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Sets of Wide-Gap Frequency-Hopping Sequence With Optimal Maximum Hamming Correlation

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ABSTRACT In application, frequency-hopping (FH) communication system often suffers various interferences, such as single frequency narrow-band interference, partial band blocking interference and tracking interference, and so on. For all that, by using optimal Wide-Gap (WG) FH sequences, FH communication system can significantly improve the anti-interference performances. In this paper, the relations between WG FH sequence theoretic bounds published in the journal IEEE Access (Peihua Li et al., 2019) are first made clear, and then five types of generalized methods are presented to design new classes of WG FH sequence sets. It is shown that all designed WG FH sequence sets are optimal according to the bound derived by Peihua Li *et al.* And by selecting appropriate original sequence sets, many of the optimal WG FH sequence sets can be obtained by our methods. Most importantly, these WG FH sequence sets have new parameters that are not covered in the literature.

INDEX TERMS Frequency-hopping sequence, hamming correlations, wide-gap, frequency-hopping communication, electromagnetic interference.

I. INTRODUCTION

There exist usually all kinds of interferences in communication system. In order to reduce or eliminate the interferences, and achieve reliable and confidential communication, frequency-hopping (FH) communication emerges as the times require [1], [2]. FH communication is realized by the hopping of carrier frequencies, the hopping rule of which is expressed by FH sequence. In FH communication system, each user is given an FH sequence, on the basis of which, each sender transmits a message along with the switching frequencies in every time slot, and the corresponding receiver receives the signals under the control of the same FH sequence. There exists signal interference measured by the so-called Hamming autocorrelation property of FH sequence if only one FH sequence is employed by all users and there exists another kind of signal interference measured by the so-called Hamming crosscorrelation property of FH sequences otherwise. In a word, Hamming correlations (HCs) influence the performances of FH communication system, such as synchronization, anti-interference, multiple access networking, and so on. Thus, it is very desired to design FH sequence with low HCs. But normally, all the parameters of FH sequence subject to mathematical inequality (the so-called theoretic bound). Many classes of optimal FH sequences according to the corresponding theoretic bound have been designed so far in the literature [3]–[10].

However, the interferences are upgrading and are appearing the properties of dynamic and changeability in communication system nowadays. It is demanded that the FH sequences not only have optimal HC properties but also have the property of Wide-Gap (WG), that is, the gap between any two frequencies adjacent to each other is greater than the given positive integer. The use of Wide-Gap FH sequences (WG FH sequences) is beneficial to resist the enemy's intentional interferences in the complicated electromagnetic

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interference conditions, such as single frequency narrowband interference, partial band blocking interference and tracking interference. Many methods to design WG FH sequences have so far been presented in the literature. For example, removing intermediate frequency bands method [11], dual frequency bands method [12], random translation substitution method and random uniform transfer substitution method [13], [14], designs of intelligent WG FH sequences [15], [16], designs of WG FH sequences based on prime number [17], constructions based on chaos theory [18], [19], constructions over the finite fields [20]–[23], and so on. In addition, in order to evaluate the performances of WG FH sequences, the WG FH sequence theoretic bounds, including the bounds [20], [21] when the maximum periodic HC is equal to 1 and the bounds [25], [26] when the maximum periodic HC is greater than 0, have been also put forwarded. Although there have existed many known designs of WG FH sequences, but until now, there have been very few WG FH sequences with optimal HCs according to the corresponding theoretic bounds in the literature [20]–[24].

To instruct the designs of optimal WG FH sequences well, one objective of this paper is to derive the relations between the WG FH sequence theoretic bounds [25]. The other objective is to present generalized methods to design new WG FH sequence sets with optimal maximum periodic HC. The rest of this paper is organized as follows. Section 2 gives the terms and definitions, studies the relations between WG FH sequence theoretical bounds; Section 3 gives the generalized methods, and bases on which to design WG FH sequence sets; and Section 4 gives the summary of full paper.

II. PRELIMINARIES

We always use the following notations in this paper:

- $(L_u, N_u, q_u, \mathcal{M}(u))$: An FH sequence set u of N_u FH sequences of length L_u over a frequency set of size q_u, with the maximum periodic Hamming correlation (PHC) $\mathcal{M}(u)$.
- $(L_w, N_w, q_w, d_w, \mathcal{M}(w))$: A WG FH sequence set w of N_w FH sequences of length L_w over a frequency set of size q_w, with the maximum PHC $\mathcal{M}(w)$ and with the minimum frequency gap $d_w > 1$.
- $(L_v, q_v, d_v, \mathcal{M}(v))$: A WG FH sequence v of length L_v over a frequency set of size q_v, with the maximum PHC $\mathcal{M}(v)$ and with the minimum frequency gap $d_v > 1$.
- $\mathcal{M}(X, \tau)$: The maximum PHC of FH sequence set X at the relative delay τ.
- gcd: The greatest common divisor.
- lcm: The least common multiple.
- $\otimes$: The Cartesian Product.
- $\langle x \rangle_L$: The least positive integer of x modulo L.
- $\lceil x \rceil$: The least integer greater than or equal to x.
- $\lfloor x \rfloor$: The greatest integer less than or equal to x.
- $|x|$: The absolute value of x.
- $\mathbb{F}_i$, $0 \le i \le k-1$: The set of $\{0, 1, \ldots, p_i - 1\}$.

A. ON THE MAXIMUM PHC

Let $s = \{s_i = (s_0^i, s_1^i, \ldots, s_{N-1}^i) | i = 0, 1, \ldots, M-1\}$ be any FH sequence set $(N, M, q, \mathcal{M}(s))$. For any two FH sequences s_i and s_j in s, at the relative delay τ, $1 \le \tau < N$ if $s_i = s_j$ and $0 \le \tau < N$ if $s_i \ne s_j$, the PHC of s_i and s_j is defined as follows:

$$H_{s_i,s_j}(\tau) = \sum_{k=0}^{N-1} h(s_k^i, s_{\langle k+\tau \rangle_N}^j) \tag{1}$$

where $h(s_k^i, s_{\langle k+\tau \rangle_N}^j) = 1$ if $s_k^i = s_{\langle k+\tau \rangle_N}^j$, $h(s_k^i, s_{\langle k+\tau \rangle_N}^j) = 0$ otherwise.

For any FH sequence set s, the maximum periodic Hamming autocorrelation $\mathcal{M}_a(s)$, the maximum periodic Hamming crosscorrelation $\mathcal{M}_c(s)$ and the maximum PHC $\mathcal{M}(s)$ are defined as follows, respectively:

$$\begin{aligned} \mathcal{M}_a(s) &= \max_{1 \le \tau \le N-1} \{H_{s_i,s_i}(\tau) | \forall s_i \in s\}, \\ \mathcal{M}_c(s) &= \max_{0 \le \tau \le N-1} \{H_{s_i,s_j}(\tau) | \forall s_i \ne s_j \in s\}, \\ \mathcal{M}(s) &= \max\{\mathcal{M}_a(s), \mathcal{M}_c(s)\}. \end{aligned}$$

The maximum PHC $\overline{\mathcal{M}}(s)$ under the relative delay $\tau \ne 0$ is defined by

$$\begin{aligned} \overline{\mathcal{M}}(s) = \max\{ &\max_{1 \le \tau \le N-1} \{H_{s_i,s_i}(\tau) | \forall s_i \in s\}, \\ &\max_{1 \le \tau \le N-1} \{H_{s_i,s_j}(\tau) | \forall s_i \ne s_j \in s\}\}. \end{aligned}$$

B. ON THE WG FH SEQUENCES

Definition 1: Assume $\mathcal{X} = (x_0, x_1, \ldots, x_{N_{\mathcal{X}}-1})$ *is any single FH sequence. Let the maximum frequency be* f_{max} *and the minimum frequency be* f_{min}*. From the broad sense,* $\mathcal{X}$ *is called special WG FH sequence if the minimum frequency gap* $d_{min} \ge 2$*, namely,*

$$\begin{aligned} \min\{|x_{k+1} - x_k|, f_{max} - f_{min} + 1 - |x_{k+1} - x_k|\} \\ = d_{min} \ge 2 \end{aligned} \tag{2}$$

for $k = 0, 1, \ldots, N_{\mathcal{X}} - 1$*. And* $\mathcal{X}$ *is called general WG FH sequence, if the minimum frequency gap* $d_{min} \ge 2$*, namely,*

$$\min\{|x_{k+1} - x_k|\} = d_{min} \ge 2 \tag{3}$$

for $k = 0, 1, \ldots, N_{\mathcal{X}} - 1$.

Definition 2: For any FH sequence set s, s is called special WG FH sequence set if any FH sequence of which is special WG FH sequence, and is called general WG FH sequence set if any FH sequence of which is general WG FH sequence.

For any FH sequence set s denoted as $(N, M, q, \mathcal{M}(s))$, besides HCs, the minimum frequency gap d_{min} is another important index used to evaluate the performance of anti-interference.

$$d_{min} = \min_{0 \le i \le M-1} \{d_{i,min}\}$$

where $d_{i,min}$ denotes the minimum frequency gap of any FH sequence s_i in s.

Lemma 1: Any WG FH sequence is also general WG FH sequence if which is special WG FH sequence, but the converse may not be true.

Lemma 2: For any WG FH sequence set s, s is also general WG FH sequence set if s is special WG FH sequence set, but the converse isn't necessarily true.

For the maximum PHC, Li et al [25] obtained the following theoretical bounds of WG FH sequence set in 2019.

Lemma 3: Let $(N, M, q, d, \mathcal{M}(s))$ be any WG FH sequence set s over $\mathbb{F}$, $I = \lfloor \frac{MN}{q} \rfloor$. Then

$$\mathcal{M}(s) \geq \left\lceil \frac{(MN-q)N}{(MN-3)q} \right\rceil \tag{4}$$

and

$$\mathcal{M}(s) \geq \left\lceil \frac{2IMN-(I+1)Iq}{(MN-3)M} \right\rceil. \tag{5}$$

There exist the relations between (4) and (5) as follows:

Theorem 1: Assume $(N, M, q, d, \mathcal{M}(s))$ is any WG FH sequence set s over $\mathbb{F}$. Let $MN = Iq + J$, $0 \leq J \leq q-1$. Then, (4) and (5) are identical, that is,

$$\mathcal{M}(s) \geq \left\lceil \frac{(MN-q)N}{(MN-3)q} \right\rceil = \left\lceil \frac{2IMN-(I+1)Iq}{(MN-3)M} \right\rceil$$

for $J = 0$. (5) is tighter than (4) in some cases when $J \neq 0$ otherwise.

Proof: Let $\lambda_0 = \frac{(MN-q)N}{(MN-3)q}$ and $N = \mathcal{M}(s)q + r$. One has

$$\lceil\lambda_0\rceil = \left\lceil \mathcal{M}(s) - (\frac{(q-3)N}{(MN-3)q} - \frac{r}{q}) \right\rceil$$
$$= \left\lceil \mathcal{M}(s) - \frac{(q-3)N-(MN-3)r}{(MN-3)q} \right\rceil.$$

Let $A = (MN-3)r$, $B = (q-3)N$, $C = (MN-3)(r+q)$ and $D = (MN-3)(r+2q)$. There exist four cases:

Case 1: We obtain $\frac{(q-3)N-(MN-3)r}{(MN-3)q} < 0$ when $(MN-3)r > (q-3)N$ i.e. $A > B$. Therefore

$$\lceil\lambda_0\rceil \geq \mathcal{M}(s) + 1.$$

Case 2: $0 \leq \frac{(q-3)N-(MN-3)r}{(MN-3)q} < 1$ when $(MN-3)r \leq (q-3)N < (MN-3)(r+q)$ i.e. $A \leq B < C$. Thus

$$\lceil\lambda_0\rceil = \mathcal{M}(s).$$

Case 3: $1 \leq \frac{(q-3)N-(MN-3)r}{(MN-3)q} < 2$ when $(MN-3)(r+q) \leq (q-3)N < (MN-3)(r+2q)$ i.e. $C \leq B < D$. So

$$\lceil\lambda_0\rceil = \mathcal{M}(s) - 1.$$

Case 4: $\frac{(q-3)N-(NM-3)r}{(NM-3)q} \geq 2$ when $(q-3)N \geq (MN-3)(r+2q)$ i.e. $B \geq D$. Hence

$$\lceil\lambda_0\rceil < \mathcal{M}(s) - 1.$$

Furthermore, let $\lambda_1 = \frac{2IMN-(I+1)Iq}{(MN-3)M}$, one can verify that

$$\lambda_0 - \lambda_1 = \frac{(MN-q)MN - 2IMNq + (I+1)Iq^2}{(MN-3)qM}$$
$$= \frac{(J-q)J}{(MN-3)qM}.$$

Obviously, (4) and (5) are identical when $J = 0$. Moreover, we can get

$$\lceil\lambda_1\rceil = \left\lceil \frac{(MN-q)N}{(MN-3)q} + \frac{(q-J)J}{(MN-3)qM} \right\rceil$$
$$= \left\lceil \mathcal{M}(s) - \frac{(q-3)MN-(MN-3)rM-(q-J)J}{(MN-3)qM} \right\rceil.$$

Thus, when only consider the case of optimal, another representations of (4) and (5) can be given respectively as

$$\mathcal{M}(s) = \begin{cases} \left\lceil \frac{(MN-q)N}{(MN-3)q} \right\rceil & A \leq B < C, \\ \left\lceil \frac{(MN-q)N}{(MN-3)q} \right\rceil + 1 & C \leq B < D, \end{cases} \tag{6}$$

and

$$\mathcal{M}(s) = \begin{cases} \left\lceil \frac{(MN-q)N}{(MN-3)q} \right\rceil & A \leq B-\triangle < C, \\ \left\lceil \frac{(MN-q)N}{(MN-3)q} \right\rceil + 1 & C \leq B-\triangle < D. \end{cases} \tag{7}$$

where $\triangle = \frac{(q-J)J}{M}$. According to (6) and (7), it is clear that (5) is tighter than (4) in some cases when $J \neq 0$. □

Definition 3: Let $(N, M, q, d, \mathcal{M}(s))$ be any WG FH sequence set s. According to the bound (5), s is said to be optimal if $\mathcal{M}(s)$ let the equality hold, and to be almost optimal if $\mathcal{M}(s) - 1$ let the equality hold.

III. NEW SETS OF OPTIMAL WG FH SEQUENCE

In this section, generalized methods will be presented to construct several types of WG FH sequence sets with optimal maximum PHC with respect to the bound (5).

A. CONSTRUCTIONS OF WG FH SEQUENCE SETS WITH MAXIMUM PHC 1 OR 2

1) THE FIRST GENERALIZED METHOD

Let $X_0 = \{x_0^i = (x_0^i(0), x_0^i(1), \ldots, x_0^i(L-1)) | i = 0, 1, \ldots, N_0 - 1\}$ be any FH sequence set $(L, N_0, p_0, \mathcal{M}(X_0))$ over frequency set $\mathbb{F}_0$ and the minimum frequency gap of which be $d_{X_0,min}$ satisfying

$\mathbb{A}$. 1. $d_{X_0,min} = 1$.

Furthermore, let $X_a = \{x_a^j = (x_a^j(0), x_a^j(1), \ldots, x_a^j(L-1)) | \ j = 0, 1, \ldots, N_a - 1\}$ be any FH sequence set $(L, N_a, p_a, \mathcal{M}(X_a))$ over frequency set $\mathbb{F}_a$, $a = 1, 2, \ldots, k-1$. And for h with $1 \leq h \leq k-1$, assume X_h satisfies

$\mathbb{A}$. 2. $|x_h^{i_h}(t) - x_h^{i_h}(t+1)| < p_h - 1$ if $(x_0^{i_0}(t) - x_0^{i_0}(t+1))(x_h^{i_h}(t) - x_h^{i_h}(t+1)) = |x_h^{i_h}(t) - x_h^{i_h}(t+1)|$ for $t = 0, 1, \ldots, L-1$, $0 \leq i_0 \leq N_0 - 1$, $0 \leq i_h \leq N_h - 1$.

Over frequency set $\mathbb{F}_0 \otimes \mathbb{F}_1 \otimes \cdots \otimes \mathbb{F}_{k-1}$, generate the desired FH sequence set $X = \{(X_l(0), X_l(1), \ldots, X_l$

$(L-1))|l = (l_0, l_1, \ldots, l_{k-1}), 0 \leq l_0 < N_0, 0 \leq l_1 < N_1, \ldots, 0 \leq l_{k-1} < N_{k-1}\}$ as

$$X_l(t) = x_0^{l_0}(t)p_1p_2\cdots p_{k-1} + x_1^{l_1}(t)p_2p_3\cdots p_{k-1} + \cdots + x_{k-2}^{l_{k-2}}(t)p_{k-1} + x_{k-1}^{l_{k-1}}(t)$$

for $t = 0, 1, \ldots, L-1$.

Theorem 2: **A**. *The maximum PHC $\mathcal{M}(X)$ satisfies*

$$\mathcal{M}(X) \leq \min\{\mathcal{M}(X_0), \mathcal{M}(X_1), \ldots, \mathcal{M}(X_{k-1})\}$$

if there exists $\max\{\mathcal{M}(X_0, 0), \mathcal{M}(X_1, 0), \ldots, \mathcal{M}(X_{k-1}, 0)\} \leq \min\{\overline{\mathcal{M}}(X_0), \overline{\mathcal{M}}(X_1), \ldots, \overline{\mathcal{M}}(X_{k-1})\}$.

B. *X is general WG FH sequence set $(L, N_0N_1\cdots N_{k-1}, p_0p_1\cdots p_{k-1}, \mathcal{M}(X))$.*

Proof: **A**. Let $\eta = (\eta_0, \eta_1, \ldots, \eta_{k-1})$, $\mu = (\mu_0, \mu_1, \ldots, \mu_{k-1})$. Assume $\mathbb{X}_\eta$ and $\mathbb{X}_\mu$ are any two sequences in X, we have

$$\begin{aligned}H_{\mathbb{X}_\eta,\mathbb{X}_\mu}(\tau) &= \sum_{t=0}^{L-1} h((x_0^{\eta_0}(t), x_1^{\eta_1}(t), \ldots, x_{k-1}^{\eta_{k-1}}(t)), (x_0^{\mu_0}(t+\tau), x_1^{\mu_1}(t+\tau), \ldots, x_{k-1}^{\mu_{k-1}}(t+\tau)))\\ &= \sum_{t=0}^{L-1} h(x_0^{\eta_0}(t), x_0^{\mu_0}(t+\tau))h(x_1^{\eta_1}(t), x_1^{\mu_1}(t+\tau)) \cdots h(x_{k-1}^{\eta_{k-1}}(t), x_{k-1}^{\mu_{k-1}}(t+\tau)).\end{aligned}$$

It can be divided into the following two cases to discuss.

Case 1: $(\eta_0, \eta_1, \ldots, \eta_{k-1}) = (\mu_0, \mu_1, \ldots, \mu_{k-1}), \tau \neq 0$.

$$\begin{aligned}H_{\mathbb{X}_\eta,\mathbb{X}_\eta}(\tau) &= \sum_{t=0}^{L-1} h(x_0^{\eta_0}(t), x_0^{\eta_0}(t+\tau))h(x_1^{\eta_1}(t), x_1^{\eta_1}(t+\tau)) \cdots h(x_{k-1}^{\eta_{k-1}}(t), x_{k-1}^{\eta_{k-1}}(t+\tau))\\ &\leq \min\{\mathcal{M}(X_0), \mathcal{M}(X_1), \cdots, \mathcal{M}(X_{k-1})\}.\end{aligned}$$

Case 2: $(\eta_0, \eta_1, \ldots, \eta_{k-1}) \neq (\mu_0, \mu_1, \ldots, \mu_{k-1}), \tau = 0$.

Case 2.1: $\mathcal{M}(X_j, 0) = 0, j = 0, 1, \ldots, k-1$. We have

$$H_{\mathbb{X}_\eta,\mathbb{X}_\mu}(0) = \sum_{t=0}^{L-1} h(x_0^{\eta_0}(t), x_0^{\mu_0}(t))h(x_1^{\eta_1}(t), x_1^{\mu_1}(t)) \cdots h(x_{k-1}^{\eta_{k-1}}(t), x_{k-1}^{\mu_{k-1}}(t)) = 0.$$

Case 2.2: Without loss of generality, assume $\mathcal{M}(X_0, 0) \neq 0$ and $\mathcal{M}(X_0, 0) = \max\{\mathcal{M}(X_j, 0), j = 0, 1, \ldots, k-1\}$.

Case 2.2.1: $\eta_0 \neq \mu_0$. For any η and μ, we have

$$\begin{aligned}H_{\mathbb{X}_\eta,\mathbb{X}_\mu}(0) &\leq \sum_{t=0}^{L-1} h(x_0^{\eta_0}(t), x_0^{\mu_0}(t))h(x_1^{\eta_1}(t), x_1^{\eta_1}(t)) \cdots h(x_{k-1}^{\eta_{k-1}}(t), x_{k-1}^{\eta_{k-1}}(t))\\ &= \mathcal{M}(X_0, 0).\end{aligned}$$

Case 2.2.2: $\eta_0 = \mu_0$. For any η and μ, one can get

$$\begin{aligned}H_{\mathbb{X}_\eta,\mathbb{X}_\mu}(0) &= \sum_{t=0}^{L-1} h(x_1^{\eta_1}(t), x_1^{\mu_1}(t)) \cdots h(x_{k-1}^{\eta_{k-1}}(t), x_{k-1}^{\mu_{k-1}}(t))\\ &\leq \max\{\mathcal{M}(X_1, 0), \cdots, \mathcal{M}(X_{k-1}, 0)\}.\end{aligned}$$

Case 2.3: $0 < \tau \leq L-1$. We assume $\overline{\mathcal{M}}(X_{k'}) = \min\{\overline{\mathcal{M}}(X_0), \ldots, \overline{\mathcal{M}}(X_{k-1})\}, 0 \leq k' < k$. It follows that

Case 2.3.1: $\eta_{k'} \neq \mu_{k'}$. For any η and μ, one can get

$$\begin{aligned}H_{\mathbb{X}_\eta,\mathbb{X}_\mu}(\tau) &\leq \sum_{t=0}^{L-1} h(x_k^{\eta_{k'}}(t), x_k^{\mu_{k'}}(t+\tau))\\ &= \overline{\mathcal{M}}(X_{k'}).\end{aligned}$$

Case 2.3.2: $\eta_{k'} = \mu_{k'}$. For any η and μ, one can have

$$\begin{aligned}H_{\mathbb{X}_\eta,\mathbb{X}_\mu}(\tau) &= \sum_{t=0}^{L-1} h(x_0^{\eta_0}(t), x_0^{\mu_0}(t+\tau)) \cdots h(x_k^{\mu_{k'}}(t), x_k^{\mu_{k'}}(t+\tau)) \cdots h(x_{k-1}^{\eta_{k-1}}(t), x_{k-1}^{\mu_{k-1}}(t+\tau))\\ &\leq \overline{\mathcal{M}}(X_{k'}).\end{aligned}$$

B. Now let $x_l(t) = x_0^{l_0}(t)p_1\cdots p_{k-1} + x_1^{l_1}(t)p_2\cdots p_{k-1} + \cdots + x_{k-2}^{l_{k-2}}(t)p_{k-1} + x_{k-1}^{l_{k-1}}(t)$ and $x_l(t+1) = x_0^{l_0}(t+1)p_1p_2\cdots p_{k-1} + x_1^{l_1}(t+1)p_2p_3\cdots p_{k-1} + \cdots + x_{k-2}^{l_{k-2}}(t+1)p_{k-1} + x_{k-1}^{l_{k-1}}(t+1)$ are any two adjacent frequencies in any FH sequence $\mathbb{X}_l \in X$, $x_v^{l_v}(t) \in \mathbb{F}_v$, $x_v^{l_v}(t+1) \in \mathbb{F}_v$, $v = 0, 1, \ldots, k-1$. Since $p_1\cdots p_{k-1}$, $p_2\cdots p_{k-1}$, …, $p_{k-2}p_{k-1}$ and p_{k-1} are linear independence, $x_l(t) = x_l(t+1)$ if and only if $x_0^{l_0}(t+1) = x_0^{l_0}(t)$, $x_1^{l_1}(t+1) = x_1^{l_1}(t)$, …, $x_{k-1}^{l_{k-1}}(t+1) = x_{k-1}^{l_{k-1}}(t)$. Thus, $x_l(t) \neq x_l(t+1)$. Suppose $x_l(t) < x_l(t+1)$ and $1 < h < k-3$, the frequency gap $d_{t+1,t}$ between $x_l(t)$ and $x_l(t+1)$ can be calculated by

$$\begin{aligned}d_{t+1,t} = &(x_0^{l_0}(t+1) - x_0^{l_0}(t))p_1p_2\cdots p_{k-1}\\ &+ (x_1^{l_1}(t+1) - x_1^{l_1}(t))p_2p_3\cdots p_{k-1} + \cdots\\ &+ (x_{h-1}^{l_{h-1}}(t+1) - x_{h-1}^{l_{h-1}}(t))p_hp_{h+1}\cdots p_{k-1}\\ &+ (x_h^{l_h}(t+1) - x_h^{l_h}(t))p_{h+1}p_{h+2}\cdots p_{k-1}\\ &+ (x_{h+1}^{l_{h+1}}(t+1) - x_{h+1}^{l_{h+1}}(t))p_{h+2}p_{h+3}\cdots p_{k-1}\\ &+ \cdots + (x_{k-1}^{l_{k-1}}(t+1) - x_{k-1}^{l_{k-1}}(t))\end{aligned}$$

It can be divided into the following three cases to discuss.

Case 1: $(x_0^{l_0}(t+1) - x_0^{l_0}(t))(x_h^{l_h}(t+1) - x_h^{l_h}(t)) = |x_h^{l_h}(t+1) - x_h^{l_h}(t)|$. Basing on the condition of $\mathbb{A}$. 2, we have

$$\begin{aligned}d_{t+1,t} &\geq p_1p_2\cdots p_{k-1} - (p_1-1)\,p_2p_3\cdots p_{k-1}\\ &\quad - \cdots - (p_{h-1}-1)\,p_hp_{h+1}\cdots p_{k-1}\\ &\quad - (x_h^{l_h}(t+1) - x_h^{l_h}(t))p_{h+1}\cdots p_{k-1}\\ &\quad - (p_{h+1}-1)\,p_{h+2}p_{h+3}\cdots p_{k-1}\\ &\quad - \cdots - (p_{k-2}-1)\,p_{k-1} - (p_{k-1}-1)\\ &= (p_h - (x_h^{l_h}(t+1) - x_h^{l_h}(t)) - 1)p_{h+1}\cdots p_{k-1} + 1\\ &> 1.\end{aligned}$$

Case 2: $(x_0^{l_0}(t+1)-x_0^{l_0}(t))(x_h^{l_h}(t+1)-x_h^{l_h}(t)) = -\,|x_h^{l_h}(t+1) - x_h^{l_h}(t)|$. It follows that

$$\begin{aligned}d_{t+1,t} &\geq p_1p_2\cdots p_{k-1} - (p_1-1)\,p_2p_3\cdots p_{k-1}\\ &\quad - \cdots - (p_{h-1}-1)\,p_hp_{h+1}\cdots p_{k-1}\end{aligned}$$

$$\begin{aligned}&+|x_h^{l_h}(t+1)-x_h^{l_h}(t)|p_{h+1}\cdots p_{k-1}\\&-(p_{h+1}-1)\,p_{h+2}p_{h+3}\cdots p_{k-1}\\&-\cdots-(p_{k-2}-1)\,p_{k-1}-(p_{k-1}-1)\\=&\ (p_h+|x_h^{l_h}(t+1)-x_h^{l_h}(t)|-1)p_{h+1}\cdots p_{k-1}+1\\>&\ 1.\end{aligned}$$

Case 3: $x_0^{l_0}(t+1)-x_0^{l_0}(t)\geq 2$. It follows again that

$$\begin{aligned}d_{t+1,t}\geq&\ 2\,p_1p_2\cdots p_{k-1}-(p_1-1)\,p_2p_3\cdots p_{k-1}\\&-\cdots-(p_{h-1}-1)\,p_hp_{h+1}\cdots p_{k-1}\\&-(p_h-1)p_{h+1}\cdots p_{k-1}\\&-(p_{h+1}-1)\,p_{h+2}p_{h+3}\cdots p_{k-1}\\&-\cdots-(p_{k-2}-1)\,p_{k-1}-(p_{k-1}-1)\\\geq&\ p_1p_2\cdots p_{k-1}\end{aligned}$$

So, X is a general WG FH sequence set. This completes the proof. □

2) THE SECOND GENERALIZED METHOD

Let $Y_0=\{y_0^i=(y_0^i(0),y_0^i(1),\ldots,y_0^i(N-1))|i=0,1,\ldots,M_0-1\}$ be any FH sequence set $(N,M_0,p_0,\mathcal{M}(Y_0))$ over frequency set $\mathbb{F}_0$ and the minimum frequency gap of which be d_{Y_0} satisfying

$\mathbb{B}$. 1. $d_{Y_0}=1$.

And select FH sequence set $Y_b=\{y_b^j=(y_b^j(0),y_b^j(1),\ldots,y_b^j(N-1))|j=0,1,\ldots,M_b-1\}$ denoted as $(N,M_b,p_b,\mathcal{M}(Y_b))$ over frequency set $\mathbb{F}_b$, $b=1,2,\ldots,k-1$. For h with $1\leq h\leq k-1$, the FH sequence set Y_h satisfies

$\mathbb{B}$. 2. $|Y_h^{j_h}(t)-Y_h^{j_h}(t+1)|<p_h-1$ if $(Y_0^{j_0}(t)-Y_0^{j_0}(t+1))\,(Y_h^{j_h}(t)-Y_h^{j_h}(t+1))=|Y_h^{j_h}(t)-Y_h^{j_h}(t+1)|$ for $t=0,1,\ldots,N-1$, $0\leq j_0\leq M_0-1$, $0\leq j_h\leq M_h-1$.

Over the frequency set $\mathbb{F}_0\otimes\mathbb{F}_1\otimes\cdots\otimes\mathbb{F}_{k-1}$, let $Y=\{(Y_e(t))_{t=0}^{N-1}|e=0,1,\ldots,M_1M_2\cdots M_{k-1}-1\}$ be the desired FH sequence set defined by

$$\begin{aligned}Y_e(t)=&\ y_0^{\langle e\rangle_{M_0}}(t)\,p_1p_2\cdots p_{k-1}+y_1^{e_1}(t)\,p_2p_3\cdots p_{k-1}\\&+\cdots+y_{k-2}^{e_{k-2}}(t)\,p_{k-1}+y_{k-1}^{e_{k-1}}(t)\end{aligned}$$

where $0\leq e_1<M_1,\ldots,0\leq e_{k-1}<M_{k-1}$.

Theorem 3: **A.** *The maximum PHC $\mathcal{M}(Y)$ satisfies*

$$\mathcal{M}(Y)\leq\min\{\mathcal{M}(Y_0),\mathcal{M}(Y_1),\ldots,\mathcal{M}(Y_{k-1})\}.$$

if $Y_0,Y_1,\ldots,Y_{k-1}$ satisfy $\max\{\mathcal{M}(Y_1,0),\ldots,\mathcal{M}(Y_{k-1},0)\}\leq\min\{\overline{\mathcal{M}}(Y_0),\ldots,\overline{\mathcal{M}}(Y_{k-1})\}$.

B. *Y is a general WG FH sequence set $(N,M_1M_2\cdots M_{k-1},p_0p_1\ldots p_{k-1},\mathcal{M}(Y))$.*

Proof: **A.** Now let $\eta=(\eta_0,\eta_1,\ldots,\eta_{k-1})$, $\mu=(\mu_0,\mu_1,\ldots,\mu_{k-1})$ and assume $\mathbb{Y}_\eta$ and $\mathbb{Y}_\mu$ are any two sequences in Y. It follows that

$$\begin{aligned}H_{\mathbb{Y}_\eta,\mathbb{Y}_\mu}(\tau)=&\sum_{t=0}^{N-1}h((y_0^{\eta_0}(t),y_1^{\eta_1}(t),\ldots,y_{k-1}^{\eta_{k-1}}(t)),\\&(y_0^{\mu_0}(t+\tau),y_1^{\mu_1}(t+\tau),\ldots,y_{k-1}^{\mu_{k-1}}(t+\tau)))\\=&\sum_{t=0}^{N-1}h(y_0^{\eta_0}(t),y_0^{\mu_0}(t+\tau))h(y_1^{\eta_1}(t),y_1^{\mu_1}(t+\tau))\\&\cdots h(y_{k-1}^{\eta_{k-1}}(t),y_{k-1}^{\mu_{k-1}}(t+\tau)).\end{aligned}$$

There exist the following three cases.

Case 1: $(\eta_0,\eta_1,\ldots,\eta_{k-1})=(\mu_0,\mu_1,\ldots,\mu_{k-1})$, $\tau\neq 0$.

$$\begin{aligned}H_{\mathbb{Y}_\eta,\mathbb{Y}_\eta}(\tau)=&\sum_{t=0}^{N-1}h(y_0^{\eta_0}(t),y_0^{\eta_0}(t+\tau))h(y_1^{\eta_1}(t),y_1^{\eta_1}(t+\tau))\\&\cdots h(y_{k-1}^{\eta_{k-1}}(t),y_{k-1}^{\eta_{k-1}}(t+\tau))\\\leq&\min\{\mathcal{M}(Y_0),\mathcal{M}(Y_1),\ldots,\mathcal{M}(Y_{k-1})\}.\end{aligned}$$

Case 2: $(\eta_0,\eta_1,\ldots,\eta_{k-1})\neq(\mu_0,\mu_1,\ldots,\mu_{k-1})$, $\tau=0$.

Case 2.1: $\mathcal{M}(Y_j,0)=0$, $j=0,1,\ldots,k-1$. We have

$$\begin{aligned}H_{\mathbb{Y}_\eta,\mathbb{Y}_\mu}(0)=&\sum_{t=0}^{N-1}h(y_0^{\eta_0}(t),y_0^{\mu_0}(t))h(y_1^{\eta_1}(t),y_1^{\mu_1}(t))\\&\cdots h(y_{k-1}^{\eta_{k-1}}(t),y_{k-1}^{\mu_{k-1}}(t))\\=&\ 0.\end{aligned}$$

Case 2.2: Assume Y_h satisfies $\mathcal{M}(Y_h,0)\neq 0$ for $h=r_0,r_1,\ldots,r_d$, $0\leq r_0,r_1,\ldots,r_d\leq k-1$. It is clear that $(\eta_1,\ldots,\eta_{k-1})\neq(\mu_1,\ldots,\mu_{k-1})$ for any η_0,μ_0. Thus, one can verify that

$$\begin{aligned}H_{\mathbb{Y}_\eta,\mathbb{Y}_\mu}(0)=&\sum_{t=0}^{N-1}h(y_0^{\eta_0}(t),y_0^{\mu_0}(t))h(y_1^{\eta_1}(t),y_1^{\mu_1}(t))\\&\cdots h(y_{k-1}^{\eta_{k-1}}(t),y_{k-1}^{\mu_{k-1}}(t))\\\leq&\max\{\mathcal{M}(Y_1,0),\ldots,\mathcal{M}(Y_{k-1},0)\}.\end{aligned}$$

Case 3: $0<\tau<N$. We suppose that $\overline{\mathcal{M}}(Y_{k'})=\min\{\overline{\mathcal{M}}(Y_0),\ldots,\overline{\mathcal{M}}(Y_{k-1})\}$, $0\leq k'<k$. One can has

$$\begin{aligned}H_{\mathbb{Y}_\eta,\mathbb{Y}_\mu}(\tau)\leq&\sum_{t=0}^{N-1}h(y_0^{\eta_0}(t),y_0^{\mu_0}(t+\tau))\\&\cdots h(y_k^{\eta_k}(t),y_k^{\mu_k}(t+\tau))\cdots h(y_{k-1}^{\eta_{k-1}}(t),\\&y_{k-1}^{\mu_{k-1}}(t+\tau))\\=&\ \overline{\mathcal{M}}(Y_{k'}).\end{aligned}$$

Thus, the statement holds. □

Construction 1: Select any FH sequence set $(L_1,N_0,p_0,\mathcal{M}(X_0))$ denoted as X_0 satisfying $\mathbb{A}$. 1. And select any FH sequence set $(L_1,N_i,p_i,1)$ denoted as X_i, $i=1,2,\ldots,k-1$. X_h satisfies $\mathbb{A}$. 2 for h with $1\leq h\leq k-1$. Based on all X_i's, a new class of WG FH sequence set $\mathcal{S}_1$ can be constructed by the first generalized method.

Theorem 4: Let $N=N_0N_1\cdots N_{k-1}$, $p=p_0p_1\cdots p_{k-1}$, $L_1N=Ip+J$ and $L_1=2p+r$, $0\leq J\leq p-1$. According to (5), one can check that

(1). $\mathcal{S}_1$ is an almost optimal general WG FH sequence set $(L_1,N,p,2)$ if $p+r<\frac{(p-3)L_1N-(p-J)J}{(L_1N-3)N}<2p+r$ and $\mathcal{M}(X_0,0)=2$.

(2). $\mathcal{S}_1$ is an optimal general WG FH sequence set $(L_1,N,p,1)$ if $\mathcal{M}(X_0,0)\leq 1$.

Construction 2: Select any FH sequence set $(L_2, M_0, p_0, 1)$ satisfying $\mathbb{A}$. 1 as base sequence set $\mathbb{X}_0$, and select any FH sequence set $(L_2, M_i, p_i, \mathcal{M}(\mathbb{X}_i))$ over $\mathbb{F}_i$ as base sequence sets $\mathbb{X}_i$, $i = 1, 2, \ldots, k-1$. Let $\mathbb{X}_h$ satisfy $\mathbb{A}$. 2 for h with $1 \le h \le k-1$. The desired WG FH sequence set $\mathcal{S}_2$ can be obtained by the first generalized method.

Theorem 5: Let $M = M_0M_1\cdots M_{k-1}$, $p = p_0p_1\cdots p_{k-1}$, $L_2M = Ip + J$ *and* $L_2 = 2p + r$, $0 \le J \le p-1$. *We have*

(1). $\mathcal{S}_2$ *is an almost optimal general WG FH sequence set* $(L_2, M, p, 2)$ *if* $\max\{\mathcal{M}(\mathbb{X}_1, 0), \mathcal{M}(\mathbb{X}_2, 0), \ldots, \mathcal{M}(\mathbb{X}_{k-1}, 0)\} = 2$ *and* $p + r < \frac{(p-3)L_2M-(p-J)J}{(L_2M-3)M} < 2p + r$.

(2). $\mathcal{S}_2$ *is an optimal general WG FH sequence set* $(L_2, M, p, 1)$ *if* $\max\{\mathcal{M}(\mathbb{X}_1, 0), \mathcal{M}(\mathbb{X}_2, 0), \ldots, \mathcal{M}(\mathbb{X}_{k-1}, 0)\} \le 1$.

Construction 3: Select any FH sequence set Y_0 denoted as $(L_3, M_0, p_0, \mathcal{M}(Y_0))$ satisfying $\mathcal{M}(Y_0) \ge 3$, $\mathcal{M}(Y_0, 0) \ge 3$ and the condition of $\mathbb{B}$. 1. And select any FH sequence set $(L_3, M_j, p_j, \mathcal{M}(Y_j))$ as base sequence sets $Y_j, j = 1, 2, \ldots, k-1$. Let Y_h satisfy $\mathbb{B}$. 2 for h with $1 \le h \le k-1$. The WG FH sequence set $\mathcal{S}_3$ can be designed by the second generalized method.

Theorem 6: Let $M = M_1M_2\cdots M_{k-1}$, $p = p_0p_1\cdots p_{k-1}$, $L_3M = Ip + J$ *and* $L_3 = 2p + r$, $0 \le J \le p-1$. *We have*

(1). $\mathcal{S}_3$ *is an almost optimal general WG FH sequence set* $(L_3, M, p, 2)$ *if* $p + r < \frac{(p-3)ML_3-(p-J)J}{(ML_3-3)M} < 2p + r$, $\min\{\mathcal{M}(Y_1), \mathcal{M}(Y_2), \ldots, \mathcal{M}(Y_{k-1})\} = 2$, $\mathcal{M}(Y_i, 0) \le 2$, $i = 1, 2, \ldots, k-1$.

(2). $\mathcal{S}_3$ *is an optimal general WG FH sequence set* $(L_3, M, p, 1)$ *if* $\min\{\mathcal{M}(Y_1), \mathcal{M}(Y_1), \ldots, \mathcal{M}(Y_{k-1})\} = 1$, $\mathcal{M}(Y_i, 0) \le 1$, $i = 1, 2, \ldots, k-1$.

Example 1: Select any FH sequence set (28,23,29,1) denoted as $X_0 = \{X_{0,0}, \ldots, X_{0,21}, X_{0,22}\}$ over F_{29}, such that

$$X_{0,0} = (1, 2, 4, 8, 16, 3, 6, 12, 24, 19, 9, 18, 7, 14, 28, 27, 25, 21, 13, 26, 23, 17, 5, 10, 20, 11, 22, 15);$$

$$\ldots\ldots$$

$$X_{0,21} = (25, 26, 28, 3, 11, 27, 1, 7, 19, 14, 4, 13, 2, 9, 23, 22, 20, 16, 8, 21, 18, 12, 0, 5, 15, 6, 17, 10);$$

$$X_{0,22} = (0, 1, 3, 7, 15, 2, 5, 11, 23, 18, 8, 17, 6, 13, 27, 26, 24, 20, 12, 25, 22, 16, 4, 9, 19, 10, 21, 14).$$

Select any FH sequence set (28,10,14,7) denoted as $X_1 = \{X_{1,0}, \ldots, X_{1,8}, X_{1,9}\}$ over $\mathbb{R} = \{0, 1, \ldots, 13\}$ as follows:

$$X_{1,0} = (12, 12, 13, 3, 0, 6, 4, 4, 4, 8, 7, 6, 5, 2, 11, 11, 10, 1, 1, 7, 8, 8, 10, 4, 6, 5, 3, 13);$$

$$\ldots\ldots$$

$$X_{1,9} = (12, 2, 3, 7, 4, 10, 8, 10, 10, 10, 11, 13, 9, 6, 1, 1, 6, 9, 13, 11, 12, 12, 0, 8, 10, 4, 7, 3).$$

It is clear that X_0 and X_1 are non WG FH sequence set, and satisfy the conditions in the first generalized method. The WG FH sequence set $\mathcal{S} = \{\mathcal{S}_{(i,j)} = (\mathcal{S}_{(i,j)}(0), \mathcal{S}_{(i,j)}(1), \ldots, \mathcal{S}_{(i,j)}(27)) | \mathcal{S}_{(i,j)}(t) = 14X_{0,i}(t) + X_{1,j}(t), 0 \le t \le 27, 0 \le i \le 22, 0 \le j \le 9\}$ over $F_{29} \otimes \mathbb{R} = \{14a + b | a \in F_{29}, b \in \mathbb{R}\}$ can be obtained:

$$\mathcal{S}_{(0,0)} = (26, 40, 69, 115, 224, 48, 88, 172, 340, 274, 133, 258, 103, 198, 403, 389, 360, 295, 183, 371, 330, 246, 80, 144, 286, 159, 311);$$

$$\ldots\ldots$$

$$\mathcal{S}_{(22,9)} = (12, 16, 45, 105, 214, 38, 78, 164, 332, 262, 123, 251, 93, 188, 379, 365, 342, 289, 181, 361, 320, 236, 56, 134, 276, 144, 301).$$

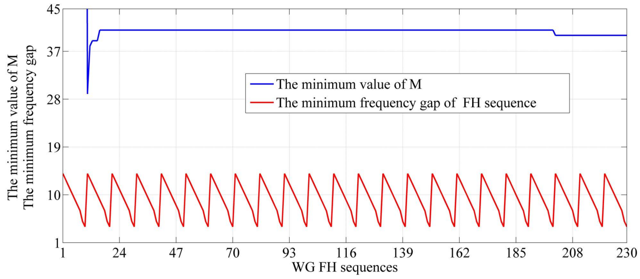

FIGURE 1. The WG property of $\mathcal{S}$ in example 1.

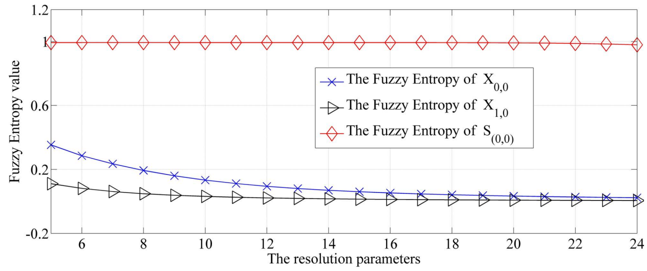

FIGURE 2. The complexity comparisons between X_0, X_1 and $\mathcal{S}$ in example 1.

For every FH sequence $\mathcal{S}_{(i,j)}$ in $\mathcal{S}$, let the maximum frequency be $\mathcal{S}_{(i,j),max}$, the minimum frequency be $\mathcal{S}_{(i,j),min}$, and $M = \mathcal{S}_{(i,j),max} - \mathcal{S}_{(i,j),min} + 1 - |\mathcal{S}_{(i,j)}(t+1) - \mathcal{S}_{(i,j)}(t)|$. The minimum value of M and the minimum frequency gap are shown in FIGURE 1, one can check that $\mathcal{S}$ is an optimal special WG FH sequence set (28, 230, 406, 4, 1).

We use Fuzzy Entropy [27] to measure respectively the complexity of any one FH sequence in X_0, X_1 and $\mathcal{S}$, such as $X_{0,0}$, $X_{1,0}$ and $\mathcal{S}_{(0,0)}$. As shown in FIGURE 2, the Fuzzy Entropy of $\mathcal{S}_{(0,0)}$ is greater than that of $X_{0,0}$ and $X_{1,0}$ when the measuring window length equates 5. So, compared with X_0 and X_1, $\mathcal{S}$ has better complexity.

3) THE THIRD GENERALIZED METHOD

Select FH sequence set $Z_\eta = \{Z_\eta^i = (z_\eta^i(0), z_\eta^i(1), \ldots, z_\eta^i(L-1)) | i = 0, 1, \ldots, M_\eta - 1\}$ over $\mathbb{F}_\eta$ as base sequence sets $(L, M_\eta, p_\eta, \mathcal{M}(Z_\eta))$, $\eta = 0, 1, \ldots, k-1$. Over $\mathbb{F}_0 \otimes \mathbb{F}_1 \otimes \cdots \otimes \mathbb{F}_{k-1}$, generate a new class of FH sequence set $(L, M_0M_1\cdots M_{k-1}, p_0p_1\cdots p_{k-1}, \mathcal{M}(Z))$ denoted as $Z = \{(Z_j(0), Z_j(1), \ldots, Z_j(L-1)) | j = (j_0, j_1, \ldots, j_{k-1}),$

$0 \le j_0 < M_0, 0 \le j_1 < M_1, \ldots, 0 \le j_{k-1} < M_{k-1}\}$ where

$$Z_j(t) = z_0^{j_0}(t)p_1p_2\cdots p_{k-1} + z_1^{j_1}(t)p_2p_3\cdots p_{k-1} + \cdots + z_{k-2}^{j_{k-2}}(t)p_{k-1} + z_{k-1}^{j_{k-1}}(t)$$

Theorem 7: **A**. *The maximum PHC $\mathcal{M}(Z)$ satisfies*

$$\mathcal{M}(Z) \le \min\{\mathcal{M}(Z_0), \mathcal{M}(Z_1), \ldots, \mathcal{M}(Z_{k-1})\}$$

if there exists $\max\{\mathcal{M}(Z_0,0), \mathcal{M}(Z_1,0), \ldots, \mathcal{M}(Z_{k-1},0)\} \le \min\{\overline{\mathcal{M}}(Z_0), \overline{\mathcal{M}}(Z_1), \ldots, \overline{\mathcal{M}}(Z_{k-1})\}$.

B. *Z is WG FH sequence set if Z_0 is WG FH sequence set.*

C. *Z is general WG FH sequence set if Z_h is general WG FH sequence set satisfying*

$\mathbb{C}$. $|z_h^{i_h}(t) - z_h^{i_h}(t+1)| < p_h - 1$ *if* $(z_0^{i_0}(t) - z_0^{i_0}(t+1))(z_h^{i_h}(t) - z_h^{i_h}(t+1)) = |z_h^{i_h}(t) - z_h^{i_h}(t+1)|$ *for* $t = 0, 1, \ldots, L-1$, $0 \le i_0 < M_0, 0 \le i_h < M_h$, $1 \le h \le k-1$.

Proof: **A**. It follows from the proofs of Theorem 2.

B. Let $z_j(t) = z_0^{j_0}(t)p_1p_2\cdots p_{k-1} + z_1^{j_1}(t)p_2p_3\cdots p_{k-1} + \cdots + z_{k-2}^{j_{k-2}}(t)p_{k-1} + z_{k-1}^{j_{k-1}}(t)$ and $z_j(t+1) = z_0^{j_0}(t+1)p_1p_2\cdots p_{k-1} + z_1^{j_1}(t+1)p_2p_3\cdots p_{k-1} + \cdots + z_{k-2}^{j_{k-2}}(t+1)p_{k-1} + z_{k-1}^{j_{k-1}}(t+1)$ are two adjacent frequencies in $\mathbb{Z}_j \in Z$, $z_v^{j_v}(t) \in \mathbb{F}_v$, $z_v^{j_v}(t+1) \in \mathbb{F}_v$, $v = 0, 1, \ldots, k-1$. Assume $z_j(t) < z_j(t+1)$, we have

$$\begin{aligned} d_{t+1,t} &= (z_0^{j_0}(t+1) - z_0^{j_0}(t))p_1p_2\cdots p_{k-1} \\ &\quad + (z_1^{j_1}(t+1) - z_1^{j_1}(t))p_2p_3\cdots p_{k-1} + \cdots \\ &\quad + (z_h^{j_h}(t+1) - z_h^{j_h}(t))p_{h+1}p_{h+2}\cdots p_{k-1} \\ &\quad + \cdots + (z_{k-1}^{j_{k-1}}(t+1) - z_{k-1}^{j_{k-1}}(t)) \end{aligned}$$

Let the minimum frequency gap of Z_0 be d_0, one can have

$$\begin{aligned} d_{t+1,t} &\ge d_0p_1p_2\cdots p_{k-1} - (p_1-1)p_2\cdots p_{k-1} \\ &\quad - \cdots - (p_{k-2}-1)p_{k-1} - (p_{k-1}-1) \\ &= (d_0-1)p_1p_2\cdots p_{k-1} + 1 \end{aligned}$$

Thus, Z is WG FH sequence set. It is easy to check that Z is special WG FH sequence set if Z_0 is special WG FH sequence set.

C. There exist the following two cases.

Case 1: $z_u^{j_u}(t+1) = z_u^{j_u}(t)$, $u = 0, \ldots, h-1$. Without loss of generality, let $1 \le h < k-3$ and the minimum frequency gap of Z_h be d_h, we can get

$$\begin{aligned} d_{t+1,t} &= (z_h^{j_h}(t+1) - z_h^{j_h}(t))p_{h+1}p_{h+2}\cdots p_{k-1} \\ &\quad + (z_{h+1}^{j_{h+1}}(t+1) - z_{h+1}^{j_{h+1}}(t))p_{h+2}p_{h+3}\cdots p_{k-1} \\ &\quad + \cdots + (z_{k-2}^{j_{k-2}}(t+1) - z_{k-2}^{j_{k-2}}(t))p_{k-1} \\ &\quad + (z_{k-1}^{j_{k-1}}(t+1) - z_{k-1}^{j_{k-1}}(t)) \\ &\ge (z_h^{j_h}(t+1) - z_h^{j_h}(t))p_{h+1}p_{h+2}\cdots p_{k-1} \\ &\quad - (p_{h+1}-1)p_{h+2}p_{h+3}\cdots p_{k-1} \\ &\quad - \cdots - (p_{k-2}-1)p_{k-1} - (p_{k-1}-1) \\ &= (d_h-1)p_{h+1}\cdots p_{k-1} + 1 \end{aligned}$$

Case 2: $z_0^{j_0}(t+1) \ne z_0^{j_0}(t)$. Basing on the condition of $\mathbb{C}$ and the proofs of the Theorem 2, we can get

$$d_{t+1,t} > 1$$

This completes the proof. □

4) THE FOURTH GENERALIZED METHOD

Let $U_\eta = \{u_\eta^j = (u_\eta^j(0), u_\eta^j(1), \ldots, u_\eta^j(N-1)) | j = 0, 1, \ldots, M_\eta - 1\}$ be any FH sequence set over frequency set $\mathbb{F}_\eta$, $\eta = 0, 1\ldots, k-1$. A new class of FH sequence set $U = \{(U_l(0), U_l(1), \ldots, U_l(N-1)) | l = 0, 1, \ldots, M_1M_2\cdots M_{k-1} - 1\}$ denoted as $(N, M_1\cdots M_{k-1}, p_0p_1\cdots p_{k-1}, \mathcal{M}(U))$ over frequency set $\mathbb{F}_0 \otimes \mathbb{F}_1 \otimes \cdots \otimes \mathbb{F}_{k-1}$ is designed by

$$U_l(t) = u_0^{\langle l\rangle_{N_0}}(t)p_1p_2\cdots p_{k-1} + u_1^{l_1}(t)p_2p_3\cdots p_{k-1} + \cdots + u_{k-2}^{l_{k-2}}(t)p_{k-1} + u_{k-1}^{l_{k-1}}(t)$$

where $0 \le l_1 < M_1, \ldots, 0 \le l_{k-1} < M_{k-1}$.

Theorem 8: **A**. *The maximum PHC $\mathcal{M}(U)$ satisfies*

$$\mathcal{M}(U) \le \min\{\mathcal{M}(U_0), \mathcal{M}(U_1), \ldots, \mathcal{M}(U_{k-1})\}.$$

if there exists $\max\{\mathcal{M}(U_1,0), \mathcal{M}(U_2,0), \ldots, \mathcal{M}(U_{k-1},0)\} \le \min\{\overline{\mathcal{M}}(U_0), \overline{\mathcal{M}}(U_1), \ldots, \overline{\mathcal{M}}(U_{k-1})\}$.

B. *U is WG FH sequence set if U_0 is WG FH sequence set.*

C. *U is general WG FH sequence set if U_h is general WG FH sequence set satisfying*

$\mathbb{D}$. $|u_h^{i_h}(t) - u_h^{i_h}(t+1)| < p_h - 1$ *if* $(u_0^{i_0}(t) - u_0^{i_0}(t+1))(u_h^{i_h}(t) - u_h^{i_h}(t+1)) = |u_h^{i_h}(t) - u_h^{i_h}(t+1)|$ *for* $t = 0, 1, \ldots, N-1$, $0 \le i_0 < M_0, 0 \le i_h < M_h$, $1 \le h \le k-1$.

Construction 4: Select any FH sequence set $(L_4, M_0, p_0, \mathcal{M}(Z_0))$ denoted as Z_0 over $\mathbb{F}_0$. And select any FH sequence set $(L_4, M_i, p_i, 1)$ denoted as Z_i over $\mathbb{F}_i$, $i = 1, 2, \ldots, k-1$. Z_0 is WG FH sequence set or Z_h is general WG FH sequence set satisfying $\mathbb{C}$, $1 \le h \le k-1$. Based on all Z_i's, a new class of WG FH sequence set $\mathcal{S}_4$ can be defined.

Theorem 9: Let $M = M_0M_1\cdots M_{k-1}$, $p = p_0p_1\cdots p_{k-1}$, $L_4M = Ip + J$, $L_4 = 2p + r$, $0 \le J \le p-1$. We have

(1). $\mathcal{S}_4$ is an almost optimal general WG FH sequence set $(L_4, M, p, 2)$ if $p + r < \frac{(p-3)L_4M-(p-J)J}{(L_4M-3)M} < 2p + r$ and $\mathcal{M}(Z_0, 0) = 2$.

(2). $\mathcal{S}_4$ is an optimal general WG FH sequence set $(L_4, M, p, 1)$ if $\mathcal{M}(Z_0, 0) \le 1$.

Construction 5: Select FH sequence set $(L_5, N_0, p_0, 1)$ as base sequence set $\mathbb{Z}_0$ over $\mathbb{F}_0$, and select FH sequence set $(L_5, N_i, p_i, \mathcal{M}(\mathbb{Z}_i))$ as base sequence sets $\mathbb{Z}_i$ over $\mathbb{F}_i$, $i = 1, 2, \ldots, k-1$. $\mathbb{Z}_0$ is WG FH sequence set or $\mathbb{Z}_h$ is general WG FH sequence set satisfying $\mathbb{C}$, $1 \le h \le k-1$. A new kind of WG FH sequence set $\mathcal{S}_5$ can be obtained by the third generalized method.

Theorem 10: Let $N = N_0N_1\cdots N_{k-1}$, $p = p_0p_1\cdots p_{k-1}$, $L_5N = Ip + J$ and $L_5 = 2p + r$, $0 \le J \le p-1$. According to (5), we can have

(1). $\mathcal{S}_5$ is an almost optimal general WG FH sequence set $(L_5, N, p, 2)$ if $\max\{\mathcal{M}(\mathbb{Z}_1, 0), \ldots, \mathcal{M}(\mathbb{Z}_{k-1}, 0)\} = 2$ and $p + r < \frac{(p-3)L_5N-(p-J)J}{(L_5N-3)N} < 2p + r$.

(2). $\mathcal{S}_5$ is an optimal general WG FH sequence set $(L_5, N, p, 1)$ if $\max\{\mathcal{M}(\mathbb{Z}_1, 0), \ldots, \mathcal{M}(\mathbb{Z}_{k-1}, 0)\} \leq 1$.

Construction 6:

Step 1: Over $\mathbb{K}_0 = \{0, 1, \ldots, p-1\}$, select any FH sequence set $(N, M_0, p, 1)$ as base sequence set $\mathbb{T}_0$ to construct FH sequence set $\mathbb{L}_0$ over $\mathbb{K}_0 \otimes \mathbb{K}_0 \otimes \cdots \otimes \mathbb{K}_0 = \{a_{k_0-1}p^{k_0-1} + a_{k_0-2}p^{k_0-2} + \cdots + a_1p + a_0 | a_i \in \mathbb{K}_0, i = 0, 1, \ldots, k_0 - 1\}$.

Step 2: Over $\mathbb{K}_1 = \{0, 1, \ldots, q-1\}$, select any WG FH sequence set $(N, M_1, q, \mathcal{M}(\mathbb{T}_1))$ as base sequence set $\mathbb{T}_1$ to construct another FH sequence set $\mathbb{L}_1$ over $\mathbb{K}_1 \otimes \mathbb{K}_1 \otimes \cdots \otimes \mathbb{K}_1 = \{b_{k_1-1}q^{k_1-1} + b_{k_1-2}q^{k_1-2} + \cdots + b_1q + b_0 | b_i \in \mathbb{K}_1, i = 0, 1, \ldots, k_1 - 1\}$.

$\mathbb{T}_0$ is WG FH sequence set or $\mathbb{T}_1$ is general WG FH sequence set satisfying $\mathbb{C}$.

Step 3: Basing on the third generalized method, use $\mathbb{L}_0$ and $\mathbb{L}_1$ to design an FH sequence set $\mathcal{S}_6$ over $\mathbb{K}_0 \otimes \cdots \otimes \mathbb{K}_0 \otimes \mathbb{K}_1 \otimes \cdots \otimes \mathbb{K}_1 = \{a_{k_0-1}p^{k_0-1}q^{k_1} + \cdots + a_1pq^{k_1} + a_0q^{k_1} + b_{k_1-1}q^{k_1-1} + \cdots + b_1q + b_0 | a_i \in \mathbb{K}_0, b_j \in \mathbb{K}_1, i = 0, \ldots, k_0 - 1, j = 0, \ldots, k_1 - 1\}$.

Theorem 11: Let $N{M_0}^{k_0}{M_1}^{k_1} = Ip^{k_0}q^{k_1} + J$ and $N = 2p^{k_0}q^{k_1} + r$, $0 \leq J \leq p^{k_0}q^{k_1} - 1$. According to (5), we have

(1). $\mathcal{S}_6$ is an almost optimal general WG FH sequence set $(N, {M_0}^{k_0}{M_1}^{k_1}, p^{k_0}q^{k_1}, 2)$ if $\mathcal{M}(\mathbb{L}_1, 0) = 2$ and $p^{k_0}q^{k_1} + r < \frac{(p^{k_0}q^{k_1}-3)N{M_0}^{k_0}{M_1}^{k_1}-(p^{k_0}q^{k_1}-J)J}{(N{M_0}^{k_0}{M_1}^{k_1}-3){M_0}^{k_0}{M_1}^{k_1}} < 2p^{k_0}q^{k_1} + r$.

(2). $\mathcal{S}_6$ is an optimal general WG FH sequence set $(N, {M_0}^{k_0}{M_1}^{k_1}, p^{k_0}q^{k_1}, 1)$ if $\mathcal{M}(\mathbb{L}_1, 0) \leq 1$.

Construction 7: Select any FH sequence set $(L_7, M_0, p_0, \mathcal{M}(\mathbb{U}_0))$ satisfying $\mathcal{M}(\mathbb{U}_0, 0) \geq 3$ and $\mathcal{M}(\mathbb{U}_0) \geq 3$ as base sequence set $\mathbb{U}_0$ over $\mathbb{F}_0$, and select FH sequence set $(L_7, M_j, q_j, \mathcal{M}(\mathbb{U}_j))$ as base sequence set $\mathbb{U}_j$ over $\mathbb{F}_j$, $j = 1, 2, \ldots, k-1$. $\mathbb{U}_0$ is WG FH sequence set or $\mathbb{U}_h$ is general WG FH sequence set satisfying $\mathbb{D}$, $1 \leq h \leq k-1$. The WG FH sequence set $\mathcal{S}_7$ can be designed by the fourth generalized method.

Theorem 12: Let $M = M_1 \cdots M_{k-1}$, $p = p_0p_1 \cdots p_{k-1}$, $L_7M = Ip + J$ and $L_7 = 2p + r$, $0 \leq J \leq p - 1$. According to (5), we have

(1). $\mathcal{S}_7$ is an almost optimal general WG FH sequence set $(L_7, M, p, 2)$ if $p + r < \frac{(p-3)ML_7-(p-J)J}{(ML_7-3)M} < 2p + r$, $\min\{\mathcal{M}(\mathbb{U}_1), \mathcal{M}(\mathbb{U}_2), \ldots, \mathcal{M}(\mathbb{U}_{k-1})\} = 2$, $\mathcal{M}(\mathbb{U}_i, 0) \leq 2$, $i = 1, 2, \ldots, k-1$.

(2). $\mathcal{S}_7$ is an optimal general WG FH sequence set $(L_7, M, p, 1)$ if $\min\{\mathcal{M}(\mathbb{U}_1), \mathcal{M}(\mathbb{U}_2), \ldots, \mathcal{M}(\mathbb{U}_{k-1})\} = 1$, $\mathcal{M}(\mathbb{U}_i, 0) \leq 1$, $i = 1, 2, \ldots, k-1$.

Example 2: Select any WG FH sequence set (28,7,7,1,17) denoted as $R = \{R_0, \ldots, R_6\}$ over F_7, such that

$$R_0 = (1, 6, 4, 2, 0, 4, 1, 5, 2, 6, 3, 0, 3, 6, 2, 5, 1, 4, 0, 4, 1, 5, 2, 6, 3, 5, 2, 5);$$
$$\cdots\cdots$$
$$R_6 = (0, 5, 3, 1, 6, 3, 0, 4, 1, 5, 2, 6, 2, 5, 1, 4, 0, 3, 6, 3, 0, 4, 1, 5, 2, 4, 1, 4).$$

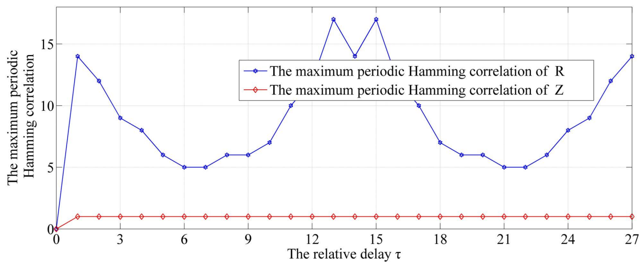

FIGURE 3. **The maximum PHC of *R* and *Z* in example 2.**

It is clear that $\mathcal{M}(R, 0) = 0$ and $\mathcal{M}(R) = 17$, the HC properties are obviously bad.

Select an FH sequence set (28,29,29,1) denoted as $T = \{T_0, \ldots, T_{27}, T_{28}\}$ over F_{29} as follows:

$$T_0 = (1, 2, 4, 8, 16, 3, 6, 12, 24, 19, 9, 18, 7, 14, 28, 27, 25, 21, 13, 26, 23, 17, 5, 10, 20, 11, 22, 15);$$
$$\cdots\cdots$$
$$T_{28} = (0, 1, 3, 7, 15, 2, 5, 11, 23, 18, 8, 17, 6, 13, 27, 26, 24, 20, 12, 25, 22, 16, 4, 9, 19, 10, 21, 14).$$

By the third generalized method, the WG FH sequence set $Z = \{Z_{(i,j)} = (Z_{(i,j)}(0), Z_{(i,j)}(1), \ldots, Z_{(i,j)}(27)) | Z_{(i,j)}(t) = 29R_i(t) + T_j(t), 0 \leq t \leq 27, 0 \leq i \leq 6, 0 \leq j \leq 28\}$ over $F_7 \times F_{29} = \{29a + b | a \in F_7, b \in F_{29}\}$ can be obtained:

$$Z_{(0,0)} = (30, 176, 120, 66, 16, 119, 35, 157, 82, 193, 96, 18, 94, 188, 86, 172, 54, 137, 13, 142, 52, \ldots);$$
$$\cdots\cdots$$
$$Z_{(6,28)} = (0, 146, 90, 36, 189, 89, 5, 127, 52, 163, 66, 191, 64, 158, 56, 142, 24, 107, 186, 112, 22, \ldots).$$

As shown in FIGURE 3 and in FIGURE 4, compared with the base sequence set R, the maximum PHC of Z reduce significantly and achieve optimal, the minimum frequency gap d increase obviously, Z is an optimal general WG FH sequence set (28, 203, 203, 49, 1). For every FH sequence $Z_i \in Z$, let the maximum frequency be $z_{i,max}$, the minimum frequency be $z_{i,min}$ and $M = z_{i,max} - z_{i,min} + 1 - |z_i(t+1) - z_i(t)|$, for any adjacent frequencies $z_i(t+1)$ and $z_i(t)$, the minimum value of M and the minimum frequency gap d are shown in FIGURE 3, and basing on which one can check that Z is an optimal special WG FH sequence set (28, 203, 203, 35, 1).

Choose any FH sequence from R, T and Z respectively and measure the complexity of which by Fuzzy Entropy [27], As shown in FIGURE 5, the Fuzzy Entropy [26] of $Z_{(0,0)}$ is greater than that of R_0 and T_0 when the measuring window length equates 3. So, Z has better complexity.

B. CONSTRUCTIONS OF OPTIMAL WG FH SEQUENCE SETS WITH ANY MAXIMUM PHC VALUE

In this section, we will introduce the fifth generalized method to design WG FH sequence sets with any maximum PHC

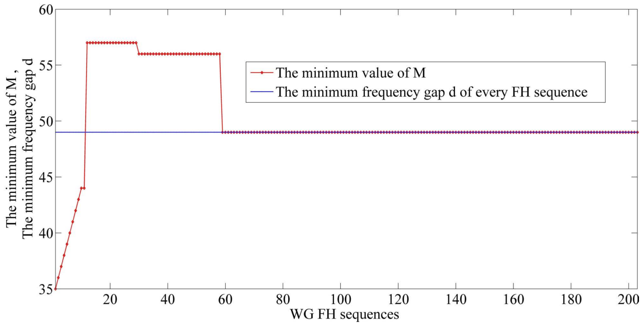

FIGURE 4. The WG property of Z in example 2.

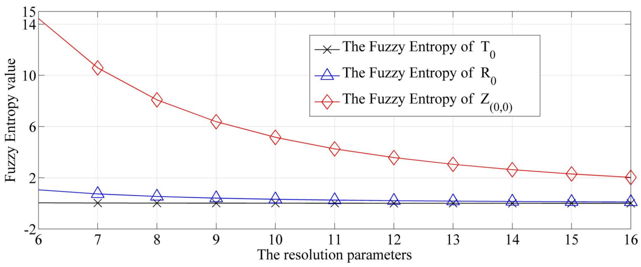

FIGURE 5. The complexity comparisons between R, T and Z in example 2.

value by choosing appropriate original FH sequence set and Solomon's FH sequence set [28]. Then we first introduce the construction of Solomon's FH sequence set.

Lemma 4 (Solomon, [28]:] Let α be a primitive element of finite field $GF(p)$ and $f(x)$ be a primitive polynomials over finite field $GF(p^n)$ with $f(\alpha) = 0$. Write down the nonzero elements of $GF(p^n)$ as the powers of α, such that

$$G_0 = (1, \alpha, \dots, \alpha^{p^n-2}).$$

Basing upon G_0, we design an FH sequence set $(p^n - 1, p^n, p^n, 1)$ denoted as $V = \{V_k = (v_k(0), v_k(1), \dots, v_k(p^n - 2))|k = 0, \dots, p^n - 1\}$ where

$$v_k(t) = G_0(t) + \beta_k, \quad \beta_k \in GF(p^n).$$

for $t = 0, 1, \dots, p^n - 2$.

It is easy to check that $\mathcal{M}_a(s) = 0, \mathcal{M}_c(s) = 1$ and $\mathcal{M}(V, 0) = 0$.

1) THE FIFTH GENERALIZED METHOD

Select any FH sequence set (N, M, q, λ) denoted as $R = \{R_j = (r_j(0), r_j(1), \dots, r_j(N-1))| j = 0, 1, \dots, M-1\}$ over frequency set $\mathbb{F}$ with size q. And R satisfies

(1) $\mathcal{M}(R, 0) = 0$;

(2) $|r_i(0) - r_i(N-1)| \geq 2$ for any i with $0 \leq i \leq M-1$.

Furthermore, select a prime number p satisfying $p^n \geq N$ and design a Solomon' FH sequence set [27] $V = \{V_k = (v_k(0), v_k(1), \dots, v_k(p^n - 2))|k = 0, 1, \dots, p^n - 1\}$. And converts every element $v_k(t)$ to decimal, $0 \leq k \leq p^n - 1$, $0 \leq t \leq p^n - 2$. Basing on R and V, for any positive integer e with $e > q$, we design a desired FHS set $W = \{W_j = (w_j(0), w_j(1), \dots, w_j(N(p^n-1)-1))|j = 0, 1, \dots, M-1\}$ where W_j is defined by (8), as shown at the bottom of the page.

Theorem 13: Let $MN(p^n - 1) = Iqp^n + J$ and $N(p^n - 1) = \lambda qp^n + r$, $0 \leq J \leq qp^n - 1$. According to (5), W is an optimal general WG FH sequence set $(N(p^n-1), M, qp^n, \lambda)$ if $r < \frac{(qp^n-3)MN(p^n-1)-(qp^n-J)J}{(MN(p^n-1)-3)M} < qp^n + r$.

Proof: Assume w_η and w_μ are any two sequences in W. Let $\tau = \tau_1 N + \tau_0$, $0 \leq \tau_0 \leq N - 1$. We have

$$\begin{aligned} H_{w_\eta,w_\mu}(\tau) &= \sum_{t_0=0}^{N-1}\sum_{t_1=0}^{p^n-2} h(r_\eta(t_0) + ev_{t_0}(t_1), \\ &\quad r_\mu(t_0+\tau_0) + ev_{t_0+\tau_0}(t_1+\tau_1)) \\ &= \sum_{t_0=0}^{N-1}\sum_{t_1=0}^{p^n-2} h(r_\eta(t_0), r_\mu(t_0+\tau_0)) \\ &\quad \times h(v_{t_0}(t_1), v_{t_0+\tau_0}(t_1+\tau_1)) \end{aligned}$$

There exist the following three cases:

Case 1: $\eta \neq \mu$, $\tau = 0$. We have

$$\begin{aligned} H_{w_\eta,w_\mu}(\tau) &= \sum_{t_0=0}^{N-1}\sum_{t_1=0}^{p^n-2} h(r_\eta(t_0), r_\mu(t_0)) \\ &\quad \times h(v_{t_0}(t_1), v_{t_0}(t_1)) \\ &= 0 \end{aligned}$$

Case 2: $\tau = iN$, $i = 1, 2, \dots, p^n - 2$. One can check that

$$\begin{aligned} H_{w_\eta,w_\mu}(\tau) &= \sum_{t_0=0}^{N-1}\sum_{t_1=0}^{p^n-2} h(r_\eta(t_0), r_\mu(t_0)) \\ &\quad \times h(v_{t_0}(t_1), v_{t_0}(t_1+\tau_1)) \\ &= 0 \end{aligned}$$

Case 3: $\tau \neq iN$, $i = 1, 2, \dots, p^n - 2$. We have

$$\begin{aligned} H_{w_\eta,w_\mu}(\tau) &= \sum_{t_0=0}^{N-1}\sum_{t_1=0}^{p^n-2} h(r_\eta(t_0), r_\mu(t_0+\tau_0)) \\ &\quad \times h(v_{t_0}(t_1), v_{t_0+\tau_0}(t_1+\tau_1)) \\ &\leq H_{r_\eta,r_\mu}(\tau_0) \end{aligned}$$

Let $t = t_1 N + t_0$, $0 \leq t_0 \leq N - 1$. Now we analyse the WG property of W.

$$W_j = \begin{array}{ccccc} (r_j(0) + ev_0(0), & r_j(1) + ev_1(0), & r_j(2) + ev_2(0), & \dots, & r_j(N-1) + ev_{N-1}(0), \\ r_j(0) + ev_0(1), & r_j(1) + ev_1(1), & r_j(2) + ev_2(1), & \dots, & r_j(N-1) + ev_{N-1}(1), \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_j(0) + ev_0(p^n-2), & r_j(1) + ev_1(p^n-2), & r_j(2) + ev_2(p^n-2), & \dots, & r_j(N-1) + ev_{N-1}(p^n-2)) \end{array} \quad (8)$$

TABLE 1. The parameters of new optimal WG FH sequence sets generated by generalized methods.

Th.	WG FH sequence sets	Constraints
Th.4	$(L_1, N_0N_1\cdots N_{k-1}, p_0p_1\cdots p_{k-1}, 2)$	$\mathcal{M}(X_0, 0) = 2$
	$(L_1, N_0N_1\cdots N_{k-1}, p_0p_1\cdots p_{k-1}, 1)$	$\mathcal{M}(X_0, 0) \le 1$
Th.5	$(L_2, M_0M_1\cdots M_{k-1}, p_0p_1\cdots p_{k-1}, 2)$	$\max\{\mathcal{M}(\mathbb{X}_1, 0), \mathcal{M}(\mathbb{X}_2, 0), ..., \mathcal{M}(\mathbb{X}_{k-1}, 0)\} = 2$
	$(L_2, M_0M_1\cdots M_{k-1}, p_0p_1\cdots p_{k-1}, 1)$	$\max\{\mathcal{M}(\mathbb{X}_1, 0), \mathcal{M}(\mathbb{X}_2, 0), ..., \mathcal{M}(\mathbb{X}_{k-1}, 0)\} \le 1$
Th.6	$(L_3, M_1M_2\cdots M_{k-1}, p_0p_1\cdots p_{k-1}, 2)$	$\min\{\mathcal{M}(Y_1), \mathcal{M}(Y_2), ..., \mathcal{M}(Y_{k-1})\} = 2$, $\mathcal{M}(Y_i, 0) \le 2, i = 1, 2, ..., k-1.$
	$(L_3, M_1M_2\cdots M_{k-1}, p_0p_1\cdots p_{k-1}, 1)$	$\min\{\mathcal{M}(Y_1), \mathcal{M}(Y_2), ..., \mathcal{M}(Y_{k-1})\} = 1$, $\mathcal{M}(Y_i, 0) \le 1, i = 1, 2, ..., k-1.$
Th.9	$(L_4, M_0M_1\cdots M_{k-1}, p_0p_1\cdots p_{k-1}, 2)$	$\mathcal{M}(Z_0, 0) = 2$
	$(L_4, M_0M_1\cdots M_{k-1}, p_0p_1\cdots p_{k-1}, 1)$	$\mathcal{M}(Z_0, 0) \le 1$
Th.10	$(L_5, N_0N_1\cdots N_{k-1}, p_0p_1\cdots p_{k-1}, 2)$	$\max\{\mathcal{M}(\mathbb{Z}_1, 0), ..., \mathcal{M}(\mathbb{Z}_{k-1}, 0)\} = 2$
	$(L_5, N_0N_1\cdots N_{k-1}, p_0p_1\cdots p_{k-1}, 1)$	$\max\{\mathcal{M}(\mathbb{Z}_1, 0), ..., \mathcal{M}(\mathbb{Z}_{k-1}, 0)\} \le 1$
Th.11	$(N, {M_0}^{k_0}{M_1}^{k_1}, p^{k_0}q^{k_1}, 2)$	$\mathcal{M}(\mathbb{L}_1, 0) = 2$
	$(N, {M_0}^{k_0}{M_1}^{k_1}, p^{k_0}q^{k_1}, 1)$	$\mathcal{M}(\mathbb{L}_1, 0) \le 1$
Th.12	$(L_7, M_1\cdots M_{k-1}, p_0p_1\cdots p_{k-1}, 2)$	$\min\{\mathcal{M}(\mathbb{U}_1), \mathcal{M}(\mathbb{U}_2), ..., \mathcal{M}(\mathbb{U}_{k-1})\} = 2$, $\mathcal{M}(\mathbb{U}_i, 0) \le 2, i = 1, 2, ..., k-1$
	$(L_7, M_1\cdots M_{k-1}, p_0p_1\cdots p_{k-1}, 1)$	$\min\{\mathcal{M}(\mathbb{U}_1), \mathcal{M}(\mathbb{U}_2), ..., \mathcal{M}(\mathbb{U}_{k-1})\} = 1$, $\mathcal{M}(\mathbb{U}_i, 0) \le 1, i = 1, 2, ..., k-1$
Th.13	$(N(p^n-1), M, qp^n, \lambda)$	$\mathcal{M}(R, 0) = 0, \|r_i(0) - r_i(N-1)\| \ge 2, i = 1, 2, ..., M-1$

Case 1: $t \ne iN-1, i = 1, 2, \ldots, p^n-2$. $w_l(t) = r_l(t_0) + ev_{t_0}(t_1)$ and $w_l(t+1) = r_l(t_0+1) + ev_{t_0+1}(t_1)$ are any two adjacent frequencies in any FH sequence $w_l \in W$, $r_l(t_0) \in \mathbb{F}$, $r_l(t_0+1) \in \mathbb{F}$, $v_{t_0}(t_1) \in GF(p^n)$, $v_{t_0+1}(t_1) \in GF(p^n)$. Suppose $w_l(t) < w_l(t+1)$, the frequency gap $d_{t+1,t}$ between $w_l(t)$ and $w_l(t+1)$ can be calculated by

$$d_{t+1,t} = r_l(t_0+1) - r_l(t_0) + ev_{t_0+1}(t_1) - ev_{t_0}(t_1)$$

Case 1. 1: $r_l(t_0+1) \ne r_l(t_0)$. One can has

$$d_{t+1,t} \ge -(q-1) + e \ge 2.$$

Case 1. 2: $r_l(t_0+1) = r_l(t_0)$. We have

$$d_{t+1,t} = ev_{t_0+1}(t_1) - ev_{t_0}(t_1) \ge e.$$

Case 2: $t = iN-1, i = 1, 2, \ldots, p^n-2$. $t_0 = \langle t\rangle_N = N-1$, $\langle t_0+1\rangle_N = 0$, $w_l(t) = r_l(N-1) + ev_{N-1}(t_1)$ and $w_l(t+1) = r_l(0) + ev_0(t_1+1)$ are any other two adjacent frequencies in any FH sequence $w_l \in W$, $r_l(N-1) \in \mathbb{F}$, $r_l(0) \in \mathbb{F}$, $v_{N-1}(t_1) \in GF(p^n)$, $v_0(t_1+1) \in GF(p^n)$. Suppose $w_l(t) < w_l(t+1)$, the frequency gap $d_{t+1,t}$ between $w_l(t)$ and $w_l(t+1)$ can also be calculated by

$$d_{t+1,t} = r_l(0) - r_l(N-1) + ev_0(t_1+1) - ev_{N-1}(t_1)$$

Case 2. 1: $v_0(t_1+1) = v_{N-1}(t_1)$. Since $|r_i(0) - r_i(N-1)| \ge 2$ for any i with $0 \le i \le M-1$, one can have

$$d_{t+1,t} \ge 2.$$

Case 2. 2: $v_0(t_1+1) \ne v_{N-1}(t_1)$. We have

$$d_{t+1,t} \ge -(q-1) + e \ge 2.$$

So, the conclusion is true. □

Example 3: Select an FH sequence set (25,4,5,5) denoted as $R = \{r_0, r_1, r_2, r_3\}$ over $GF(5)$ as follows:

$r_0 = (0, 1, 1, 1, 1, 1, 1, 0, 2, 3, 4, 1, 2, 4, 0, 3, 1, 3, 0, 4, 2, 1, 4, 3, 2);$

......

$r_3 = (1, 4, 4, 0, 0, 4, 4, 1, 3, 4, 3, 4, 0, 1, 2, 2, 4, 2, 2, 1, 0, 4, 3, 4, 3).$

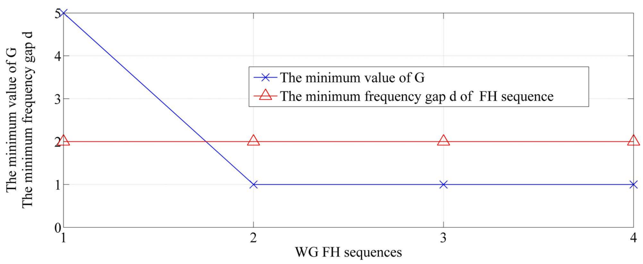

FIGURE 6. The WG property of W in example 3.

Let α be a primitive element of the finite field $GF(3)$ and $f(x) = x^3 + 2x^2 + 1$ be a primitive polynomials over the finite field $GF(3^3)$ with $f(\alpha) = 0$. Design an FH sequence set [28] (26,27,27,1) denoted as $V = \{v_0, v_1, \ldots, v_{26}\}$, and the decimal version of which is

$v_0 = (1, 3, 9, 5, 15, 23, 13, 17, 20, 4, 12, 14, 11, 2, 6, 18, 7, 21, 16, 26, 22, 10, 8, 24, 25, 19);$

......

$v_{26} = (20, 22, 1, 21, 7, 12, 5, 6, 9, 23, 4, 3, 0, 18, 25, 10, 26, 13, 8, 15, 14, 2, 24, 16, 17, 11).$

By the fifth generalized method, the desirable WG FH sequence set $W = \{w_0, w_1, w_2, w_3\}$ can be obtained:

$w_0 = (10, 21, 41, 101, 31, 161, 211, 140, 152, 183, 54, 131, 122, \ldots);$

......

$w_3 = (11, 24, 44, 100, 30, 164, 214, 141, 153, 184, 53, 134, 120, \ldots).$

For every FH sequence $w_i \in W$, let the maximum frequency be $w_{i,max}$, the minimum frequency be $w_{i,min}$ and

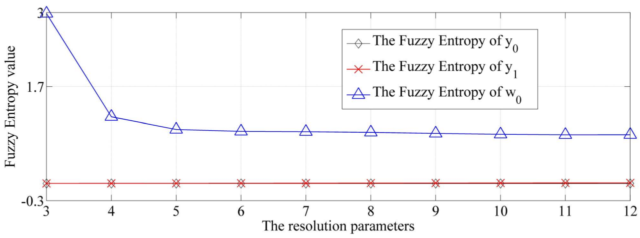

FIGURE 7. The complexity comparisons between y_0, y_1 and w in example 3.

$G = w_{i,max} - w_{i,min} + 1 - |w_i(t+1) - w_i(t)|$, for any two adjacent frequencies $w_i(t+1)$ and $w_i(t)$, the minimum value of G and the minimum frequency gap d are shown in FIGURE 4. It is easy to check that W is an optimal general WG FH sequence set $(650, 4, 135, 2, 5)$ according to the bound (5).

We design single WG FH sequence y_0 [20] with parameter (653,653,4,1) and single WG FH sequence y_1 [23] with parameter (653,653,317,1). When the measuring window length equates 3, the Fuzzy Entropy [27] of y_0, y_1 and w_0 are shown in FIGURE 7, basing on which, one can check that w has better complexity even though the minimum frequency gap of which is less than that of y_0 and y_1.

IV. CONCLUSION

In this paper, we first made clear the relationships between the WG FH sequence theoretical bounds [27], and according to which, presented five generalized methods to construct new classes of WG FH sequence sets with optimal maximum periodic HC. All designed WG FH sequence sets have new parameters, as shown in Table 1. All these designs can provide more optimal WG FH sequences for secure and reliable FH communication system.

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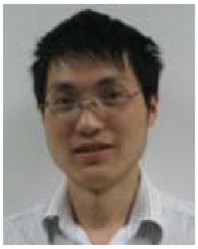

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