the following distributed observers:

$$\begin{aligned} \dot{w}_{12} &= L(-w_{12} - y_1 + u_1), \\ \dot{w}_{22} &= L(-w_{22} - y_2 + y_1 + u_2), \\ \dot{w}_{32} &= L(-w_{32} - y_3 + u_3), \end{aligned} \tag{52}$$

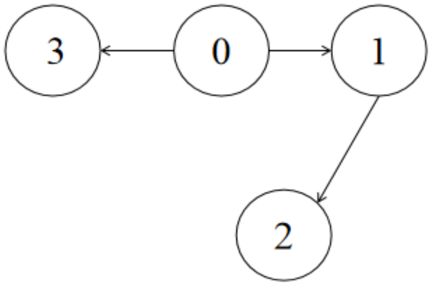

FIGURE 1. Directed topology $\bar{\mathcal{G}}$.

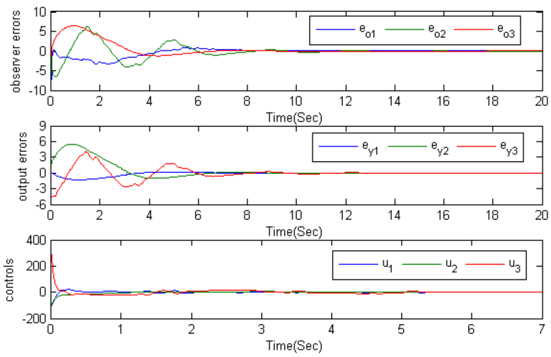

FIGURE 2. The response of the closed-loop system (52)–(55).

and the following distributed controllers:

$$u_1 = -51.8x_{11} - 23.02w_{12} + \frac{51.8}{1+t},$$
$$u_2 = -355.5x_{21} - 50.625x_{11} - 22.5w_{22} + \frac{406.125}{1+t},$$
$$u_3 = -7.2x_{31} - 3.2w_{32} + \frac{7.2}{1+t}. \qquad (53)$$

Letting $e_{i2} = z_{i2} - z_{i1} - w_{i2}$, we can get the following observer errors:

$$e_{o1} = -x_{11} - w_{12} - \frac{1}{38}x_{12} + \frac{1}{1+t},$$
$$e_{o2} = x_{11} - x_{21} - w_{22} + \frac{1}{38}x_{22},$$
$$e_{o3} = -x_{31} - w_{32} + \frac{1}{38}x_{32} + \frac{1}{1+t}. \qquad (54)$$

The tracking errors is defined by

$$e_{y1} = x_{11} - \frac{1}{1+t},$$
$$e_{y2} = x_{21} - \frac{1}{1+t},$$
$$e_{y3} = x_{31} - \frac{1}{1+t}. \qquad (55)$$

By choosing initial conditions $x_{11}(0) = 2.6, x_{12}(0) = 10, w_{12}(0) = 1, x_{21}(0) = 1, x_{22}(0) = 0.5, w_{22}(0) = 1, x_{31}(0) = 0.01, x_{32}(0) = 0.01, w_{32}(0) = -0.1$. Fig. 2 shows the responses of the closed-loop system (52)–(55).

V. CONCLUSION

For stochastic nonlinear MASs, a new distributed output-feedback tracking scheme is proposed in this paper. The MASs simultaneously consider time-varying delays, unmeasurable states, and Hessian terms. We construct a L-K functional to deal with the time-delays terms. Distributed observers and distributed output-feedback controllers are designed to solve the output tracking problem.

In the future work, we will consider generalizing the results in this paper to more general systems such as those in [28]–[30].

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