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A Computationally Efficient Connectivity Index for Weighted Directed Graphs With Application to Underwater Sensor Networks

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ABSTRACT This paper investigates the global connectivity of complex networks with random links. An expected communication graph with weighted edges is used to model the network. The notion of weighted vertex connectivity (WVC) introduced in the literature as a generalization of the notion of vertex connectivity, is known to be effective in measuring the connectivity of this type of network. However, given the computational complexity of the WVC, a numerically efficient approximate measure for that is more desirable. In this paper, a polynomial-time approximation to the WVC is derived, which is less conservative than the previously introduced approximate measure. It is shown that under some conditions the proposed approximation is identical to the WVC. Simulation results demonstrate the usefulness of the proposed measure.

INDEX TERMS Graph connectivity, directed weighted graphs, sensor networks.

I. INTRODUCTION

Sensor networks are increasingly used in a broad range of civilian and military applications such as environmental monitoring [1]–[4], target tracking [5]–[9], and surveillance [10]–[13]. In some applications, the network comprises a combination of static and mobile sensors exchanging data without the support of a pre-existing infrastructure [14]. Recent research has focused on sensor networks modeling and analysis, as well as the efficient design of algorithms operating on such networks in order to achieve a desired global objective [15]–[17]. As an emerging application, underwater acoustic sensor networks (UASN) consist of a number of fixed and mobile sensors. UASNs are capable of exchanging data using acoustic communication over long ranges [18]–[20]. A typical objective for such networks is to aggregate data for a variety of applications including underwater exploration, ocean sampling, climate reporting, and disaster prevention, to name only a few [21]–[25].

A particle swarm optimization (PSO) algorithm is used in [26] for adaptive equalization of underwater acoustic communication channels. The independent from channel characteristics and faster convergence are recognized as two benefits of the PSO algorithm compared to other methods. Different approaches based on machine learning for identification of underwater acoustic communication channels are proposed in [27]–[29]. A neural network-based method is employed in [27] for signal processing of underwater acoustic communication channels. The authors in [28] investigate a measured sea trial data-set to find the suitable link adaptation procedure using a rule-based strategy. The logistic regression algorithm is used in [29] to predict the communication channel quality between a transmitter and a receiver in the underwater environment. The predicted communication quality is then used to minimize the energy consumption of underwater acoustic transmitters and prolong the network lifetime. A variety of adaptive routing algorithms for underwater acoustic communication channels are presented in [30]–[32], which utilize different machine learning procedures. An adaptive deep Q-network-based energy- and the latency-aware routing protocol to prolong network lifetime in UASNs is proposed in [30] with less energy consumption and strict latency limitations. Moreover, a reinforcement learning-based congestion-avoided routing protocol is developed

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in [31] to reduce the end-to-end delay and energy consumption in UASNs by employing a hybrid broadcast and unicast communication mechanism to reduce network overhead.

The connectivity of underwater sensors is critical in marine activities such as scientific data collection, underwater exploration, and environmental monitoring. Hence, measuring the network connectivity is very important and helps the operators take the appropriate actions (e.g. replacing the weak sensors or increasing the transmission power of sensors, etc.) for improving network connectivity and, therefore, data/information diffusion. Note that the complexity of a good algorithm for calculating network connectivity should be low such that it can be used in real-time applications because this type of sensor is mainly battery-powered. Unlike terrestrial networks, UASNs suffer from a highly uncertain and unpredictable communication channel, which is strongly influenced by multi-path propagation, temperature and salinity fluctuations, scattering and reverberation, variation of sound speed profile, and underwater currents [33]–[36]. Moreover, these sources of uncertainty cannot be measured globally and they vary both over time and space, resulting in highly temporal, spatially variable, and uncontrollable acoustic communication links between nodes [33]. Such dynamic and unmeasured perturbations to the communication links make random graphs good candidates to model underwater acoustic sensor networks [37], [38]. Besides their variable communication links, sensors in such networks are usually battery-operated; thus, node deletion due to limited battery life, and likewise node addition will occur. As a result, the structure and composition of this type of networks may be subject to abrupt changes [39].

On the other hand, consensus, distributed estimation, data aggregation, and many other algorithms running on networks (and specially their convergence time) are highly dependent on the connectivity of the network's expected communication graph [40]–[43]. Various measures of connectivity with distinct features are proposed for different types of networks. Note that, in the field of underwater acoustics, the relationship between the distance between two nodes and their connectivity is neither linear nor straightforward. For example, there are shadow zones preventing communications within 1 km, there are ducts allowing communications up to 3000 km, and anything between those two extreme cases will exist. Also, there are seasonal dependencies and continuous temporal changes. Therefore, the requested correlation is beyond the scope of this paper as it would require years of effort while considering multiple locations and time-of-year. Moreover, with the ongoing melting of polar ice caps, the heat and material exchanges between the oceans and the Earth's atmosphere may make such correlation rapidly obsolete. An important characteristic of network vertex connectivity is that it reflects the robustness of the network to node failure (or removal), making it a very useful connectivity measure for underwater sensor networks. Vertex connectivity can be computed in polynomial-time [44] by finding multiple vertex-disjoint paths between any pair of nonadjacent vertices of the graph representing the network [45]. Thus, the efficient characterization of multiple disjoint paths in the graph can potentially improve network energy conservation, load balancing, and robustness to failure [46]. In [47], Cardei *et al.* propose a fault-tolerant topology control procedure which minimizes the total power consumption of a sensor network while maintaining a certain degree of vertex connectivity.

The authors in [48] introduce the notion of the *weighted vertex connectivity (WVC)*, extending the results of [49] to the expected communication graph of random networks. An approximate weighted vertex connectivity (AWVC) measure was also introduced in [48], which provides a lower bound on the WVC by sequentially applying a polynomial-time shortest path algorithm between different pairs of vertices. However, the AWVC measure of [48] has some shortcomings in networks with certain topologies. For instance, it can be conservative and not accurate. To address these shortcomings, another approximate weighted vertex connectivity measure is proposed in this paper with a polynomial-time computation, which is less conservative than that in [48]. The paper also introduces a new weight matrix whose update depends on the network size after the removal of the internal nodes. The new approximation measure is compared with that in [48] in terms of accuracy and computational efficiency. Evaluation results confirm that the algorithm proposed in this work is more accurate and computationally efficient, making it suitable for marine activities using underwater sensor networks.

The paper is organized as follows. Section II introduces preliminaries and useful definitions. The problem formulation and motivation are presented in Section III. Section IV introduces the new connectivity measure and provides the algorithm to compute it. Section V elaborates on the effectiveness of the proposed connectivity measure compared to other measures via simulations. Some concluding remarks are given in Section VI.

II. PRELIMINARIES

Sets $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, represent positive reals and non-negative reals numbers, respectively. The finite set of integers $\{1, 2, \ldots, n\}$ is denoted by $\mathbb{N}_n$. Given a finite set Φ, ϕ^i is its i-th element, $|\Phi|$ is its cardinality, and $\mathcal{P}(\Phi)$ is its power set (i.e. the set of all of its subsets).

A. GRAPH THEORY

Definition 1 ([48]): Consider a random directed graph (digraph) $G = (V, E)$ composed of a finite set of vertices V and a finite set of edges E. Let matrix $\mathbf{P}_G = [p_{ij}]$ represent the existence probability of all directed edges in G, where $p_{ij} \in [0, 1]$ is the probability of the existence of the edge $(j, i) \in E$. Matrix $\mathbf{P}_G$ is referred to as the probability matrix of the random graph G. Define $\mathbf{A} = [a_{ij}]$ as the adjacency matrix of G, where a_{ij} is a binary random variable characterized as:

$$a_{ij} = \begin{cases} 1, & \text{with probability } p_{ij}, \\ 0, & \text{with probability } 1 - p_{ij}. \end{cases} \quad (1)$$

Definition 2 ([48]): The expected graph of a random digraph $G = (V, E)$ is a digraph $\hat{G} = (V, \hat{E})$ whose weighted adjacency matrix is represented by $\hat{\mathbf{A}} = [\hat{a}_{ij}]$, where $\hat{a}_{ij} = p_{ij}$ for any pair of distinct nodes $i, j \in V$. Note that $\hat{G}$ and G have the same vertex set, and

$$\hat{E} = \{(i,j) \in V \times V \mid p_{ji} \neq 0\}. \tag{2}$$

The *communication graph* of a network composed of n sensors is modeled as a random digraph $G = (V, E)$ with node set V and edge set E given by:

$$V = \mathbb{N}_n, \tag{3a}$$

$$E = \{(i,j) \in V \times V \mid a_{ji} = 1\}. \tag{3b}$$

Definition 3 ([50]): Consider a directed graph $G = (V, E)$ and two vertices $i, j \in V$. The *shortest path* from i to j is defined as a directed path from vertex i to vertex j with the minimum sum of edge weights.

B. ESTIMATION OF THE EXPECTED COMMUNICATION GRAPH

A distributed adaptive procedure is introduced next for estimating the expected graph of a random communication network (expected communication graph) from the viewpoint of each sensor. In the UASN under study, at most one sensor at any time instant is allowed to broadcast its data in order to avoid interference. Then, a time interval with a pre-specified length is considered from which distinct time slots are assigned to different nodes to broadcast their estimate of the expected communication graph. The acoustic propagation time from any node to its neighboring nodes is taken into account by appropriately selecting the length of each time slot. Moreover, the estimate of $\hat{G}$ is updated by each node accordingly before its broadcasting time starts, using its previous estimate and the information it has received from other nodes since its last broadcast. Given that the expected communication graph $\hat{G}$ is completely characterized by the probability matrix $\mathbf{P}$, an efficient procedure needs to be developed to estimate the probability matrix of the network from the view point of each node.

Let $X(q)$ be a binary random variable with a Bernoulli distribution at discrete time instant $q \in \mathbb{N}$, i.e. $X(q) \in \{0, 1\}$. Assume that $p_0(q)$ and $p_1(q)$ denote, respectively, the probabilities of two complementary events $X = 0$ and $X = 1$ at time q. The binary random variable $X(q)$ can then be described as:

$$X(q) = \begin{cases} 1, & \text{with probability } p_1(q), \\ 0, & \text{with probability } p_0(q), \end{cases} \tag{4}$$

where $0 \leq p_i(q) \leq 1$, $i \in \{0, 1\}$, and $p_0(q) + p_1(q) = 1$. Let also $\hat{X}(q)$ denote an estimate of $X(q)$ such that:

$$\hat{X}(q) = \begin{cases} 1, & \text{with probability } \hat{p}_1(q), \\ 0, & \text{with probability } \hat{p}_0(q), \end{cases} \tag{5}$$

where the expectation of the estimated probabilities $\hat{p}_0(q)$ and $\hat{p}_1(q)$ are guaranteed to converge asymptotically to $p_0(q)$ and $p_1(q)$, respectively, using the estimation procedure in [51]. The corresponding update rule is given by:

$$\hat{p}_i(q+1) = \begin{cases} (1-\alpha)\hat{p}_i(q) + \alpha, & \text{if } X(q) = i, \\ (1-\alpha)\hat{p}_i(q), & \text{if } X(q) \neq i, \end{cases} \tag{6}$$

for $i \in \{0, 1\}$, where $\alpha \in (0, 1)$ represents the learning rate of the estimation method [51]. Note that the learning rate numerically specifies the impact of new information on estimating the elements of the probability matrix, and it accepts a real value greater than zero and less than 1. A larger α means that the newly-observed information has a higher impact on the overall estimate of the probability matrix elements than the information in the previous steps, and vice-versa. Based on the update rule (6), Algorithm 4 proposed in [48] is employed in this work to obtain an estimation of the probability matrix $\mathbf{P}$ from the viewpoint of every node. Algorithm 5 from [48] is then utilized by each node to update its estimate of the expected communication graph according to the information it receives from its neighbors during every broadcasting cycle.

III. PROBLEM STATEMENT

Vertex connectivity (VC) was originally introduced as a measure of network robustness to node failure [49]. For a strongly connected graph, vertex connectivity is, in fact, the minimum number of vertices that need to be removed in order for the graph to lose strong connectivity. The *local* VC associated with the pair of nodes $i, j \in V$, where $i \neq j$, is denoted by $\kappa_{i,j}(G)$, and is defined as the minimum number of nodes whose removal eliminates all directed paths from node i to j. Menger's theorem [49] also stipulates that $\kappa_{i,j}(G)$ is the maximum number of vertex-disjoint paths from node i to node j in G. With $N_{i,j}(G)$ denoting the maximum number of vertex-disjoint paths from i to j in graph G, one can write:

$$\kappa_{i,j}(G) = \begin{cases} N_{i,j}(G), & \text{if } (i,j) \notin E, \\ |V| - 1, & \text{if } (i,j) \in E. \end{cases} \tag{7}$$

The *global* VC, denoted by $\kappa(G)$, is then defined as the minimum *local* VC associated with every pair of distinct nodes, i.e.: In other words:

$$\kappa(G) = \min_{i,j \in V,\, i \neq j} \kappa_{i,j}(G). \tag{8}$$

Consider now a random network modeled by graph G in which a binary random variable describes every communication link. The corresponding expected communication graph $\hat{G}$ is a deterministic weighted graph where the weight associated with the link from node i to node j is the (j, i) element of the probability matrix $\mathbf{P}$. More precisely, p_{ji} represents the existence probability of an edge from vertex v_i to vertex v_j. Note that if there is no directed edge from v_i to v_j, then the corresponding element of matrix $\mathbf{P}$ is zero. Note that, in the UASN application, the weight of each link depends on different parameters such as the transmitters' power, the distance between nodes, environmental conditions, etc. The weighted vertex connectivity (WVC) defined in [48] reflects the joint effect of path reliability and robustness to node failure.

Remark 1: It is worth mentioning that there is no universally-accepted underwater acoustic-communication channel model. This adds another layer of approximation to the problem under investigation. Furthermore, any communication model has a stochastic/probabilistic component, which reflects the probabilities of node-to-node connection at a higher level of abstraction (while purposefully associating communications with errors to the "no communication" cases). Note that random graphs are a good candidate for modelling underwater acoustic sensor networks. In this type of graph, a binary random variable is used to model the successful transfer of a data package from a transmitter node to a receiver node in an unpredictable and uncertain marine environment.

Definition 4: [48] Given an expected communication graph $\hat{G}$ with probability matrix $\mathbf{P} = [p_{ij}]$, let $\Pi_{i,j}$ represent the set of all directed paths from node i to node j whose lengths are greater than one. Let also $\pi^k_{i,j} = \{v^k_0, v^k_1, \ldots, v^k_{m_k-1}, v^k_{m_k}\}$ be the k-th elements of $\Pi_{i,j}$. Note that $\pi^k_{i,j}$ is a directed path of length $m_k > 1$ from node i to node j such that $v^k_0 = i$, $v^k_{m_k} = j$, and $(v^k_{l-1}, v^k_l) \in \hat{E}$ for all $l \in \mathbb{N}_{m_k}$. Define the multiplicative weight of path $\pi^k_{i,j}$, denoted by $W(\pi^k_{i,j})$, as:

$$W(\pi^k_{i,j}) = \prod_{l=1}^{m_k} p_{v^k_l v^k_{l-1}}. \tag{9}$$

where $p_{v^k_l v^k_{l-1}}$ represents the existence probability of an edge from node v^k_{l-1} to node v^k_l. Because all edges are characterized by a set of independent binary random variables, the multiplicative weight defined by (9) can be interpreted as the operational probability of a directed path from node i to node j.

Define $\hat{\kappa}_{i,j}(\hat{G})$ as the maximum of the summation of the multiplicative weights of all the vertex-disjoint paths from node i to node j. With $\Pi_{i,j}$ from Definition 4 and its power set $\mathcal{P}(\Pi_{i,j})$, let $\hat{\mathcal{P}}(\Pi_{i,j}) \subseteq \mathcal{P}(\Pi_{i,j})$ represent the set of all nonempty subsets of $\Pi_{i,j}$ forming vertex-disjoint paths from node i to node j. Then $\hat{\kappa}_{i,j}(\hat{G})$ can mathematically be expressed as:

$$\hat{\kappa}_{i,j}(\hat{G}) = \begin{cases} \displaystyle\sum_{k=1}^{|\hat{\Pi}_{i,j}|} W(\hat{\pi}^k_{i,j}), & \text{if } (i,j) \notin \hat{E}, \\ \max\Big((|\hat{V}|-1)p_{ji}, p_{ji} \\ \quad + \displaystyle\sum_{k=1}^{|\hat{\Pi}_{i,j}|} W(\hat{\pi}^k_{i,j})\Big), & \text{if } (i,j) \in \hat{E}, \end{cases} \tag{10}$$

where

$$\hat{\Pi}_{i,j} = \underset{\Pi \in \hat{\mathcal{P}}(\Pi_{i,j})}{\operatorname{argmax}} \sum_{k=1}^{|\Pi|} W(\pi^k). \tag{11}$$

The path set Π in the above summation is defined as $\{\pi^k \mid k \in \mathbb{N}_{|\Pi|}\}$ where $|\Pi|$ denotes the cardinality of Π. As defined above, $\hat{\kappa}_{i,j}(\hat{G})$ is the local WVC measure between nodes i and j in graph $\hat{G}$. The weighted vertex connectivity degree of the expected graph $\hat{G}$, represented by $\hat{\kappa}(\hat{G})$, is the global connectivity measure of the graph, and is described as:

$$\hat{\kappa}(\hat{G}) = \min_{i,j \in \hat{V}, i \neq j} \hat{\kappa}_{i,j}(\hat{G}). \tag{12}$$

Using the above development, the computation of the local WVC measure reduces to a maximum weight clique problem, which is known to be NP-hard. Thus, a computationally efficient approximate local WVC measure is needed. To this end, the notion of the *most reliable path* is defined next.

Definition 5: The most reliable path from node i to node j in $\hat{G}$ is defined as a path belonging to $\Pi_{i,j}$ with the largest multiplicative weight. This path will hereafter be denoted by $\pi^r_{i,j}$ [52].

Analogously to the notion of the local WVC, the approximate local weighted vertex connectivity (AWVC) measure associated with the directed paths from node i to node j in the expected communication graph $\hat{G}$ is introduced next. To this end, it is required as the first step to find the multiplicative weight of the most reliable path from i to j. Using the properties of the logarithm function, it can be shown that the most reliable path from i to j in $\hat{G}$ with the weight matrix $\mathbf{P} = [p_{ij}]$ is equivalent to the shortest path connecting i to j in $\hat{G}$ with the modified weight matrix $\overline{\mathbf{P}} = [\overline{p}_{ij}]$, where $\overline{p}_{ij} = -\ln(p_{ij})$ for all $i, j \in \hat{V}$, $i \neq j$ [48].

The next step involves removing from $\hat{G}$ the most reliable path's internal nodes along with their adjacent edges. If no path exists from i to j in the resultant reduced graph, then the local AWVC measure is the multiplicative weight of $\pi^r_{i,j}$. Otherwise, the remaining most reliable path from i to j in the reduced graph is identified and its multiplicative weight is calculated. This procedure continues iteratively until no more path exists from i to j in the modified $\hat{G}$. The local AWVC is then obtained as the sum of the multiplicative weights of all the previously obtained most reliable paths denoted by $\bar{\kappa}_{i,j}(\bar{G})$.

Let $W(\pi^{r,k}_{i,j})$ denote the multiplicative weight of the most reliable path from i to j after removing the internal nodes (and their corresponding adjacent edges) of $k-1$ previously found most reliable paths $\pi^{r,l}_{i,j}$, $l \in \mathbb{N}_{k-1}$, from $\hat{G}$. Let also l_{ij} denote the maximum number of the vertex-disjoint most reliable paths directed from i to j with length greater than one such that after deletion of their internal nodes no path will remain from i to j in $\hat{G}$. Then, one can formulate the local AWVC measure as:

$$\overline{\kappa}_{i,j}(\hat{G}) = \begin{cases} \displaystyle\sum_{k=1}^{l_{ij}} W(\pi^{r,k}_{i,j}), & \text{if } (i,j) \notin \hat{E}, \\ \max\Big((|\hat{V}|-1)p_{ji}, p_{ji} \\ \quad + \displaystyle\sum_{k=1}^{l_{ij}} W(\pi^{r,k}_{i,j})\Big), & \text{if } (i,j) \in \hat{E}. \end{cases} \tag{13}$$

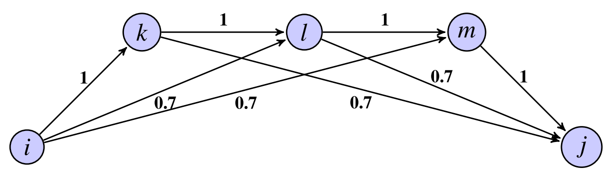

FIGURE 1. The expected graph of the random network of Example 1.

The global AWVC measure of $\hat{G}$, $\overline{\kappa}(\hat{G})$, is accordingly defined as:

$$\overline{\kappa}(\hat{G}) = \min_{i,j \in \hat{V},\, i \neq j} \overline{\kappa}_{i,j}(\hat{G}). \tag{14}$$

The AWVC defined in [48] (and described by (13)) is, in fact, a lower bound on the WVC given by (10). However, this lower bound can be conservative as illustrated by the next example.

Example 1: The expected communication graph $\hat{G}$ of a random network given in Figure 1. In this graph representation of the network, nodes i, j, k, l and m represent underwater sensors and the weight values 0.7 and 1 represent the probability of the existence of communication links between such sensors. Moreover, the direction of the arrowheads indicates the directivity of such node-to-node communications, as described above. More precisely, for example, weight 0.7 of the edge connecting node i to node m means that there is a communication link from sensor i to sensor m with the probability of 0.7, and weight 1 of the edge connecting node i to node k implies that there is certainly a communication link from sensor i to sensor k. The objective is to obtain the local AWVC measure $\overline{\kappa}_{i,j}(\hat{G})$, given by equation (13), associated with the directed paths from node i to node j. For this example $\Pi_{i,j} = \{\pi^1_{i,j}, \pi^2_{i,j}, \pi^3_{i,j}, \pi^4_{i,j}, \pi^5_{i,j}, \pi^6_{i,j}\}$, where $\pi^1_{i,j} = \{i, k, j\}$, $\pi^2_{i,j} = \{i, l, j\}$, $\pi^3_{i,j} = \{i, m, j\}$, $\pi^4_{i,j} = \{i, k, l, j\}$, $\pi^5_{i,j} = \{i, l, m, j\}$, and $\pi^6_{i,j} = \{i, k, l, m, j\}$. It can also be verified that $\pi^6_{i,j}$ is the most reliable path from i to j in $\hat{G}$, i.e., $\pi^r_{i,j} = \pi^6_{i,j}$, with $W(\pi^r_{i,j}) = 1$. After removing the internal nodes of $\pi^r_{i,j}$ in $\hat{G}$, no more path is left from i to j. Therefore, according to (13), $\overline{\kappa}_{i,j}(\hat{G}) = 1$. On the other hand, from equation (10), $\hat{\Pi}_{i,j} = \{\pi^1_{i,j}, \pi^2_{i,j}, \pi^3_{i,j}\}$, and hence $\hat{\kappa}_{i,j}(\hat{G}) = W(\pi^1_{i,j}) + W(\pi^2_{i,j}) + W(\pi^3_{i,j}) = 0.7 + 0.49 + 0.7 = 1.89$.

The above example demonstrates that the AWVC defined in [48] can be a conservative approximation of the WVC, for the network of Figure 1. In the next section, a less conservative approximation of the local WVC metric is presented to address this shortcoming. In addition, two algorithms are developed to measure the AWVC of the network with a fixed topology via a low complexity technique such that it can be used in real-time applications.

IV. AN ALTERNATIVE APPROACH TO APPROXIMATING THE WVC MEASURE

Algorithm 1 is employed first to obtain a new local measure of connectivity associated with the distinct nodes i and j, denoted by $\tilde{\kappa}_{i,j}(\hat{G})$.

Algorithm 1 A Procedure to Compute the Proposed Local AWVC for the Pair (i, j)

---

begin

1: *Initialization*: Set $\xi = |\hat{V}| - 2$ and $\tilde{\kappa}_{i,j}(\hat{G}) = 0$.
2: *Termination verification step*: If there is no path from node i to node j, then set $\tilde{\kappa}_{i,j}(\hat{G}) = \max\{\tilde{\kappa}_{i,j}(\hat{G}), (|\hat{V}| - 1)p_{ji}\}$ and stop the procedure; otherwise, go to the next step.
3: If there is a direct edge from i to j, then $\tilde{\kappa}_{i,j}(\hat{G}) = p_{ji}$ Update $\hat{G}$ by removing this edge.
4: If $\xi = 1$, then $c = ln(2)$; otherwise set $c = \frac{\ln(\xi)}{\xi - 1}$.
5: Construct the modified weight matrix $\tilde{\mathbf{P}} = [\tilde{p}_{ij}]$, where $\tilde{p}_{ij} = -\ln(p_{ij}) + c$ for all $i, j \in \hat{V}$, $i \neq j$.
6: Find the shortest path $\pi^s_{i,j}$ (from i to j) in $\hat{G}$ according to the modified weight matrix $\tilde{\mathbf{P}}$.
7: Check the length of the shortest path.
  i) If it is equal to $\xi + 1$, then
    ◇ add the multiplicative weight of this path (with the weight matrix $\mathbf{P} = [p_{ij}]$) to $\tilde{\kappa}_{i,j}(\hat{G})$;
    ◇ update $\hat{G}$ by removing the internal nodes of the shortest path and their corresponding adjacent edges from it;
    ◇ update $\xi = \xi - |Int(\pi^s_{i,j})|$, where $Int(\pi^s_{i,j})$ denotes internal nodes of $\pi^s_{i,j}$.
  ii) If it is less than $\xi + 1$, then update $\xi = \xi - 1$ and go to step 2.

After computing the local approximate weighted vertex connectivity for every pair of distinct nodes (i, j), the global AWVC measure of $\hat{G}$, denoted by $\tilde{\kappa}(\hat{G})$, can be derived as follows:

$$\tilde{\kappa}(\hat{G}) = \min_{i,j \in \hat{V},\, i \neq j} \tilde{\kappa}_{i,j}(\hat{G}). \tag{15}$$

The procedure to find the AWVC measure described above is summarized in Algorithm 2.

Note that if the shortest path $\pi^s_{i,j}$ obtained in step 6 of the local connectivity measure procedure of Algorithm 1 satisfies the condition given in step 7(i), then it is guaranteed that no set of vertex-disjoint paths from i to j whose node set belongs to $\{i, j, Int(\pi^s_{i,j})\}$ can be found such that the sum of their multiplicative weights is greater than the multiplicative weight of $\pi^s_{i,j}$ (with the weight matrix $\mathbf{P} = [p_{ij}]$).

In the sequel, it is shown that the approximate weighted vertex connectivity measure introduced here is closer to the exact WVC measure, compared to the approximate measure given in [48].

Lemma 1: Consider two positive integers p and n. If $0 < p \leq n$, then:

$$n^{p-1} \leq p^{n-1}. \tag{16}$$

Proof: A proof by induction is given. As the first step, one can write:

$$n = p \Longrightarrow p^{p-1} \leq p^{p-1}. \tag{17}$$

Algorithm 2 A Procedure to Compute the Proposed AWVC

begin

$\tilde{\kappa}(\hat{G}) = |\hat{V}| - 1$
for all $i, j \in V$ and $i \neq j$ **do**
  $\xi = |\hat{V}| - 2$; $Flag = 1$; $V_{ij} = \hat{V}$; $E_{ij} = \hat{E}$
  $G_{ij} = (V_{ij}, E_{ij})$; $\tilde{\kappa}_{i,j}(G_{ij}) = 0$
  while $Flag = 1$ **do**
    if *there is no path from node i to node j* **then**
      $\tilde{\kappa}_{i,j}(G_{ij}) = \max\{\tilde{\kappa}_{i,j}(G_{ij}), (|\hat{V}| - 1)p_{ji}\}$
      $Flag = 1$
    end if
    if $\xi = 1$ **then**
      $c = ln(2)$
    else
      $c = \frac{\ln(\xi)}{\xi - 1}$
    end if
    Construct the modified weight matrix $\tilde{\mathbf{P}} = [\tilde{p}_{ij}]$, where $\tilde{p}_{ij} = -\ln(p_{ij}) + c$
    Find $\pi^s_{i,j}$ in G_{ij} by considering the modified weight matrix $\tilde{\mathbf{P}}$
    if $|\pi^s_{i,j}| = \xi + 1$ **then**
      $\tilde{\kappa}_{i,j}(G_{ij}) = \tilde{\kappa}_{i,j}(G_{ij}) + W(\pi^s_{i,j})$
      $V_{ij} = V_{ij} \backslash Int(\pi^s_{i,j})$
      Construct G_{ij} as a graph induced by the updated vertex set V_{ij}
      $\xi = \xi - |Int(\pi^s_{i,j})|$
    else if $|\pi^s_{i,j}| < \xi + 1$ **then**
      $\xi = \xi - 1$
    end if
  end while
  $\tilde{\kappa}(\hat{G}) = \min\{\tilde{\kappa}(\hat{G}), \tilde{\kappa}_{i,j}(G_{ij})\}$
end for
return $\tilde{\kappa}(\hat{G})$

Assume that for $n = k \geq p$:

$$k^{p-1} \leq p^{k-1}. \tag{18}$$

Note also that:

$$(k+1)^{p-1} = \sum_{i=0}^{p-1} C(p-1, i) k^{p-1-i}. \tag{19}$$

On the other hand, it can be shown that:

$$C(p-1, i) \leq k^i \quad \forall\, 0 \leq i \leq p-1. \tag{20}$$

It is concluded from (19) and (20) that:

$$(k+1)^{p-1} \leq p k^{p-1}. \tag{21}$$

Finally, from (18) and (21), one arrives at:

$$(k+1)^{p-1} \leq p^k, \tag{22}$$

which is the same as (16). □

Lemma 2: Consider two positive integers m and n. If $1 < m < n$, *then*:

$$\frac{\ln(m)}{(m-1)} > \frac{\ln(n)}{(n-1)}. \tag{23}$$

Proof: From Lemma 1:

$$m^{n-1} > n^{m-1} \Longrightarrow (n-1)\ln(m) > (m-1)\ln(n),$$

or equivalently:

$$\frac{\ln(m)}{(m-1)} > \frac{\ln(n)}{(n-1)}.$$

□

Lemma 3: Consider two positive integers n and k such that $\sum_{i=1}^{k} m_i = n$, *where* $m_i \in \mathbb{N}$ *for all* $i \in \mathbb{N}_k$. *Consider also a real number c greater than or equal to* $\frac{\ln(n)}{n-1}$. *Define the following function*:

$$f(m_1, m_2, \ldots, m_k, n, c) = \sum_{i=1}^{k} \frac{1}{\exp[(n - m_i)c]}. \tag{24}$$

Then the inequality below holds:

$$f(m_1, m_2, \ldots, m_k, n, c) \leq 1. \tag{25}$$

Proof: Since $c \geq \frac{\ln(n)}{n-1}$, it follows from the definition of function f that:

$$f(m_1, m_2, \ldots, m_k, n, c) \leq f(m_1, m_2, \ldots, m_k, n, \frac{\ln(n)}{n-1}) = \sum_{i=1}^{k} \frac{1}{n^{\frac{n-m_i}{n-1}}} \tag{26}$$

Also, since $m_i \leq n$ for all $i \in \mathbb{N}_k$, according to Lemma 1:

$$n^{m_i - 1} \leq {m_i}^{n-1} \Longrightarrow n^{\frac{m_i - 1}{n-1}} \leq m_i \Longrightarrow n^{\frac{m_i - n}{n-1}} \leq \frac{m_i}{n}, \tag{27}$$

or equivalently:

$$\frac{1}{n^{\frac{n-m_i}{n-1}}} \leq \frac{m_i}{n}, \quad \forall i \in \mathbb{N}_k. \tag{28}$$

From (26) and (28) one can conclude that:

$$f(m_1, \ldots, m_i, n, c) \leq \sum_{i=1}^{k} \frac{m_i}{n} = 1, \tag{29}$$

and this completes the proof. □

Theorem 1: Consider graph $\hat{G} = (\hat{V}, \hat{E})$, where $\hat{V} = \mathbb{N}_{n+2}$ $(n > 1)$, $\hat{E} = \{\hat{e}_{ij}\}$ and $\hat{e}_{ij} = p_{ij}$ for $i, j \in \hat{V}$. Construct graph $G = (V, E)$ where $V = \hat{V}$ and $E = \{e_{ij}\}$ such that $e_{ij} = \ln(\frac{1}{p_{ij}}) + c$ and real number $c \geq \frac{\ln(n)}{n-1}$. If the length of the shortest path $\pi^s_{i_0, i_{n+1}}$ between nodes i_0 and i_{n+1} of G is equal to $n+1$, then the local AWVC and WVC measures from i_0 to i_{n+1} in $\hat{G}$ are exactly the same.

Proof: We first prove that the following inequality holds:

$$W(\pi^s_{i_0, i_{n+1}}) \geq \sum_{i=1}^{k} W(\pi_i), \tag{30}$$

where $W(\pi)$ denotes the multiplicative weight of path π in $\hat{G}$, and $\pi_1, \ldots, \pi_k$, are vertex-disjoint paths between i_0 and i_{n+1}.

Let the shortest path between nodes i_0 and i_{n+1} in $\hat{G}$ be denoted by $\pi^s_{i_0,i_{n+1}} = \{i_0, i_1, \ldots, i_n, i_{n+1}\}$. Consider k vertex-disjoint paths between i_0 and i_{n+1} as follows:

$$\pi_1 = \{i_0, i^1_1, i^1_2, \ldots, i^1_{m_1}, i_{n+1}\},$$
$$\pi_2 = \{i_0, i^2_1, i^2_2, \ldots, i^2_{m_2}, i_{n+1}\},$$
$$\vdots$$
$$\pi_k = \{i_0, i^k_1, i^k_2, \ldots, i^k_{m_k}, i_{n+1}\},$$

where $m_1 + m_2 + \cdots + m_k = n$. Let the sum of edge weights in path π of graph G be denoted by $S_W(\pi)$. Since $\pi^s_{i_0,i_{n+1}}$ is the shortest path between nodes i_0 and i_{n+1} in graph G, one can write:

$$S_W(\pi^s_{i_0,i_{n+1}}) \leq S_W(\pi_j), \quad (31)$$

for any $j \in \mathbb{N}_k$, or equivalently:

$$\ln(\frac{1}{p_{i_0 i_1}}) + \ln(\frac{1}{p_{i_1 i_2}}) + \cdots + \ln(\frac{1}{p_{i_n i_{n+1}}}) + (n+1)c \leq \ln(\frac{1}{p_{i_0 i^j_1}}) + \cdots + \ln(\frac{1}{p_{i^j_{m_j} i_{n+1}}}) + (m_j+1)c. \quad (32)$$

It follows from the above inequality that:

$$\ln(\frac{1}{W(\pi^s_{i_0,i_{n+1}})}) + (n - m_j)c \leq \ln(\frac{1}{W(\pi_j)}), \quad \forall j \in \mathbb{N}_k, \quad (33)$$

which can be rewritten as:

$$W(\pi_j) \leq \frac{1}{\exp[(n - m_j)c]} W(\pi^s_{i_0,i_{n+1}}), \quad \forall j \in \mathbb{N}_k. \quad (34)$$

From (34), one arrives at:

$$\sum_{j=1}^{k} W(\pi_j) \leq [\sum_{i=1}^{k} \frac{1}{\exp[(n - m_i)c]}] W(\pi^s_{i_0,i_{n+1}}). \quad (35)$$

The above inequality along with Lemma 3 yields:

$$\sum_{j=1}^{k} W(\pi_j) \leq W(\pi^s_{i_0,i_{n+1}}). \quad (36)$$

From (36) and Algorithm 1, it is now straightforward to show that the local AWVC and WVC measures from i_0 to i_{n+1} in $\hat{G}$ are the same. □

Remark 2: Consider a scenario where the condition given in step 7(i) of Algorithm 1 is satisfied in the first round, i.e., the length of $\pi^s_{i,j}$ is equal to $|\hat{V}| - 1$. According to Theorem 1, the local AWVC and WVC measures from i to j are exactly the same. Note that the problem of finding the local WVC from i to j is NP-hard, in general, while the local AWVC measure can be obtained in polynomial time. Note that the number of different pairs of nodes in a graph with n vertices is of order n^2, and also, for each pair of vertices, we should find the shortest path between them. In addition, the complexity of the shortest path algorithm (e.g., the Dijkstra's algorithm with Fibonacci heap) is of order $O(E + VlogV)$ [53], where V and E are the number of vertices and edges of the digraph, respectively. As a result, the complexity of the proposed algorithm for computing the AWVC measure of a graph with n vertices is at most of order $O(n^4)$.

Lemma 4: Let the length of the shortest path between two distinct nodes i and j in graph G introduced in Theorem 1 with positive integer $x > 1$ and real number $c = \frac{\ln(x)}{x-1}$ be denoted by $L_x(\pi^s_{i,j})$. If positive integer $y > x$, then $L_y(\pi^s_{i,j}) \geq L_x(\pi^s_{i,j})$.

Proof: Let the sum of the edge weights in path π of graph G with $c = \frac{\ln(x)}{x-1}$ be denoted by $S_W(\pi, x)$. Let also the shortest path between nodes i and j of this graph be denoted by $\pi^s_{i,j}(x)$. From the definition of the shortest path, one arrives at the following inequalities:

$$S_W(\pi^s_{i,j}(x), x) \leq S_W(\pi^s_{i,j}(y), x), \quad (37a)$$
$$S_W(\pi^s_{i,j}(y), y) \leq S_W(\pi^s_{i,j}(x), y), \quad (37b)$$

which yield:

$$S_W(\pi^s_{i,j}(x), x) - S_W(\pi^s_{i,j}(x), y) \leq S_W(\pi^s_{i,j}(y), x) - S_W(\pi^s_{i,j}(y), y), \quad (38)$$

or equivalently:

$$L_x(\pi^s_{i,j})[\frac{\ln(x)}{x-1} - \frac{\ln(y)}{y-1}] \leq L_y(\pi^s_{i,j})[\frac{\ln(x)}{x-1} - \frac{\ln(y)}{y-1}]. \quad (39)$$

On the other hand, since $x < y$, it follows from Lemma 2 that:

$$\frac{\ln(x)}{x-1} - \frac{\ln(y)}{y-1} > 0. \quad (40)$$

From (39) and (40), it results that:

$$L_x(\pi^s_{i,j}) \leq L_y(\pi^s_{i,j}).$$

□

Corollary 1: Let the length of the shortest path obtained in the sixth step of Algorithm 1, denoted by $L_\xi(\pi^s_{i,j})$, be equal to ϑ. If $\vartheta < \xi$, then $L_x(\pi^s_{i,j}) < x + 1$ for all $\vartheta \leq x \leq \xi$.

Proof: The proof follows immediately from Lemma 4. □

Remark 3: According to Corollary 1, if $L_x(\pi^s_{i,j}) = \vartheta < \xi$, then one can set $\xi = \vartheta - 1$ in step 7(ii) of Algorithm 1 instead of decreasing ξ one by one and running the procedure repeatedly. This reduces the time complexity of the proposed algorithm significantly.

V. SIMULATION RESULTS

Example 2: The following example demonstrates the procedure to find the AVWC measure for a random network composed of five nodes. Let Fig. 2 represent the expected communication graph $\hat{G}$ of the network.

The values of the local AWVC measures for all pairs of distinct nodes $i, j \in \hat{V}$ in this example are given in Table 1, using the method proposed in the present paper ($\tilde{\kappa}_{i,j}(\hat{G})$) and the one introduced in [48] ($\overline{\kappa}_{i,j}(\hat{G})$). To demonstrate how the value of the last row of Table 1 are obtained, note that $\Pi_{4,5} = \{\pi^1_{4,5}, \pi^2_{4,5}, \ldots, \pi^{14}_{4,5}, \pi^{15}_{4,5}\}$, where

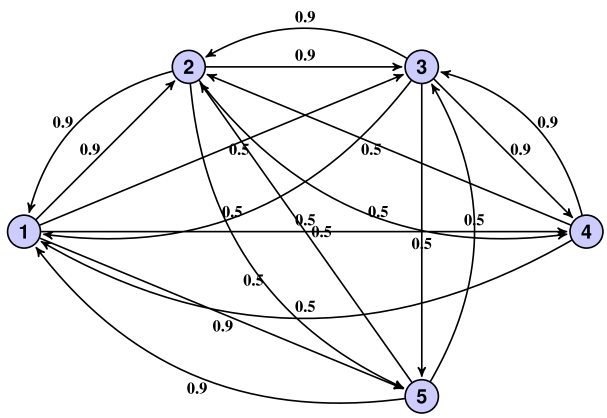

FIGURE 2. The expected graph of the random network in Example 2.

TABLE 1. The local AWVC measures of expected graph $\hat{G}$, using the methods proposed in this work and the one in [48].

(i, j)	$\tilde{\kappa}_{i,j}(\hat{G})$	$\overline{\kappa}_{i,j}(\hat{G})$
$(1, 2)$ and $(2, 1)$	3.6	3.6
$(1, 3)$ and $(3, 1)$	2	2
$(1, 4)$ and $(4, 1)$	2	2
$(1, 5)$ and $(5, 1)$	2	2
$(2, 3)$ and $(3, 2)$	3.6	3.6
$(2, 4)$ and $(4, 2)$	2	2
$(2, 5)$ and $(5, 2)$	2	2
$(3, 4)$ and $(4, 3)$	3.6	3.6
$(3, 5)$ and $(5, 3)$	2	2
$(4, 5)$ and $(5, 4)$	1.15	0.6561

$\pi^1_{4,5} = \{4, 1, 5\}$, $\pi^2_{4,5} = \{4, 2, 5\}$, $\pi^3_{4,5} = \{4, 3, 5\}$, $\pi^4_{4,5} = \{4, 2, 1, 5\}$, $\pi^5_{4,5} = \{4, 1, 2, 5\}$, $\pi^6_{4,5} = \{4, 3, 1, 5\}$, $\pi^7_{4,5} = \{4, 1, 3, 5\}$, $\pi^8_{4,5} = \{4, 3, 2, 5\}$, $\pi^9_{4,5} = \{4, 2, 3, 5\}$, $\pi^{10}_{4,5} = \{4, 3, 2, 1, 5\}$, $\pi^{11}_{4,5} = \{4, 3, 1, 2, 5\}$, $\pi^{12}_{4,5} = \{4, 2, 3, 1, 5\}$, $\pi^{13}_{4,5} = \{4, 2, 1, 3, 5\}$, $\pi^{14}_{4,5} = \{4, 1, 2, 3, 5\}$, *and* $\pi^{15}_{4,5} = \{4, 1, 3, 2, 5\}$. *From the definition of the local WVC measure in [48],* $\hat{\Pi}_{4,5} = \{\pi^1_{4,5}, \pi^2_{4,5}, \pi^3_{4,5}\}$, *which yields* $\hat{\kappa}_{4,5}(\hat{G}) = W(\pi^1_{4,5}) + W(\pi^2_{4,5}) + W(\pi^3_{4,5}) = 0.45 + 0.25 + 0.45 = 1.15$. *By applying a shortest path algorithm to* $\hat{G}$ *with the modified weight matrix* $\overline{\boldsymbol{P}}$ *defined earlier, the most reliable path from node 4 to node 5 is obtained to be* $\pi^{10}_{4,5}$. *Since no path exists from 4 to 5 after removing the internal nodes of* $\pi^{10}_{4,5}$ *from* $\hat{G}$, *thus* $\overline{\kappa}_{4,5}(\hat{G}) = W(\pi^{10}_{4,5}) = (0.9)^4 = 0.6561$. *Using Algorithm 1, the local AWVC measure* $\tilde{\kappa}_{4,5}(\hat{G})$ *is obtained by summing up the multiplicative weights of three paths* $\pi^1_{4,5}$, $\pi^2_{4,5}$ *and* $\pi^3_{4,5}$. *The same procedure can be employed to compute all other local connectivity measures in graph* $\hat{G}$ *as well.*

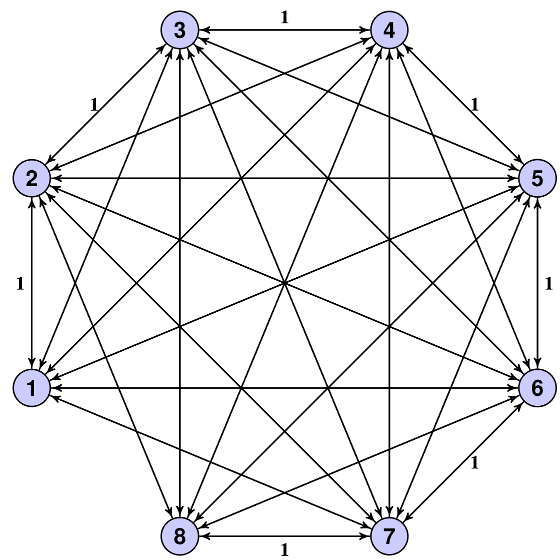

FIGURE 3. The expected graph $\hat{G}$ introduced in Example 3 for the special case of $n = 8$.

From Table 1, one gets that $\overline{\kappa}(\hat{G}) = 0.6561$ *and* $\tilde{\kappa}(\hat{G}) = 1.15$. *Moreover,* $\hat{\kappa}(\hat{G}) = \tilde{\kappa}(\hat{G})$, *i.e., the AWVC measure in this example is equal to the WVC measure.*

Example 3: Consider the expected communication graph $\hat{G} = (\hat{V}, \hat{E})$ *for a network, where* $\hat{V} = \{1, 2, \ldots, n\}$, $\hat{E} = \{(i, j) \in \hat{V} \times \hat{V} \mid p_{ji} \neq 0\}$, *and* $n > 3$, *and let the probability matrix* $\boldsymbol{P} = [p_{ji}]$ *be given by:*

$$p_{ji} = \begin{cases} 1 & \text{if } |j - i| = 1, \\ 0 & \text{if } |j - i| = n - 1, \\ x & \text{otherwise}, \end{cases}$$

where $0.5 < x < 1$. *Fig. 3 shows the expected graph* $\hat{G}$ *in the special case where* $n = 8$.

The AWVC measure for the above-mentioned graph is equal to $\tilde{\kappa}(\hat{G}) = 2x + (n - 4)x^2$. *The global AWVC and WVC measures and the corresponding computation times using a personal computer with Intel(R) Core(TM) i5 @ 1.60 GHz processor and 8 GB RAM are given in Table 2 for* $n = 4, 5, 6, 7$ *with* $x = 0.9$. *As can be seen from this table, the global AWVC in this example is exactly the same as the global WVC for all four cases, and more importantly, the computation time for the global AWVC is significantly less than that for the global WVC. It is worth noting that since the computation of the global WVC is an NP-hard problem, the time required for finding the global WVC in this example can be too long for* $n \geq 8$. *For instance, such a computation can take several hours for* $n = 8$, *whereas the global AWVC can be computed in less than 6 seconds. Finally, note that using the method given in [48], the global AWVC measure of* $\hat{G}$ *for all* $n > 3$ *is obtained to be* $\overline{\kappa}(\hat{G}) = 1$, *which is significantly different from* $\hat{\kappa}(\hat{G})$.

Example 4: Consider a network composed of five sensors which broadcast their data periodically as described in [38], [48]. Let the existence probability of the communication links

TABLE 2. The global WVC and AWVC measures for the expected graph $\hat{G}$ of Example 3 along with the corresponding computation times.

n	$\tilde{\kappa}(\hat{G})$	$\hat{\kappa}(\hat{G})$	run time for $\tilde{\kappa}(\hat{G})$	run time for $\hat{\kappa}(\hat{G})$
4	1.8	1.8	0.0343s	0.0406s
5	2.61	2.61	0.0351s	0.7025s
6	3.42	3.42	0.0603s	8.4785s
7	4.23	4.23	0.0996s	293.3370s

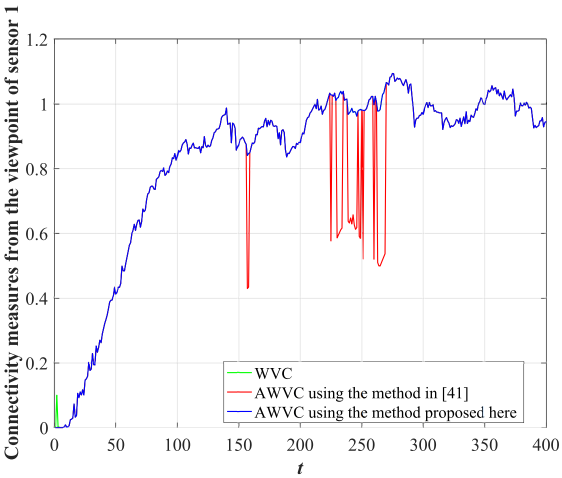

FIGURE 4. Global connectivity measures $\hat{\kappa}(\hat{G})$, $\overline{\kappa}(\hat{G})$, and $\tilde{\kappa}(\hat{G})$ of the network in Example 4, from the viewpoint of sensor 1.

of the network be given by a time-varying probability matrix $\boldsymbol{P}(t)$ *as:*

$$\boldsymbol{P}(t) = \begin{bmatrix} 0 & 0.9 & 0 & 0.8 & 0 \\ 0.6 & 0 & 0.6 & 0 & 0.8 \\ 0 & 0.6 & 0 & 0.8 & 0 \\ 0.8 & p_{42}(t) & 0.8 & 0 & p_{45}(t) \\ 0 & 0.7 & 0 & 0.9 & 0 \end{bmatrix}, \tag{41}$$

where $p_{42}(t) = 0.6 + 0.1\sin(0.005\ t)$ *and* $p_{45}(t) = 0.5 + 0.2\sin(0.005\ t)$. *Note that due to the numerous factors impacting the existence probabilities of underwater communication links, two time-varying trigonometric functions are employed in this example to model* p_{42} *and* p_{45} *as functions of time and to represent the cyclical behavior of the underwater communication due to seasonal and tidal changes in the marine environment. The elements of the probability matrix* $\boldsymbol{P}(t)$ *and the topology of the expected communication graph* $\hat{G}$ *are estimated by each sensor in a distributed fashion using, respectively, Algorithms 4 and 5 proposed in [48]. Figs. 4 and 5 depict the WVC metric for* $\hat{G}$ *along with the approximate measures using the methods proposed in [48] and the present work, from the viewpoint of sensors 1 and 3, respectively, after applying the corresponding algorithms to the estimated probability matrix* $\boldsymbol{P}(t)$.

It is implied from Figs. 4 and 5 that the AWVC measure introduced in the present work is closer to the exact WVC measure compared to the one proposed in [48]. In fact, the AWVC measure proposed in this paper is exactly the same as WVC in this example after a certain time. Moreover, the

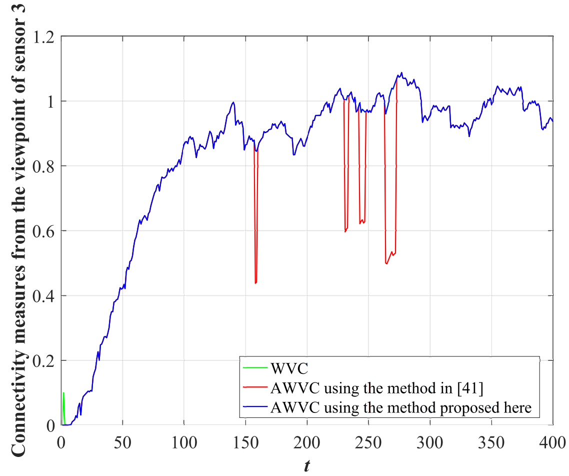

FIGURE 5. Global connectivity measures $\hat{\kappa}(\hat{G})$, $\overline{\kappa}(\hat{G})$, and $\tilde{\kappa}(\hat{G})$ of the network in Example 4, from the viewpoint of sensor 3.

time complexity of the algorithm used to compute the AWVC measure is considerably lower than that of the WVC measure.

VI. CONCLUSION

A quantitative measure of connectivity in random networks is investigated in this paper. An approximate weighted vertex connectivity (AWVC) measure is introduced to evaluate the connectivity of a weighted digraph representing the expected communication graph of a random network. The measure is used as a computationally efficient alternative to the weighted vertex connectivity (WVC) of the network and to a more conservative approximation of that metric introduced in an earlier work. It is shown that under certain conditions the proposed approximation is exactly equal to the WVC measure. Numerical examples demonstrate the efficacy of the proposed technique in finding the WVC measure.

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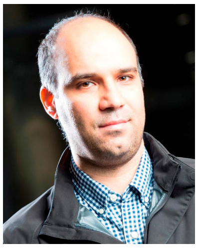

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