Stability Analysis and Synchronization Control of Fractional-Order Inertial Neural Networks With Time-Varying Delay

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ABSTRACT This paper mainly investigates the stability analysis and synchronization control of a fractional-order time-varying delay inertial neural network. Firstly, a time-varying delay inertial neural network model is established, which is easy to implement in engineering applications. Secondly, based on the properties of the Caputo fractional derivative and the proposed lemma, the original inertial system is transferred into conventional system through the proper variable substitution, and a synchronous control strategy for the time varying delay inertial network is then established. In addition, the stability conditions of a class of Caputo fractional-order time-delay inertial neural networks are given. Finally, three simulation examples are given to verify the rationality and effectiveness of the method proposed in this paper.

INDEX TERMS Fractional-order, inertial neural network, time-varying delay, synchronization control, stability analysis.

I. INTRODUCTION

In 1986, Babcock and Westervelt introduced the inductance into the neural network for the first time to reveal the inertia of the system, and constructed a second-order inertial neural network model, which gave birth to the inertial neural network. Based on the structured network model, they analyzed the dynamic behavior of the model such as chaos and bifurcation [1]. Due to the limited transmission speed between neurons, Babcock and Westervelt introduced an inertial neural network with time delay [2]. It is proved that inertial neural networks have a wide range of practical applications in the fields of associative memory, image processing, and signal processing. However, there is a inertial term in the neural network, which leads to the occurrence of very complex dynamic behaviors such as instability, chaos, and bifurcation.

There are many significant findings in the research on integer-order inertial neural networks. Cui *et al.* [3] discussed an inertial neural network with linear proportional inequality, and obtained the global asymptotic stability conditions and

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global robust stability conditions of the system through linear matrix inequality. Rakkiyappan *et al.* [4] studied the periodicity and synchronization of inertial resistive neural networks with time delay by using matrix measure method and Halany inequality technology. In addition, Lakshmanan *et al.* [5] proposed an image encryption algorithm based on a piecewise linear chaotic graph and chaotic inertial neural network. Prakash *et al.* [6] proposed a synchronization criteria for Markovian jumping time delayed bidirectional associative memory inertial neural networks and their applications in secure image communications. Alimi *et al.* [7] investigated the finite-time and fixed-time synchronization problem for a class of inertial neural networks with multi-proportional delays. Zhang *et al.* [8] considered the global exponential dissipation of memristive inertial neural networks with discrete and distributed time-varying delays. Wang *et al.* [9] derived several constrained conditions to ensure the global Lagrange stability of inertial neural networks with discrete and distributed time-varying delays, and gave a global exponential attraction set. Dharani *et al.* [10] explored the sampling data synchronization of coupled inertial neural networks with reaction-diffusion terms and time-varying delays.

Li et al. [11] studied the asymptotic stability and synchronization of a class of time-delayed inertial neural networks without converting the second-order inertial neural network to the first-order differential system through variable substitution. Zhang et al. [12] studied the global asymptotic synchronization of a class of inertial delay neural networks, by using constructed integral inequality and inequality techniques instead of using conventional global exponential and asymptotic synchronization research methods: linear matrix inequality method, matrix measurement strategy and stability theory method. Zhang et al. [13] proposed a class of state-dependent switching neural networks with inertial terms and distributed time delays. Chaouki et al. [14] proposed the driver response synchronization in a finite time and a fixed time of an inertial neural network with time-varying and distributed time-delay. Chen et al. [15] considered the fixed-time synchronization control of inertial memristor-based neural networks with discrete delay. Yotha et al. [16] proposed the delay-dependent passivity analysis issue for uncertain neural networks with discrete interval and distributed time-varying delays.

Fractional calculus is the extension of integers and integer derivatives to any order, and the history of the field can be traced back to around 300 years ago. Fractional differential equations are considered to be a powerful tool for modeling practical problems in biology, chemistry, physics, medicine, economics and other sciences. In view of the hereditary and memory properties of fractional-order calculus, many scholars apply fractional operators to neural networks to establish a fractional neural network model, e.g. [17]–[24]. With the development of theory on fractional-order, the study of synchronization control for fractional-order neural networks (NNS) has received more and more attention and some interesting results have been obtained, such as asymptotical synchronization [25], exponential synchronization [26], finite-time synchronization [27], fixed-time synchronization [28], robust synchronization [29], and adaptive synchronization [30].

At present, many studies mainly focused on fractional-order neural networks with single fractional-order derivative of the states. It is also of significant importance to introduce an inertial term, which is considered as a powerful tool to generate complicated bifurcation behavior and chaos. Therefore, it is meaningful and valuable to investigate the dynamics of fractional-order inertial neural network. Gu et al. [31] investigated the stability and synchronization of Riemann-Liouville fractional-order inertial neural networks with time delays. Li et al. [32] investigated the boundedness and the global Mittag-Leffler synchronization of fractional-order inertial Cohen-Grossberg neural networks with time delays. Ke [33] investigated the stability for a class fractional-order inertial neural networks with time-delay. Zhang et al. [34] studied the synchronization of a Riemann-Liouville-type fractional inertial neural network with a time delay and two inertial terms. Yang et al. [35] investigated the quasi-synchronization problem of fractional order memristor-based inertial neural

networks. However, the above literatures mainly discuss either synchronous control of fractional-order inertial neural network in the sense of Riemann-Liouville or the quasi-synchronization problem of fractional order memristor-based inertial neural networks. It should be point out that the fractional-order systems in the sense of Caputo have greater physical meaning. On the other hand, quasi-synchronization does not ensure that the synchronization error tends to zero.

Motivated by these, the asymptotic synchronization control of a Caputo fractional time-varying delay inertial neural network are investigated in this paper. The main contributions of this paper are summarized as follows.

- The Caputo fractional time-varying delay inertial neural network model is established. This model is easy to implement and practically significant in engineering applications.
- A lemma on the composition properties of Caputo fractional-order derivative and integral are given, which provide critical tools for fractional-order system.
- Based on the properties of the Caputo fractional derivative, the original inertial system is transferred into conventional system through the proper variable substitution.
- A synchronization control strategy for fractional time-varying time-delay inertial neural networks is proposed. In addition, the stability conditions of a class of fractional-order time-varying inertial neural networks with time delay are also given.

The structure of this paper is organized as follows. In the second section, the model formulation and some preliminaries are presented. In third section, some criteria for asymptotic synchronization of the fractional order inertial neural networks with time-varying delay are derived. In addition, the stability conditions of a class of fractional-order delayed inertial neural networks with are also given. In the fourth section, three numerical examples and its simulations are given to illustrate the effectiveness of our theoretical results. Finally, a brief discussion and future research topic are given in the fifth section.

Notations: The symbols are used in the article as follows. Z^+ represents positive integers. $C^m[a, b]$ represents the continuous function set of the m derivative function on the closed interval $[a, b]$. R represents a real number. R^+ represents a positive real number. R^n represents n-dimension vector space.

II. PRELIMINARIES AND PROBLEM FORMULATION
A. PRELIMINARIES
In the research and application of fractional derivatives and integrals, there are three common types of definition: Grunwald-Letnikov, Rieman-Liouville, and Caputo definitions. Among these, the initial system values given by Caputo definitions are the same those of an integer-order system and have greater physical meaning. Therefore, the Caputo fractional differential is used to define the model in this article.

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In this section, we recall some basic defnitions about fractional calculus and introduce some useful lemmas.

Definition 1 ' [36]': The fractional integral of the function $f(t)$ is defined as

$$
{}_{t_0}^{C}D_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1}f(\tau)d\tau, \tag{1}
$$

where $t > t_0$, $\alpha > 0$, and $\Gamma(\cdot)$ is the gamma function.

Definition 2 ' [36]': The fractional derivative of the function $f(t)$ is defined as

$$
{}_{t_0}^{C}D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau, \tag{2}
$$

where $n-1 \le \alpha < n, n \in Z^+, t > t_0$. If $0 < \alpha < 1$, then

$$
{}_{t_0}^{C}D_t^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f'(\tau)}{(t-\tau)^{\alpha}}d\tau. \tag{3}
$$

We will denote ${}_{0}^{C}D_t^{\alpha}x(t)$ with $D_t^{\alpha}x(t)$ for simplicity. Some important properties of the fractional derivative and integral are listed as follows.

Lemma 1 ' [36]': If $x(t)$ is continuous for $t > t_0$ then integration of arbitrary real order defined by Eq.(1) has the following important property.

$$
D_t^{-\alpha}D_t^{-\beta}x(t) = D_t^{-\beta}D_t^{-\alpha}x(t) = D_t^{-\alpha-\beta}x(t). \tag{4}
$$

where $\alpha, \beta \in R^+$.

Lemma 2 ' [36]': Suppose that $x(t) \in C^1[0, T]$, where T is a positive constant and the fractional order $\alpha \in (0, 1]$. then the following equation holds.

$$
D_t^{\alpha}D_t^{-\alpha}x(t) = x(t). \tag{5}
$$

Lemma 3 ' [37]': If $\lambda, \mu \in R$, and $\forall \alpha > 0$, then Eq. (6) holds.

$$
D_t^{\alpha}(\lambda x_1(t) + \mu x_2(t)) = \lambda D_t^{\alpha}x_1(t) + \mu D_t^{\alpha}x_2(t). \tag{6}
$$

Lemma 4 ' [38]': If $x(t) \in C^1[0, T]$, $T > 0$, $\alpha_1, \alpha_2 > 0$, and $\alpha_1 + \alpha_2 \le 1$, then Eq. (7) holds.

$$
D_t^{\alpha_1}D_t^{\alpha_2}x(t) = D_t^{\alpha_1+\alpha_2}x(t). \tag{7}
$$

Lemma 5 ' [38]': if $x(t) \in C^m[0, T]$, $T > 0$, then Eq. (8) holds.

$$
D_t^{\alpha}x(t) = D_t^{\alpha_n} \cdots D_t^{\alpha_2}D_t^{\alpha_1}x(t), t \in [0, T]. \tag{8}
$$

where $\alpha = \sum_{i=1}^{n} \alpha_i$, $\alpha_i \in (0, 1]$, and $m-1 \le \alpha < m \in Z^+$, and there exist $i_k < n$ s.t. $\sum_{j=1}^{i_k} \alpha_j = k$ $(k = 1, 2, \cdots m-1)$

Lemma 6 ' [39]': Suppose $x(t) : [0, +\infty] \to R^n$ is bounded and differentiable. If $0 < \alpha \le 1$, then Eq. (9) holds.

$$
D_t^{\alpha}x^T(t)x(t) \le 2 x^T(t)D_t^{\alpha}x(t). \tag{9}
$$

Lemma 7 ' [40]': If $z_1, z_2 \in R$ and $\rho \in R^+$ is a constant. then Eq. (10) holds.

$$
2z_1 z_2 \le \rho z_1^2 + \rho^{-1}z_2^2. \tag{10}
$$

Lemma 8 ' [41]': Let $x = 0$ be an equilibrium point for $D_t^{\alpha}x(t) = f(x(t))$ or $D_t^{\alpha}x(t) = f(t, x(t))$, and $D \subset R$ be the

region of attraction containing $x = 0$, where $f(t, x)$ is locally Lipschitz to $x(t)$. Let the proposed $V(t, x) : [0, +\infty) \times D \to R$ be a continuously differentiable function such that $V(t, x)|_{(x=0)} = 0$, $V(t, x) > 0$ in $D \setminus \{0\}$ and $\dot{V}(t, x) \le 0$ in D. Then, $x = 0$ is stable. Moreover, if $\dot{V}(t, x) < 0$ in D, then $x = 0$ is asymptotically stable.

On the basis of the above lemma, the following property is given.

Lemma 9: If $\alpha, \beta \in (0, 1], x(t) \in C^1[0, T]$ and $T > 0$, then the following equation holds.

$$
D_t^{\alpha}D_t^{-\beta}x(t) = D_t^{\alpha-\beta}x(t). \tag{11}
$$

Proof: Three cases must be considered.
(i) While $0 < \alpha < \beta \le 1$, according to Lemma 1 and Lemma 2, we have

$$
\begin{aligned}
D_t^{\alpha}D_t^{-\beta}x(t) &= D_t^{\alpha}D_t^{-\alpha-(\beta-\alpha)}x(t) \\
&= D_t^{\alpha}D_t^{-\alpha}D_t^{-(\beta-\alpha)}x(t) \\
&= D_t^{\alpha-\beta}x(t).
\end{aligned}
$$

(ii) While $0 < \beta < \alpha \le 1$, according to Lemma 2 and Lemma 4, we obtain

$$
\begin{aligned}
D_t^{\alpha}D_t^{-\beta}x(t) &= D_t^{\alpha-\beta+\beta}D_t^{-\beta}x(t) \\
&= D_t^{\alpha-\beta}D_t^{\beta}D_t^{-\beta}x(t) \\
&= D_t^{\alpha-\beta}x(t).
\end{aligned}
$$

(iii) While $0 < \beta = \alpha \le 1$, according to Lemma 2, we have

$$
D_t^{\alpha}D_t^{-\beta}x(t) = x(t) = D_t^{\alpha-\beta}x(t).
$$

The proof is completed.

B. PROBLEM FORMULATION

In this section, we will give the mathematical model of the fractional-order inertial neural networks with time-varying delay. The drive system is defined as:

$$
\begin{cases}
D_t^{\alpha}x_i(t) = -a_i D_t^{\beta}x_i(t) - b_i x_i(t) + \sum_{j=1}^{n} c_{ij}f_j(x_j(t)) \\
\quad + \sum_{j=1}^{n} d_{ij}g_j(x_j(t-\tau_j(t))) + I_i(t) \\
x_i(s) = \varphi_i(s) \\
D_t^{\beta}x_i(s) = \psi_i(s), s \in [-\tau_i(t), 0]; \ i = 1, 2 \ldots n, t > 0.
\end{cases} \tag{12}
$$

where $0 < \beta \le 1, \beta < \alpha \le 1+\beta$. n represents the number of neurons of the neural network. $x_i(t)$ represents the state of the ith neuron, and $f_j(\cdot)$ and $g_j(\cdot)$ represents the output of the jth neuron at the times t and $t-\tau_j(t)$, respectively. $a_i > 0, b_i > 0$, c_{ij} and d_{ij} are the connection weights of the neurons. $\tau_j(t)$ is the delay, and $I_i(t)$ represents the disturbance outside the network. $\varphi_i(s)$ and $\psi_i(s)$ are the initial values of the system, which are continuous and bounded function.

Remark 1: When $\alpha = \beta$, the drive system (12) degenerates into a general fractional neural network in the following manner:

$$D_t^\alpha x_i(t) = -\frac{b_i}{a_i + 1} x_i(t) + \frac{1}{a_i + 1} \sum_{j=1}^n c_{ij} f_j(x_j(t))$$

$$+ \frac{1}{a_i + 1} \sum_{j=1}^n d_{ij} g_j(x_j(t - \tau_j(t))) + \frac{1}{a_i + 1} I_i(t). \tag{13}$$

When $\alpha < \beta$, the drive system(12) is transformed into:

$$D_t^\beta x_i(t) = -\frac{1}{a_i} D_t^\alpha x_i(t) - \frac{b_i}{a_i} x_i(t) + \frac{1}{a_i} \sum_{j=1}^n c_{ij} f_j(x_j(t))$$

$$+ \frac{1}{a_i} \sum_{j=1}^n d_{ij} g_j(x_j(t - \tau_j(t))) + \frac{1}{a_i} I_i(t). \tag{14}$$

Therefore, this article focuses only on the case of $\alpha > \beta$. The corresponding response system is given by

$$\begin{cases} D_t^\alpha y_i(t) = -a_i D_t^\beta y_i(t) - b_i y_i(t) + \sum_{j=1}^n c_{ij} f_j(y_j(t)) \\ \quad + \sum_{j=1}^n d_{ij} g_j(y_j(t - \tau_j(t))) + I_i(t) + u_i(t) \\ y_i(s) = \varphi_i'(s) \\ D_t^\beta y_i(s) = \psi_i'(s), s \in [-\tau_i(t), 0]; \ i = 1, 2 \ldots n, t > 0. \end{cases} \tag{15}$$

where $u_i(t)$ is the designed controller, and $\varphi_i'(s)$ and $\psi_i'(s)$ are the initial functions of the system. The meaning of other parameters is consistent with the system(12).

Assumption 1: The activation functions $f_i(\cdot)$ and $g_i(\cdot)$ satisfy the Lipschitz condition, and there are constants l_i, $m_i > 0$, establishing the inequality given in Eq. (16):

$$|f_i(y) - f_i(x)| \le l_i|y - x|$$
$$|g_i(y) - g_i(x)| \le m_i|y - x|, \quad i = 1, 2, \ldots n. \tag{16}$$

Remark 2: In order to ensure the existence and uniqueness of the solutions of the fractional-order time-varying delay inertial neural network, we must first put forward relevant assumptions about the activation function of the considered inertial neural network. Therefore, Assumption 1 is rational, and one can also see the reference [42].

Remark 3: Many previous studies mainly focused on fractional-order neural networks with single fractional-order derivative of the states [17]–[30]. However, it is important to introduce an inertia term, which is considered as a powerful tool to generate complex bifurcation behavior and chaos. Moreover, fractional-order inertial neural network is a more approximate simulation of the dynamical behavior of the neurons. Therefore, it is meaningful and valuable to study the dynamics of fractional-order inertial neural networks. In this paper, we study the stability analysis and synchronization control of a fractional inertial neural network with time-varying delays.

III. MAIN RESULTS

In this section, the synchronization between the drive system (12) and the response system (15) is investigated, and the respective synchronization control strategy is proposed. Firstly, we perform a system equivalent transformation.

Let $p_i(t) = D_t^\beta x_i(t) + x_i(t)$. According to Lemmas 3 and 5, if $0 < \beta \le 1$, and $\beta < \alpha \le 1 + \beta$, we have

$$D_t^{\alpha-\beta} p_i(t) = D_t^{\alpha-\beta} (D_t^\beta x_i(t) + x_i(t))$$
$$= D_t^{\alpha-\beta} D_t^\beta x_i(t) + D_t^{\alpha-\beta} x_i(t)$$
$$= D_t^\alpha x_i(t) + D_t^{\alpha-\beta} x_i(t). \tag{17}$$

Then, the drive system (12) can be transformed into

$$\begin{cases} D_t^{\alpha-\beta} p_i(t) = -a_i p_i(t) - (b_i - a_i) x_i(t) \\ \quad + \sum_{j=1}^n c_{ij} f_j(x_j(t)) + \sum_{j=1}^n d_{ij} g_j(x_j(t - \tau_j(t))) \\ \quad + D_t^{\alpha-\beta} x_i(t) + I_i(t) \\ D_t^\beta x_i(t) = -x_i(t) + p_i(t), \quad i = 1, 2 \ldots n; t > 0. \end{cases} \tag{18}$$

Similarly, let $q_i(t) = D_t^\beta y_i(t) + y_i(t)$. If $0 < \beta \le 1$, and $\beta < \alpha \le 1 + \beta$, then the response system (15) is transformed into:

$$\begin{cases} D_t^{\alpha-\beta} q_i(t) = -a_i q_i(t) - (b_i - a_i) y_i(t) \\ \quad + \sum_{j=1}^n c_{ij} f_j(y_j(t)) + \sum_{j=1}^n d_{ij} g_j(y_j(t - \tau_j(t))) \\ \quad + D_t^{\alpha-\beta} y_i(t) + I_i(t) + u_i(t) \\ D_t^\beta y_i(t) = -y_i(t) + q_i(t), \quad i = 1, 2 \ldots n; t > 0. \end{cases} \tag{19}$$

Denote synchronization error as follows:

$$\begin{cases} e_i(t) = y_i(t) - x_i(t) \\ z_i(t) = q_i(t) - p_i(t), \quad i = 1, 2 \ldots n. \end{cases} \tag{20}$$

If $\lim_{t \to \infty} e(t) = 0$, and $\lim_{t \to \infty} z(t) = 0$, then the drive system (12) or (18) and the response system (15) or (19) will achieve synchronization.

According to the drive system (18) and the response system (19), we can redescribe the error system as follows:

$$\begin{cases} D_t^{\alpha-\beta} z_i(t) = D_t^{\alpha-\beta} q_i(t) - D_t^{\alpha-\beta} p_i(t) \\ \quad = -a_i z_i(t) - (b_i - a_i) e_i(t) + \sum_{j=1}^n c_{ij} \triangle f_j \\ \quad + \sum_{j=1}^n d_{ij} \triangle g_j + D_t^{\alpha-\beta} e_i(t) + u_i(t) \\ D_t^\beta e_i(t) = -e_i(t) + z_i(t), \end{cases} \tag{21}$$

where $\triangle f_j = f_j(y_j(t)) - f_j(x_j(t))$, and $\triangle g_j = g_j(y_j(t - \tau_j(t))) - g_j(x_j(t - \tau_j(t)))$.

A. SYNCHRONIZATION OF FRACTIONAL INERTIAL NEURAL NETWORKS WITH TIME-VARYING DELAY

The synchronization controller $u_i(t)$ is designed as follows:

$$u_i(t) = -\rho_i e_i(t) - k_i z_i(t) - D_t^{\alpha-\beta} e_i(t) \tag{22}$$

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where ρ_i, k_i and γ_i are the design parameters of the controller.

$$\begin{cases} -\omega_i + \gamma_i + \dfrac{|\omega_i + a_i - b_i - \rho_i|}{2} + \displaystyle\sum_{j=1}^{n} \dfrac{|c_{ij}l_i|}{2} < 0 \\ -a_i - k_i + \dfrac{|\omega_i + a_i - b_i - \rho_i|}{2} + \displaystyle\sum_{j=1}^{n} \dfrac{|c_{ij}l_j| + |d_{ij}m_j|}{2} < 0 \\ \displaystyle\sum_{j=1}^{n} \dfrac{|d_{ij}m_i|}{2} - \gamma_i(1 - \dot{\tau}_i(t)) < 0. \end{cases} \tag{23}$$

where $\dot{\tau}_i(t)$ represents the derivative of the delay function $\tau_i(t)$ and satisfies $\dot{\tau}_i(t) < 1$.

Theorem 1: Assumed that Assumption 1 is satisfied. Under the controller (22), the synchronization of the response system (15) or (19) and the drive system (12) or (18) are achieved if the control parameters $\rho_i, k_i, \gamma_i, \omega_i > 0, l_i > 0$, and $m_i > 0$ are selected such that Eq. (23) holds.

Proof: Select a Lyapunov functional as follows:

$$V_i(t) = \frac{1}{2}D_t^{\beta-1}(\omega_i e_i^2(t)) + \frac{1}{2}D_t^{\alpha-\beta-1}z_i^2(t) \\ + \int_{t-\tau_i(t)}^{t} \gamma_i e_i^2(s)ds, \tag{24}$$

where the constant $\omega_i > 0$, $\gamma_i > 0$.

According to Lemma 6 and Lemma 9, calculating the derivative of $V_i(t)$, we have

$$\begin{aligned} \dot{V}_i(t) &= \frac{1}{2}D_t^{\beta}(\omega_i e_i^2(t)) + \frac{1}{2}D_t^{\alpha-\beta}z_i^2(t) + \gamma_i e_i^2(t) \\ &\quad - \gamma_i(1 - \dot{\tau}_i(t))e_i^2(t - \tau_i(t)) \\ &\leq \omega_i e_i(t)D_t^{\beta}e_i(t) + z_i(t)D_t^{\alpha-\beta}z_i(t) + \gamma_i e_i^2(t) \\ &\quad - \gamma_i(1 - \dot{\tau}_i(t))e_i^2(t - \tau_i(t)) \\ &= \omega_i e_i(t)\Big(- e_i(t) + z_i(t)\Big) \\ &\quad + z_i(t)\Big(- a_i z_i(t) - (b_i - a_i)e_i(t) + \sum_{j=1}^{n} c_{ij}\bigtriangleup f_j \\ &\quad + \sum_{j=1}^{n} d_{ij}\bigtriangleup g_j + D_t^{\alpha-\beta}e_i(t) + u_i(t)\Big) + \gamma_i e_i^2(t) \\ &\quad - \gamma_i(1 - \dot{\tau}_i(t))e_i^2(t - \tau_i(t)) \\ &= (-\omega_i + \gamma_i)e_i^2(t) - (a_i + k_i)z_i^2(t) \\ &\quad + (\omega_i + a_i - b_i - \rho_i)e_i(t)z_i(t) \\ &\quad + \sum_{j=1}^{n} c_{ij}z_i(t)\bigtriangleup f_j + \sum_{j=1}^{n} d_{ij}z_i(t)\bigtriangleup g_j \\ &\quad - \gamma_i(1 - \dot{\tau}_i(t))e_i^2(t - \tau_i(t)) \\ &\leq (-\omega_i + \gamma_i)e_i^2(t) - (a_i + k_i)z_i^2(t) \\ &\quad + |\omega_i + a_i - b_i - \rho_i||e_i(t)z_i(t)| \\ &\quad + \sum_{j=1}^{n} |c_{ij}l_j||e_j(t)z_i(t)| \end{aligned}$$

$$+ \sum_{j=1}^{n} |d_{ij}m_j||z_i(t)e_j(t - \tau_j(t))| \\ - \gamma_i(1 - \dot{\tau}_i(t))e_i^2(t - \tau_i(t)). \tag{25}$$

According to Lemma 7, we have

$$\begin{cases} e_i(t)z_i(t) \leq \dfrac{1}{2}e_i^2(t) + \dfrac{1}{2}z_i^2(t) \\ e_j(t)z_i(t) \leq \dfrac{1}{2}e_j^2(t) + \dfrac{1}{2}z_i^2(t) \\ e_j(t - \tau_j(t))z_i(t) \leq \dfrac{1}{2}e_j^2(t - \tau_j(t)) + \dfrac{1}{2}z_i^2(t) \\ e_i(t - \tau_i(t))z_i(t) \leq \dfrac{1}{2}e_i^2(t - \tau_i(t)) + \dfrac{1}{2}z_i^2(t). \end{cases} \tag{26}$$

Substituting Eq. (26) into Eq. (25), we obtain

$$\begin{aligned} \dot{V}_i(t) &\leq -(\omega_i - \gamma_i)e_i^2(t) - (a_i + k_i)z_i^2(t) \\ &\quad + \frac{|\omega_i + a_i - b_i - \rho_i|}{2}(e_i^2(t) + z_i^2(t)) \\ &\quad + \sum_{j=1}^{n} \frac{|c_{ij}l_j|}{2}(e_j^2(t) + z_i^2(t)) \\ &\quad + \sum_{j=1}^{n} \frac{|d_{ij}m_j|}{2}(z_i^2(t) + e_j^2(t - \tau_j(t))) \\ &\quad - \gamma_i(1 - \dot{\tau}_i(t))e_i^2(t - \tau_i(t)) \\ &= \Big(-\omega_i + \gamma_i + \frac{|\omega_i + a_i - b_i - \rho_i|}{2} + \sum_{j=1}^{n} \frac{|c_{ij}l_i|}{2}\Big)e_i^2(t) \\ &\quad + \Big(-a_i - k_i + \frac{|\omega_i + a_i - b_i - \rho_i|}{2} \\ &\quad + \sum_{j=1}^{n} \frac{|c_{ij}l_j| + |d_{ij}m_j|}{2}\Big)z_i^2(t) \\ &\quad + \Big(\sum_{j=1}^{n} \frac{|d_{ij}m_i|}{2} - \gamma_i(1 - \dot{\tau}_i(t))\Big)e_i^2(t - \tau_i(t)). \end{aligned} \tag{27}$$

Select the parameter values fulfilling $\rho_i, k_i, \gamma_i, \omega_i > 0$, $l_i > 0$, and $m_i > 0$ to satisfy Eq. (23). We have

$$\dot{V}_i(t) < 0.$$

According to Lemma 8, it can be seen that the synchronization error $e_i(t)$, $z_i(t)$ asymptotically approaches 0. i.e., the drive system(12) or (18) and the response system(15) or (19) can be synchronized. The proof is completed.

Remark 4: When $\tau_i(t)$ is a constant, the system given in Eq. (12) degenerate into a fractional-order inertial neural network with fixed time delay. Therefore, compared with the reference [31], our method is more versatile.

Remark 5: In this paper, we use the Caputo fractional differential and integral properties and the Lyapunov direct method to construct a Lyapunov function with integral terms. The main advantage is to avoid calculating the fractional derivative of Lyapunov functional to discuss the synchronization stability conditions.

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B. STABILITY ANALYSIS OF FRACTIONAL INERTIAL NEURAL NETWORK WITH TIME-VARYING DELAY

When $0 < \beta < 1$, and $\alpha = 2\beta$, a fractional-order inertial neural network system with time-varying delay can be defined as follows:

$$D_t^{2\beta} x_i(t) = -a_i D_t^{\beta} x_i(t) - b_i x_i(t) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t))$$

$$+ \sum_{j=1}^{n} d_{ij} g_j(x_j(t - \tau_j(t))) + I_i(t),$$

$$i = 1, 2 \ldots n. t > 0. \quad (28)$$

Next, we will discuss the global uniform stability of fractional-order time-delayed inertial neural network described by (28)

Suppose that $x_i(t)$ and $\hat{x}_i(t)$ are two different solutions of the system (28) at different initial values $x_i(s) = \varphi_i(s), D_t^{\beta} x_i(s) = \psi_i(s)$ and $\hat{x}_i(s) = \varphi_i(s), D_t^{\beta} \hat{x}_i(s) = \psi_i(s)$, $s \in [-\tau_i(t), 0]$.

Denote $\check{e}_i(t) = \hat{x}_i(t) - x_i(t)$. Then we have

$$D_t^{2\beta} \check{e}_i(t) = -a_i D_t^{\beta} \check{e}_i(t) - b_i \check{e}_i(t)$$

$$+ \sum_{j=1}^{n} c_{ij} (f_j(\hat{x}_j(t)) - f_j(x_j(t)))$$

$$+ \sum_{j=1}^{n} d_{ij} (g_j(\hat{x}_j(t - \tau_j(t))) - g_j(x_j(t - \tau_j(t)))). \quad (29)$$

Denote $\check{z}_i(t) = D_t^{\beta} \check{e}_i(t) + \check{e}_i(t)$. Then system (29) can be transformed as follows:

$$\begin{cases} D_t^{\beta} \check{z}_i(t) = -(a_i - 1)\check{z}_i(t) - (b_i - a_i + 1)\check{e}_i(t) \\ \quad + \sum_{j=1}^{n} c_{ij} (f_j(\hat{x}_j(t)) - f_j(x_j(t))) \\ \quad + \sum_{j=1}^{n} d_{ij} (g_j(\hat{x}_j(t - \tau_j(t))) - g_j(x_j(t - \tau_j(t)))) \\ D_t^{\beta} \check{e}_i(t) = -\check{e}_i(t) + \check{z}_i(t), \quad i = 1, 2 \ldots n, t > 0. \end{cases} \quad (30)$$

Corollary 1: The system (28) is globally uniformly stable with any different initial value under without controller, if the parameters $\tilde{\omega}_i, \tilde{\gamma}_i, \tilde{l}_i > 0$, and $\tilde{m}_i > 0$ are chosen such that the follow conditions are satisfied.

$$\begin{cases} -\tilde{\omega}_i + \tilde{\gamma}_i + \dfrac{|\tilde{\omega}_i + a_i - b_i - 1|}{2} + \sum_{j=1}^{n} \dfrac{|c_{ij}\tilde{l}_i|}{2} < 0 \\ -a_i + 1 + \dfrac{|\tilde{\omega}_i + a_i - b_i - 1|}{2} + \sum_{j=1}^{n} \dfrac{|c_{ij}\tilde{l}_j| + |d_{ij}\tilde{m}_j|}{2} < 0 \\ \sum_{j=1}^{n} \dfrac{|d_{ij}\tilde{m}_i|}{2} - \tilde{\gamma}_i(1 - \dot{\tau}_i(t)) < 0. \end{cases} \quad (31)$$

where $\dot{\tau}_i(t)$ represents the derivative of the delay function $\tau_i(t)$ and satisfies $\dot{\tau}_i(t) < 1$.

Proof: Select a Lyapunov functional as follows:

$$V_i(t) = \frac{1}{2} D_t^{\beta-1} (\tilde{\omega}_i \check{e}_i^2(t)) + \frac{1}{2} D_t^{\beta-1} \check{z}_i^2(t)$$

$$+ \int_{t-\tau_i(t)}^{t} \tilde{\gamma}_i \check{e}_i^2(s) ds \quad (32)$$

where the constant $\tilde{\omega}_i > 0$, and $\tilde{\gamma}_i > 0$.

According to Lemma 6 and Lemma 9, calculating the derivative of $V_i(t)$, we have

$$\dot{V}_i(t) = \frac{1}{2} D_t^{\beta} (\tilde{\omega}_i \check{e}_i^2(t)) + \frac{1}{2} D_t^{\beta} \check{z}_i^2(t) + \tilde{\gamma}_i \check{e}_i^2(t)$$

$$- \tilde{\gamma}_i(1 - \dot{\tau}_i(t))\check{e}_i^2(t - \tau_i(t))$$

$$\leq \tilde{\omega}_i \check{e}_i(t) D_t^{\beta} \check{e}_i(t) + \check{z}_i(t) D_t^{\beta} \check{z}_i(t) + \tilde{\gamma}_i \check{e}_i^2(t)$$

$$- \tilde{\gamma}_i(1 - \dot{\tau}_i(t))\check{e}_i^2(t - \tau_i(t))$$

$$= \tilde{\omega}_i \check{e}_i(t) \left(-\check{e}_i(t) + \check{z}_i(t) \right) + \check{z}_i(t) \left(-(a_i - 1)\check{z}_i(t) \right.$$

$$- (b_i - a_i + 1)\check{e}_i(t) + \sum_{j=1}^{n} c_{ij} (f_j(\hat{x}_j(t)) - f_j(x_j(t)))$$

$$+ \sum_{j=1}^{n} d_{ij} (g_j(\hat{x}_j(t - \tau_j(t))) - g_j(x_j(t - \tau_j(t)))) \bigg)$$

$$+ \tilde{\gamma}_i \check{e}_i^2(t) - \tilde{\gamma}_i(1 - \dot{\tau}_i(t))\check{e}_i^2(t - \tau_i(t))$$

$$= (-\tilde{\omega}_i + \tilde{\gamma}_i)\check{e}_i^2(t) - (a_i - 1)\check{z}_i^2(t)$$

$$(\tilde{\omega}_i + a_i - b_i - 1)\check{e}_i(t)\check{z}_i(t)$$

$$+ \sum_{j=1}^{n} c_{ij} (f_j(\hat{x}_j(t)) - f_j(x_j(t)))$$

$$+ \sum_{j=1}^{n} d_{ij} (g_j(\hat{x}_j(t - \tau_j(t))) - g_j(x_j(t - \tau_j(t))))$$

$$- \tilde{\gamma}_i(1 - \dot{\tau}_i(t))\check{e}_i^2(t - \tau_i(t))$$

$$\leq (-\tilde{\omega}_i + \tilde{\gamma}_i)\check{e}_i^2(t) - (a_i - 1)\check{z}_i^2(t)$$

$$+ |\tilde{\omega}_i + a_i - b_i - 1||\check{e}_i(t)\check{z}_i(t)|$$

$$+ \sum_{j=1}^{n} |c_{ij}\tilde{l}_j||\check{e}_j(t)\check{z}_i(t)|$$

$$+ \sum_{j=1}^{n} |d_{ij}\tilde{m}_j||\check{z}_i(t)\check{e}_j(t - \tau_j(t))|$$

$$- \tilde{\gamma}_i(1 - \dot{\tau}_i(t))\check{e}_i^2(t - \tau_i(t)). \quad (33)$$

Similar to the proof method of Theorem 1, we have

$$\dot{V}_i(t) \leq \left(-\tilde{\omega}_i + \tilde{\gamma}_i + \frac{|\tilde{\omega}_i + a_i - b_i - 1|}{2} + \sum_{j=1}^{n} \frac{|c_{ij}\tilde{l}_i|}{2} \right) e_i^2(t)$$

$$+ \left(-a_i + 1 + \frac{|\tilde{\omega}_i + a_i - b_i - 1|}{2} \right.$$

$$+ \sum_{j=1}^{n} \frac{|c_{ij}\tilde{l}_j| + |d_{ij}\tilde{m}_j|}{2} \bigg) z_i^2(t)$$

$$+ \left(\sum_{j=1}^{n} \frac{|d_{ij}\tilde{m}_i|}{2} - \tilde{\gamma}_i(1 - \dot{\tau}_i(t)) \right) e_i^2(t - \tau_i(t)). \quad (34)$$

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Selected the values of the parameters $\tilde{\gamma}_i > 0$, $\tilde{\omega}_i > 0$, $\tilde{l}_i > 0$, and $\tilde{m}_i > 0$ to satisfy Eq. (31).

We have

$$\dot{V}_i(t) < 0.$$

According to Lemma 8, it follows that the system synchronization error $\check{e}_i(t)$, $\check{z}_i(t)$ asymptotically diminishes to 0. i.e., the system (28) is globally consistent and stable. The proof is completed.

Remark 6: Although $\alpha = 2\beta$ is included in the Theorem 1, it should be point out that a controller needs to be designed in Theorem 1, while Corollary 1 need not.

Remark 7: When $\alpha = 2$ and $\beta = 1$, the fractional-order inertial neural network will be simplified to an integer-order inertial neural network. Therefore, the integer-order inertial neural network can be regarded as a special case of the fractional-order inertial neural network. Theorem 1 and Corollary 1 proposed in this paper can be used to solve such problems with high versatility.

IV. NUMERICAL SIMULATIONS

In this section, we present three numerical examples to verify the validity of the results of Theorem 1 and Corollary 1.

Since most fractional differential equations do not have analytical solutions, so approximation and numerical techniques must be used. a numerical algorithm is proposed for solving fractional-order differential equations in [43]. Actually, this scheme is the generalization of Adams-Bashforth-Moulton Method. Consider the Caputo fractional-order differential equation as follows.

$$\begin{cases} D_t^\alpha x_i(t) = f(t, x(t)), & 0 \le t \le T \\ x^{(k)}(0) = y_0^{(k)}, & k = 0, 1, 2, \ldots, m-1. \end{cases} \quad (35)$$

The Eq.(35) is equivalent to the following Volterra integral equation.

$$x(t) = \sum_{k=0}^{\lceil \alpha \rceil - 1} x_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds, \quad (36)$$

where $\lceil \alpha \rceil - 1$ denote the first integer which is no less than α. Set $h = \frac{T}{N}$, $t_n = nh$, $n = 0, 1, 2, \ldots, N \in Z^+$, then Eq.(36) can be written as

$$x_h(t_{n+1}) = \sum_{k=0}^{\lceil \alpha \rceil - 1} x_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, x_h^p(t_{n+1}))$$
$$+ \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, x_h^p(t_j)) \quad (37)$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(N+1)^\alpha, & j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} \\ \quad - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n, \\ 1, & j = n+1. \end{cases} \quad (38)$$

$$x_h^p(t_{n+1}) = \sum_{k=0}^{\lceil \alpha \rceil - 1} x_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, x_h(t_j)), \quad (39)$$

where

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}((n+1-j)^\alpha - (n-j)^\alpha). \quad (40)$$

The estimation error is $e = \max_{j=0,1,2,\ldots,N} |x(t_j) - x_h(t_j)| = O(h^p)$, where $p = \min(2, 1+\alpha)$. Based on the above method, we can get discretization of fractional-order differential equations and numerical solutions.

Example 1: Consider a two-dimensional fractional-order inertial neural network system with time-varying delay.

The drive system is as follows:

$$D_t^\alpha x_i(t) = -a_i D_t^\beta x_i(t) - b_i x_i(t) + \sum_{j=1}^{2} c_{ij} f_j(x_j(t))$$
$$+ \sum_{j=1}^{2} d_{ij} g_j(x_j(t - \tau_j(t))) + I_i(t), \quad i = 1, 2. t > 0. \quad (41)$$

The response system is as follows.

$$D_t^\alpha y_i(t) = -a_i D_t^\beta y_i(t) - b_i y_i(t) + \sum_{j=1}^{2} c_{ij} f_j(y_j(t))$$
$$+ \sum_{j=1}^{2} d_{ij} g_j(y_j(t - \tau_j(t))) + I_i(t)$$
$$+ u_i(t), i = 1, 2. t > 0. \quad (42)$$

where $f_j(\cdot) = tanh(\cdot)$, and $g_j(\cdot) = sin(\cdot)$.

The system parameters are set as follows.

$$\begin{cases} a_1 = 1.1, a_2 = 0.8; b_1 = 1, b_2 = 1.7 \\ c_{11} = 3, c_{12} = 3, c_{21} = -0.9, c_{22} = -0.9 \\ d_{11} = 3, d_{12} = -1, d_{21} = 1, d_{22} = 2 \\ \alpha = 1.8, \beta = 0.95; I_1 = 0.1, I_2 = 0.2 \end{cases}$$

The initial value of the system and the time-varying delay $\tau_i(t)$ are set as follows:

$$\begin{cases} x_1(s) = 5, x_2(s) = -0.3 \\ p_1(s) = 0.4, p_2(s) = 0.7, s \in [-\tau, 0] \\ y_1(s) = -1.1, y_2(s) = 1 \\ q_1(s) = 1.1, q_2(s) = 2, s \in [-\tau, 0] \\ \tau_j(t) = e^t/(1+e^t). \end{cases}$$

The nonnegative constants l_i and m_i are set as follows, satisfying Assumption 1.

$$l_1 = l_2 = 1, m_1 = m_2 = 1.$$

The controller design is as follows:

$$u_i(t) = -\rho_i e_i(t) - k_i z_i(t) - D_t^{\alpha-\beta} e_i(t), \quad i = 1, 2. \quad (43)$$
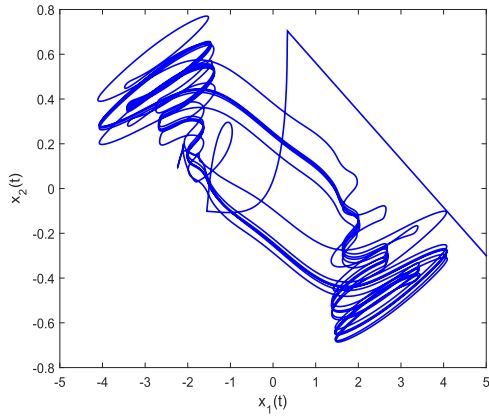
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FIGURE 1. Phase diagram of the drive system(41).
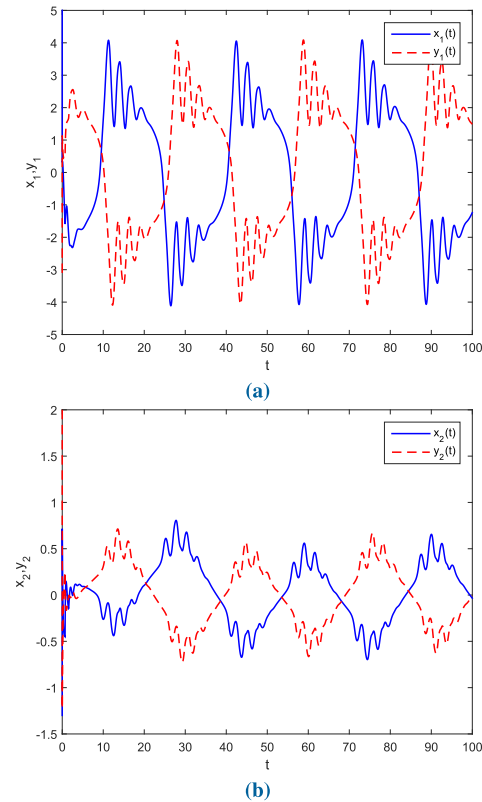
FIGURE 2. Trajectories of *x* and *y* without a controller.

FIGURE 3. Trajectories of *x* and *y* with the controller(43).
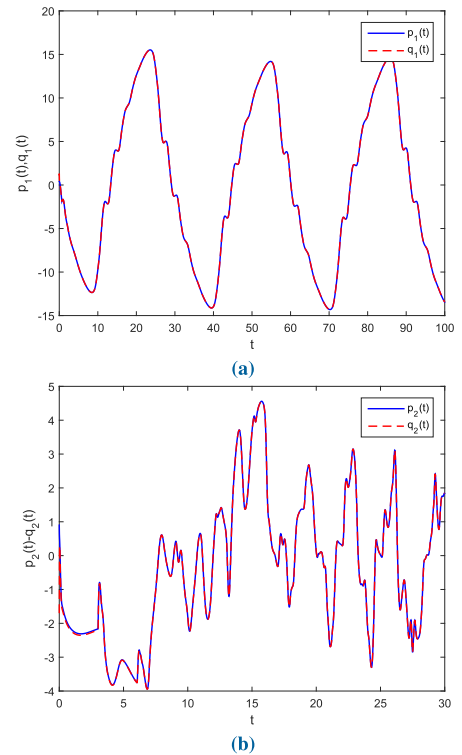
FIGURE 4. Trajectories of *p* and *q* with the controller(43).

The controller parameters ρ_i, k_i, ω_i, and γ_i are chosen as follows:

$$\begin{cases} \rho_1 = 4, \rho_2 = 4; k_1 = 12, k_2 = 10 \\ \omega_1 = 22, \omega_2 = 22; \gamma_1 = 8, \gamma_2 = 12 \end{cases}$$

Obviously, this satisfies the condition of Eq. (23). Thus, according to Theorem 1, the drive system(41) and the response system(42) can be synchronized. Figure 1 shows the phase diagram of the drive system(41), which indicates that the system is chaotic. Figure 2 shows that the drive system and the response system are not synchronized without a controller. Figure 3 and 4 show that the drive system(41) and the response system(42) achieve synchronization under the controller(43)quickly. Figure 5 presents the evolution of
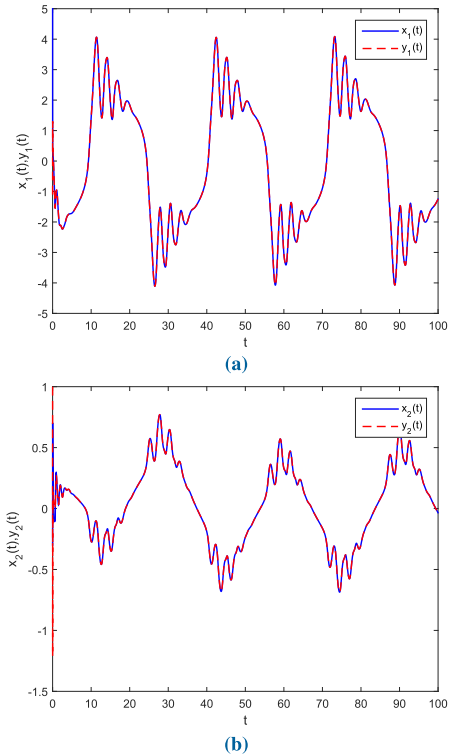
the error system. Figure 3, 4, and 5 show that the simulations are consistent with Theorem 1.

In order to show that our method is more effective and more accurate, we have performed a comparison with the results given in [35]. Figure 6 shows the states trajectories
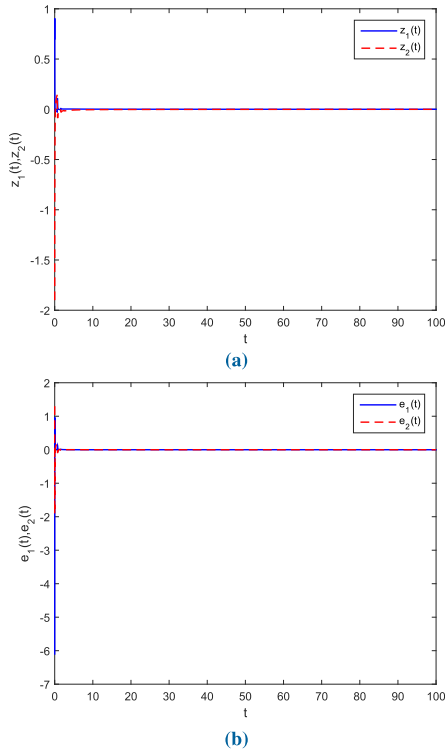
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FIGURE 5. Evolution of the synchronization error of *z* and *e* with the controller(43).

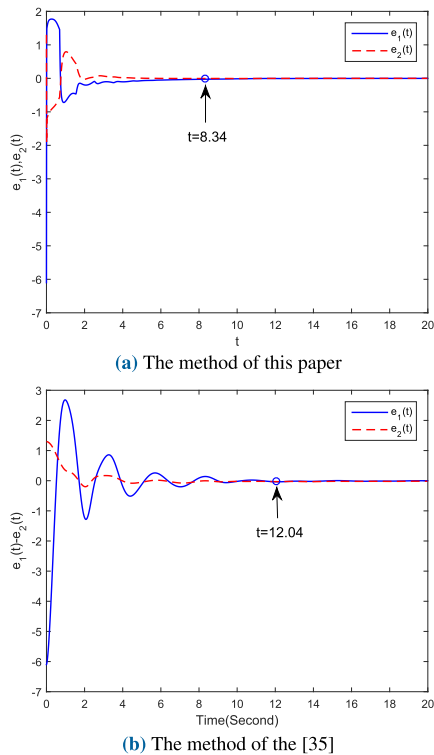**(a)** The method of this paper

(b) The method of the [35]

FIGURE 6. Evolution of the synchronization error of this paper and the [35].

of the error system under the control strategy given in [35] and in this paper. From Figure 6, we can see that the control strategy proposed in our paper makes the system achieve faster synchronization time and less overshoot.
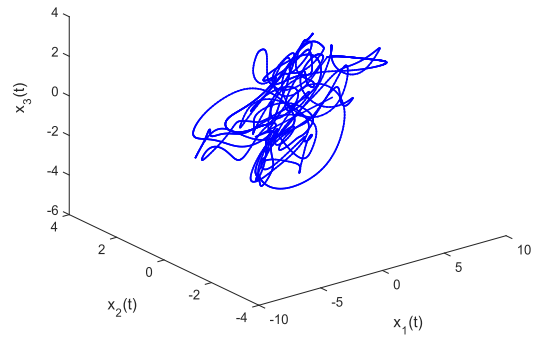
FIGURE 7. Phase diagram of the drive system(44).

Example 2: Consider a three-dimensional fractional-order inertial neural network system with time-varying delay.

The drive system is as follows:

$$D_t^\alpha x_i(t) = -a_i D_t^\beta x_i(t) - b_i x_i(t) + \sum_{j=1}^{2} c_{ij} f_j(x_j(t))$$

$$+ \sum_{j=1}^{2} d_{ij} g_j(x_j(t - \tau_j(t))) + I_i(t),$$

$$i = 1, 2, 3. \ t > 0. \quad (44)$$

The response system is as follows:

$$D_t^\alpha y_i(t) = -a_i D_t^\beta y_i(t) - b_i y_i(t) + \sum_{j=1}^{2} c_{ij} f_j(y_j(t))$$

$$+ \sum_{j=1}^{2} d_{ij} g_j(y_j(t - \tau_j(t))) + I_i(t)$$

$$+ u_i(t), \quad i = 1, 2, 3, \ t > 0. \quad (45)$$

where $f_j(\cdot) = tanh(\cdot)$, and $g_j(\cdot) = sin(\cdot)$.

The system parameters are set as follows:

$$\begin{cases} a_1 = 2, a_2 = 3.8, a_3 = 2.1 \\ b_1 = 2, b_2 = 3.9, b_3 = 2.7 \\ I_1 = 0.1, I_2 = 0.2, I_3 = 0.1 \\ \alpha = 1.7, \beta = 0.9 \end{cases}$$

$$C = (c_{ij})_{3\times3} = \begin{bmatrix} 11 & -2.5 & 4.2 \\ 3.9 & 3 & -3.2 \\ -6.1 & 8 & 3.3 \end{bmatrix},$$

$$D = (d_{ij})_{3\times3} = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 3 & -3.9 \\ 8 & -2 & 12 \end{bmatrix}$$

The initial value of the system and the fixed time delay $\tau_i(t)$ are set as follows:

$$\begin{cases} x_1(s) = 1.1, x_2(s) = -1.3, x_3(s) = 1 \\ p_1(s) = -0.8, p_2(s) = 0.9, p_3(s) = -1.9, s \in [-\tau, 0] \\ y_1(s) = 2.1, y_2(s) = -1.5, y_2(s) = 2 \\ q_1(s) = 0.4, q_2(s) = -1.7, q_3(s) = -0.9, s \in [-\tau, 0] \\ \tau_j(t) = 3. \end{cases}$$
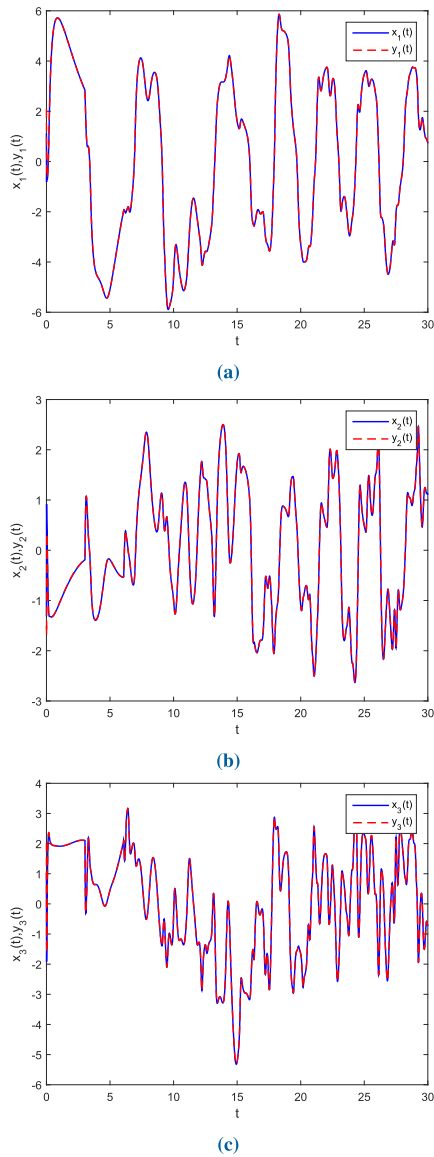
FIGURE 8. Trajectories of *x* and *y* with the controller(46).

FIGURE 9. Trajectories of *p* and *q* with the controller(46).

The nonnegative constants l_i and m_i are set as follows, satisfying Assumption 1:

$$l_1 = l_2 = l_3 = 1, m_1 = m_2 = m_3 = 1.$$

The design controller is as follows:

$$u_i(t) = -\rho_i e_i(t) - k_i z_i(t) - D_t^{\alpha-\beta} e_i(t), \quad i = 1, 2, 3. \quad (46)$$

The controller parameters are set as follows:

$$\begin{cases} \rho_1 = 18, \rho_2 = 12, \rho_3 = 15; k_1 = 12, k_2 = 10, k_3 = 8 \\ \omega_1 = 20, \omega_2 = 25, \omega_3 = 35; \gamma_1 = 5, \gamma_2 = 6, \gamma_3 = 15 \end{cases}$$

Obviously, this satisfies the condition of Eq. (23). Thus, according to Theorem 1, the drive system (44) and the response system (45) can be synchronized. Figure 7 shows a phase diagram of the drive system (44), which indicates this system is chaotic. Figure 8 and 9 demonstrate that the drive system (44) and the response system (45) achieved
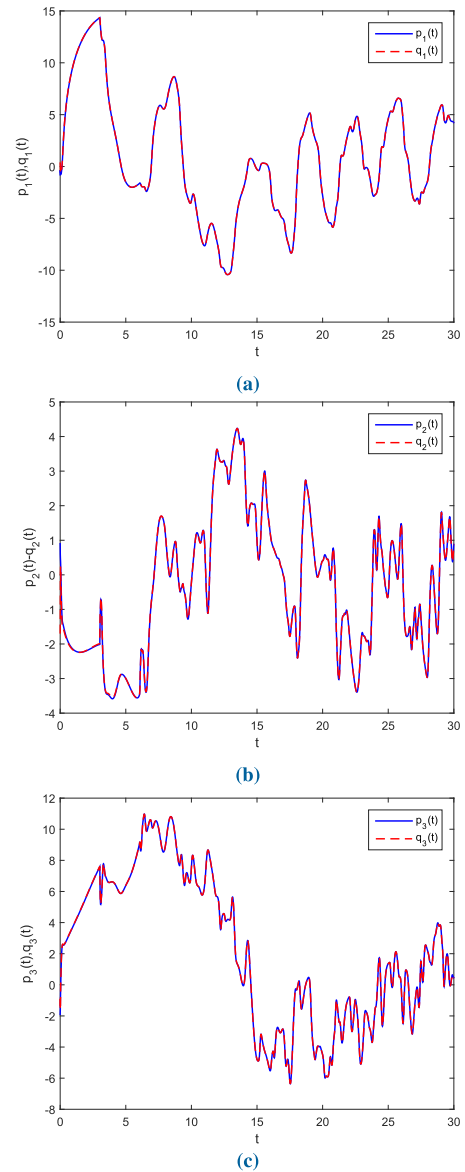
synchronization under the controller (46) quickly. Figure 10 presents the evolution of the error system. Figure 11 shows the states trajectories of the error system under the control strategy given in [35] and in this paper.

Similarly, we compare the results of this paper with the results in [35], and it is not difficult to find that the control strategy proposed in our paper makes the system achieve faster synchronization time and less overshoot.

Remark 8: Compared with the results given in [35], the control strategy proposed in this paper makes the system achieve faster synchronization time and less overshoot. On the other hand, from a theoretical analysis point of view, quasi-synchronization discussed in [35], while asymptotic synchronization is studied in our paper, which means that there may exist errors in [35]. In addition, the delays are constant in [35], while the delays are time-varying in this paper.
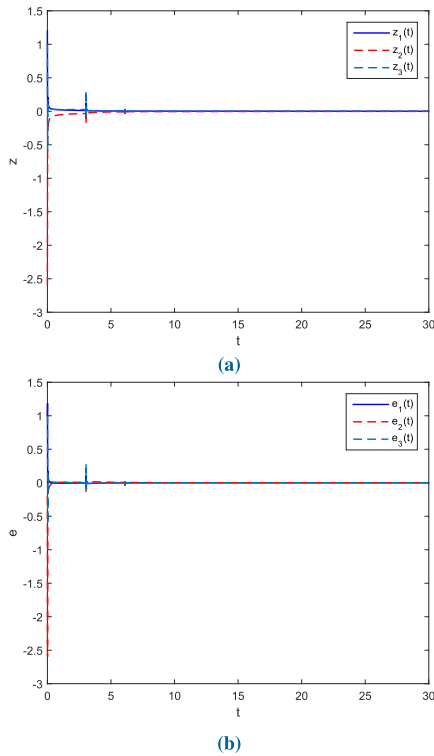
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FIGURE 10. Evolution of the synchronization error of z and e with the controller(46).
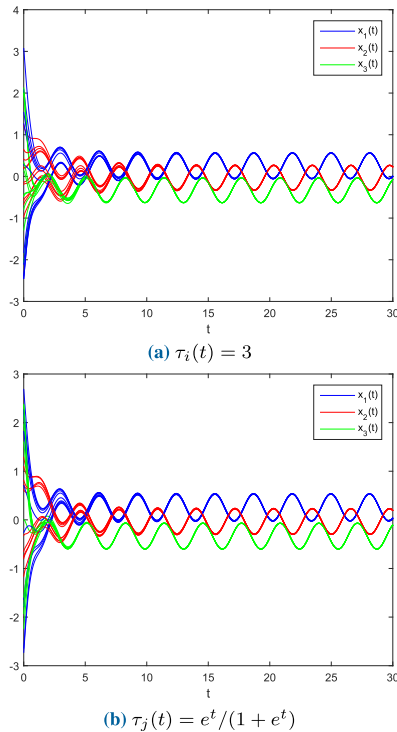

FIGURE 11. Evolution of the synchronization error of this paper and the [35].

Example 3: Consider a three-dimensional fractional-order inertial neural network system with time-varying delay:

$$D_t^{2\beta} x_i(t) = -a_i D_t^\beta x_i(t) - b_i x_i(t) + \sum_{j=1}^n c_{ij} f_j(x_j(t))$$
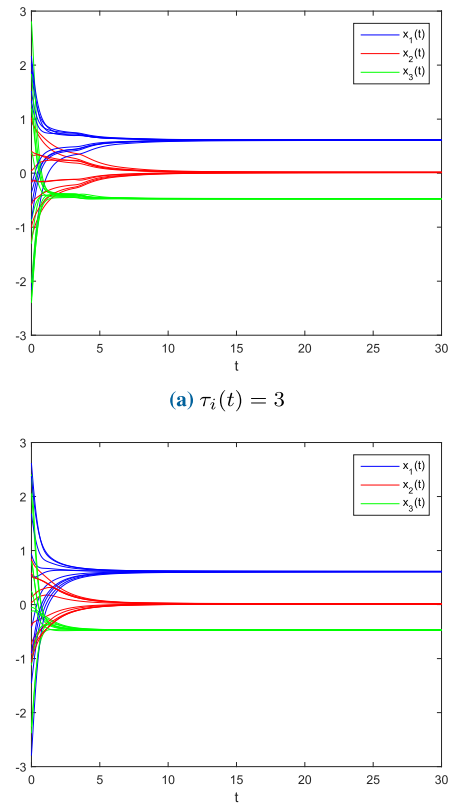

FIGURE 12. Uniform stability of the system (47) with fixed external disturbance and different initial values.

$$+ \sum_{j=1}^n d_{ij} g_j(x_j(t - \tau_j(t))) + I_i(t),$$
$$i = 1, 2, 3. t > 0. \tag{47}$$

where $f_j(\cdot) = tanh(\cdot)$, and $g_j(\cdot) = sin(\cdot)$.

The system parameters are set as follows:

$$\begin{cases} \beta = 0.95 \\ a_1 = 8, a_2 = 9, a_3 = 7 \\ b_1 = 12, b_2 = 10, b_3 = 14 \\ C = (c_{ij})_{3\times3} = \begin{bmatrix} 1 & -1.5 & -2 \\ -1.59 & 2 & -0.5 \\ -2 & 0.5 & -1 \end{bmatrix}, \\ D = (d_{ij})_{3\times3} = \begin{bmatrix} 1 & -0.2 & 0.3 \\ -1 & 0.4 & -0.6 \\ -1 & -0.6 & 0.9. \end{bmatrix} \end{cases}$$

Then, we randomly generate 10 sets of initial system values, with the generating function defined as follows:

$$\begin{cases} x_1(s) = 3.1 * (-1)^i * rand(\cdot), s \in [-\tau, 0], i = 1, 2 \dots 10. \\ x_2(s) = -1.3 * (-1)^i * rand(\cdot), s \in [-\tau, 0], i = 1, 2 \dots 10. \\ x_3(s) = 3 * (-1)^i * rand(\cdot), s \in [-\tau, 0], i = 1, 2 \dots 10. \end{cases}$$

The nonnegative constants l_i and m_i are set as follows, satisfying Assumption 1:

$$l_1 = l_2 = l_3 = 1, m_1 = m_2 = m_3 = 1$$

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(a) $\tau_i(t) = 3$

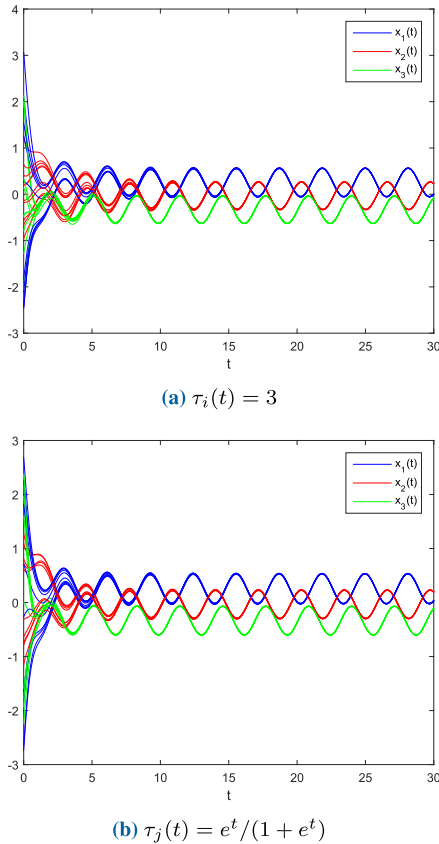**(b)** $\tau_j(t) = e^t/(1 + e^t)$

FIGURE 13. Uniform stability of the system (47) with time-varying external disturbance and different initial values.

The values of $\tau_i(t)$ are selected as follows:

$$\tau_i(t) = 3 \tag{48}$$

or

$$\tau_i(t) = e^t/(1 + e^t). \tag{49}$$

The values of I_i are selected as follows:

$$I_1 = 5.5, I_2 = 1, I_3 = -5 \tag{50}$$

or

$$\begin{cases} I_1 = -5.5sin(2t) + 2 \\ I_2 = 5sin(2t) \\ I_3 = -4.5cos(2t) - 4 \end{cases} \tag{51}$$

The parameters ω_i and γ_i are selected as follows:

$$\begin{cases} \omega_1 = 5.5, \omega_2 = 8, \omega_3 = 8 \\ \gamma_1 = 3, \gamma_2 = 1.5, \gamma_3 = 2 \end{cases}$$

Obviously, this satisfies the condition of Eq. (31). Figure 12 shows the uniform stability evolution process of the system (47) under fixed external disturbances and different initial values with different time delays. Figure 13 shows the uniform stability evolution process of the system (47) given time-varying external disturbances, different initial values, and different time delays. Numerical simulations demonstrate that the fractional inertial neural network with time

delay is uniformly stable, which satisfies the conditions of Corollary 1.

V. CONCLUSION

Stability analysis and synchronization control of a class of fractional-order inertial neural network with time-varying delay is investigated in this paper. The original inertial neural network system is transformed into a conventional system through variable substitution.Based on the Lyapunov's direct method, by constructing a simple synchronous controller and choosing a novel Lyapunov function, some sufficient conditions which are easy to verify are obtained to ensure that the fractional-order inertial neural networks with time-varying delay synchronization achieve synchronization. Finally, three numerical examples are given to illustrate the feasibility and effectiveness of the method proposed in this paper. In the future, we will discuss the practical application of fractional-order inertial neural network and the more general model of fractional-order inertial neural network.

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