

Received May 29, 2022, accepted June 10, 2022, date of publication June 14, 2022, date of current version June 20, 2022.

Digital Object Identifier 10.1109/ACCESS.2022.3183149

New Fixed-Time Stability Theorems for Delayed Fractional-Order Systems and Applications

UDHAYAKUMAR KANDASAMY¹, FATHALLA A. RIHAN¹,
RAKKIYAPPAN RAJAN², AND MAHMOUD M. EL-KHOULY³

¹Department of Mathematical Sciences, College of Science, United Arab Emirates (UAE) University, Al Ain 15551, United Arab Emirates

²Department of Mathematics, Bharathiar University, Coimbatore 641 046, India

³Department of Information Technology, Faculty of Computers and Artificial Intelligence, Helwan University, Cairo 11795, Egypt

Corresponding author: Fathalla A. Rihan (frihan@uaeu.ac.ae)

This work was supported by United Arab Emirates (UAE) University, United Arab Emirates, under Project 12S005-UPAR-5-2020.

ABSTRACT This paper examines the problem of improved fixed-time stability for generalized delayed fractional-order systems (FOSs). As a first step, some stability conditions are presented in two theorems to verify fixed-time stability (F-TS) of FOSs by using Lyapunov stability theory and indefinite Lyapunov functionals, where the fractional-order derivative of the indefinite functions may not exist. Furthermore, the corresponding estimated settling time of FOSs is also provided. Second, the results are extended to study fractional-order neural networks (FONNs) with time-delays in fixed-time synchronization. Using the Dirac delta functions, we propose an explicit saturated impulsive controller to synchronize the master and slave systems. Moreover, by constructing suitable indefinite Lyapunov-Krasovskii functions (LKF), we derive algebraic conditions to guarantee the fixed-time synchronization of the addressed FONNs. The simulation results demonstrate the feasibility and efficacy of the proposed method.

INDEX TERMS Fractional-order, time-delay, fixed-time stability, indefinite functional, Lyapunov function, saturation, impulsive control.

I. INTRODUCTION

The literature indicates that arbitrary order calculus (or fractional calculus) appeared around the same time as classical calculus. Fractional calculus did not attract attention until recent years when scholars found that classical calculus cannot explain many random errors, whereas fractional calculus will be able to describe some unusual diffusion processes. Because of its memory and hereditary properties, fractional calculus has become widely used in a variety of fields. The flow of blood, electrolysis, viscosity, and similar phenomena are examples of these phenomena [1]–[3]. Many promising results have been published in the past few years, particularly in the area of stability analysis of FOSs. Yang *et al.* [4] study impulsive fractional-order nonlinear systems with quadratic Lyapunov functions. In their paper, Liu *et al* [5] apply the linear matrix inequality method to the case of uncertain FOSs to extend the results obtained by Yang *et al* [4].

The neural networks (NNs) are computational models that contain interconnected nodes that perform functions

similar to the brain's neurons. In particular, they are useful for signal processing, pattern classification, convex classification, cognitive control, etc. [6]–[8]. Note that NN internal dynamics, such as stability, multistability, synchronization, etc. [9]–[13], each of which has become increasingly important in recent decades. FONNs have therefore been investigated in recent years to determine their stability over a finite time period, see [14]–[16]. FONNs, which are a type of fractional-order dynamical system, have recently been discussed extensively. Since FOSs contain memory and heredity, they have an advantage in information processing, parameter estimations, and a variety of other artificial intelligence tasks. Many great and fascinating discoveries have been made using FONNs in the areas of stability [17], stabilization [18], and synchronization [19].

It is inevitable that there will be delays in many electronic networks. Due to the finite switching speed of amplifiers and the inherent time required for information transmission between neurons, the time-delay produces nonlinear dynamics such as oscillation, bifurcation, chaos, and even instability [20]. A delay in the response of a neuron might affect the network's stability, resulting in oscillatory and

The associate editor coordinating the review of this manuscript and approving it for publication was Norbert Herencsar^{id}.

unstable characteristics. Therefore, it is necessary to study the networks stability. Thus FONNs with various stability has been studied [21]–[25], such as global stability, Mittag-Leffler stability, finite-time stability (FN-TS) and asymptotic stability. Additionally, a great deal of attention has been paid to synchronization as a major and intriguing phenomenon of NNs by a large number of scientists from a variety of different domains. Synchronization means that two or more systems share a common dynamical behavior by coupling or external force. Because of its wide application in the fields of secure communication, biological system, information processing and so on, synchronization control has been extensively studied and many good results have been achieved [27]–[30]. Among them, the global stability analysis of FONNs is investigated in [21]. By utilizing the geometrical properties of activation functions and algebraic properties of nonsingular M-matrix, the coexistence and multistability of multiple equilibrium points of FONNs are obtained in [22]. By using the generalized reciprocally convex inequality and novel LKfs, several stability criteria for the considered NNs are investigated in [26]. Based on the maximum modulus principle and the spectral radii of matrices, [29] studies the delay independent stability criteria for the FONNs. Some results for FONNs with unbounded time-varying delays are derived to ensure that the equilibrium points of the nonlinear system is asymptotically stable in [30].

Synchronization can be divided into two broad categories based on synchronization time: asymptotic synchronization and finite-time synchronization (FN-TSY). As time passes, driving and response systems achieve asymptotic synchronization. There are several types of synchronization, including exponential synchronization, hybrid synchronization, lag synchronization, and cluster synchronization. The practice of synchronization in indefinite time is often unrealistic, we consider only one example: in secure communication, the longer the synchronization period, the greater the probability that the information will be broken, and that more information will not be successfully received during transmission.

Additionally, in some engineering domains, it is always expected that the synchronization will be completed within a short period of time, which is commonly referred to as FN-TSY. While asymptotic synchronization is better when it comes to robustness and anti-interference properties, FN-TSY is also better in terms of synchronization time since it achieves optimality. However, one disadvantage of FN-TSY is that its settling time is dependent on the initial condition of the system. The fact that many engineering systems' initial values are no longer available may result in significant inconvenience for their practical implementation. A solution to this problem has been established by [31], where the settling time is restricted at its upper limit by a fixed number that is independent of the initial value. Calculating the settling time is a major challenge in F-TSY, as it depends on the initial values of the original system. Generally, different initial values will result in different convergence times and different forecasts. From the perspective of the application, it is obvious that it

is problematic. Therefore, it is necessary to learn about the convergence times of solutions with initial values. Moreover, many practical systems, such as power systems and spacecraft dynamics, also require FN-TS for some realistic reasons and needs. It is therefore important to examine both the theory and the applications of F-TS of dynamic systems.

Lyapunov function plays an important role in analysing systems stability, because it does not require the explicit solution of the corresponding ordinary difference or differential equations. By this method, the asymptotic stability is guaranteed if the time-derivative of the Lyapunov function (a positive definite function) along the solutions is negative definite [32]. When the time-derivative of the Lyapunov function is negative semi-definite, stability rather than asymptotic stability follows [33]. Thus, construction of Lyapunov functions has been a fascinating subject. However, the results of previous work [34]–[37] have a common requirement that the time derivatives of Lyapunov functions of systems (or subsystems) must be negative in the convergence process. To relax this restriction, Chen *et al* [38] gave an improved result that the time derivatives of Lyapunov functions are allowed to be indefinite. To relax these restrictions, we will use an analysis tool named indefinite Lyapunov function, which is introduced in the work of Ning *et al* [40] and used to analyze the input-to-state stability of nonlinear time-varying systems. In recent decades, indefinite derivative LFs have been proposed as a framework for investigating the synchronization of various types of dynamical systems, such as nonlinear systems [39]–[41], switched nonlinear time varying systems [42], [43], inertial NNs [41], etc., illustrates how indefinite Lyapunov derivatives can be used to study the F-TS of nonlinear time-varying systems with switched perturbations, where positive definite Lyapunov functions have an indefinite derivative to study the F-TS property. Comparing with the traditional Lyapunov function, this tool allows the time derivatives of Lyapunov functions of subsystems to be indefinite and have a tighter upper bound. This approach has the advantage that, as we have already stated, the derivative of the Lyapunov function need not be negative or negative semi-definite.

Another critical concern is the synchronization control of FONNs to the desired trajectory, which has been well-formulated as derive-response synchronization, which is accomplished using linear feedback control [44], and active control [45]. Impulsive control generally has lower control costs, greater confidentiality, and greater resilience than other types of control. It has been widely applied in a variety of fields, including financial markets, static multisynchronization, stability, and information security. Accordingly, the majority of existing results on the design of impulsive controllers [4], [23] are based on the assumption that impulsive strength is not restricted. In order to achieve the desired control performance, we can artificially increase the impulsive strength of the system. From an applications perspective, it is actually very difficult to accomplish the design aim for every control input since actuator saturation is ubiquitous in

practically all control systems, making it difficult to achieve the design objective for every control input. When a car travels over speed bumps at a high rate of speed, a reduction in speed may be experienced. Speed bumps serve as impulses, and the reduction in speed may lead to saturation. As a consequence, the saturation of impulses can have a significant impact on the dynamics of NNs. If such restrictions are not managed carefully, or even if the appropriate controllers fail to take them into consideration, then detrimental behaviors may result. Therefore, when investigating the impulsive control of FONNs, it is essential to consider the influence of saturated impulses.

Based on the above discussion, the primary goal of this paper is to study the novel F-TS theorems for delayed FOS and to obtain a more accurate settling time as a result of this study. Second, using the newly discovered F-TS theorems, the F-TSY of the zero solution of a class of FONNs with time-delays is investigated, and some new F-TSY results are derived. Among the contributions of the paper are:

- 1) In light of the preliminary results of fractional calculus, some new results are imposed on the indefinite LKF and the framework of a Caputo fractional-order derivative introduced in [46], which leads to the new F-TS theorems. The well-known results presented in [47] are significantly improved over the previously published results.
- 2) The newly established new F-TS theorems can be applied to achieve the F-TSY of delayed FONNs with time-delays by incorporating the newly established F-TS theorems. For delayed FONNs, we propose an explicit saturated impulsive controller based on delta functions. The developed saturated impulsive FOSs reveal a significant result that impulsive control has an influence on controlled FOSs that is both dependent on impulsive function and related to the FOSs' order. Certain prior conclusions about the FN-TS and F-TS of FONNs can be extended.

The rest of this paper is organized as follows: Section II explains the FOS architecture and provides some lemmas. In Section III, the F-TS theorems for considered FOSs with an indefinite LKF approach are presented. In section IV, the saturated impulsive controller is used to analyze the F-TSY of FONNs. The effectiveness and correctness of the presented results are demonstrated by some simulations, and the conclusion of this paper is briefly discussed in Section V and Section VI, respectively.

II. PRELIMINARIES AND MATHEMATICAL MODEL DESCRIPTION

There are some fundamental definitions of fractional calculus in this section, as well as some lemmas that are required during the demonstration of the theory and problem statement.

A. FRACTIONAL CALCULUS BACKGROUND

A fractional derivative can be defined in a number of ways, but three commonly used definitions are Grunwald-Letnikov,

Riemann-Liouville, and Caputo (see [46]). In general, these three definitions are not equivalent.

Definition 1 ([46]): The Caputo fractional-order derivative of order q of a continuous function $w(x)$ is defined as follows:

$$D^q w(x) = \frac{1}{\Gamma(1-q)} \int_0^x (x-\theta)^{-q} f'(\theta) d\theta.$$

where $0 < q < 1$, the term D^q stands for the Caputo fractional-order derivative and the Gamma function is defined in the same manner as before

$$\Gamma(q) = \int_0^\infty e^{-\theta} \theta^{q-1} d\theta.$$

Definition 2 ([46]): For a Lebesgue-integrable function $w(x)$, the fractional-order integral of order q is defined as follows:

$$I_{x_0}^q w(x) = \frac{1}{\Gamma(q)} \int_{x_0}^x (x-\theta)^{q-1} w(\theta) d\theta,$$

where $q > 0, x \geq x_0$.

The Caputo definition is the one that is most frequently encountered in Physical-Chemistry applications, out of the three definitions listed above. Because it simply requires initial conditions in terms of integer-order derivatives, the Caputo derivative is more applicable to real-world problems because it is easier to understand.

Lemma 1 ([48]): For $q \in (0, 1), p \in \mathbb{R}$, there holds

$$D^q w^p(x) = \frac{\Gamma(1+p)}{\Gamma(1+p-q)} w^{p-q}(x) D^q w(x).$$

Lemma 2 ([46]): Suppose that the Caputo fractional-order derivative $D^q w(x)$ is integrable, then

$$I^q D^q w(x) = w(x) - \sum_{l=0}^{n-1} \frac{w^{(l)}(x_0)}{l!} (x-x_0)^l.$$

Particularly, for $0 < q < 1$, one has

$$I^q D^q w(x) = w(x) - w(x_0).$$

Remark 1: The fractional-order operator D^q is a nonlocal operator since its fractional-order derivative depends on the integral's lower boundary. In contrast, the derivative of integer order is unquestionably a local operator.

B. MATHEMATICAL MODEL DESCRIPTION

Consider the following fractional-order differential system with time-delay:

$$\begin{cases} D^q w(t) = g(t, w(t), w(t-\sigma)), & t \geq 0, \\ w(\theta) = \gamma(\theta), & -\sigma \leq \theta \leq 0. \end{cases} \quad (1)$$

Here, $w(t) = (w_1(t), w_2(t), \dots, w_n(t))^T \in \mathbb{R}^n$ is the state variable; $\gamma(\theta) = (\gamma_1(\theta), \gamma_2(\theta), \dots, \gamma_n(\theta))^T \in \mathbb{R}^n$ is the initial function; $w(t-\sigma) = (w_1(t-\sigma), w_2(t-\sigma), \dots, w_n(t-\sigma))^T$ is the delayed state variable; σ is a constant time-delay; let $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be Lebesgue

measurable and essentially locally bounded; $\|\gamma\| = \sup_{\theta \in [-\sigma, 0]} \|\gamma(\theta)\|$.

Definition 3: The origin of (1) is said to achieve fixed-time (F-T) stable, if it is finite-time (FN-T) stable and there is a constant $T' > 0$ such that the settling time $T^*(t_0, w_0) \leq t_0 + T'$, for any initial condition $w_0 \in \mathbb{R}^n$.

Definition 4: A functional $V(w) : \mathbb{R}^n \rightarrow \mathbb{R}$ is C -regular if $V(w)$ satisfies the following:

- a) $V(w) > 0$ for any $w \neq 0$ and $V(0) = 0$;
- b) regular in \mathbb{R}^n ;
- c) $V(w) \rightarrow +\infty$ as $\|w\| \rightarrow +\infty$, i.e $V(w)$ is radially unbounded.

It is important to note that a C -regular Lyapunov function $V(w)$ is not always differentiable in the usual sense.

Our goal in this article is to establish new criteria for the F-TSY of delayed fractional-order differential systems (FDSs) based on indefinite functionals, which will be discussed further below.

III. FIXED-TIME STABILITY/SYNCHRONIZATION THEOREM'S

This section presents some novel F-TS or synchronization theorems for the delayed FDS (1).

Theorem 1: Suppose there exists a C -regular function $V(w(t)) : \mathbb{R}^n \rightarrow \mathbb{R}$, for any indefinite function $\mathcal{A}(t)$ and non-positive function $\mathcal{B}(t)$ such that any solution $w(t)$ of (1) satisfies the inequality

$$D^q V(w(t)) \leq \mathcal{A}(t)V^\alpha(w(t)) + \mathcal{B}(t)V^\beta(w(t)), w(t) \neq 0, \tag{2}$$

where $0 < \alpha < \beta < q$, and $\mathcal{A}(t), \mathcal{B}(t)$ satisfy the following inequalities

$$\begin{cases} \frac{1}{\Gamma(q)} \int_0^t \frac{\mathcal{A}^-(\theta)}{(t-\theta)^{-(q-1)}} d\theta \leq -\delta_1 t^q + \varpi_1, \\ \frac{1}{\Gamma(q)} \int_0^t \frac{\mathcal{B}(\theta)}{(t-\theta)^{-(q-1)}} d\theta \leq -\delta_2 t^q + \varpi_2, \\ \frac{1}{\Gamma(q)} \int_0^t \frac{\mathcal{A}^+(\theta)}{(t-\theta)^{-(q-1)}} d\theta \leq \varpi, \mathcal{B}(t) \leq \mathcal{A}^-(t) \end{cases} \tag{3}$$

for all $t > 0$, where $\mathcal{A}^+(\theta) = \mathcal{A}(\theta) \vee 0, \mathcal{A}^-(\theta) = \mathcal{A}(\theta) \wedge 0$, and $\varpi, \varpi_1, \varpi_2, \delta_1, \delta_2$ are positive constants. Then $V(w(t))$ satisfies the following inequality

$$V^{q-\alpha}(w(t)) \leq V^{q-\alpha}(w(t_0)) + \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} [\varpi + \varpi_1 - \delta_1(t-t_0)^q] \tag{4}$$

for $t_0 \leq t \leq T^*$ and $V(w(t)) = 0$ for all $t \geq T^*$. Then the zero solution of the FDS (1) is F-T stable and the corresponding settling-time is

$$T^* = \left[\frac{(\varpi + \varpi_2)\Gamma(1+q-\beta) - \Gamma(1-\beta)}{\delta_2\Gamma(1+q-\beta)} \right]^{\frac{1}{q}} + \left[\frac{\Gamma(1-\beta) + (\varpi + \varpi_1)\Gamma(1+q-\alpha)}{\delta_1\Gamma(1+q-\alpha)} \right]^{\frac{1}{q}}.$$

Proof 1: To begin, suppose there is a t_0 such that

$$V(w(t_0)) \leq 1, \tag{5}$$

and

$$t_0 \leq \left[\frac{(\varpi + \varpi_2)\Gamma(1+q-\beta) - \Gamma(1-\beta)}{\delta_2\Gamma(1+q-\beta)} \right]^{\frac{1}{q}} = T_0. \tag{6}$$

If it is not true, suppose that $V(w(t)) > 1, \forall t \in [0, T_0]$, then which implies that $V(w(T_0)) > 1$ and we also have $V^\alpha(w(t)) \leq V^\beta(w(t)), 0 < \alpha < \beta$.

By the condition (2) and $\mathcal{A}^+(\theta) = \mathcal{A}(\theta) \vee 0$, we have

$$\begin{aligned} D^q V(w(t)) &\leq \mathcal{A}(t)V^\alpha(w(t)) + \mathcal{B}(t)V^\beta(w(t)), \\ &\leq \mathcal{A}(t)V^\beta(w(t)) + \mathcal{B}(t)V^\beta(w(t)), \\ &\leq \mathcal{A}^+(t)V^\beta(w(t)) + \mathcal{B}(t)V^\beta(w(t)). \end{aligned}$$

If inequality (2) is true, then a nonnegative function $\mathcal{C}(t)$ exists such that

$$D^q V(w(t)) + \mathcal{C}(t)V^{-\beta}(w(t)) = \mathcal{A}^+(t)V^\beta(w(t)) + \mathcal{B}(t)V^\beta(w(t)). \tag{7}$$

Multiplying (7) by $\frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)}V^{-\beta}(w(t))$ on both sides yields

$$\begin{aligned} \frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)}V^{-\beta}(t)D^q V(w(t)) + \frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)}\mathcal{C}(t) \\ = \mathcal{A}^+(t)\frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)} + \mathcal{B}(t)\frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)}. \end{aligned}$$

From Lemma 1, it follows that

$$\begin{aligned} D^{q-\beta} V(w(t)) &= -\frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)}\mathcal{C}(t) + \mathcal{A}^+(t)\frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)} \\ &\quad + \mathcal{B}(t)\frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)}. \end{aligned} \tag{8}$$

Using Lemma 2, we can integrate (8) from 0 to T_0 and get

$$\begin{aligned} V^{q-\beta}(w(T_0)) - V^{q-\beta}(w(0)) \\ = -I_t^q \frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)}\mathcal{C}(t) + I_t^q \frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)}\mathcal{A}^+(t) \\ + I_t^q \frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)}\mathcal{B}(t). \end{aligned} \tag{9}$$

We can deduce from fractional integral Definition 2 and $(t-\theta)^{-(q-1)} > 0, \Gamma(q) > 0, \forall \theta > 0 \mathcal{C}(\theta) \geq 0$, then we can easily obtained that

$$\begin{aligned} I_t^q \frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)}\mathcal{C}(t) \\ = \frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)\Gamma(q)} \int_0^{T_0} \frac{\mathcal{C}(\theta)}{(t-\theta)^{-(q-1)}} d\theta \geq 0. \end{aligned} \tag{10}$$

In addition,

$$\begin{aligned} I_t^q \frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)}\mathcal{A}^+(t) \\ = \frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)\Gamma(q)} \int_0^{T_0} \frac{\mathcal{A}^+(\theta)}{(t-\theta)^{-(q-1)}} d\theta \leq \frac{\varpi\Gamma(1+q-\beta)}{\Gamma(1-\beta)} \end{aligned} \tag{11}$$

and

$$I_t^q \frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)} \mathcal{B}(t) = \frac{\Gamma(1+q-\beta)}{\Gamma(1-\beta)\Gamma(q)} \int_0^{T_0} \frac{\mathcal{B}(\theta)}{(t-\theta)^{-(q-1)}} d\theta \leq \frac{(-\delta_2 T_0 + \varpi_2)\Gamma(1+q-\beta)}{\Gamma(1-\beta)}. \quad (12)$$

From (9)-(12), one has

$$V^{q-\beta}(w(T_0)) \leq \frac{\Gamma(1+q-\beta)(\varpi + \varpi_2 - \delta_2 T_0^q)}{\Gamma(1-\beta)}. \quad (13)$$

Thus, we have $V(w(T_0)) < 1$, it leads to a contradiction to $V(w(T_0)) > 1$. On the other hand, because $\mathcal{B}(t)$ is negative, $\mathcal{B}(t) \leq \mathcal{A}^-(t)$ and $\mathcal{A}^+(\theta) = \mathcal{A}(\theta) \vee 0$, $\mathcal{A}^-(\theta) = \mathcal{A}(\theta) \wedge 0$, (2) implies that

$$\begin{aligned} D^q V(w(t)) &\leq \mathcal{A}(t)V^\alpha(w(t)) + \mathcal{B}(t)V^\beta(w(t)) \\ &\leq \mathcal{A}(t)V^\alpha(w(t)) + \mathcal{B}(t)V^\alpha(w(t)) \\ &\leq \mathcal{A}(t)V^\alpha(w(t)) + \mathcal{A}^-(t)V^\alpha(w(t)) \\ &= [\mathcal{A}^+(t) + \mathcal{A}^-(t)]V^\alpha(w(t)). \end{aligned}$$

Then, a nonnegative function $\mathcal{C}'(t)$ exists such that

$$\begin{aligned} D^q V(w(t)) + \mathcal{C}'(t)V^{-\alpha}(w(t)) \\ = [\mathcal{A}^+(t) + \mathcal{A}^-(t)]V^\alpha(w(t)). \end{aligned} \quad (14)$$

Multiplying (14) by $\frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)}V^{-\alpha}(w(t))$ on both sides and using Lemma 1 we can get

$$\begin{aligned} D^{q-\alpha} V(w(t)) &= -\frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} \mathcal{C}'(t) + \mathcal{A}^+(t) \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} \\ &\quad + \mathcal{A}^-(t) \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)}. \end{aligned} \quad (15)$$

Using Lemma 2, we can integrate (15) from t_0 to t and get

$$\begin{aligned} V^{q-\alpha}(w(t)) - V^{q-\alpha}(w(t_0)) \\ = -I_t^q \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} \mathcal{C}'(t) + I_t^q \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} \mathcal{A}^+(t) \\ + I_t^q \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} \mathcal{A}^-(t) \end{aligned}$$

similar to (10)-(12), we can obtain

$$\begin{aligned} V^{q-\alpha}(w(t)) &\leq V^{q-\alpha}(w(t_0)) \\ &\quad + \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} [\varpi + \varpi_1 - \delta_1(t-t_0)^q]. \end{aligned}$$

Therefore, we get $V(w(t)) = 0$ for all

$$t > t_0 + \left[\frac{\Gamma(1-\beta) + (\varpi + \varpi_1)\Gamma(1+q-\alpha)}{\delta_1\Gamma(1+q-\alpha)} \right]^{\frac{1}{q}}.$$

Since $t_0 \leq T_0$, we can have that $V(w(t)) = 0$ for all

$$\begin{aligned} t > \left[\frac{(\varpi + \varpi_2)\Gamma(1+q-\beta) - \Gamma(1-\beta)}{\delta_2\Gamma(1+q-\beta)} \right]^{\frac{1}{q}} \\ + \left[\frac{\Gamma(1-\beta) + (\varpi + \varpi_1)\Gamma(1+q-\alpha)}{\delta_1\Gamma(1+q-\alpha)} \right]^{\frac{1}{q}} = T^* \end{aligned} \quad (16)$$

The proof of the Theorem is completed. \square

Corollary 1: Suppose there exists a C -regular function $V(w(t)) : \mathbb{R}^n \rightarrow \mathbb{R}$, for any non-positive function $\mathcal{B}(t)$ such that any solution $w(t)$ of (1) satisfies the inequality

$$D^q V(w(t)) \leq \mathcal{B}(t)V^\beta(w(t)), w(t) \neq 0,$$

where $0 < \beta < q$ and $\mathcal{B}(t)$ satisfies the following inequality

$$\frac{1}{\Gamma(q)} \int_0^t \frac{\mathcal{B}(\theta)}{(t-\theta)^{-(q-1)}} d\theta \leq -\delta_2 t^q + \varpi_2$$

where ϖ_2, δ_2 are positive constants. Then the zero solution of the FDS (1) is F-T stable and the corresponding settling-time is calculated by:

$$\begin{aligned} T^* &= \left[\frac{\Gamma(1+q-\beta)\varpi_2 - \Gamma(1-\beta)}{\delta_2\Gamma(1+q-\beta)} \right]^{\frac{1}{q}} \\ &\quad + \left[\frac{\Gamma(1-\beta) + \varpi_2\Gamma(1+q-\beta)}{\delta_2\Gamma(1+q-\beta)} \right]^{\frac{1}{q}}. \end{aligned}$$

Based on the proof of Theorem 1, the conclusion is obvious.

Remark 2: Evidently, Theorem 1 presents a novel result for delayed FDS's F-TS, which differs from earlier results [47]. When $\mathcal{A}(t) \equiv 0$ and $\mathcal{B}(t) = -a$, where 'a' is a nonnegative constant, the inequality (2) will reduce to the inequality proposed in [47] as following:

$$D^q V(w(t)) \leq -aV^\beta(w(t)).$$

Therefore, Theorem 1 includes and extends the results of [47]. Theorem 1 is more generalized than the Lemma proposed in [47]. Hence, the F-TS/F-TSY of the FDS is extended.

Remark 3: In particular, the settling-time of Theorem 1 could be mathematically calculated by the following equality

$$\begin{aligned} T^* &= \left[\frac{(\varpi + \varpi_2)\Gamma(1+q-\beta) - \Gamma(1-\beta)}{\delta_2\Gamma(1+q-\beta)} \right]^{\frac{1}{q}} \\ &\quad + \left[\frac{\Gamma(1-\beta) + (\varpi + \varpi_1)\Gamma(1+q-\alpha)}{\delta_1\Gamma(1+q-\alpha)} \right]^{\frac{1}{q}} \end{aligned}$$

from the above equality, the settling-time T^* depending on the parameters $\beta, \alpha, \varpi, \varpi_2, \varpi_1, \delta_1, \delta_2$ and fractional-order q .

Theorem 2: For the FDS (1), if there exists a C -regular function $\hat{V}(w(t)) : \mathbb{R}^n \rightarrow \mathbb{R}$, two functions $\mathcal{A}'(t), \mathcal{B}'(t)$ such that the following inequality is hold:

$$\begin{aligned} D^q \hat{V}(w(t)) &\leq \left(\mathcal{A}'(t)\hat{V}^{\bar{\alpha}}(w(t)) + \mathcal{B}'(t)\hat{V}^{\bar{\beta}}(w(t)) \right)^{\kappa'}, \\ w(t) &\neq 0, \end{aligned} \quad (17)$$

where $\bar{\alpha}, \bar{\beta}, \kappa' > 0$ and satisfying $0 < \bar{\alpha}\kappa' < \bar{\beta}\kappa' < q$ and $\mathcal{A}'(t), \mathcal{B}'(t)$ satisfies the following inequalities

$$\frac{1}{\Gamma(q)} \int_0^t \frac{\mathcal{A}'(\theta)}{(t-\theta)^{-(q-1)}} d\theta \leq -\delta_1' t^q + \varpi_1', \quad (18)$$

$$\frac{1}{\Gamma(q)} \int_0^t \frac{\mathcal{B}'(\theta)}{(t-\theta)^{-(q-1)}} d\theta \leq -\delta_2' t^q + \varpi_2', \mathcal{B}'(t) \leq \mathcal{A}'(t) \quad (19)$$

where $\varpi'_1, \varpi'_2, \delta'_1, \delta'_2$ are positive constants. Then the zero solution of the FDS (1) is F-T stable and the corresponding settling-time is estimated by:

$$T^* = \left[\frac{2\varpi'_2\Gamma(1+q-\bar{\beta}\kappa' - \Gamma(1-\bar{\beta}\kappa'))}{2\Gamma(1+q-\bar{\beta}\kappa')} \right]^{\frac{1}{q}} + \left[\frac{\Gamma(1-\bar{\beta}\kappa') + 2\varpi'_1\Gamma(1+q-\bar{\alpha}\kappa')}{\delta'_1\Gamma(1+q-\bar{\alpha}\kappa')} \right]^{\frac{1}{q}}.$$

Proof 2: To begin, suppose there is a t_0 such that $\hat{V}(w(t_0)) \leq 1$, and

$$t_0 \leq \left[\frac{2\varpi'_2\Gamma(1+q-\bar{\beta}\kappa' - \Gamma(1-\bar{\beta}\kappa'))}{2\Gamma(1+q-\bar{\beta}\kappa')} \right]^{\frac{1}{q}} = T_0. \quad (20)$$

If not, suppose that $\hat{V}(w(t)) > 1, \forall t \in [0, T_0]$, then which implies that $\hat{V}(w(T_0)) > 1$ and we also have $\hat{V}^{\hat{\alpha}}(w(t)) \leq \hat{V}^{\hat{\beta}}(w(t)), 0 < \bar{\alpha}\kappa' < \bar{\beta}\kappa'$. By the condition (17), we can have,

$$\begin{aligned} D^q \hat{V}(w(t)) &\leq \left(\mathcal{A}'(t)\hat{V}^{\hat{\alpha}}(w(t)) + \mathcal{B}'(t)\hat{V}^{\hat{\beta}}(w(t)) \right)^{\kappa'}, \\ &\leq \left(\mathcal{A}'(t)\hat{V}^{\hat{\beta}}(w(t)) + \mathcal{B}'(t)\hat{V}^{\hat{\beta}}(w(t)) \right)^{\kappa'}, \\ &\leq \left(2\mathcal{B}'(t)\hat{V}^{\hat{\beta}}(w(t)) \right)^{\kappa'}. \end{aligned}$$

For a nonnegative function $\mathcal{S}(t)$ and multiplying by $\frac{\Gamma(1+q-\bar{\beta}\kappa')}{\Gamma(1-\bar{\beta}\kappa')} \hat{V}^{-\bar{\beta}\kappa'}(w(t))$ on both sides above formula gives that

$$\begin{aligned} \frac{\Gamma(1+q-\bar{\beta}\kappa')}{\Gamma(1-\bar{\beta}\kappa')} \hat{V}^{-\bar{\beta}\kappa'}(t) D^q \hat{V}(w(t)) + \frac{\Gamma(1+q-\bar{\beta}\kappa')}{\Gamma(1-\bar{\beta}\kappa')} \mathcal{S}(t) \\ = 2\mathcal{B}'(t) \frac{\Gamma(1+q-\bar{\beta}\kappa')}{\Gamma(1-\bar{\beta}\kappa')} \end{aligned}$$

and from Lemma 1

$$\begin{aligned} D^{q-\bar{\beta}\kappa'} \hat{V}(w(t)) &= -\frac{\Gamma(1+q-\bar{\beta}\kappa')}{\Gamma(1-\bar{\beta}\kappa')} \mathcal{S}(t) \\ &\quad + 2\mathcal{B}'(t) \frac{\Gamma(1+q-\bar{\beta}\kappa')}{\Gamma(1-\bar{\beta}\kappa')}. \quad (21) \end{aligned}$$

Using Lemma 2, we can integrate (21) from 0 to T_0 and get

$$\begin{aligned} \hat{V}^{q-\bar{\beta}\kappa'}(w(T_0)) - \hat{V}^{q-\bar{\beta}\kappa'}(w(0)) \\ = -I_t^q \frac{\Gamma(1+q-\bar{\beta}\kappa')}{\Gamma(1-\bar{\beta}\kappa')} \mathcal{S}(t) + 2I_t^q \frac{\Gamma(1+q-\bar{\beta}\kappa')}{\Gamma(1-\bar{\beta}\kappa')} \mathcal{B}'(t). \end{aligned}$$

we can deduce from fractional integral Definition 2 and $(t-\theta)^{-(q-1)} > 0, \Gamma(q) > 0, \forall \theta > 0 \mathcal{S}(\theta) \geq 0$ and similar to (10)-(12), we can obtain

$$\hat{V}^{q-\bar{\beta}\kappa'}(w(T_0)) \leq \frac{2\Gamma(1+q-\bar{\beta}\kappa')(\varpi'_2 - \delta'_2 T_0^q)}{\Gamma(1-\bar{\beta}\kappa')}$$

thus, we have $\hat{V}(w(T_0)) < 1$, which is a contradiction to $\hat{V}(w(T_0)) > 1$. On the other hand,

$$D^q \hat{V}(w(t)) \leq \left(\mathcal{A}'(t)\hat{V}^{\hat{\alpha}}(w(t)) + \mathcal{B}'(t)\hat{V}^{\hat{\beta}}(w(t)) \right)^{\kappa'},$$

$$\begin{aligned} &\leq \left(\mathcal{A}'(t)\hat{V}^{\hat{\alpha}}(w(t)) + \mathcal{B}'(t)\hat{V}^{\hat{\alpha}}(w(t)) \right)^{\kappa'}, \\ &\leq \left(2\mathcal{A}'(t)\hat{V}^{\hat{\alpha}}(w(t)) \right)^{\kappa'} \end{aligned}$$

then a nonnegative function $\mathcal{S}'(t)$ exists and similar to above process we can integrate from t_0 to t , one has

$$\hat{V}^{q-\bar{\alpha}\kappa'}(w(t)) \leq 1 + \frac{2\Gamma(1+q-\bar{\alpha}\kappa')(\varpi'_1 - \delta'_1(t-t_0)^q)}{\Gamma(1-\bar{\alpha}\kappa')}$$

therefore, we get $\hat{V}(w(t)) = 0$ for all.

$$t > t_0 + \left[\frac{\Gamma(1-\bar{\beta}\kappa') + 2\varpi'_1\Gamma(1+q-\bar{\alpha}\kappa')}{\delta'_1\Gamma(1+q-\bar{\alpha}\kappa')} \right]^{\frac{1}{q}}.$$

Since $t_0 \leq T_0$, we can have that $V(w(t)) = 0$ for all

$$\begin{aligned} t > \left[\frac{2\varpi'_2\Gamma(1+q-\bar{\beta}\kappa' - \Gamma(1-\bar{\beta}\kappa'))}{2\Gamma(1+q-\bar{\beta}\kappa')} \right]^{\frac{1}{q}} \\ + \left[\frac{\Gamma(1-\bar{\beta}\kappa') + 2\varpi'_1\Gamma(1+q-\bar{\alpha}\kappa')}{\delta'_1\Gamma(1+q-\bar{\alpha}\kappa')} \right]^{\frac{1}{q}}. \quad (22) \end{aligned}$$

The proof of Theorem 2 is completed. \square

Corollary 2: For the FDS (1), if there exists a C -regular function $\hat{V}(w(t)) : \mathbb{R}^n \rightarrow \mathbb{R}$, and function $\mathcal{B}'(t)$ such that the following inequality is hold:

$$D^q \hat{V}(w(t)) \leq \left(\mathcal{B}'(t)\hat{V}^{\hat{\beta}}(w(t)) \right)^{\kappa'}, w(t) \neq 0,$$

where $\bar{\beta}, \kappa' > 0$ and satisfying $0 < \bar{\beta}\kappa' < q$ and $\mathcal{B}'(t)$ satisfies the following inequality

$$\frac{1}{\Gamma(q)} \int_0^t \frac{\mathcal{B}'(\theta)}{(t-\theta)^{-(q-1)}} d\theta \leq -\delta'_2 t + \varpi'_2,$$

where ϖ'_2, δ'_2 are positive constants. Then the zero solution of the FDS (1) is F-T stable and the corresponding settling-time calculated by:

$$\begin{aligned} T^* &= \left[\frac{\Gamma(1+q-\bar{\beta}\kappa')\varpi'_2 - \Gamma(1-\bar{\beta}\kappa')}{\delta'_2\Gamma(1+q-\bar{\beta}\kappa')} \right]^{\frac{1}{q}} \\ &\quad + \left[\frac{\Gamma(1-\bar{\beta}\kappa') + \varpi'_2\Gamma(1+q-\bar{\beta}\kappa')}{\delta'_2\Gamma(1+q-\bar{\beta}\kappa')} \right]^{\frac{1}{q}}. \end{aligned}$$

Proof 3: Based on the proof of Theorem 2, the conclusion is obvious.

Remark 4: Previous publications have extensively utilized Lyapunov function theory to analyze the stability and synchronization of FONNs and delayed FONNs. Their conducted Lyapunov functions, however, have all negative-definite derivatives. The fractional derivative of function $V(w(t))$ in Theorem 1 is indefinite, that is, it can be unbounded, negative definite, or positive definite. Accordingly, Theorem 1 has a better chance of achieving the F-TS/F-TSY for delayed FONNs. It is possible to improve previous findings on FN-T and F-TS/F-TSY of FONNs.

IV. FIXED-TIME SYNCHRONIZATION OF DELAYED FRACTIONAL-ORDER NEURAL NETWORKS

FDSs have received considerable attention to date because they have a number of practical applications. Few studies have been conducted on the dynamics of FONNs with saturated impulsive control schemes. Up until now, only a few investigations have been conducted on the F-TSY theorem for FONNs. Moreover, many earlier studies of the stability or synchronization of FONNs were based on the negative definite of the derivative of the Lyapunov functional. Therefore, the F-TSY of FONNs with saturated impulsive control must be studied in more depth, and the derivative of the Lyapunov functionals must be indefinite. In this section, we consider the F-TSY of delayed FONNs based on Theorems 1 and 2.

A. SYSTEM DESCRIPTION

The dynamics of the p th neuron of FONNs with time-delay is represented by

$$\begin{cases} D^q w_p(t) = -d_p w_p(t) + \sum_{r=1}^n a_{pr} g_r(w_r(t)) \\ + \sum_{r=1}^n b_{pr} q_r(w_r(t - \tau_r)) + I_p, \\ w(\theta) = \gamma(\theta), \theta \in [-\tau, 0] \end{cases} \quad (23)$$

where $p = 1, 2, \dots, n, d_p > 0$, indicate the rates of neuron self inhibition in a p th neuron; $w_p(t)$ denote the state variable; $g_r(\cdot)$ is the activation function; I_p represent external vector; τ_r correspond to the time-delay and satisfies $0 \leq \tau_r \leq \tau$, where τ is a constant. Let $w(t) = (w_1(t), w_2(t), \dots, w_n(t))^T$. System (23) is referred to as the driving system, while following system (24) is the associated response system

$$\begin{cases} D^q \bar{w}_p(t) = -d_p \bar{w}_p(t) + \sum_{r=1}^n a_{pr} g_r(\bar{w}_r(t)) \\ + \sum_{r=1}^n b_{pr} g_r(\bar{w}_r(t - \tau_r)) \\ + I_p + v_p(t), \\ \bar{w}(\theta) = \bar{\gamma}(\theta), \theta \in [-\tau, 0] \end{cases} \quad (24)$$

where $v_p(t)$ is controller input to be designed later. Let $\bar{w}(t) = (\bar{w}_1(t), \bar{w}_2(t), \dots, \bar{w}_n(t))^T$.

The appropriate controller $v_p(t)$ is designed in the following way:

$$v_p(t) = -\text{sgn}(e_p(t)) \left[\eta + \mathcal{A}(t) |e_p(t)|^\alpha + \mathcal{B}(t) |e_p(t)|^\beta \right] + \sum_{h=1}^{\infty} \mathbb{S}_p(t) \delta(t - t_h), \quad (25)$$

where $h \in \mathbb{N}_+, \mathbb{S}_p(t) = \mathbb{K}_p e_p(t), e_p(t) = \bar{w}_p(t) - w_p(t)$ is the synchronization error. \mathbb{K}_p is the impulsive control gain to be determined. $\mathcal{A}(t)$ is an indefinite function, $\mathcal{B}(t)$ is a negative function and $0 < \alpha < \beta < q$. $\delta(\cdot)$ is the delta function with the time sequence $\{t_h, h \in \mathbb{N}_+\}$, it is a strictly increasing sequence of impulsive instances and assumed that $\lim_{h \rightarrow \infty} t_h = +\infty, e_p(t_h) = e_p(t_h^-) = \lim_{t \rightarrow t_h^-} e(t)$. However, in practice, controllers

can only deliver a certain amount of signal due to physical or safety limitations, resulting in an inability to attain the required results. This issue can best be addressed by implementing an impulsive controller with actuator saturation: $\sum_{k=1}^{\infty} \text{sat}(\mathbb{S}_p(t)) \delta(t - t_h)$, where $\text{sat}(\mathbb{S}_p(t)) = [\text{sat}(\mathbb{S}_1(t)), \text{sat}(\mathbb{S}_2(t)), \dots, \text{sat}(\mathbb{S}_n(t))]^T \in \mathbb{R}^n$ is the saturation function, $\text{sat}(\mathbb{S}_p(t)) = \text{sgn}(\mathbb{S}_p(t)) \min\{\Delta, |\mathbb{S}_p(t)|\}$ where $\mathbb{S}_p(t) \in \mathbb{R}$ and $\Delta > 0$ is the known saturation level. Then the controller $v_p(t)$ is designed as follows:

$$v_p(t) = -\text{sgn}(e_p(t)) \left[\eta + \mathcal{A}(t) |e_p(t)|^\alpha + \mathcal{B}(t) |e_p(t)|^\beta \right] + \sum_{k=1}^{\infty} \text{sat}(\mathbb{S}_p(t)) \delta(t - t_h). \quad (26)$$

The proposed controller (26) represents the synchronization error systems as

$$\begin{cases} D^q e_p(t) = -d_p e_p(t) + \sum_{r=1}^n \{a_{pr} g_r(\bar{w}_r(t)) - a_{pr} g_r(w_r(t))\} + \sum_{r=1}^n \{b_{pr} g_r(\bar{w}_r(t - \tau_r)) - b_{pr} g_r(w_r(t - \tau_r))\} - \text{sgn}(e_p(t)) \left[\eta + \mathcal{A}(t) |e_p(t)|^\alpha + \mathcal{B}(t) |e_p(t)|^\beta \right], \\ \Delta e_p(t_h) = e_p(t_h^+) - e_p(t_h^-) = \text{sat}(\mathbb{K}_p e_p(t_h^-)), \\ p = 1, 2, \dots, n; h = 1, 2, \dots \\ e_p(\theta) = \bar{\gamma}(\theta) - \gamma(\theta), \theta \in [-\tau, 0]. \end{cases} \quad (27)$$

The initial conditions of above synchronization error system (27) is $e_p(\theta) = \bar{\gamma}(\theta) - \gamma(\theta)$.

To obtain the F-TSY criteria for the delayed FONNs (27), the following Assumptions are given

Assumption 1: Assume that for any positive constant G_r that meets the following condition:

$$|g_r(\bar{w}) - g_r(w)| \leq G_r |\bar{w} - w|, \quad \forall \bar{w}, w \in \mathbb{R}.$$

Assumption 2: The controller is perturbed by the control parametric uncertainty $\Delta \mathbb{K}_p$, which meets the following condition: $\Delta \mathbb{K}_p = \rho \psi(t_h) \mathbb{K}_p$. where $\rho > 0$ is a known constant, $|\psi(t_h)| < 1$.

Define a parameter that changes over time $u_p(t_h)$ as

$$u_p(t_h) = \begin{cases} \frac{\Delta}{|(1 + \rho \psi(t_h)) \mathbb{K}_p e_p(t_h)|}, & |(1 + \rho \psi(t_h)) \mathbb{K}_p e_p(t_h)| > \Delta, \\ 1, & |(1 + \rho \psi(t_h)) \mathbb{K}_p e_p(t_h)| \leq \Delta. \end{cases}$$

Without a doubt, it possesses $u_p(t) \in (0, 1]$, and the saturation input can be stated as

$$\text{sat}((1 + \rho \psi(t_h)) \mathbb{K}_p e_p(t_h)) = (1 + \rho \psi(t_h)) \mathbb{K}_p u_p(t_h) e_p(t_h).$$

The synchronization error system (27) can then be described as follows:

$$\begin{cases} D^q e_p(t) = -d_p e_p(t) + \sum_{r=1}^n \{a_{pr} g_r(\bar{w}_r(t)) \\ - a_{pr} g_r(w_r(t))\} + \sum_{r=1}^n \{b_{pr} g_r(\bar{w}_r(t - \tau_r)) \\ - b_{pr} g_r(w_r(t - \tau_r))\} - \text{sgn}(e_p(t)) \left[\eta \right. \\ \left. + \mathcal{A}(t) |e_p(t)|^\alpha \right. \\ \left. + \mathcal{B}(t) |e_p(t)|^\beta \right], \\ e_p(t_h^+) = ((1 + \rho\psi(t_h)) \mathbb{K}u_p(t_h) + 1)e_p(t_h). \\ e_p(\theta) = \bar{\gamma}(\theta) - \gamma(\theta), \theta \in [-\tau, 0]. \end{cases} \quad (28)$$

Lemma 3: For any function $e(t) \in C^1[(0, +\infty), \mathbb{R}]$ and $0 < q < 1$, then $D^q |e(t)| \leq \text{sign}(e(t))D^q e(t)$, $t \geq t_0$ holds almost everywhere.

In the next theorem, a new F-TSY criteria will be developed for the delayed FONNs (28).

Theorem 3: Suppose that the Assumptions 1 and 2 holds, for any impulsive sequence $\{t_h, h \in \mathbb{N}_+\}$, if there exists impulsive control gain \mathbb{K} , any $\mathbb{U}(t_h) = \text{diag}\{u_1(t_h), u_2(t_h), \dots, u_n(t_h)\}$ with time varying parameter $u_p(t_h)$, $|\psi(t_h)| < 1$, and positive constants ρ, η and $0 < \bar{\phi} < 1$, as a result of which such that the inequalities listed below are established

$$-\eta + \sum_{r=1}^n b_{pr} G_r (|\bar{\gamma}_r(\theta) - \gamma_r(\theta)|) \leq 0, \quad (29)$$

$$-d_p + \sum_{r=1}^n (a_{pr} + b_{pr}) G_r \leq 0, \quad (30)$$

$$(1 + \rho\psi(t_h)) \mathbb{K}\mathbb{U}(t_h) + \mathbb{I}_n \leq \bar{\phi} \mathbb{I}_n \quad (31)$$

then the response system (24) is synchronized with derive system (23) in a F-T via the following controller

$$v_p(t) = -\text{sgn}(e_p(t)) \left[\eta + \mathcal{A}(t) |e_p(t)|^\alpha \right. \\ \left. + \mathcal{B}(t) |e_p(t)|^\beta \right] + \sum_{h=1}^{\infty} \text{sat}(\mathbb{S}_p(t)) \delta(t - t_h), \quad (32)$$

where $0 < \alpha < \beta < q$, and $\mathcal{A}(t), \mathcal{B}(t)$ satisfies the inequalities $\frac{1}{\Gamma(q)} \int_0^t \frac{\mathcal{A}^-(\theta)}{(t-\theta)^{(q-1)}} d\theta \leq -\delta_1 t^q + \varpi_1$, $\frac{1}{\Gamma(q)} \int_0^t \frac{\mathcal{B}(\theta)}{(t-\theta)^{(q-1)}} d\theta \leq -\delta_2 t^q + \varpi_2$, $\frac{1}{\Gamma(q)} \int_0^t \frac{\mathcal{A}^+(\theta)}{(t-\theta)^{(q-1)}} d\theta \leq \varpi$, $\mathcal{B}(t) \leq \mathcal{A}^-(t)$ where $\varpi, \varpi_1, \varpi_2, \delta_1, \delta_2$ are positive constants. Then the settling time is estimated as

$$T^* = \left[\frac{(\varpi + \varpi_2)\Gamma(1 + q - \beta) - \Gamma(1 - \beta)}{\delta_2 \Gamma(1 + q - \beta)} \right]^{\frac{1}{q}} \\ + \left[\frac{\Gamma(1 - \beta) + (\varpi + \varpi_1)\Gamma(1 + q - \alpha)}{\delta_1 \Gamma(1 + q - \alpha)} \right]^{\frac{1}{q}}.$$

Proof: Consider the following Lyapunov functional candidate:

$$V(e(t)) = \sum_{p=1}^n |e_p(t)|. \quad (33)$$

Based on Lemma 3, the Caputo fractional-order derivative of $V(e(t))$ can be determined for $t \neq t_h$ by deriving the above expression as

$$D^\alpha V(e(t)) \leq \sum_{p=1}^n \text{sgn}(e_p(t)) D^q e_p(t)$$

from Assumption 1, we have that

$$\begin{aligned} D^\alpha V(e(t)) &\leq \sum_{p=1}^n \text{sgn}(e_p(t)) \left[-d_p e_p(t) + \sum_{r=1}^n \{a_{pr} g_r(\bar{w}_r(t)) \right. \\ &\quad - a_{pr} g_r(w_r(t))\} + \sum_{r=1}^n \{b_{pr} g_r(\bar{w}_r(t - \tau_r)) \\ &\quad - b_{pr} g_r(w_r(t - \tau_r))\} \\ &\quad \left. - \text{sgn}(e_p(t)) \left[\eta + \mathcal{A}(t) |e_p(t)|^\alpha + \mathcal{B}(t) |e_p(t)|^\beta \right] \right], \\ &\leq \sum_{p=1}^n \left\{ -d_p |e_p(t)| + \sum_{r=1}^n a_{pr} G_r |e_r(t)| \right. \\ &\quad \left. + \sum_{r=1}^n b_{pr} G_r |e_r(t - \tau_r)| + \eta \right. \\ &\quad \left. + \mathcal{A}(t) |e_p(t)|^\alpha + \mathcal{B}(t) |e_p(t)|^\beta \right\}. \end{aligned} \quad (34)$$

In contrast, on the other side,

$$\begin{aligned} e_r(t - \tau_r) &\leq \sup_{-\tau \leq \theta \leq t} |e_r(\theta)| \\ &\leq \sup_{-\tau \leq \theta \leq 0} |e_r(\theta)| + \sup_{0 \leq \theta \leq t} |e_r(\theta)| \\ &= |\bar{\gamma}_r(\theta) - \gamma_r(\theta)| + |e_r(t)|, \end{aligned} \quad (35)$$

where, $e_p(\theta) = (e_1(\theta), e_2(\theta), \dots, e_n(\theta))^T = (\bar{\gamma}_1(\theta) - \gamma_1(\theta), \bar{\gamma}_2(\theta) - \gamma_2(\theta), \dots, \bar{\gamma}_n(\theta) - \gamma_n(\theta))^T$, $\theta \in [-\tau, 0]$ and as a result of (34), (35), and condition (29)

$$\begin{aligned} D^\alpha V(e(t)) &\leq \sum_{p=1}^n \left\{ (-d_p) |e_p(t)| + \sum_{r=1}^n (a_{pr} + b_{pr}) G_r |e_r(t)| \right. \\ &\quad \left. + \sum_{r=1}^n b_{pr} G_r (|\bar{\gamma}_r(\theta) - \gamma_r(\theta)|) - \eta \right. \\ &\quad \left. + \mathcal{A}(t) |e_p(t)|^\alpha + \mathcal{B}(t) |e_p(t)|^\beta \right\}, \\ &\leq \sum_{p=1}^n \left\{ (-d_p) |e_p(t)| + \sum_{r=1}^n (a_{pr} + b_{pr}) G_r |e_r(t)| \right. \\ &\quad \left. + \mathcal{A}(t) |e_p(t)|^\alpha + \mathcal{B}(t) |e_p(t)|^\beta \right\} \end{aligned} \quad (36)$$

then, the fractional derivative of $V(e(t))$ along the trajectories of the error system (28) produces for $t \neq t_h$

$$D^\alpha V(e(t)) \leq \sum_{p=1}^n \left\{ (-d_p + \sum_{r=1}^n (a_{pr} + b_{pr}) G_r) |e_p(t)| \right.$$

$$+\mathcal{A}(t)|e_p(t)|^\alpha + \mathcal{B}(t)|e_p(t)|^\beta \}. \quad (37)$$

Let $-d_p + \sum_{r=1}^n(a_{pr} + b_{pr})G_r \leq 0$, from (37), we can obtain

$$D^\alpha V(e(t)) \leq \mathcal{A}(t) \sum_{p=1}^n |e_p(t)|^\alpha + \mathcal{B}(t) \sum_{p=1}^n |e_p(t)|^\beta. \quad (38)$$

From inequality (38), for any nonnegative function $\mathcal{C}'(t)$ it can be deduced from Theorem 1 that

$$D^{q-\alpha} V(e(t)) = -\frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} \mathcal{C}'(t) + \mathcal{A}^+(t) \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} + \mathcal{A}^-(t) \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)}. \quad (39)$$

Using Lemma 2, we can integrate (39) from t_{h-1} to t and get

$$\begin{aligned} &V^{q-\alpha}(e(t)) - V^{q-\alpha}(e(t_{h-1})) \\ &= -I_t^q \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} \mathcal{C}'(t) + I_t^q \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} \mathcal{A}^+(t) \\ &\quad + I_t^q \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} \mathcal{A}^-(t) \end{aligned}$$

from Theorem 1, one has

$$\begin{aligned} V^{q-\alpha}(e(t)) &\leq V^{q-\alpha}(e(t_{h-1})) \\ &\quad + \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} [\varpi + \varpi_1 - \delta_1(t - t_{h-1})^q]. \end{aligned} \quad (40)$$

When $t = t_h$, from (28) it follows that

$$e(t_h^+) = ((1 + \rho\psi(t_h)) \mathbb{K} \cup (t_h) + \mathbb{I})e(t_h).$$

then from (31) demonstrates that

$$V(e(t_h^+)) \leq \bar{\phi} V(e(t_h)). \quad (41)$$

From the above formula and $0 < q - \alpha < 1$, one has

$$V^{q-\alpha}(e(t_h^+)) \leq \bar{\phi}^{q-\alpha} V^{q-\alpha}(e(t_h)). \quad (42)$$

When $t_0 < t \leq t_1$, from (40), one can easily get

$$\begin{aligned} V^{q-\alpha}(e(t)) &\leq V^{q-\alpha}(e(t_0^+)) \\ &\quad + \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} [\varpi + \varpi_1 - \delta_1(t - t_0)^q] \end{aligned}$$

which leads to

$$\begin{aligned} V^{q-\alpha}(e(t_1)) &\leq V^{q-\alpha}(e(t_0^+)) \\ &\quad + \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} [\varpi + \varpi_1 - \delta_1(t_1 - t_0)^q] \end{aligned}$$

hence from (41), one has

$$\begin{aligned} V^{q-\alpha}(e(t_1^+)) &\leq \bar{\phi}^{q-\alpha} \left[V^{q-\alpha}(e(t_0^+)) \right. \\ &\quad \left. + \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} [\varpi + \varpi_1 - \delta_1(t_1 - t_0)^q] \right]. \end{aligned}$$

When $t_1 < t \leq t_2$, similarly, we have

$$V^{q-\alpha}(e(t)) \leq \bar{\phi}^{q-\alpha} \left[V^{q-\alpha}(e(t_0^+)) \right.$$

$$\begin{aligned} &\left. + \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} [\varpi + \varpi_1 - \delta_1(t_1 - t_0)^q] \right] \\ &\quad + \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} [\varpi + \varpi_1 - \delta_1(t - t_1)^q], \\ &\leq V^{q-\alpha}(e(t_0^+)) \\ &\quad + \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} [\varpi + \varpi_1 - \delta_1(t - t_0)^q], \\ &\leq \bar{\phi}^{q-\alpha} \left[V^{q-\alpha}(e(t_0^+)) \right. \\ &\quad + \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} [\varpi + \varpi_1 - \delta_1(t - t_1)^q] \\ &\quad \left. + \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} [\varpi + \varpi_1 - \delta_1(t_1 - t_0)^q] \right], \\ &= \bar{\phi}^{q-\alpha} \left[V^{q-\alpha}(e(t_0^+)) \right. \\ &\quad \left. + \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} [\varpi + \varpi_1 - \delta_1(t - t_0)^q] \right] \end{aligned}$$

since $\bar{\phi}^{q-\alpha} V^{q-\alpha}(e(t)) \leq V^{q-\alpha}(e(t))$, we can obtain that

$$\begin{aligned} V^{q-\alpha}(e(t)) &\leq V^{q-\alpha}(e(t_0^+)) \\ &\quad + \frac{\Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} [\varpi + \varpi_1 - \delta_1(t - t_0)^q]. \end{aligned}$$

In general, for any $t_h < t \leq t_{h+1}$, we can obtain that

$$\begin{aligned} &V^{q-\alpha}(e(t)) \\ &\leq \bar{\phi}^{hq-h\alpha} V^{q-\alpha}(e(t_0^+)) \\ &\quad + \frac{\bar{\phi}^{hq-h\alpha} (\varpi + \varpi_1 - \delta_1(t - t_h)^q) \Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} \\ &\quad + \frac{\bar{\phi}^{hq-h\alpha} (\varpi + \varpi_1 - \delta_1(t_h - t_{h-1})^q) \Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} \\ &\quad + \frac{\bar{\phi}^{hq-h\alpha} (\varpi + \varpi_1 - \delta_1(t_{h-1} - t_{h-2})^q) \Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} \\ &\quad + \dots \\ &\quad + \frac{\bar{\phi}^{hq-h\alpha} (\varpi + \varpi_1 - \delta_1(t_2 - t_1)^q) \Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} \\ &\quad + \frac{\bar{\phi}^{hq-h\alpha} (\varpi + \varpi_1 - \delta_1(t_1 - t_0)^q) \Gamma(1+q-\alpha)}{\Gamma(1-\alpha)}, \\ &\leq \bar{\phi}^{hq-h\alpha} \left[V^{q-\alpha}(e(t_0^+)) \right. \\ &\quad \left. + \frac{(\varpi + \varpi_1 - \delta_1(t - t_0)^q) \Gamma(1+q-\alpha)}{\Gamma(1-\alpha)} \right] \end{aligned}$$

therefore for any $t \in (t_h, t_{h+1}]$, we can have

$$\begin{aligned} V^{q-\alpha}(e(t)) &\leq V^{q-\alpha}(e(t_0)) \\ &\quad + \frac{(\varpi + \varpi_1 - \delta_1(t - t_0)^q) \Gamma(1+q-\alpha)}{\Gamma(1-\alpha)}. \end{aligned}$$

It follows from Theorem 1 that the derive and response delayed FONNs (23) and (24) can achieve the F-TSY under

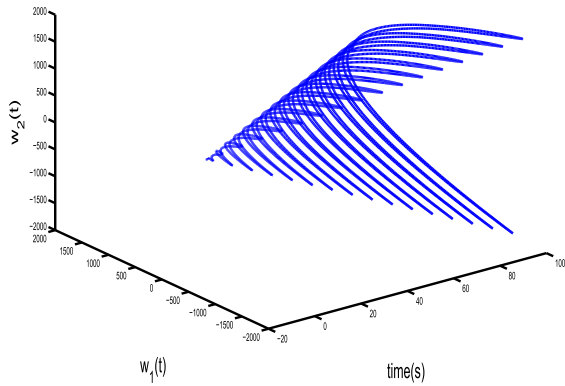


FIGURE 1. Phase plot of fractional-order derive system (23) for example 1.

the controller (32) and the settling time can be estimated by

$$T^* = \left[\frac{(\varpi + \varpi_2)\Gamma(1 + q - \beta) - \Gamma(1 - \beta)}{\delta_2\Gamma(1 + q - \beta)} \right]^{\frac{1}{q}} + \left[\frac{\Gamma(1 - \beta) + (\varpi + \varpi_1)\Gamma(1 + q - \alpha)}{\delta_1\Gamma(1 + q - \alpha)} \right]^{\frac{1}{q}}.$$

This completes the proof.

V. NUMERICAL SIMULATIONS

To demonstrate the validity and effectiveness of our theoretical findings, we provide two numerical examples.

Example 1: Let's consider the following two-dimensional FONNs with time-delays:

$$\begin{cases} D^q \bar{w}_1(t) = -d_1 \bar{w}_1(t) + a_{11}g_1(\bar{w}_1(t)) + a_{12}g_2(\bar{w}_2(t)) \\ \quad + b_{11}g_1(\bar{w}_1(t - \tau_1)) + b_{12}g_2(\bar{w}_2(t - \tau_2)) \\ \quad + I_1 + v_1(t), \\ D^q \bar{w}_2(t) = -d_2 \bar{w}_2(t) + a_{21}g_1(\bar{w}_1(t)) + a_{22}g_2(\bar{w}_2(t)) \\ \quad + b_{21}g_1(\bar{w}_1(t - \tau_1)) + b_{22}g_2(\bar{w}_2(t - \tau_2)) \\ \quad + I_2 + v_2(t), \end{cases} \quad (43)$$

where, $d_1 = 0.1, d_2 = 0.1, a_{11} = -0.3, a_{12} = 0, a_{21} = 0.3, a_{22} = -0.9, b_{11} = 1, b_{12} = 0, b_{21} = 0, b_{22} = 1, I_1 = \sin(2t), I_2 = -2\cos(t)$. Let $g_1(\bar{w}(t)) = g_2(\bar{w}(t)) = \tanh(\bar{w}(t)), g_1(\bar{w}(t - \tau)) = g_2(\bar{w}(t - \tau)) = \tanh(\bar{w}(t - \tau)), \tau = 1$. Thus, Assumption 1 is satisfied with $G_1 = 0.5, G_2 = 0.5$. Let (43) be the response system and the corresponding derive system is defined in (23) with the parameters of (43). From (23) and (43), we can design an fractional-order error system in such a way that

$$\begin{cases} D^q e_1(t) = -e_1(t) - 0.3 \tanh(e_1(t)) + \tanh(e_1(t - \tau_1)) \\ \quad + v_1(t), \\ D^q e_2(t) = -e_2(t) + 3 \tanh(e_1(t)) + 0.3 \tanh(e_2(t)) \\ \quad - \tanh(e_2(t - \tau_2)) + v_2(t), \end{cases} \quad (44)$$

If $v_1(t) = v_2(t) = 0$, Figure 2 depicts the results of a numerical simulation of state trajectories of (43) and (44) with initial values $\bar{w}_1(\theta) = -0.4, \bar{w}_2(\theta) = 2, e_1(\theta) = -0.45, e_2(\theta) = 2.05, \theta \in [-1, 0]$, in two-dimensional state

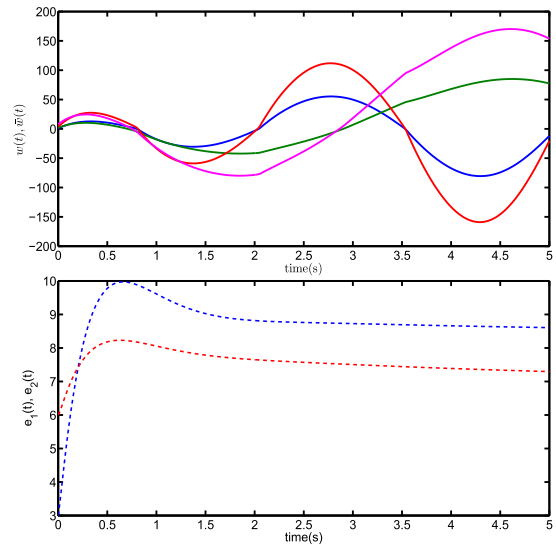


FIGURE 2. The state trajectories $w_p(t)$ and $\bar{w}_p(t), p = 1, 2$ of drive-response system (23)-(43) and synchronization errors $e_p(t), p = 1, 2$ of (44) without control inputs i.e $v_1(t) = 0, v_2(t) = 0$.

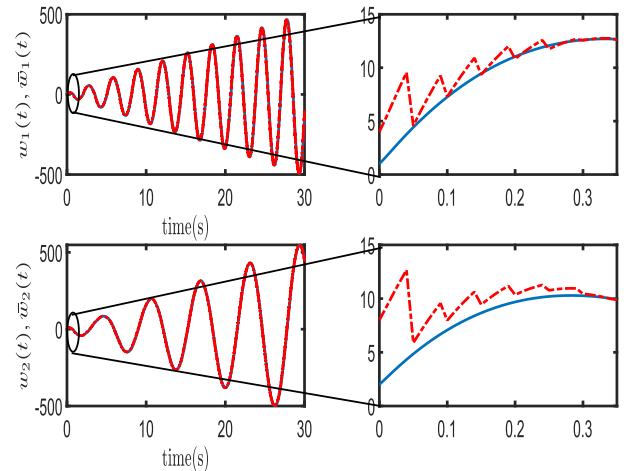


FIGURE 3. The state trajectories $w_p(t)$ and $\bar{w}_p(t), p = 1, 2$ of drive-response system (23)-(43) with control input (45).

space respectively. Now let us take the control gain parameters $q = 0.995, \alpha = 0.3, \beta = 0.7, \eta = 1.2, \bar{\phi} \in (0, 1), \rho = 0.8, \mathcal{A}(t) = -\frac{t}{2}|\cos(t)| + \frac{1}{1+t^2}, \mathcal{B}(t) = -\frac{1}{t}$. Since the conducted Lyapunov function becomes more generalized as a result of the fact that the derivative of it is indefinite almost everywhere. It has the potential to significantly improve the standard Lyapunov function, which is negative and is used to determine the stability results of FONNs. By the direct computations, we can easily verify the conditions (29)-(34) holds with the corresponding impulsive controller gain determined as $\mathbb{K}_1 = 0.899, \mathbb{K}_2 = 1.899$ and the discontinuous controller inputs $v_1(t) v_2(t)$ can be designed as bellow:

$$v_1(t) = -\text{sgn}(e_1(t)) \left[1.2 - \frac{t}{2} |\cos(t)| + \frac{1}{1+t^2} |e_1(t)|^{0.3} - \frac{1}{t} |e_1(t)|^{0.7} \right] + \sum_{h=1}^{\infty} \text{sat}(\mathbb{S}_1(t)) \delta(t - t_h),$$

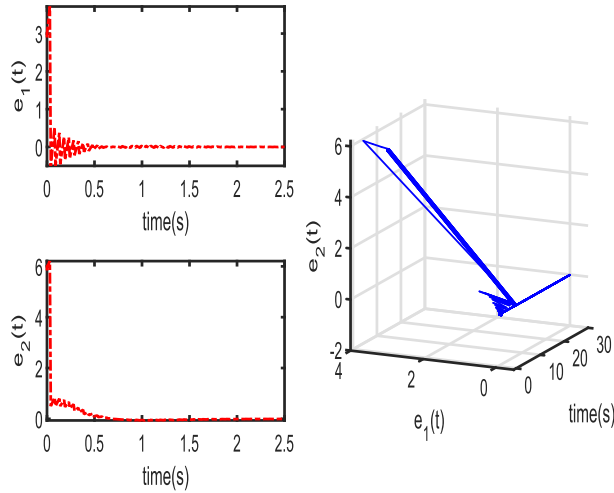


FIGURE 4. The time evolution of state trajectories and phase plot of error system (44).

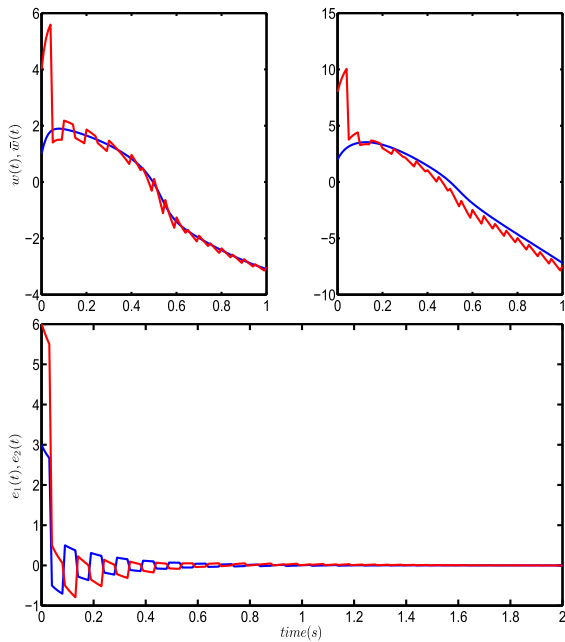


FIGURE 5. The time evolution of trajectories of state and errors in example 1 with discontinuous activation functions in (46).

$$v_2(t) = -\text{sgn}(e_2(t)) \left[1.2 - \frac{t}{2} |\cos(t)| + \frac{1}{1+t^2} |e_2(t)|^{0.3} - \frac{1}{t} |e_2(t)|^{0.7} \right] + \sum_{h=1}^{\infty} \text{sat}(\mathbb{S}_2(t)) \delta(t - t_h), \quad (45)$$

Therefore, we may conclude that the fractional-order drive-response system (23)-(43) can accomplish F-TSY in accordance with Theorem 3. This is illustrated in Figures 3-4 with the initial values $(-0.4, 2)$, and $(-0.45, 2.05)$.

In [49], the bipartite fixed-time synchronization problem for fractional-order signed neural networks with discontinuous activation is discussed. In which the Filippove multi-map is used to convert the F-TS of the fractional-order general

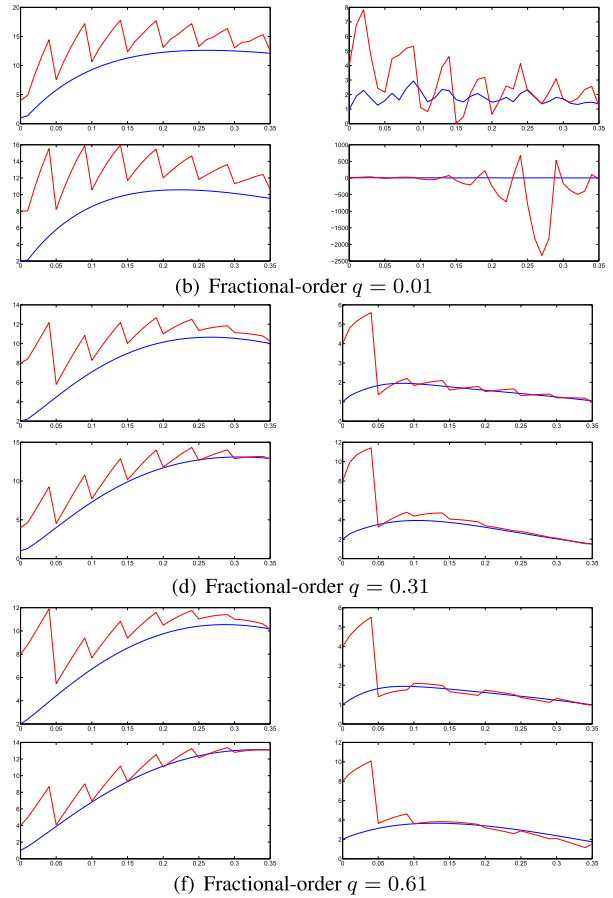


FIGURE 6. State responses of FONNs (23) (blue line) and (43) (red line) with different fractional orders. Left column figures are simulated for continuous case and right column figures are simulated for discontinuous case.

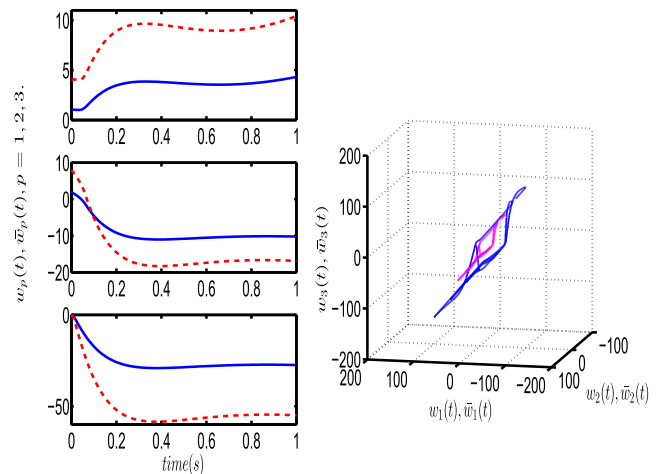


FIGURE 7. The state trajectories $w_p(t)$ and $\bar{w}_p(t)$, $p = 1, 2, 3$ of drive-response system (47)-(48) without control input $v_p(t) = 0$.

solution into the zero solution of the fractional-order differential inclusions. In our work [49], we state and illustrate the F-TS lemmas of the delayed discontinuous systems, where the results are derived by using the second mean-value theorem for definite integrals [50] for the fractional-order

$0 < \rho < 1$. In the present paper, we considers the improved fixed-time stability problem of generalized delayed FOSs by using well known Lemma 1 in [48] and the concept of Gamma functions. In both research results are imposed on the indefinite LKF and the framework of a Caputo fractional-order derivative introduced in [46]. Comparing with the results of [49], in this paper we propose an explicit saturated impulsive controller based on delta functions. The developed saturated impulsive FOSs reveal a significant result that impulsive control has an influence on controlled FOSs that is both dependent on impulsive function and related to the FOSs' order. The fractional-order delayed discontinuous system is proposed in in [49]. Consider an time-delayed 2n-dimensional FONNs (43) with the following discontinuous activations:

$$g_p(\bar{w}_p(t)) = \begin{cases} \tanh(\bar{w}_p(t)) - \bar{w}_p(t) + 1, & w_p(t) > 0, \\ \tanh(\bar{w}_p(t)) - \bar{w}_p(t) - 1, & w_p(t) \leq 0. \end{cases} \quad (46)$$

Figure 5, depicts the results of the numerical simulation of state trajectories of (43) and (44) with the discontinuous activation (46). Figure 6, demonstrates the state responses of FONNs (23) and (43) with different fractional-orders. When fractional-order q is increases then convergence between master and slave system is suddenly occurs see Figure 6(d) and 6(f). But in case fractional-order q decreases the master-slave systems are not converges exactly see Figure 6(b). This shows that the fractional-order parameter q play an important role in the problem of F-TSY.

Example 2: For $p = 1, 2, 3$, we investigate the following time-delayed fractional-order drive NNs:

$$D^q w_p(t) = -d_p w_p(t) + \sum_{r=1}^n a_{pr} g_r(w_r(t)) + \sum_{r=1}^n b_{pr} q_r(w_r(t - \tau_r)) + I_p \quad (47)$$

correspondingly, and the response system is given by the fractional-order differential equations shown below:

$$D^q \bar{w}_p(t) = -d_p \bar{w}_p(t) + \sum_{r=1}^n a_{pr} g_r(\bar{w}_r(t)) + \sum_{r=1}^n b_{pr} g_r(\bar{w}_r(t - \tau_r)) + I_p + v_p(t) \quad (48)$$

with the controllers being designed in the following ways:

$$v_p(t) = -\text{sgn}(e_p(t)) \left[\eta + \mathcal{A}(t) |e_p(t)|^\alpha + \mathcal{B}(t) |e_p(t)|^\beta \right] + \sum_{k=1}^{\infty} \text{sat}(\mathcal{S}_p(t)) \delta(t - t_h), \quad (49)$$

where, $d_1 = 0.2, d_2 = 0.2, d_3 = 0.2, a_{11} = 2, a_{12} = 0.1, a_{13} = 0.1, a_{21} = 0.7, a_{22} = 1.9, a_{23} = 0, a_{31} = 0, a_{32} = 0.2, a_{33} = 2, b_{11} = -1.5, b_{12} = -0.4, b_{13} = 0, b_{21} = -0.3, b_{22} = -1.1, b_{23} = 0, b_{31} = 0, b_{32} = 0, b_{33} = 1, I_1 = 0, I_2 = 0, I_3 = 0$. Let

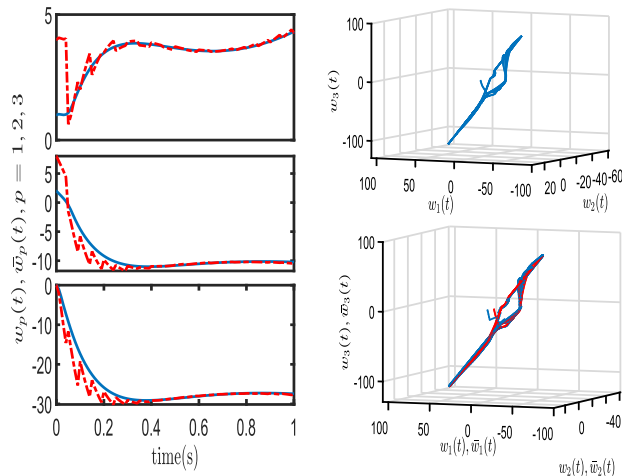


FIGURE 8. The state trajectories $w_p(t)$ and $\bar{w}_p(t), p = 1, 2, 3$ of drive-response system (47)-(48) with control input (49).

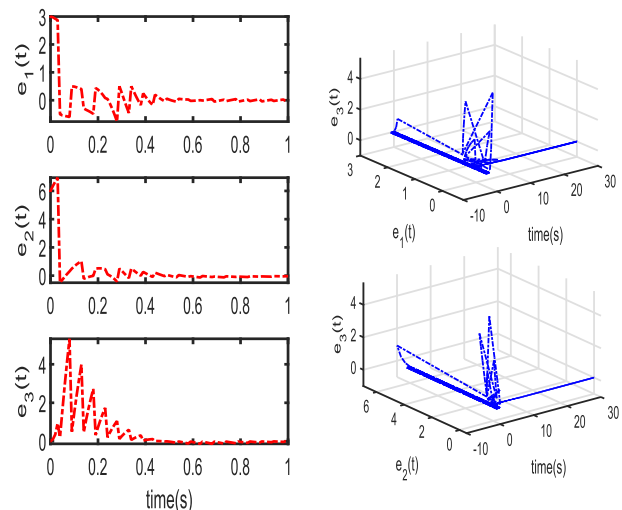


FIGURE 9. The time evolution of state trajectories and phase plots of error system (50) under discontinuous controller input (49).

$g_1(\bar{w}(t)) = g_2(\bar{w}(t)) = g_3(\bar{w}(t)) = \tanh(\bar{w}(t)), g_1(\bar{w}(t - \tau)) = g_2(\bar{w}(t - \tau)) = g_3(\bar{w}(t - \tau)) = \tanh(\bar{w}(t - \tau)), \tau = 0.01$. Thus, Assumption 1 is satisfied with $G_1 = 0.5, G_2 = 0.5, G_3 = 0.5$. Let (48) be the response systems and the corresponding drive system is defined in (47). From (47) and (48), we can design an fractional-order error system in such a way that

$$D^q e_p(t) = -d_p e_p(t) + \sum_{r=1}^3 \tanh(e_p(t)) + \sum_{r=1}^p \tanh(e_p(t - \tau_r)) + v_p(t), \quad (50)$$

If $v_p(t) = 0$, Figure 5 depicts the results of a numerical simulation of state trajectories of (47) and (48) with initial values $\bar{w}_1(\theta) = 0.2, \bar{w}_2(\theta) = -0.4, \bar{w}_3(\theta) = 0.02, \theta \in [-1, 0]$, in two and three-dimensional state spaces. Now let us take the control gain parameters $q = 0.9, \alpha = 0.2, \beta = 0.4, \eta = 1, \phi \in (0, 1), \rho = 0.6, \mathcal{A}(t) = -\frac{t}{2} |\cos(t)| + \frac{1}{1+t^2}, \mathcal{B}(t) = -\frac{1}{t}$. Since the conducted Lyapunov function

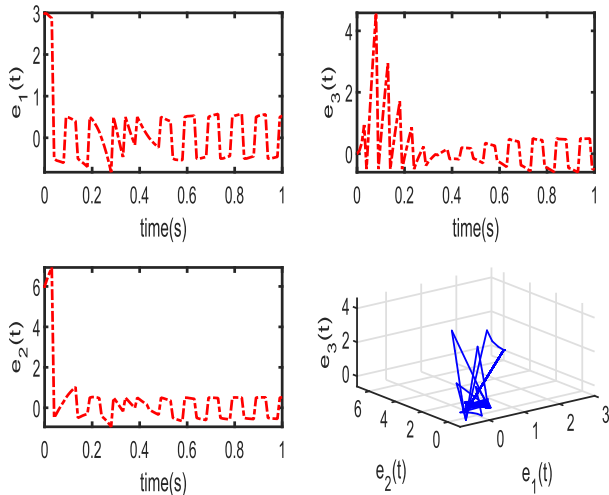


FIGURE 10. The time evolution of state trajectories and phase plot of error system (50).

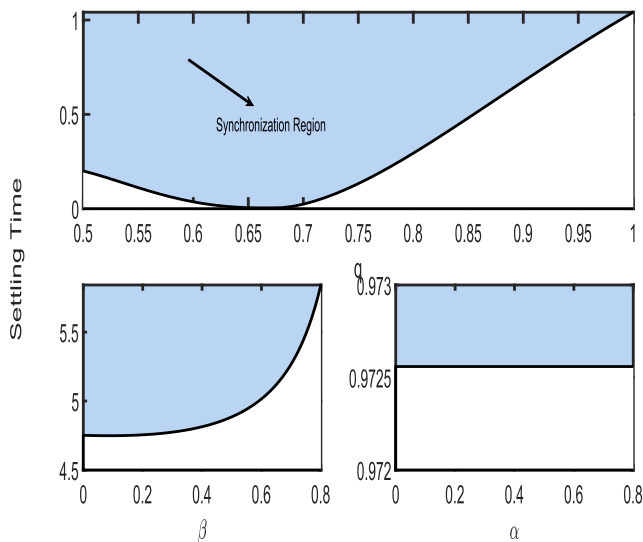


FIGURE 11. Synchronization region with response to the parameters \$q, \beta, \alpha\$.

becomes more generalized as a result of the fact that the derivative of it is indefinite almost everywhere. It has the potential to significantly improve the standard Lyapunov function, which is negative and is used to determine the stability results of FONNs. By the direct computations, we can easily verify the conditions (29)-(34) holds with the corresponding impulsive controller gain determined as \$\mathbb{K} = 0.499\mathbb{I}_3\$ and the discontinuous controller inputs \$v_p(t)\$ can be designed as bellow:

$$v_p(t) = -\text{sgn}(e_p(t)) \left[\eta - \frac{t}{2} |\cos(t)| + \frac{1}{1+t^2} |e_p(t)|^{0.2} - \frac{1}{t} |e_p(t)|^{0.4} \right] + \sum_{h=1}^{\infty} \text{sat}(\mathbb{S}_p(t)) \delta(t - t_h), \quad (51)$$

Therefore, we may conclude that the fractional-order drive-response system (47)-(48) can accomplish F-TSY in accordance with Theorem 3. This is illustrated in

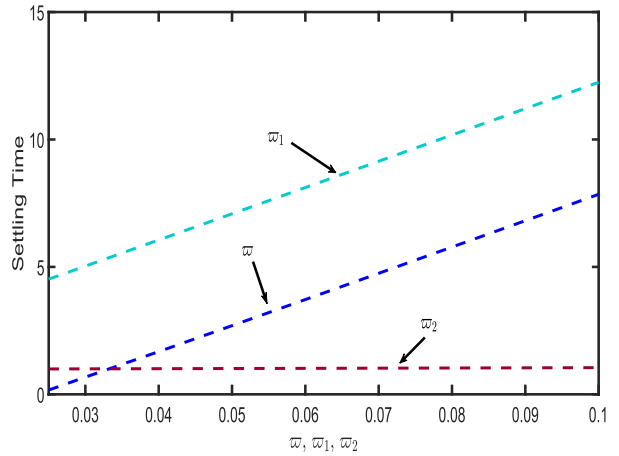


FIGURE 12. The settling time parameters \$\varpi, \varpi_1, \varpi_2\$.

Figures 6-7 with the initial values (0.2, -0.4, 0.02). If we take the impulsive gain \$\mathbb{K} = 0.99\mathbb{I}_3\$, then it does not converges, it goes against the control protocol which is shown in Figure 8. Figures 9 and 10 shows when other parameters are fixed, we have that the relationship between the settling time and the settling time parameters \$q, \alpha, \beta, \varpi, \varpi_1\$ and \$\varpi_2\$.

VI. CONCLUSION

In this paper, we investigated improved F-TS and F-TSY problems for delayed FOSs. We have obtained F-TS theorems for delayed FOSs with indefinite functionals. The fixed-stability theorems for FOSs using indefinite Lyapunov functionals have been derived using a fractional-order Caputo derivative operator. Our goal is to determine whether F-TS theorems hold for FONNs with time-delay. Based on a delta function for delayed FONNs, we have developed an explicit saturated impulsive controller. Numerical examples illustrate our theoretical findings.

Our next research will be focus on how to design periodically intermittent control and sampled data control to realize finite- and fixed-time synchronization criterion for delayed FONNs.

CONFLICT OF INTEREST STATEMENT

The author declares that the research was conducted in the absence of every commercial or financial relationships that could be construed as a potential conflict of interest.

ACKNOWLEDGMENT

The authors would like to thank the editor and reviewers for their valuable comments which improved the quality of the manuscript.

REFERENCES

- [1] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*. Singapore: World Sci., 2010.
- [2] C. A. Monje, Y. Chen, B. M. Vinagre, D. Xue, and V. Feliu-Battle, *Fractional-Order Systems and Controls: Fundamentals and Applications*. London, U.K.: Springer, 2010.
- [3] H. Sun, W. Chen, and Y. Chen, "Variable-order fractional differential operators in anomalous diffusion modeling," *Phys. A, Stat. Mech. Appl.*, vol. 388, pp. 4586-4592, Nov. 2009.

- [4] X. Yang, C. Li, T. Huang, and Q. Song, "Mittag-Leffler stability analysis of nonlinear fractional-order systems with impulses," *Appl. Math. Comput.*, vol. 293, pp. 416–422, Jan. 2017.
- [5] R.-J. Liu, Z.-Y. Nie, M. Wu, and J. She, "Robust disturbance rejection for uncertain fractional-order systems," *Appl. Math. Comput.*, vol. 322, pp. 79–88, Apr. 2018.
- [6] W. Chen, D. Ding, H. Dong, and G. Wei, "Distributed resilient filtering for power systems subject to denial-of-service attacks," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 49, no. 8, pp. 1688–1697, Aug. 2019.
- [7] H. Che and J. Wang, "A two-timescale duplex neurodynamic approach to biconvex optimization," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 30, no. 8, pp. 2503–2514, Aug. 2019.
- [8] D. Ding, Z. Wang, and Q.-L. Han, "A set-membership approach to event-triggered filtering for general nonlinear systems over sensor networks," *IEEE Trans. Autom. Control*, vol. 65, no. 4, pp. 1792–1799, Apr. 2020.
- [9] X.-M. Zhang and Q.-L. Han, "Global asymptotic stability analysis for delayed neural networks using a matrix-based quadratic convex approach," *Neural Netw.*, vol. 54, pp. 57–69, Jun. 2014.
- [10] X.-M. Zhang, Q.-L. Han, Z. Wang, and B.-L. Zhang, "Neuronal state estimation for neural networks with two additive time-varying delay components," *IEEE Trans. Cybern.*, vol. 47, no. 10, pp. 3184–3194, Oct. 2017.
- [11] X.-M. Zhang, Q.-L. Han, X. Ge, and D. Ding, "An overview of recent developments in Lyapunov–Krasovskii functionals and stability criteria for recurrent neural networks with time-varying delays," *Neurocomputing*, vol. 313, pp. 392–401, Nov. 2018.
- [12] L. Wang, S. Jiang, M.-F. Ge, C. Hu, and J. Hu, "Finite-/fixed-time synchronization of memristor chaotic systems and image encryption application," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 68, no. 12, pp. 4957–4969, Dec. 2021.
- [13] L. Wang, S. Jiang, M.-F. Ge, C. Hu, and J. Hu, "Finite-/fixed-time synchronization of memristor chaotic systems and image encryption application," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 68, no. 12, pp. 4957–4969, Dec. 2021, doi: [10.1109/TCSI.2021.3121555](https://doi.org/10.1109/TCSI.2021.3121555).
- [14] Z. Yang, J. Zhang, J. H. Hu, and J. Mei, "New results on finite-time stability for fractional-order neural networks with proportional delay," *Neurocomputing*, vol. 442, no. 1, pp. 327–336, 2021.
- [15] F. Du and J. Lu, "New results on finite-time stability of fractional-order Cohen–Grossberg neural networks with time delays," *Asian J. Control*, Jul. 2021, doi: [10.1002/asjc.2641](https://doi.org/10.1002/asjc.2641).
- [16] N. T. Thanh, P. Niamsup, and V. N. Phat, "New results on finite-time stability of fractional-order neural networks with time-varying delay," *Neural Comput. Appl.*, vol. 33, no. 24, pp. 17489–17496, Dec. 2021.
- [17] R. Rakkayappan, G. Velmurugan, and J. Cao, "Stability analysis of fractional-order complex-valued neural networks with time delays," *Chaos, Solitons Fractals*, vol. 78, pp. 297–316, Sep. 2015.
- [18] A. Wu and Z. Zeng, "Global Mittag-Leffler stabilization of fractional-order memristive neural networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 28, no. 1, pp. 206–217, Jan. 2017.
- [19] G. Velmurugan, R. Rakkayappan, and J. Cao, "Finite-time synchronization of fractional-order memristor-based neural networks with time delays," *Neural Netw.*, vol. 73, pp. 36–46, Jan. 2016.
- [20] F. A. Rihan, *Delay Differential Equations and Applications to Biology*. Singapore: Springer, 2021, doi: [10.1007/978-981-16-0626-7](https://doi.org/10.1007/978-981-16-0626-7).
- [21] S. Zhang, Y. Yu, and J. Yu, "LMI conditions for global stability of fractional-order neural networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 28, no. 10, pp. 2423–2433, Oct. 2017.
- [22] P. Liu, Z. G. Zeng, and J. Wang, "Multiple Mittag-Leffler stability of fractional-order recurrent neural networks," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 8, pp. 2279–2288, Aug. 2017.
- [23] K. Udhayakumar, R. Rakkayappan, J.-D. Cao, and X.-G. Tan, "Mittag-Leffler stability analysis of multiple equilibrium points in impulsive fractional-order quaternion-valued neural networks," *Frontiers Inf. Technol. Electron. Eng.*, vol. 21, no. 2, pp. 234–246, Feb. 2020.
- [24] C. Xu and P. Li, "On finite-time stability for fractional-order neural networks with proportional delays," *Neural Process. Lett.*, vol. 50, no. 2, pp. 1241–1256, Oct. 2019.
- [25] Y. Xu, J. Yu, W. Li, and J. Feng, "Global asymptotic stability of fractional-order competitive neural networks with multiple time-varying-delay links," *Appl. Math. Comput.*, vol. 389, Jan. 2021, Art. no. 125498.
- [26] H.-C. Lin, H.-B. Zeng, X.-M. Zhang, and W. Wang, "Stability analysis for delayed neural networks via a generalized reciprocally convex inequality," *IEEE Trans. Neural Netw. Learn. Syst.*, early access, Feb. 2, 2022, doi: [10.1109/TNNLS.2022.3144032](https://doi.org/10.1109/TNNLS.2022.3144032).
- [27] J. Cao and Y. Wan, "Matrix measure strategies for stability and synchronization of inertial BAM neural network with time delays," *Neural Netw.*, vol. 53, pp. 165–172, May 2014.
- [28] Q. Gan, "Global exponential synchronization of generalized stochastic neural networks with mixed time-varying delays and reaction-diffusion terms," *Neurocomputing*, vol. 89, pp. 96–105, Jul. 2012.
- [29] L. Chen, J. Cao, R. Wu, J. T. Machado, A. M. Lopes, and H. Yang, "Stability and synchronization of fractional-order memristive neural networks with multiple delays," *Neural Netw.*, vol. 94, pp. 76–85, Oct. 2017.
- [30] F. Zhang and Z. Zeng, "Asymptotic stability and synchronization of fractional-order neural networks with unbounded time-varying delays," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 51, no. 9, pp. 5547–5556, Sep. 2021.
- [31] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Trans. Autom. Control*, vol. 57, no. 8, pp. 2106–2110, Aug. 2012.
- [32] W. J. Rugh, *Linear System Theory*. Upper Saddle River, NJ, USA: Prentice-Hall, 1996.
- [33] D. Aeyels and J. Peuteman, "A new asymptotic stability criterion for nonlinear time-variant differential equations," *IEEE Trans. Autom. Control*, vol. 43, no. 7, pp. 968–971, Jul. 1998.
- [34] Y. Orlov, "Finite time stability and robust control synthesis of uncertain switched systems," *SIAM J. Control Optim.*, vol. 43, no. 4, pp. 1253–1271, Jan. 2004.
- [35] S. Li, H. Du, and X. Lin, "Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics," *Automatica*, vol. 47, no. 8, pp. 1706–1712, Aug. 2011.
- [36] G. Zong, H. Ren, and L. Hou, "Finite-time stability of interconnected impulsive switched systems," *IET Control Theory Appl.*, vol. 10, no. 6, pp. 648–654, Mar. 2016.
- [37] H. Ren, G. Zong, L. Hou, and Y. Yi, "Finite-time control of interconnected impulsive switched systems with time-varying delay," *Appl. Math. Comput.*, vol. 276, pp. 143–157, Mar. 2016.
- [38] G. Chen, Y. Yang, and J. Li, "Finite time stability of a class of hybrid dynamical systems," *IET Control Theory Appl.*, vol. 6, no. 1, pp. 8–13, Jan. 2012.
- [39] B. Zhou, "Stability analysis of non-linear time-varying systems by Lyapunov functions with indefinite derivatives," *IET Control Theory Appl.*, vol. 11, no. 9, pp. 1434–1442, Jun. 2017.
- [40] C. Ning, Y. He, M. Wu, Q. Liu, and J. She, "Input-to-state stability of nonlinear systems based on an indefinite Lyapunov function," *Syst. Control Lett.*, vol. 61, no. 12, pp. 1254–1259, Dec. 2012.
- [41] C. Ning, Y. He, M. Wu, and S. Zhou, "Indefinite derivative Lyapunov–Krasovskii functional method for input to state stability of nonlinear systems with time-delay," *Appl. Math. Comput.*, vol. 270, pp. 534–542, Nov. 2015.
- [42] G. Chen and Y. Yang, "Finite-time stability of switched nonlinear time-varying systems via indefinite Lyapunov functions," *Int. J. Robust Nonlinear Control*, vol. 28, no. 5, pp. 1901–1912, Mar. 2018.
- [43] P. Li, X. Li, and J. Cao, "Input-to-state stability of nonlinear switched systems via Lyapunov method involving indefinite derivative," *Complexity*, vol. 2018, pp. 1–8, Jan. 2018.
- [44] C. Rajivganthi, F. A. Rihan, S. Lakshmanan, R. Rakkayappan, and P. Muthukumar, "Synchronization of memristor-based delayed BAM neural networks with fractional-order derivatives," *Complexity*, vol. 21, no. S2, pp. 412–426, 2016.
- [45] R. Rakkayappan, R. Sivasamy, and J. H. Park, "Synchronization of fractional-order different memristor-based chaotic systems using active control," *Can. J. Phys.*, vol. 92, no. 12, pp. 1688–1695, Dec. 2014.
- [46] I. Podlubny, *Fractional Differential Equations*. New York, NY, USA: Academic, 1999.
- [47] L. Zhang and Y. Yang, "Bipartite finite time synchronization for general Caputo fractional-order impulsive coupled networks," *Neural Comput. Appl.*, vol. 33, no. 7, pp. 2459–2470, Apr. 2021.
- [48] H. Wu, L. Wang, P. Niu, and Y. Wang, "Global projective synchronization in finite time of nonidentical fractional-order neural networks based on sliding mode control strategy," *Neurocomputing*, vol. 235, pp. 264–273, Apr. 2017.
- [49] K. Udhayakumar, F. A. Rihan, R. Rakkayappan, and J. Cao, "Fractional-order discontinuous systems with indefinite LKFs: An application to fractional-order neural networks with time delays," *Neural Netw.*, vol. 145, pp. 319–330, Jan. 2022.
- [50] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*. New York, NY, USA: Academic, 2014.



UDHAYAKUMAR KANDASAMY received the B.Sc. degree in mathematics from the Government Arts College, Salem, India, in 2012, the M.Sc. degree in 2014, the M.Phil. degree in 2017, and the doctoral degree in mathematics from Bharathiar University, Coimbatore, India, in 2022. Among his current research interests include fractional-order systems, time-delay systems, and complex networks. Under the supervision of Prof. F. A. Rihan, he is currently working as a Postdoctoral Researcher with United Arab Emirates University.



FATHALLA A. RIHAN received the B.Sc. degree in mathematics from the Faculty of Science, Alexandria University, Egypt, in 1986, the Ph.D. degree in numerical treatment of delay differential equations in biosciences from the School of Mathematics, University of Manchester, U.K., in 2000, and the Doctor of Science (D.Sc.) degree in features of delay differential equations and their application, in 2018. He is currently a Professor of mathematics with United Arab Emirates (UAE) University. He has published a significant number of articles in highly regarded journals and participated in more than 100 international conferences. Moreover, he has published a book entitled *Delay Differential Equations and Applications to Biology* (Springer, 2021). His research interests include numerical analysis, mathematical biology, and mathematical modeling of real-life phenomena with memory, such as cell division, population dynamics, infectious diseases, parameter estimation, sensitivity analysis, and qualitative and quantitative analysis of delay differential equations. He is on the editorial boards of several international journals and a reviewer for *Journal of the American Mathematical Society*. He is the Principal Investigator of many mathematical biology research projects. In addition, he supervises a number of Ph.D. students. He was ranked among the top 2% of researchers by Stanford University, in 2019 and 2020. He also specializes in quality assurance and accreditation of higher education institutions.



RAKKIYAPPAN RAJAN received the B.Sc. degree in mathematics from the Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore, India, in 2002, the M.Sc. degree in mathematics from the PSG College of Arts and Science, Bharathiar University, Coimbatore, in 2004, and the Ph.D. degree in delay differential equations from the Department of Mathematics, Gandhigram Rural Institute, Gandhigram, India, in 2011. From 2017 to 2018, he was a Research Professor with the Research Center for Wind Energy Systems, Kunsan National University, Republic of Korea. In 2019, he was a Visiting Research Fellow with the School of Artificial Intelligence and Automation, Huazhong University of Science and Technology, Wuhan, China. He is currently an Assistant Professor with the Department of Mathematics, Bharathiar University. He has authored or coauthored more than 190 articles in international journals. His current research interests include the field of neural networks, complex systems, fractional-order systems, and time-delay systems. He was a recipient of the Highly Cited Researcher Award in Mathematics from the Clarivate Analysis, Web of Science Group, for 2019 and 2020.



MAHMOUD M. EL-KHOULY received the B.Sc. and first master's degrees from Helwan University, Egypt, in 1983 and 1994, respectively, the second master's degree in computer sciences from Cairo University, Egypt, in 1995, and the Doctorate of Philosophy degree in computer sciences from Saitama University, Japan, in 2000. He is an Associate Professor of information technology with Helwan University. His research interests include e-learning, cloud computing, and networks security software.

...