IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received May 30, 2022, accepted June 7, 2022, date of publication June 10, 2022, date of current version June 17, 2022.

Digital Object Identifier 10.1109/ACCESS.2022.3182046

Frequency Based Transit Assignment Models: Graph Formulation Study

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ABSTRACT Transit network design problem (TNDP) usually needs a recursive solution to successive transit assignment problems. Interestingly, the transit assignment problem is complicated with several unique criteria. In this study, we comprehensively review two well-known graphical transit assignment models from the literature. The first model is based on the hypergraph theory by Spiess and Florian (1989), and the second is the section transit network representation of De Cea and Fernandez (1993). The two assignment approaches are formulated in a single mathematical notation framework for the first time in the literature to understand the inherent differences better. We aim to bring attention again to these approaches for the upcoming TNDP studies since the most used transit assignment models in the TNDP are deficient in their basic assumptions compared with the considered models.

INDEX TERMS Hyperpath, transit assignment, section assignment, network equilibrium.

NOMENCLATURE

$\boldsymbol{o}$	Origin node.
$\boldsymbol{d}$	Destination node.
i, j	Generic nodes in V.
s	Start bus stop.
r	End bus stop.
m	Bus line index.
$\boldsymbol{w}$	Demand pair index.
h	Reference time.
k	Elementary path index.
R	Path R is composed of a set of k paths.
$\boldsymbol{d_w}$	The number of transit trips from o to d.
$\boldsymbol{O^u}$	The number of total trips for users of class (u) and purpose (z).
$\boldsymbol{p^u}$	Distriubiton share based on d, m, and h.
$\boldsymbol{\tau_i}$	Waiting time at node i.
e_{ij}	Edge of an ordered pair of indexes (i, j).
c_{ij}	Aggregate impedance on link e_{ij}.
$\boldsymbol{f_{ij}}$	Link $\boldsymbol{ij}$ flow.
$\boldsymbol{\bar{f}_{ij}}$	The competing flow of other sections that contain common lines of section $\boldsymbol{ij}$.
$\boldsymbol{h_R}$	Path $\boldsymbol{R}$ flow.
$\boldsymbol{\varphi_m}$	Line m frequency.
lcm	Line m nominal capacity.
v_m	Line m vehicle capacity, including the loading factor.
$\boldsymbol{UCm}$	User total equivalent travel time cost.
$\lambda_{\boldsymbol{k}}$	The conditional probability of choosing k.
$\boldsymbol{\alpha_i^k}$	Incident symbol equals 1 if path k traverses i, 0 otherwise.
$\boldsymbol{\beta_{i/R}}$	The conditional probability of passing the node i given the configuration R.
$\boldsymbol{\Omega_R^w}$	Path $\boldsymbol{R}$ choice proportion for $\boldsymbol{w}$.
$\boldsymbol{g_R^w}$	The average cost of $\boldsymbol{R}$.
$\boldsymbol{\delta_{ij}^{kw}}$	Incident symbol equals 1 if the $\boldsymbol{e_{ij}}$ is part of k, 0 otherwise.
$\boldsymbol{n_R^w}$	None-additive path R cost.
$\mathscr{J}$	Graph of V and E.
$\boldsymbol{AT}$	Access time.
$\boldsymbol{WT}$	Waiting time.
$\mathscr{B}/\tilde{A}$	Boarding/alighting time.
IVT	In-vehicle time.
TS	Transfer number.
£	Weight factor.
$\ell\ \mathcal{P}$	Calibrated factors.

The associate editor coordinating the review of this manuscript and approving it for publication was Dominik Strzalka.

V	Set of vertices (nodes).
E	Set of edges.
L	Set of lines defines the transit system.
Φ	Bus lines frequencies set.
LC	Bus lines capacities set.
W	Node pairs set.
H	Path flow set.
F	Link Flow set.
C	Link cost set.
G	Path cost set.
$A(i^+)$	Set of arcs directed out of node i.
$A(i^-)$	Set of arcs directed to node i.
BPR	Bureau of Public Roads.
FIFO	First In First Out order.
IIA	Independence of Irrelevant Alternatives.
MSA	Method of Successive Averages.
SE	Socie economic.
T	The existing transport system.
TNDP	Transit network design problem.

I. INTRODUCTION

Transit assignment models are fundamental tools in the evaluation process of transit network design problems (TNDP). TNDP is one of the most intractable problems to be solved in the transportation research field. This is due to its high degree of complexity. The sources that often hinder finding a unique optimal solution are; non-linearity & non-convexity, Bi-level problem formulation, combinatorial/NP hard complexity, and the multi-objective nature of the problem. The quality measure of a TNDP solution relies on the use of those models for predicting how each transit user selects a route from the origin to the destination [1], [2]. Also, in the design process, TNDP could be formulated as a bi-level programming model in which a transit assignment model is the lower model. Therefore, the TNDP solution methodology needs a recursive solution to successive transit assignment problems [3].

The reason for the transit assignment models expansion is the fact that there are many assumptions regarding users' strategies and information for transit line selection at the stops level where both users and vehicles arrive at different probability distributions, in addition to the level of aggregation, which is undertaken by each model [4]. The core difference between conventional traffic assignment models and transit ones is that transit users, in almost all cases, have to deal with overlapping transit routes where some routes share sections and stops, see Fig. 1. Interestingly, this turns the problem into a multi-path assignment even in its uncongested cases [5].

The small transit network example in Fig. 1 depicts the core that shapes the outcome of transit assignment models. In real-life action, a transit user always seeks a cost-efficient path to reach his/her destination. Rationally, passengers would consider line 1 (L1) in their route choice when boarding at

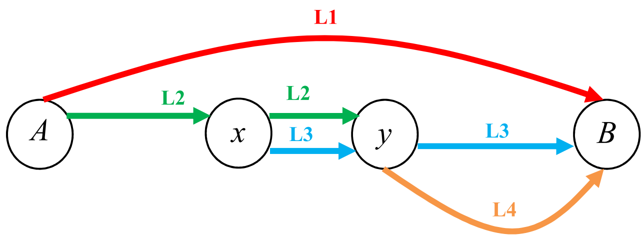

FIGURE 1. Illustrative transit network reproduced from [5].

Stop (A) destining to Stop (B). In other words, the direct service between their origin-destination. However, L2 (with transfer at x or y) would be considered in their attractive choice set only if they recognized it minimizes the total travel time. In that case, passengers would board the first incoming bus of the two lines.

Provided that the remaining waiting time for L2 is available with additional information on the expected waiting time at both stops (x) and (y), L1 could become no longer attractive to passengers. In other words, all passengers would board L2 then either L3 or L3 & L4 leaving L1 even if the first arrival bus is from L1. This problem is known as the common lines problem [6], where passengers will always choose to board the firstly arriving bus of predefined alternative services if the main/target service is absent. The common lines are determined as; whether to board the coming bus of a line or stay at the station waiting for the next bus of another line or walk to another station seeking better choices. It all depends on how much information would be available at the stops during the decision-making process.

More dimensions are added when the line's capacity is involved. Even if L1 became the only attractive choice, some passengers might fail to board the first incoming bus of L1 due to insufficient capacity. The choice would be to keep waiting for a space in L1 or change totally to L2 or consider both decisions. Also, L2 passengers may not be able to know precisely that choice cost (i.e., travel time). When they are traveling onboard, they need to decide whether to stay onboard to the stop (y) and then choose from L3 or both L3 & L4 or alight-to-transfer at the stop (x) [4].

These concepts of rational users are considered the basis of all transit assignment models. Users minimize the sum of waiting times and in-vehicle times in their boarding strategies, where the strategy is a set of rules (i.e., consecutive line selection) defined by the user to reach the destination. In a nutshell, if more than one route serves an origin node (o) and destination node (d), this would lead the users who wish to travel from (o) to (d), to determine a subset of the routes (attractive lines) boarding the first incoming bus of these routes taking into consideration that some or all lines may be involving transfers (i.e., strategy).

As a result of the assignment models, the analyst could predict the volume of the lines in addition to the crucial factor of the design of total time spent by the users in the supplied system. Operators make necessary decisions on the transit planning aspects to balance the total operation cost and users' cost to achieve an aimed level of service. Generally speaking, the product of the assignment model is expressed as;

$$UC_m = \pounds_1 AT + \pounds_2 WT + \pounds_3 \mathscr{B}/\tilde{A} + \pounds_4 IVT + \pounds_5 TS \quad (1)$$

where; UC_m is the user of mode (m) cost associated with the total travel time. The Eq. (1) four terms are access time *(AT)*, waiting time *(WT)*, boarding/alighting time ($\mathscr{B}/\tilde{A}$), in-vehicle time *(IVT),* and transfer number *(TS)*. £is a weight factor representing each term's relative significance in the user's route choice [7]–[9]. *WT* and *TS,* for example, are more significantly valued as the discomfort of travel compared with *IVT* [10]. While the UC_m is predicted from the assignment model, each of its terms is controlled by a corresponding aspect of the TNDP. For example, the *IVT* and *TS* depend on designing the itinerary of the routes. Transit vehicles' frequency setting (headways inverse) determines the average *WT* values. *AT* is controlled by the location and spacing of the stops. $\mathscr{B}/\tilde{A}$ times directly reflect the demand/supply ratio [2].

Generally speaking, the transit assignment models could be classified into three main categories, namely; frequency-based, schedule-based, and simulation-based. The frequency-based models consider the aggregated frequencies on transit lines where they are interested in calculating the ridership share percentage of each line. In contrast, the schedule-based deals with each vehicle trip independently in which its modeling scheme is capable of representing each vehicle departure time through a diachronic graph. In simulation models, transit users' real-time route choice is tracked. Vehicles and users are denoted as separated individuals in the models, which gives much more realistic modeling capability.

Theoretically, the TNDP solution algorithms would appreciate the frequency-based models due to their capability of performing assignments on real-size networks with tractable execution time. This would help in the solution process, which, as mentioned before, requires multiple transit assignments to be performed and mitigate the inherent complexity within the TNDP.

Technically speaking, there is no study until now that has calibrated any of the existing frequency-based models on real-world data. In addition, most TNDP studies have relied on simplistic (i.e., unrealistic) assumptions of their used transit assignment models (this will be discussed in detail in the next section). The reason for resorting to such deficient models is two-fold; first, the complexity of the TNDP and the need mentioned above to run several assignments before reaching a solution. Second, the missing reliable calibrated model to base all the TNDP solutions on.

Thus, this article discusses transit assignment frequency-based models using two popular literature graph equilibrium-based models from a graph formulation perspective. The study assumes that users decide according to the concept of optimal strategies from a static perspective that suits the TNDP solution's strategic stage. The remainder of this article is organized structure as follows. Section *II* draws a concise state of the art for the frequency-based models, whereas section *III* provides the basic adopted concepts in the selected models, whereas section *IV* gives the graphical representation of each model. The algorithms for reaching equilibrium are reported in section V. Finally, section *VI* presents the conclusion and discussion.

II. STATE OF THE ART

Transit assignment models based on frequency distribution have been the subject of many articles in the last five decades. While various models have been proposed to predict the users' behavior in selecting their routes, the concept known as attractive/common lines at the stops remains the main assumption in such models. Initially, the concept is first introduced by Chriqui and Robillard (1975) [6], where each user selects a set of lines to minimize the sum of average waiting time and in vehicles time (i.e., total expected travel time) in his/her strategy of reaching the destination. In [11], the focus is given to modeling the waiting time for that problem considering different user and vehicle arrival probability distributions. Interestingly, Spiess and Florian [12], [13] introduced the notion of optimal strategies in their work as a set of pre-determined rules taken by the user from the origin to the destination. They incorporated the common lines problem in a single mathematical formulation for the whole transit network assignment problem resulting in a non-linear mixed-integer programming model. Fortunately, the model has a relaxed linear version which eases finding a solution. Also, the presented formulation could be extended to incorporate the congestion effect as in the equilibrium models of the traditional transportation assignment models. The optimal strategies approach is transformed into a graph-theoretic representation by Nguyen and Pallottino [14], where the concept of hyper-paths is introduced. For each o/d, a hyper-path is generated in which some elementary paths are included. At each stop, there are outgoing links that include passengers' distribution across the elementary paths. The distribution portions are determined by the frequencies at the stop while being summed up to the unity. Each hyper-path has a total travel cost where the shortest hyper-path is equivalent to the optimal strategies obtained by the Spiess and Florian formulation. To easily adapt Bellman equation of optimality [15], the bus headways are assumed to follow the exponential distribution with random passengers' arrival. However, finding optimal hyperpaths in large transit networks with other headway distributions using label-setting or label-correcting algorithm is found to be more challenging task in [16].

De Cea and Fernandez [5] presented another graph representation that depends on what is called line sections. In that representation, the common lines problem is inherently implemented in the graph framework. The hyperbolic equation presented by Chriqui and Robillard [6] is recalled at every section formulation to select the attractive line set corresponding to that section. Interestingly the hyperbolic equation can be solved efficiently by heuristics presented in [5].

The congestion effect has become an exciting topic in frequency models studies [17]. It is an indispensable problem in many transit networks worldwide [18]. Passengers usually encounter fully congested stops that change their line selection due to either an increase in travel time impedance (mild capacity) or incapability of boarding the desired line due to capacity insufficiency (strict capacity). Passenger choice modeling becomes a more complicated task since it depends on not only individual preferences but also the congestion levels in the network [19].

The usefulness of congestion modeling is apparent in adapting Wardrop principles of equilibrium [20]. For example, in [12] and [5], the transit network assignment is solved under the deterministic user equilibrium principles, while Lam *et al.* [21] used the stochastic user equilibrium under the multinomial logit assumption for the route choice. Alternatively, Nielsen [22] used the probit-based model to model the route selection to escape from the Independence of Irrelevant Alternatives (IIA) property found in the multinomial logit model [23], [24].

Formulating a congested equilibrium assignment model requires a well-defined congestion-cost formula. Some congestion models may lead to transit assignment models that are difficult to study or use. To help overcome this, the congestion model should have nice mathematical properties (e.g., monotone increasing function) [25].

The cost is usually a function of the flow by presenting the link costs as flow-dependent. It reflects the disutility/discomfort corresponding to the crowded vehicles [12]. It can also be reflected by longer waiting times due to the possibility of not being able to board the first vehicle (i.e., full vehicles) or the bunching phenomenon (i.e., onboard travel time increase). The increase in waiting times could be formulated by reducing the nominal line frequency to attain the effective line frequency [5]. Whereas the two previous models are closely related to the well-known Bureau of Public Roads (BPR) model [26], Cominetti and Correa [27] presented a new model for congested transit assignment based on the hyperpath graph representation that incorporates queuing models.

The inclusion of lines' strict capacity has also been tackled in the transit assignment models. Cepeda *et al.* [28] extended the same model presented in [27] to incorporate the strict capacities while the primary concern is to prove the model conditions for solution existence and uniqueness. Then, they solved the model heuristically through the method of successive averages (MSA) by minimizing a newly developed gap function. Alternatively, Karauchi *et al.* [29] developed a different approach for considering the congestion. Their formulation included users' risk aversion of failing to board the next vehicle. In their graph representation, failure to board nodes and arcs are added to each bus stop. They assigned the probability of failing to board depending on the residual capacity of the vehicle and used Markov chains to obtain the line flows. Schmhocker *et al.* [30] extended the work in [29] by considering seat availability. Instead of the "failing to board" term, they used "fail to sit" to reflect the route choice according to the discomfort of standing.

All the aforementioned models did not pay attention to the queuing phenomena of First In First Out (FIFO), where they assumed users mingle at the stops. In [31], a frequency-based capacitated model is formulated by considering the FIFO discipline. The hyperpath graph is extended to the dynamic scenario while the common lines problem is embedded explicitly in the route choice modeling. Congestion is modeled as a bottleneck queue model with time-varying exit capacity [32]. The model allows overtaking among users with different attractive sets while queuing at a single stop.

Recently, none equilibrium assignment models have become a hot research topic due to the spread of online travel information. That makes users more aware of the operational conditions of the transit network and has suggested routes to follow. Cheung and Shalaby [33] proposed a heuristic assignment model to find the optimum system framework. It aims to minimize the total congestion in the transit network. Oliker and Bekhor [34] developed a heuristic assignment to consider online information that would lead to none equilibrated line flows. They extended their work in [35] to consider lines' strict capacity.

On the other hand, in the TNDP literature, the reviewed transit assignment models have received little attention as an evaluation tool. In [36]–[38], an analysis procedure called TRUST is used to evaluate the set of route configurations produced by a route generation algorithm. TRUST uses simple rules to assign the demand between o/d pairs in the transit network where the common lines problem is tackled differently. Users prioritize direct routes (i.e., without transfers) even if they are longer. The passenger is always assumed to attempt to reach his/her destination by following a set of routes that are within a prespecified range of the shortest path and has the fewest possible number of transfers. Many studies like [1], [39]–[41] followed the same rules: users would choose the path group with the least possible number of transfers and then select the first vehicle that arrived among that group.

Similarly, [42]–[47] used all or nothing assignment techniques to capture line flows where each user is assigned to the shortest path in total time. More relaxed assumptions are used in [48]–[52], in which only in-vehicle times are considered to determine served passengers' choices. Waiting times are not tackled in their objective functions.

On the contrary, in [53], [54], multiple path assignment was done when performing passenger assignment, and the frequency share method was used. It is assumed that all users would use, at maximum, two lines if available. The frequency share method was incorporated with the multinomial logit model in [55]. For each o/d pair, it was assumed that the passenger initially searches for direct route alternatives, where the frequency share is applied. If no direct options were found, the passengers were distributed among routes with transfers (up to two transfers) with the multinomial logit share function.

Some studies focused on the route network configuration without considering the assignment problem. The criteria of evaluation could be route directness (i.e., route length compared with the shortest path) [56], the number of transfers [57]–[60], and the network demand coverage [61]. Also, the construction cost values could be considered where the system is designated to be underground [58], [62]. Alternatively, in [63], [64], non-equilibrium assignment models were integrated into mathematical programming objective functions to optimize the transit line configuration simultaneously with the passengers' line assignment.

It should be noted that most of the none equilibrium based TNDP studies used the capacity-free assignment [65]. They would argue that the transit network design aims to identify the routes' capabilities through the total number of potential boarding users without any restriction. Besides, capacity-free assignment models are proved to be efficient and fast in large-scale networks [66].

A few numbers of TNDP studies have used equilibrium assignment models. In [67], [68], the conventional Wardrop's user equilibrium principle [69] is tackled without considering the common lines problem. Although [70]–[72] used all or nothing assignment in the lower level of their hierarchical design system, they used the EMME transit planning software in their final design evaluation stage. In that stage, they based their evaluation on Spiess and Florian's formulation. Also, in [73], [74], they used the original Spiess and Florian model in their lower level of design of a bi-level design approach. Similarly, in [3], [75], the De Cea and Fernandez model is used instead in the lower level of design. It is apparent in most of the TNDP literature that the conditions for the existence and uniqueness of equilibrium-based frequency models stand as a hindrance for being used in proposed solution algorithms. Finding plausible and quick solution methods for these two graph representations becomes necessary.

To this end, we could demonstrate the motivation of this study. For all reviewed frequency-based models, there are two basic graph representations; the hyper-path representation in [12]–[14] and the section-based representation in [5], where the different research achievements are built up based on them. Until now, there has been no study that reviewed the two model formulations simultaneously.

This study would stand alone in the literature by the following salient contributions;

- It presents a comprehensive review and analysis of the two most well-known transit assignment models in the literature.
- For the first time, the two assignment approaches are formulated in a single mathematical notation framework to deliver a better understanding of the inherent differences.
- The graphics representation and solution algorithms for the two models are illustrated in detail.
- The limitations of the models to be incorporated in the TNDP solution methodologies are described.
- The possible directions for upcoming studies are drawn at the manuscript's end.

We aim to bring attention again to these approaches for the upcoming TNDP studies since, as has been reviewed above, they are rarely used in the TNDP solution frameworks. In addition, the most used transit assignment models are deficient in their basic assumptions compared with the considered models.

III. PROBLEM FORMULATION

A. SUPPLY MODEL

A set of lines defines the transit system; $L = \{l_1, l_2, \ldots l_n\}$ with corresponding lines frequency set $\Phi = \{\varphi_1, \varphi_2, \ldots\ldots . \varphi_n\}$. The frequency of each line, in addition to vehicle loading capabilities, determine the line capacity set, $\text{LC} = \{lc_m : m \in L, lc_m = \varphi_m v_m\}$ since v_m is the capacity of the vehicle running on line m, including the loading factor. The transit network is constructed by assembling these lines and then is represented under an augmented graph framework. Passenger flows are transmitted between decision nodes via different functional arcs/edges (i.e., walking, accessing, waiting, hauling, and egressing arcs). The graph is denoted as $\mathscr{J} = (V, E)$, where V is the set of vertices (nodes) that are connected by the set of edges ($E = \{(i, j) : i, j \in V, c_{ij} \neq \infty\}$). The symbols i, j will represent generic nodes in V, and the index e_{ij} represents an edge as a shorthand for an ordered pair of indexes (i, j) where c_{ij} is the aggregate impedance to pass this link through the augmented network, which depends on the link functionality. $A(i^+) = \{(i, j)|(i, j) \in E\}$ set of the arcs emanating directly with from node i, whereas $A(i^-)$ is the set of arcs directed to node i. Each edge $(i, j) \equiv e_{ij}$ corresponds to a transit line segment or certain function which defines its attributes (i.e., cost and capacity). The generic path (i.e., hyperpath or segment path) R cost could be defined as follows:

$$g_R^w = \sum\nolimits_{k \in R} \mathcal{J}_k \sum\nolimits_{ij \in E} c_{ij} \delta_{ij}^{kw} + n_R^w \tag{2}$$

where; g_R^w is the average cost of the set of elementary paths (k) that constitute user pre-trip/en-route choice (according to the common line assumption). $\mathcal{J}_k$ is conditional probability of choosing the elementary path k, if $R \in \mathfrak{R}$ is the choice set for the users of w. It should be differentiated between two terms; path and elementary path. In each time we refer to a path (R), it would be considered the collective of elementary paths (k) constitute user pre-trip/en-route choice set. Although it is common in the literature to call this path a hyperpath, we could not do that in this study since another equivalent graph representation is presented (i.e., segment path). So, it would suffice to call it a path R. δ_{ij}^{kw} is an incident symbol equals one if the e_{ij} is part of the elementary path k, 0 otherwise. n_R^w is the none-additive path R cost which cannot be obtained as sum of links specific costs. In other words, it cannot be defined except after the complete configuration of the path. In transit network terminology, the non-additive performance variables are the waiting time at different stops. Interestingly, any edge e_{ij} could be selected by more than one elementary path within the same collective path. Therefore, we could obtain the conditional probability of e_{ij} being selected given the path R.

$$\alpha_{ij/R} = \sum\nolimits_{k \in R} \mathcal{J}_k \delta_{ij}^{kw} \tag{3}$$

B. DEMAND MODEL

Transit demand is the product of the transport activity system and the current transit supply system, where complex relationships between both associated by users' socio-economic characteristics result in the number of transit trips from o to d (d_w). Note that a subscript is not required to define the demand mode since transit is the only tackled mode in this research. The set of none zero origin-destination pairs $W = \{ w \triangleq (o, d), w \subseteq \mathrm{V} \times \mathrm{V} | d_w > 0\}$.

$$d_w(SE, T) = \sum_s \sum_u O^u(x/o, z, h) p^u (d/o, z, h)\, p^u(m/o, d, z, h) \tag{4}$$

The trip demand model in Eq. (4) estimates the average number d_w, which is a function of the socie economic (SE) characteristics and the existing transport system (T). O^u is the number of total trips for users of class (u) and purpose (z). p^u is the distriubiton share based on the destination (d), mode (m), and reference time (h).

As transit planning is targeted in the strategic stage, it would be appropriate to consider the rigid demand type with respect to the destination, mode (transit only), and enough reference period (i.e., hour) to construct static traffic equilibrium in the network. The only considered elasticity for the demand is path choice (R). It is assumed to result from a sequence of decisions made at different nodes in the network to have mixed pre-trip/en-route behavior. It corresponds to the basic common-lines problem where each d_w is assigned to start bus stop (s) and end bus stop (r).

C. NETWORK LOADING ASSUMPTIONS

To describe transit network loading assumptions, we need to define the set of feasible paths flow and links flows as follows:

$$h_R = \sum_{w \in W} h_R^w = \sum_{w \in W} d_w \Omega_R^w(g_R^w), \quad \forall R \in \mathfrak{R} \tag{5}$$

$$f_{ij} = \sum_{R \in \mathfrak{R}} \alpha_{ij/R} h_R \quad \forall ij \in E \tag{6}$$

where; Ω_R^w is the path choice proportion for demand pair (w), which is a function of the path cost g_R^w and consequently, both paths flow (h_R) and links flow (f_{ij}) are conditioned to it. Eq.s () & (6) define the two sets S_h and S_f for which any feasible solution to the assignment problem should satisfy. To compare the two suggested transit graph representations, the deterministic equilibrium assignment model would be only considered in this research. The considered model results in paths or links flow correspond to the equilibrium conditions expressed by the Wardrop principle, which states "for each O-D pair, the path cost used is equal, and is less than or equal to the cost of each unused path".

Due to the asymmetric property of the transit assignment model, the network loading model cannot be reduced to an equivalent mathematical optimization problem. Therefore, it is solved using a stated variational inequality in terms of path flows as follows;

$$\begin{aligned} & G^t(H - H_D) \geq 0, \quad \forall H \in S_h \\ & s.t.\, H_D \in S_h \end{aligned} \tag{7}$$

or in terms of link flows:

$$\begin{aligned} & C^t(F - F_D) + N^t(H - H_D) \geq 0, \quad \forall F \in S_f \& H \in S_h \\ & s.t.\, F_D \in S_f \& H_D \in S_h \end{aligned} \tag{8}$$

where; the capital notation is the vector of all corresponding small notation variables, the superscript t *is* for vector transpose, and the subscript D is for the deterministic equilibrium solution. If the costs of the links are independent of the flow, the equilibrium would simply turn into all-or-nothing assignment.

IV. TRANSIT NETWORK REPRESENTATIONS

A. COMMON LINES PROBLEM

The transit stop problem, which consists of estimating the passenger distribution between the attractive lines and the expected passenger waiting times at bus stops, is usually called the common lines problem. As mentioned before, it is the core sub-problem in any transit assignment formulation. As it would turn out, it forms the basis of the two considered models. The assumptions considered in that level of the general assignment problem control mainly the way of network loading stage. Let us consider the basic transit network of single start/end stops connecting by n lines. Now a passenger at the stop can choose between several lines that differ in their "in-vehicle" travel times. Intuitively, the passenger would take the least IVT line. However, the arisen question, would he/her change his choice if the first arriving vehicle was from a longer IVT line. Normally, each passenger is thought to determine a set of attractive lines which he/she would board the first arriving vehicle of this set.

To determine this set, we need to make some assumptions about lines' headway, passengers arrival rates, and lines' IVT probability distributions besides passengers' choice model and their real-time information. To build up the mathematical formulation, let us assume that the headway of the different lines is an independent random variable with exponential distribution. The passengers arrive randomly following the Poisson distribution "they do not adjust the arrival time" while the IVT times are deterministic. The passengers would choose the set of lines that minimize their total travel time (WT+IVT in that case) while they are fully aware of lines' IVT and expecting WT.

The solution to the following hyperbolic problem will define the set of attractive lines $\bar{L}_{s,r} \subseteq L_{s,r}$:

$$\begin{aligned} & \arg min_{x_l} UC = \frac{\Psi}{\sum_{l=1}^n \varphi_l x_l} + \frac{\sum_{l=1}^n IVT_l \varphi_l x_l}{\sum_{l=1}^n \varphi_l x_l} \\ & s.t.\, x_l \in \{0, 1\} \quad \forall l \in L_{s,r} \\ & \quad £_2 \;\&\; £_4 = 1 \\ & \quad £_1 \;\&\; £_3 \;\&\; £_5 = 0 \end{aligned} \tag{9}$$

The set of lines with $x_l = 1$ would be added to the $\bar{L}_{s,r}$ set. Ψ is a parameter that captures the variability of both passengers' and vehicles' arrival processes (e.g., for that study $\Psi = 1$) [4]. Interestingly, Eq. (9) could be solved with efficient heuristics in which the lines are ordered in an increasing manner according to their IVT, and then each line is inspected sequentially to be added in the $\bar{L}_{s,r}$ if they would contribute to no increase/decrease in the UC value.

The users are distributed among the lines according to each line frequency:

$$f_l = d_w \frac{\varphi_l}{\sum_{l \in \bar{L}\mathrm{s,r}} \varphi_l} \quad \forall l \in \bar{L}_{s,r} \tag{10}$$

Note that more complexity is added to Eq. (9) when lines capacity is considered. Even if lines in $\overline{L_{s,r}}$ are the only attractive choice, some passengers may fail to board the first incoming bus of them due to insufficient capacity. Logically, they would change to choose out of the set. However, they may find it is more profitable to wait for more until they board in a vacant bus of the set $\bar{L}_{s,r}$. Therefore, lines may exceed their nominal capacity (lc_m). This assumption is called a mild capacity constraint which is the chosen congestion to be dealt with in the larger framework of the assignment models in the next sections.

B. HYPERNETWORK

In the conventional formulation of the hyperpath method, it is assumed that the strategy is chosen before the trip starts and, beginning from the origin, it involves a sequence of walking to the stop/the destination, selecting the optimal lines to board and, for each of them, the stop where to alight.

To represent the graph $\mathcal{J}$ in hypernetwork terminology context, consider the urban transit network consisting of a set of transit lines where each line is defined by a set of stops (N_l). The distinct stops in all these lines are set as the basic stop set (S). Therefore, each line itinerary is associated with a subset of S where it would be connected by boarding arcs and alighting arcs. See the transformation of the transit network in Fig. 1 to the auxiliary network in Fig. 2 for the sake of hyperpath assignm ent. As we assumed that the demand the d_w is assigned to $(s, r) \in S$, we would not incorporate walking links for compact representation.

Now the graph nodes $V = (\cup_{l=1}^{n} N_l\ ) \cup S$ and $E = (\cup_{l=1}^{n} A_l) \cup A_b \cup A_a$ *where* A_l, A_b, A_a are the lines, boarding, and alighting arcs, respectively. The core of this representation is the ability to define subsets of graphs; each one is called a hypergraph (i.e., path R). R_w in this representation is associated with one start and end stop to carry certain d_w with a combination of different elementary paths, k. Every combination of paths k connecting s & r would constitute a distinct R identity ε (i.e., $R_{\varepsilon w}$), in other words, the set size $|\varepsilon| = (2^k - 1)$ for the k paths connecting w.

$\mathcal{J}_R(V_R, E_R)$ ⍰ $\mathcal{J}$ is an acyclic graph associated with the hyperpath $R_{\varepsilon w}$ in which each arc and node is defined with selection probability as follows:

$$\pi_{ij/R} = \frac{\varphi_{ij}}{\sum_{ij \in A(i^+)} \varphi_{ij}} \quad \forall ij \in E_R \tag{11}$$

$$\sum_{ij \in A(i^+)} \pi_{ij/R} = 1 \quad \forall i \in V_R - \{r\} \tag{12}$$

$$\pi_{ij/R} \geq 0 \quad \forall ij \in E_R \tag{13}$$

Now $\mathcal{K}_k$ (the probability of choosing path k in R as in Eq. (3)) is defined as follows:

$$\mathcal{K}_k = \prod_{ij \in E_R} \pi_{ij/R} \delta_{ij}^{k}, \quad \forall k \in \tag{14}$$

$$\sum_{k \in R} \mathcal{K}_\mathrm{k} = 1 \tag{15}$$

Let $\beta_{i/R}$ is the conditional probability of passing the node i given the configuration R and α_i^k is the incident symbol which is equal 1 if path k traverses i, 0 otherwise.

$$\beta_{i/R} = \sum_{k \in R} \mathcal{K}_k \alpha_i^k \quad \forall i \in V_R \tag{16}$$

It is clearly:

$$\beta_{s/R} = \beta_{r/R} = 1 \tag{17}$$

Now Eq. (2) could be used to estimate a hyperpath (R) cost contains a set of elementary paths (k) as follows:

$$g_R^w = \sum_{k \in R} \mathcal{K}_k \sum_{ij \in E} c_{ij} \delta_{ij}^{kw} + \sum_{i \in V} \beta_{i/R} \tau_{i/R}$$
$$s.t.\ c_{ij} = \begin{cases} IVT_l, & if\ ij \in A_l \\ b_l & if\ ij \in A_b \\ a_l & if\ ij \in A_a \end{cases}$$
$$\tau_{i/R} = \begin{cases} \dfrac{1}{\sum_{ij \in A(i^+)} \varphi_{ij}}, & if\ i \in S - \{r\} \\ 0 & if\ i \in N_l \cup \{r\} \end{cases} \tag{18}$$

where; τ_i is the waiting time at node i, if it is a stop in the hypergraph connected to boarding links. Each boarding/ alighting link is connected to a single transit line from which their impedances (b_l&a_l) could be estimated as a function of boarding flow to that line. Also, φ_{ij} for the boarding link ($ij \in A_a$) is equal to the frequency of the connected line while the line links themselves are associated with large frequencies ($\varphi_{ij} \approx +\infty \forall\ ij \in A_l$).

Now the transit loading parameters in Eq.s (5 & 6) are defined as follows:

$$\Omega_R^w = \begin{cases} 1, & if\ g_R^w = \min\{R_{\varepsilon w}\} \\ 0, & otherwise \end{cases} \tag{19}$$

$$\alpha_{ij/R} = \sum_k \mathcal{K}_k \delta_{ij}^k \quad \forall ij \in E_R \tag{20}$$

Eq. (19) states the optimal strategies assignment defined in [12]. Also, it matches the path choice assumed in this study, where passengers use the shortest hyperpath.

C. SEGMENT NETWORK

The second model is first presented in [5], which is thought to be much simpler than the hypernetwork context. It aims to represent the transit assignment exactly as the ordinary traffic assignment problem by a new graph representation where the common lines are dealt with inherently. For the set of lines L, we identify the distinct stops in all these lines. Then we pick each pair of stops and their corresponding lines and solve the hyperbolic problem at Eq. (9) to get the set of attractive lines connecting this pair of stops ($\bar{L}_{ij}$). This set is considered as link (i.e., segment/section) in the graph $\mathcal{J} = (V, E)$. Since $V \equiv S$ and $E = \{\bar{L}_{ij}$: i, $j \in S$, $c_{ij} \neq \infty\}$. The transformation of lines to the segment network representation is illustrated in Fig 3. Each segment cost could be defined as follows:

$$c_{ij} = \frac{1 + \sum_{l \in \bar{L}ij} IVT_l \varphi_l}{\sum_{l \in \bar{L}ij} \varphi_l} \quad \forall e_{ij} \in E \tag{21}$$

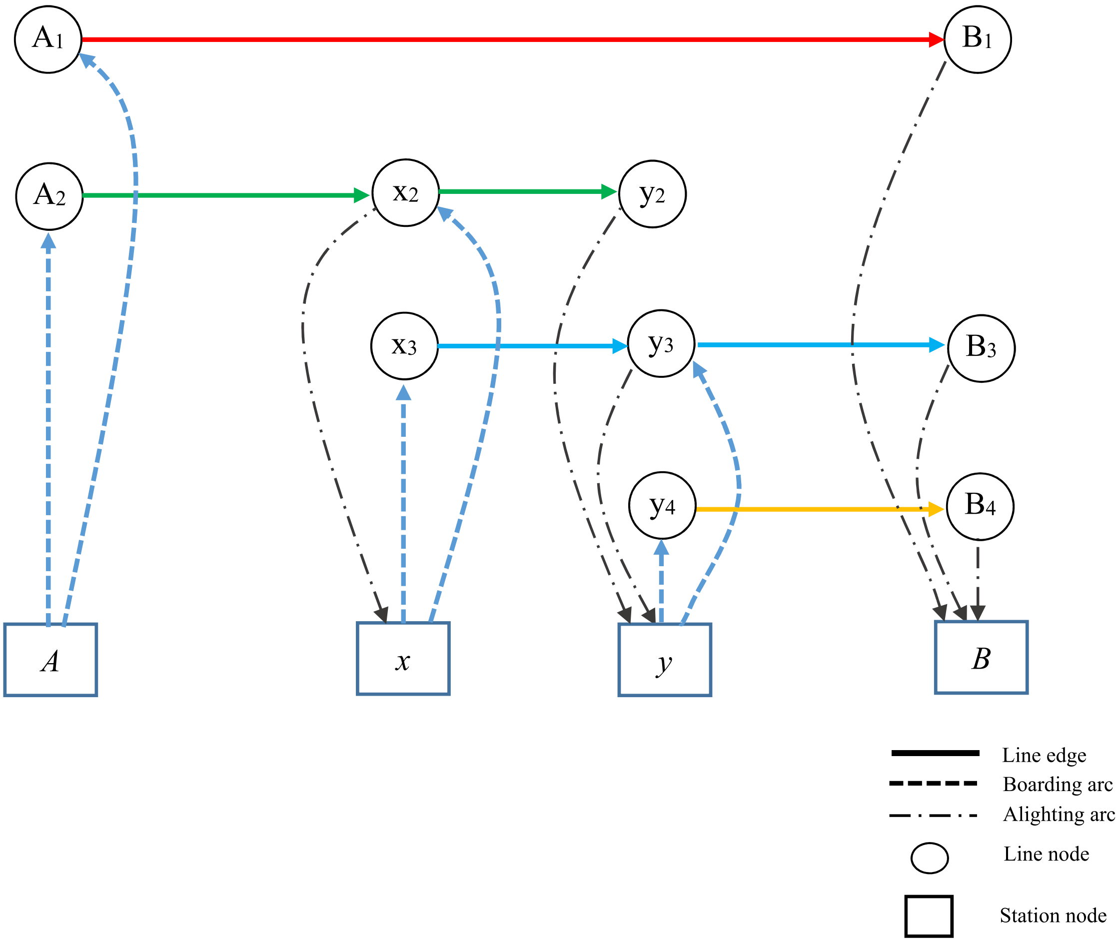

FIGURE 2. Hypernetwork graph representation for the small, reported transit network.

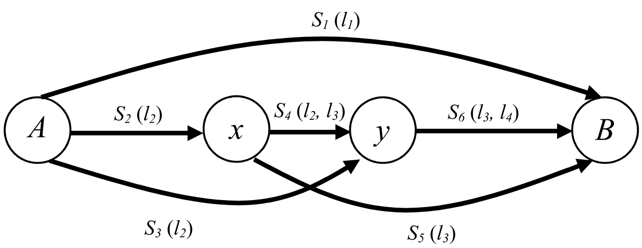

FIGURE 3. Segment network graph representation for the small, reported transit network.

Interestingly the collective/hyperpath R is collapsed in this representation to an elementary path k where the path cost from Eq. (2) is estimated as follows:

$$\begin{aligned} g_R^w &= \sum\nolimits_{ij\in E} c_{ij}\delta_{ij}^{kw} + n_R^w \\ s.t.\ R &= \{k\} \\ n_R^w &= 0 \end{aligned} \tag{22}$$

In other words, any sequence of connected links (segments) is equivalent to a hyperpath in the hypernetwork context of this representation.

Now the transit loading parameters in Eq.s (5 & 6) are defined as follows:

$$\alpha_{ij/R} = \Omega_R^w = \begin{cases} 1, & if\ g_R^w = \min\{R_{kw}\} \\ 0, & otherwise \end{cases} \tag{23}$$

V. TRANSIT ASSIGNMENT

The ultimate purpose of a transit assignment model is to produce both Ω_R^w & $\alpha_{ij/R}$ values which would define transit lines ridership. This section gives the steps and the assumptions needed to conduct the passengers' assignment over the two stated network models. Before going deep into the components of the transit assignment model using the presented formulations, Fig. 4 provides a graphical representation of the considered system models.

A. PATH CHOICE

As stated before, the path choice of transit users is a mix between pre-trip decisions and en-route decisions. Fortunately, this behavior could be modeled implicitly in both described models in the previous section in defining the path R. Even when we consider single path R for certain demand (d_w), it will incorporate a multi-line selection. Modeling path choice has two approaches, namely;

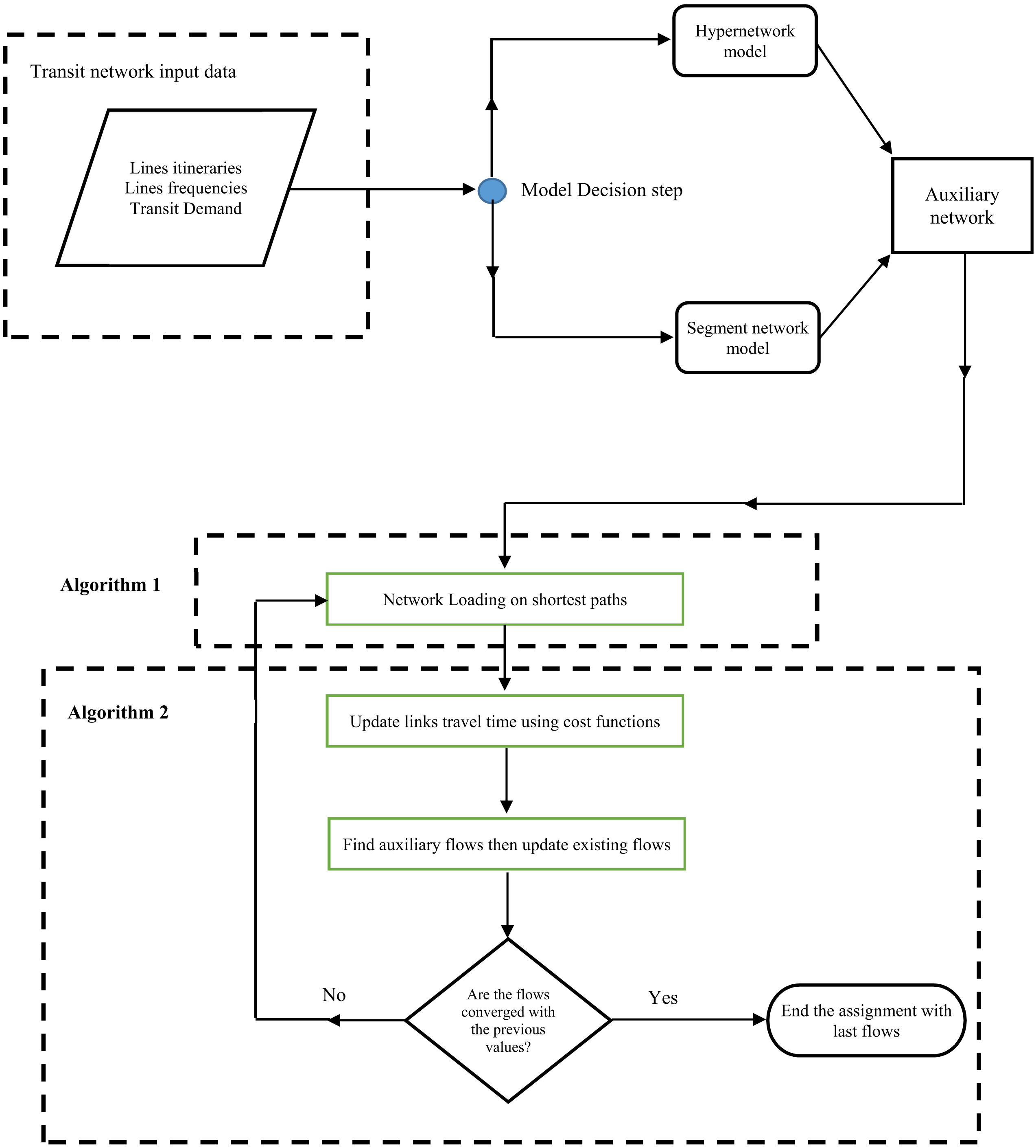

FIGURE 4. A graphical representation of the considered system models.

deterministic and stochastic. In the deterministic approach, users are assumed to be accurately aware of paths' generalized costs, so they would choose the least cost path. If the cost of the paths is flow-independent, they will choose the least cost path (i.e., the shortest path).

On the contrary, the stochastic approach distributes the users among the several paths, considering the perceived path cost as a random variable. This assumption directs most random utility choice models to assign no zero-selection probability for each path. The selection probably of each path would depend on the systematic (actual) cost, error distribution assumption, and cost-flow dependence assumption [21], [76].

This study would consider the deterministic (flow – cost dependent) approach to present the two assignment algorithms. However, extending the comparison to other path choice models is a straightforward task on the same network representations.

Algorithm (1): Shortest Transit Path $\boldsymbol{R}$

Pre-condition: *connected* $\mathcal{J}$
Post-condition: *shortest* $\mathfrak{R}$ *set*

1. For each s & $r \in W$
2. Initialization: Set $q_r := 0$; $q_i := +\infty \forall i \in V$
3. *(updating label step)*
   3.1 *If* $q_i > q_j + c_{ij}$
   then $q_i := q_j + c_{ij}$

   in hypernetwork representation update $q_i \forall i \in S$ *as follows:*

   3.2 *If* $q_i > q_i^* = min_{A_i \subseteq A(i^+)} \left\{ \frac{\left(\sum_{ij\in A_i} \varphi_{ij} q_j + 1\right)}{\sum_{ij\in A_i} \varphi_{ij}} \right\}$
   then $q_i = q_i^*$
4. *Repeat* step 3 until no label can be further improved.
5. *Return path R connecting s and r by backtracking* q_i
6. *Add R to* $\mathfrak{R}$
7. *End for*
8. *End algorithm*

For the considered deterministic model, calculating shortest path R (for each s and r) would be recalled recursively, and it can be stated under Generalized Bellman's equation:

For hypernetwork representation

$$q_i = \begin{cases} 0 \\ \qquad if\ i = r \\ min_{ij\in A(i^+)} \{q_j + c_{ij}\} \\ \qquad if\ i \in V - \{S\} - \{r\} \\ min_{A_i \subseteq A(i^+)} \left\{ \dfrac{\left(\sum_{ij\in A_i} \varphi_{ij} q_j + 1\right)}{\sum_{ij\in A_i} \varphi_{ij}} \right\} \\ \qquad if\ i \in \{S\} - \{r\} \end{cases} \tag{24}$$

For segment *representation*

$$q_i = \begin{cases} 0 & if\ i = r \\ min_{ij\in A(i^+)} \{q_j + c_{ij}\} & if\ i \in S - \{r\} \end{cases} \tag{25}$$

where q_i is set to be the length of a shortest $\bar{R}(\subseteq R)$ *path* from an intermediate node i to the destination r. The following iterative procedure could be used for computing the shortest R for both representations:

In the hypernetwork representation, to update q_i at stops, it is needed to find the optimal subset links outgoing form i (i.e., A_i) to minimize the following:

$$q_i^* = min_{x_{ij}} \left\{ \frac{\left(\sum_{ij\in A(i^+)} \varphi_{ij} q_j x_{ij} + 1\right)}{\sum_{ij\in A(i^+)} \varphi_{ij} x_{ij}} \right\}$$
$$s.t.\ x_{ij} \in \{0, 1\} \quad \forall ij \in A\left(i^+\right)$$
$$x_{ij} = 1 \quad \forall ij \in A_i \tag{26}$$

Note that solving the subset A_i would be repeated at each stop label update iteration. Therefore, using the same heuristic defined for the hyperbolic problem at Eq. (9), replacing lines in-vehicle time by node labels, would be efficient. Also, one may notice that segment representation does not need special treatment to update the stop labels by removing step 3.2. the algorithm would convert to the conventional shortest path algorithm.

B. PERFORMANCE FUNCTIONS

It is crucial to incorporate volume-delay functions in the assignment process to reproduce the effects of increasing waiting times due to the inability (or inconvenience) to board the first arriv ed vehicle(s). This evaluation would help the analyst figure out the sufficiency of examined design of a transit network more effectively in the strategic stage as the proposed function may not represent the actual waiting times. However, it could reflect the relative efficiency among different evaluated solutions.

In this study, the effect of the increased cost would be represented by resembling the well-known Bureau of Public Roads (BPR) formula for the two models as follows:

For hypernetwork representation

$$c_{ij} = \begin{cases} IVT_l, & if\ ij \in A_l \\ \mathscr{b} \left(\dfrac{f_{ij}}{\varphi_l v_l}\right)^P & if\ ij \in A_b \\ a_l = 0 & if\ ij \in A_a \end{cases} \tag{27}$$

For segment representation

$$c_{ij} = \frac{1 + \sum_{l\in\bar{L}ij} IVT_l \varphi_l}{\sum_{l\in\bar{L}ij} \varphi_l} + \mathscr{b} \left(\frac{f_{ij} + \sum_{ij\in E} \bar{f}_{ij}}{\sum_{m\in\bar{L}ij} lcm}\right)^P \tag{28}$$

where; $\mathscr{b}$ and P are factors that to be calibrated to determine how the flow affects the travel time. $\bar{f}_{ij}$ is the competing flow of other sections that contain common lines of section ij. In both models, the increase in the impedance is assumed due to the increase in waiting times. In hypernetwork representation, the waiting times are calculated separately as additive costs, see Eq. (18).

The solution flows for these cost functions may override the lines' physical bounds (i.e., capacities). This may be the case when using volume-delay functions directly from calibration. The adoption of these formulas could be argued in two ways. First, the line's physical capacity could be over exceeded in real life by increasing the waiting time until getting a vacant place in that line. Second, these formulas have the required criteria to ensure convergence and uniqueness for most assignment models [23], [24]. Therefore, they are the most appropriate way to convey the supply-demand interaction in the strategic stage of the design.

C. ASSIGNMENT ALGORITHM

The assignment algorithm aims to solve the variational inequalities presented in Eq.s (7 or 8). While it is proved that the solution in terms of paths - as in Eq. (7) - is not unique, Eq. (8) could provide a unique solution in terms of links flow. This uniqueness is guaranteed by the monotonicity assumption of the performance function in Eq. (27 & 28), and the assumption of the non-additive cost is independent of the flow.

The two representations could be solved by either a variational inequality problem or a fixed-point problem using

Algorithm (2): Transit Assignment Equilibrium

Pre-condition: *connected* $\mathcal{J}$
Post-condition: *set of link flows* (F)

1. **Initialization**:
   1.1 $u := 0$
   1.2 compute a feasible arc flow F^u *through all or nothing using Algorithm (1) and the costs associated with none flow on the links, then compute the associated non-additive costs* (N^u) *if any.*
2. **Auxiliary flow estimation step:** *diagonalize the C vector* (Sheffi [77]) *to compute the auxiliary arc flow* $\bar{F}^u$ *by using one iteration for the diagonalized C associated with* F^u *and* N^u *(if any).*
3. *Set* $u := u+1$
4. ***Find the updated flow vector (MSA):*** $F^u := F^i + 1/(\mathrm{u})[\,F^u - \bar{F}^u]$
5. **Check the stopping criterion:** Test the current flow: If $\frac{\sum_{ij \in E} (f_{ij}{}^{u} - f_{ij}{}^{u-1})^2}{|E|} \leq \kappa$ then stop and return F^u *as the solution, otherwise go to step 2.*
6. *End algorithm*

the maximum successive average (MSA). As the segment representation, a has a nonseparable cost function structure with an asymmetric Jacobian matrix. It would need a diagonalization step in which the Jacobian matrix would be approximate to a diagonal one considering only the variation of the link cost at each diagonal cell. Also, we would apply the streamlined method suggested by Sheffi [77] to reduce the number of required iterations in the whole algorithm.

It is worth noting that in step 4, computation of an improving direction and optimal step length would be required for better convergence performance. However, if the assignment's target is to evaluate the transit network with respect to the TNDP solution algorithms, it would be sufficient to use the MSA method.

VI. CONCLUSION AND DISCUSSION

This study presents a comprehensive review of two well-known transit assignment models; Spiess and Florian (1989) and De Cea and Fernandez (1993), to give a profound understanding of them. Many studies come after adding modifications to either the user's behavior assumptions or equilibrium solution algorithms. To the best of our knowledge, no study reviews the two models simultaneously under one single framework from a TNDP analyst perspective. Despite the profound logic in their user choice behavior, they are used rarely in the TNDP literature. This may return for many reasons, where the two revised graph formulations suffer from major complexity issues.

First, both models require an augmented/auxiliary network representation. Hypernetwork duplicates bus stops nodes as many as lines passing the stops with additional boarding/alighting arcs in addition to walking arcs connecting these nodes. Even for one O/D pair, the network will have at least a number of hyperpaths $= 2^n - 1$. While the segment network representation is more concise with the original V nodes, the number of links does not exceed $[|V|(|V| + 1)]/2$. It would imply considerable CPU time while yielding more practical travel choices [19], [78]. In terms of running time, the best-reported CPU time for a hypernetwork representation was 6.59 min using a network of 571 stops and 35 lines [78], whereas [19] reported 0.25 min for a network consisting of 24 nodes and 5 lines only. These running times still put limits on using the presented models as subroutines in the TNDP solution frameworks. Obviously, TNDP solutions algorithms that need several consecutive assignments would counter running time explosion using the newest contributions.

Second, they do not restrict the number of transfers made to reach the destination. According to equilibrium conditions, the user would make the trip if there were capacity, and the time impedance is plausible regardless of any number of transfers. Transfers could be included only as an addition impedance in the cost function. Consequently, these types of models do not allow the planner to control or track the number of transfers on the final TNDP solution. In a survey done by [79] in the United States, about 60% of the respondents of several transit agencies believed that transit users are willing to make only one transfer per trip.

Third, as ordinary assignment problems on road networks, they can be formulated as path-based or link-based models depending on the required transit flow information (i.e., path flow or link flow). Equilibrium models are mainly link-based to achieve relatively quick solutions, which obviates the combinatorial process of complete path enumeration. Unfortunately, link-based techniques do not enable the analyst to track users' trajectories within the network. In contrast, path-based models provide path flow information, enabling the analyst to evaluate the presented TNDP design effect on a specific group of users.

Fourth, the uniqueness of the UE solution further requires special conditions on the assumed cost function and the stops modeling stage. The mapping cost function should be strictly monotonic, which needs to be proved. In addition, cost functions are asymmetric in nature, and therefore, an equivalent optimization problem cannot be formulated. It is usually expressed implicitly, which makes the implementation of a diagonalization algorithm a challenging problem. At the level of transit stops, it assumes that the flow split between attractive services would be according to the nominal frequencies of the services instead of the effective ones. Also, distinguishing users aboard arriving lines at a stop from passengers waiting on the platform at the same stop causes a high complexity at the modeling stage.

Moreover, there is no study has calibrated any of the assignment models with the actual user's flow on a transit network. This makes the real benefit of using a more complicated/sophisticated assignment model questionable. Developing a transit assignment model would still be challenging with an open gate for new contributions.

For further studies, the focus should be given to investigating;

- Much simpler representations of the two models regarding as much as possible how users choose the services to reach their destination.

- The inclusion of new concepts to reduce the time complexity of the solutions algorithms to reach few seconds while tracking users' travel routes.
- Developing a method to stipulate the constraints on the maximal number of transfers.
- The capability of incorporating variable demand without an excessive increase in the computational time complexity.

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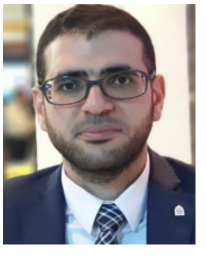

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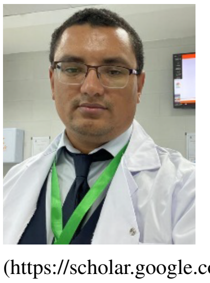

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