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Fully Discrete Interpolation Coefficients Mixed Finite Element Methods for Semi-Linear Parabolic Optimal Control Problem

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ABSTRACT We use the fully discrete interpolation coefficient mixed finite element methods to solve the semi-linear parabolic optimal control problems. The space discretization of the state variable is separated using interpolation coefficient mixed finite elements. We approximate the state and the co-state by Raviart-Thomas mixed finite elements on the lowest order, which is approximated by piecewise constant elements. By applying mixed finite element methods to estimate errors, we get priori errors estimates for both the coupled state and the control approximations. We finally confirm the theoretical results numerically by a numerical example.

INDEX TERMS Priori error estimate, interpolation coefficients, mixed finite element methods, fully discrete, semi-linear parabolic optimal control problem.

I. INTRODUCTION

O*ptimal control problems* (OCPs) are extremely important topics on engineering and science numerical simulation, and there exists a large number of references related to OCPs by *finite element methods* (FEMs). Falk [1] derived a priori error estimate of the standard FEMs for linear elliptic OCPs. Chen and Liu [2], [5] and Chen and Lu [4] studied errors estimates for nonlinear OCPs and established the finite element approximation of them. Arada *et al.* [6] estabilshed the finite element approximation of semi-linear elliptic OCPs, and they obtained error estimates for OCPs with the maximum norm. Kammann and Tröltzsch [7] solved nonlinear parabolic boundary control problems and gave estimation of posteriori error. For the parabolic OCPs, the scholars studied the errors estimates of the multiscale FEMs in [11] and *mixed finite element methods* (MFEMs) in [12]. Meidner and Vexler [38] and U. Langer *et al.* [16] used FEMs to solve priori error estimate about discretization of OCPs on the space-time. Jiang *et al.* studied the optimal applications of the multiscale finite element method in [18], [19]. Sun
textit *et al.* [32] gave an advanced ALE-mixed finite element method for a cardiovascular fluid-structure interaction problem with multiple moving interfaces. Neittaanmaki and Tiba [39] gave several improvements to the parabolic OCPs. Some progress in parabolic optimal control problems can be found in [20]–[23], but there was only few published results on a priori error estimates of fully discrete interpolation coefficients mixed finite element methods for semi-linear parabolic optimal control problem.

The existing literature rarely involves the use of MFEMs to solve priori estimates of its fully discrete interpolation coefficients. MFEMs are based on the mixed variational principle for FEMs. The characteristic of MFEMs is simultaneous to select two basic unknown functions, namely, the displacement function and the force function. The mixed energy principle is used to derive the basic equations of MFEMs. In the past decades, some superconvergence and error estimates about quadratic OCPs by MFEMs have been derived from [3], [13], [24], [37]–[47]. Kwon and Milner [28] Milner

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studied L^{∞}-error estimates for mixed methods for semilinear second-order elliptic equations. Shi and Liu [29] gave the superconvergent analysis of a nonconforming MFEMs for non-stationary conduction-convection problem. Akram textit *et al.* [30] presents MFES for solving R-type Pythagorean fuzzy linear programming problems. Some useful references for better understanding this paper can be found in [8], [9], [14], [25], [28], [34].

At the same time, a lot of works have been done by many scholars on study for OCPs by interpolation coefficient finite element methods. Larson *et al.* [33] studied the semi-discrete for nonlinear heat equation. Keil textit *et al.* [36] solved chance constrained OCPs by using biased kernel density estimators. Xiong and Chen [45] presented the superconvergence of rectangular finite volume element on semi-linear elliptic problems. Zhang and Han [26] solved compressible miscible displacement problem by a new discontinuous Galerkin mixed finite element. Wang [42] used interpolated coefficients FEMs to solve the semi-linear parabolic equations and got a nonlinear model reduction. According to the above literature, there are some obvious differences between our paper and the above references, such as, different questions and different methods. Liu and Li [47] presented a numerical method for interval multi-objective mixed-integer OCPs based on quantum heuristic algorithm. Therefore, MFEMs with interpolation coefficients provide scholars with a more efficient computational approach.

The novelty of this paper is that we add the FEM to MFEMs for semi-linear parabolic equations, and analyze the priori error of their fully discrete interpolation coefficients. There are two differences between our paper and the reference [17]. Firstly, the research problems are different. In [17], the author studied the nonlinear parabolic equations, while, we mainly discuss the semi-linear parabolic OCPs. Secondly, we mainly use the fully discrete interpolation coefficient method, while [17] used the semi-discrete MFEMs to approximate parabolic equations. By comparing, we can find that our method in computation speed of competely discrete interpolation coefficients is better than that of [17].

Now we consider the semi-linear parabolic OCP as follows

$$\min_{u(t)\in K\subset U}\left\{\frac{1}{2}\int_0^T\left(||\boldsymbol{p}-\boldsymbol{p}_d||^2+||y-y_d||^2+\alpha||u||^2\right)dt\right\} \tag{1}$$

such that

$$y_t(x,t)+\mathrm{div}\boldsymbol{p}(x,t)+\phi(y(x,t))=f(x,t)+u(x,t), \tag{2}$$
$$\boldsymbol{p}(x,t)=-A(x)\nabla y(x,t), \tag{3}$$
$$y(x,t)=0,\ x\in\partial\Omega,\qquad y(x,0)=y_0(x), \tag{4}$$

where $\alpha>0$, $t\in J$ for $J=(0,T]$, $x\in\Omega$, $u(t)$ is the control variable in control space U, and K is a closed convex set. $\boldsymbol{p}$ denotes adjoint state variable, and y denotes state variable. Ω is contained in $\mathbb{R}^2$ with smooth boundary, and it denotes an open set which is regular bounded convex. $T>0$ is a constant.

We suppose that f belongs to $L^2(J;L^2(\Omega))$, whereas $\boldsymbol{p}_d, y_d$ are continuously differentiable. For any $\Re>0$ the function $\phi(\cdot)\in W^{2,\infty}(-\Re,\Re)$, where $\phi'(y)\in L^2(\Omega)$ and $\phi'(y)\geq 0$. We define the symmetric coefficient 2×2 matrix $A(x)=(a_{i,j}(x))_{2\times 2}\in L^{\infty}(\Omega;\mathbb{R}^{2\times 2})$. Then there exists $c>0$ which satisfies the conditions: for any $\mathbf{X}\in\mathbb{R}^2$, $\mathbf{X}^tA\mathbf{X}\geq c\|\mathbf{X}\|^2_{\mathbb{R}^2}$.

In addition, we present the control variable G which satisfies:

$$G=\left\{u(x,t)\in L^2(J;L^2(\Omega)):\ u(x,t)\geq 0\ a.e.\ x\in\Omega,\ t\in J\right\}. \tag{5}$$

Now, we define $W_0^{m,p}(\Omega)=\{v\in W^{m,p}(\Omega),\ v|_{\partial\Omega}=0\}$ as the standard notation on Sobolev spaces. There exists the norm $||\cdot||_{m,p}$ given by $||v||^p_{m,p}=\sum_{|\alpha|\leq m}||D^{\alpha}v||^p_{L^p(\Omega)}$ and the semi-norm $|\cdot|_{m,p}$ given by $|v|^p_{m,p}=\sum_{|\alpha|=m}||D^{\alpha}v||^p_{L^p(\Omega)}$. For $p=2$, we denote $H^m(\Omega)$ as $W^{m,2}(\Omega)$, and $H_0^m(\Omega)$ as $W_0^{m,2}(\Omega)$, and use $||\cdot||_m$ to express $||\cdot||_{m,2}$, and use $||\cdot||$ to express $||\cdot||_{0,2}$. Let $L^s(0,T;W^{m,p}(\Omega))$ to express the Banach space with all L^s integrable functions, where $W^{m,p}(\Omega)$ is equipped with norm $||v||_{L^s(J;W^{m,p}(\Omega))}=\left(\int_0^T||v||^s_{W^{m,p}(\Omega)}dt\right)^{\frac{1}{s}}$ for $s\in[1,\infty)$, and the standard modification can be made for $s=\infty$. From [35] we can get more details about OCPs.

The remaining paper is structured as follows. Section II constructs interpolation coefficient mixed finite element discretization for the semi-linear OCP. Section III derives the intermediate variables for its priori error estimates. Section IV derives detailed steps to solve the elemental approximation control problem of semi-linear parabolic optimality using MFEMs. Section V gives a specific numerical example.

II. INTERPOLATION COEFFICIENTS MIXED METHODS

Firstly, we give the parabolic equation of co-state as follows

$$-z_t-\mathrm{div}(A(\nabla z+\boldsymbol{p}-\boldsymbol{p}_d))+\phi'(y)z=y-y_d, \tag{6}$$

which satisfies

$$z(x,t)=0,\ x\in\partial\Omega,\ t\in J,$$
$$z(x,T)=0,\ x\in\Omega.$$

We shall give a completely description of OCP. Set $H(\mathrm{div})$ be $\{v\in(L^2(\Omega))^2, \mathrm{div}v\in L^2(\Omega)\}$ with the norm $||v||_{H(\mathrm{div})}=(\|\ v\|^2_0+||\mathrm{div}v||^2_0)^{1/2}$, we denote $\boldsymbol{V}=L^2(J,H(\mathrm{div}))$ and $W=L^2(J,L^2(\Omega))$.

We know that $(\boldsymbol{p},y,u)$ belongs to $\boldsymbol{V}\times W\times G$ when (1)-(4) are rewritten into the following weak form

$$\min_{u_h\in K_h}\left\{\frac{1}{2}\int_0^T\left(||\boldsymbol{p}-\boldsymbol{p}_d||^2+||y-y_d||^2+\alpha||u||^2\right)dt\right\}, \tag{7}$$
$$(A^{-1}\boldsymbol{p},v)-(y,\mathrm{div}v)=0, \tag{8}$$
$$(y_t,\omega)+(\phi(y),\omega)+(\mathrm{div}\boldsymbol{p},\omega)=(f+u,\omega_h), \tag{9}$$

$$y(x,0) = y_0(x), \tag{10}$$

where $\nu \in \boldsymbol{V}$, $\omega \in W$ and $x \in \Omega$.

We know the conclusion (see, e.g., [35]) that OCPs (7)-(10) have at least a solution $(\boldsymbol{p}, y, u)$. There exists a co-state variable $(\boldsymbol{q}, z)$ belongs to $\boldsymbol{V} \times W$, so $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the following conditions

$$(A^{-1}\boldsymbol{p}, \nu) - (y, \mathrm{div}\nu) = 0, \tag{11}$$
$$(y_t, \omega) + (\mathrm{div}\boldsymbol{p}, \omega) + (\phi(y), \omega) = (f + u, \omega), \tag{12}$$
$$y(x,0) = y_0(x), \tag{13}$$
$$(z, \mathrm{div}\nu) - (A^{-1}\boldsymbol{q}, \nu) = (\boldsymbol{p} - \boldsymbol{p}_d, \nu), \tag{14}$$
$$(\mathrm{div}\boldsymbol{q}, \omega) - (z_t, \omega) + (\phi'(y)z, \omega) = (y - y_d, \omega), \tag{15}$$
$$z(x, T) = 0, \tag{16}$$
$$\int_0^T (\alpha u + z, \widehat{u} - u)dt \geq 0, \tag{17}$$

for $t \in (0, T]$, where $\nu \in \boldsymbol{V}$, $\omega \in W$, $\widehat{u} \in G$ and $x \in \Omega$.

We suppose that the parabolic equations (2) and (6) have sufficient regularity, where y and z belong to $L^2(H^2(\Omega))$, $\boldsymbol{p}$ and $\boldsymbol{q}$ belong to $(L^2(H^2(\Omega)))^2$, and $u \in L^2(W^{1,\infty}(\Omega))$.

Set $\mathcal{T}_h$ as a regular triangulation on Ω, and $\tau \in \mathcal{T}_h$ is assumed to satisfy the angle condition, which means that there is a positive constant C such that

$$C^{-1}h_\tau^2 \leq |\tau| \leq Ch_\tau^2, \quad \forall\, \tau \in \mathcal{T}_h,$$

where $|\tau|$ means the area of τ, h_τ is the diameter of τ, h is equal to $\max h_\tau$.

Let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ denotes the Raviart-Thomas space [15] of the lowest order associated with the triangulation $\mathcal{T}_h$ of Ω, namely, $\boldsymbol{V}(\tau) = \{\nu \in P_0^2(\tau) + x \cdot P_0(\tau)\}$. Let $P_0(\sigma)$ equal $W(\tau)$ for any $\tau \in \mathcal{T}_h$, we have

$$\boldsymbol{V}_h := \{\nu_h \in \boldsymbol{V} : \forall\, \tau \in \mathcal{T}_h,\ \nu_h|_\tau \in L^2(J, \boldsymbol{V}(\tau))\},$$
$$W_h := \{\omega_h \in W : \forall\, \tau \in \mathcal{T}_h,\ \omega_h|_\tau \in L^2(J, W(\tau))\},$$
$$G_h := \{\widehat{u}_h \in K : \forall\, \tau \in \mathcal{T}_h,\ \widehat{u}_h|_\tau \in L^2(J, P_0(\tau))\},$$

where P_γ means total degree for most γ on the space polynomials.

By using (7)-(10) with the mixed finite element discretization, we let $(\boldsymbol{p}_h, y_h, u_h)$ be the solution of

$$\min_{u_h \in K_h} \left\{ \frac{1}{2} \int_0^T \left(||\boldsymbol{p}_h - \boldsymbol{p}_d||^2 + ||y_h - y_d||^2 + \alpha ||u_h||^2 \right) dt \right\}, \tag{18}$$
$$(A^{-1}\boldsymbol{p}_h, \nu_h) - (y_h, \mathrm{div}\nu_h) = 0, \tag{19}$$
$$(y_{ht}, \omega_h) + (\mathrm{div}\boldsymbol{p}_h, \omega_h) + (\phi(y_h), \omega_h) = (f + u_h, \omega_h), \tag{20}$$
$$y_h(x,0) = Y(x,0), \tag{21}$$

for $\nu_h \in \boldsymbol{V}_h$, $\omega_h \in W_h$, $x \in \Omega$, and $Y(x, 0)$ is projected by the elliptic mixed method for $y_0(x)$ on finite dimensional space W_h. Then OCPs (18)-(21) have at least a solution $(\boldsymbol{p}_h, y_h, u_h)$, and that $(\boldsymbol{p}_h, y_h, u_h)$ is the solution of (18)-(21), so we set $(\boldsymbol{q}_h, z_h)$ as the co-state, and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ satisfies the equations as follows

$$(A^{-1}\boldsymbol{p}_h, \nu_h) - (y_h, \mathrm{div}\nu_h) = 0, \tag{22}$$
$$(y_{ht}, \omega_h) + (\mathrm{div}\boldsymbol{p}_h, \omega_h) + (\phi(y_h), \omega_h) = (f + u_h, \omega_h), \tag{23}$$
$$y_h(x,0) = Y(x,0), \tag{24}$$
$$(A^{-1}\boldsymbol{q}_h, \nu_h) - (z_h, \mathrm{div}\nu_h) = -(\boldsymbol{p}_h - \boldsymbol{p}_d, \nu_h), \tag{25}$$
$$-(z_{ht}, \omega_h) + (\mathrm{div}\boldsymbol{q}_h, \omega_h) + (\phi'(y_h)z_h, \omega_h) = (y_h - y_d, \omega_h), \tag{26}$$
$$z_h(x, T) = 0, \tag{27}$$
$$(z_h + \alpha u_h, \widehat{u}_h - u_h) \geq 0, \tag{28}$$

where $\nu_h \in \boldsymbol{V}_h$, $\omega_h \in W_h$, $\widehat{u}_h \in G_h$, and $x \in \Omega$. Define interpolation operator $I_h : L^2(J; C(\bar{\Omega})) \rightarrow W_h$ by

$$I_h v(x,t) = \sum_{j=1}^{M} v_j \varphi_j(x,t),$$

where $\{\varphi_j(x,t)\}_{j=1}^M$ is the standard Lagrangian nodal basis of W_h.

We assume $y_h(x,t) = \sum_{j=1}^{M} y_j \varphi_j(x,t)$, then

$$\phi(y_h(x,t)) = \phi\left(\sum_{j=1}^{M} y_j \varphi_j(x,t) \right).$$

By definition of the interpolating operator I_h, we get

$$I_h \phi(y_h(x,t)) = \sum_{j=1}^{M} \phi(y_j) \varphi_j(x,t), \tag{29}$$

while the interpolation error estimate [33] for $1 \leq p \leq \infty$ is

$$\|v - I_h v\|_{0,p} \leq Ch \|v\|_{1,p}, \tag{30}$$

where v belongs to $L^2(J; C(\bar{\Omega})) \cap L^2(J; W^{1,p}(\tau_h))$ for $\tau_h \in \mathcal{T}_h$. By substituting $I_h\phi(y_h)$ for $\phi(y_h)$ in (20), we have

$$(A^{-1}\boldsymbol{p}_h, \nu_h) - (y_h, \mathrm{div}\nu_h) = 0, \tag{31}$$
$$(y_{ht}, \omega_h) + (\mathrm{div}\boldsymbol{p}_h, \omega_h) + (I_h\phi(y_h), \omega_h) = (f + u_h, \omega_h), \tag{32}$$
$$y_h(x,0) = Y(x,0), \tag{33}$$
$$(A^{-1}\boldsymbol{q}_h, \nu_h) - (z_h, \mathrm{div}\nu_h) = -(\boldsymbol{p}_h - \boldsymbol{p}_d, \nu_h), \tag{34}$$
$$-(z_{ht}, \omega_h) + (\mathrm{div}\boldsymbol{q}_h, \omega_h) + (\phi'(y_h)z_h, \omega_h) = (y_h - y_d, \omega_h), \tag{35}$$
$$z_h(x, T) = 0, \tag{36}$$
$$(z_h + \alpha u_h, \widehat{u}_h - u_h) \geq 0, \tag{37}$$

for any $\nu_h \in \boldsymbol{V}_h$, $\omega_h \in W_h$, $\widehat{u}_h \in G_h$, and $x \in \Omega$.

Now, we construct different methods about time discretization. Define $N = T/\Delta t$ for $\Delta t > 0$, and $t^n = n\Delta t$ for $n \in \mathbb{Z}$, then set

$$\Psi^n = \Psi(x, t^n), \quad d_t\Psi^n = \frac{\Psi^n - \Psi^{n-1}}{\Delta t}.$$

Moreover, we give a definition of the discrete time norm as

$$|||\Psi|||_{L^p(H^s(\Omega))} := \left(\sum_{n=1}^{N} \Delta t||\Psi^n||_s^p\right)^{\frac{1}{p}},$$

for $1 \le p < \infty$, while the standard modification can be made for $p = \infty$. Let the solution $(\boldsymbol{p}_h^n, y_h^n, \boldsymbol{q}_h^{n-1}, z_h^{n-1}, u_h^n)$ satisfy

$$(A^{-1}\boldsymbol{p}_h^n, v) - (y_h^n, \mathrm{div} v) = 0, \quad (38)$$
$$(d_t y_h^n, \omega) + (\mathrm{div}\boldsymbol{p}_h^n, \omega) + (I_h\phi(y_h^n), \omega) = (f + u_h^n, \omega), \quad (39)$$
$$y_h^0(x) = Y(x, 0), \quad (40)$$
$$(A^{-1}\boldsymbol{q}_h^{n-1}, v) - (z_h^{n-1}, \mathrm{div} v) = -(\boldsymbol{p}_h^n - \boldsymbol{p}_d, v), \quad (41)$$
$$-(d_t z_h^n, \omega) + (\mathrm{div}\boldsymbol{q}_h^{n-1}, \omega) + (\phi'(y_h^n)z_h^{n-1}, \omega) = (y_h^n - y_d, \omega), \quad (42)$$
$$z_h^N(x) = 0, \quad (43)$$
$$(z_h^n + \alpha u_h^n, \widehat{u} - u_h^n) \ge 0, \quad (44)$$

for $n = 0, 1, \cdots, N$, where $v_h \in \boldsymbol{V}_h$, $\omega_h \in W_h$, $\widehat{u}_h \in G_h$, and $x \in \Omega$.

Note that Q_h is a standard L^2-orthogonal projection, we can get

$$(\widehat{u} - Q_h\widehat{u}, \widehat{u}_h)_U = 0, \quad \forall\, \widehat{u}_h \in G_h, \quad (45)$$
$$||\widehat{u} - Q_h\widehat{u}||_{-s} \le Ch^{1+s}|\widehat{u}|_1, \quad \widehat{u} \in H^1(\Omega), \quad (46)$$

for $s = 0, 1$, where $G \to G_h$, $\widehat{u} \in G$.

III. ERROR ESTIMATES OF INTERMEDIATE VARIABLES

In this section, let's consider adding intermediate variables to the error estimation of problems. We first define $(\boldsymbol{p}(\widehat{u}), y(\widehat{u}), \boldsymbol{q}(\widehat{u}), z(\widehat{u}))$ as the state solution, such that

$$(A^{-1}\boldsymbol{p}(\widehat{u}), v) - (y(\widehat{u}), \mathrm{div} v) = 0, \quad (47)$$
$$(y_t(\widehat{u}), \omega) + (\mathrm{div}\boldsymbol{p}(\widehat{u}), \omega) + (\phi(y(\widehat{u})), \omega) = (f + \widehat{u}, \omega), \quad (48)$$
$$y(\widehat{u})(x, 0) = y_0(x), \quad (49)$$
$$(A^{-1}\boldsymbol{q}(\widehat{u}), v) - (z(\widehat{u}), \mathrm{div} v) = -(\boldsymbol{p}(\widehat{u}) - \boldsymbol{p}_d, v), \quad (50)$$
$$-(z_t(\widehat{u}), \omega) + (\mathrm{div}\boldsymbol{q}(\widehat{u}), \omega) + (\phi'(y(\widehat{u}))z(\widehat{u}), \omega) = (y(\widehat{u}) - y_d, \omega), \quad (51)$$
$$z(\widehat{u})(x, T) = 0, \quad (52)$$

for any $v \in \boldsymbol{V}$, $\omega \in W$, and $x \in \Omega$.

Since we have assumed the domain Ω is a 2-dimensional regular one, for $\lambda > 0$, the Dirichlet problem

$$L_\lambda\beta = -\mathrm{div}(A(x)\nabla\beta) + \lambda\beta = g, \quad x \in \Omega, \quad (53)$$
$$\beta = 0, \quad x \in \partial\Omega, \quad (54)$$

is uniquely solvable for $g \in L^2(\Omega)$ and $||\beta||_2 \le C||g||_0$ for all $g \in L^2(\Omega)$.

Then we give the definition $(P^n(\widehat{u}), Y^n(\widehat{u}), Q^n(\widehat{u}), Z^n(\widehat{u}))$ as an elliptic for $\widehat{u} \in G$, mixed method projection of the solution, which solves the differential problem in the finite dimensional space $\boldsymbol{V}_h \times W_h$, that is the map $(P(\widehat{u}), Y(\widehat{u}), Q(\widehat{u}), Z(\widehat{u}))$ given by

$$(A^{-1}(\boldsymbol{p}^n(\widehat{u}) - P^n(\widehat{u})), v) - (y^n(\widehat{u}) - Y^n(\widehat{u}), \mathrm{div} v) = 0, \quad (55)$$
$$(\mathrm{div}(\boldsymbol{p}^n(\widehat{u}) - P^n(\widehat{u})), \omega) + \lambda(y^n(\widehat{u}) - Y^n(\widehat{u}), \omega) = 0, \quad (56)$$
$$(A^{-1}(\boldsymbol{q}^n(u) - Q^n(\widehat{u})), v) - (z^n(\widehat{u}) - Z^n(\widehat{u}), \mathrm{div} v) = 0, \quad (57)$$
$$(\mathrm{div}(\boldsymbol{q}^n(\widehat{u}) - Q^n(\widehat{u})), \omega) + \lambda(z^n(\widehat{u}) - Z^n(\widehat{u}), \omega) = 0, \quad (58)$$

for any $v_h \in \boldsymbol{V}_h$ and $\omega_h \in W_h$.

Set $\lambda > 0$ be fully large in order to get the bilinear form associated with $L_\lambda(\cdot)$ on $H_0^1(\Omega)$. From [10], we know that λ can be chosen as

$$C\left(||\xi||_0^2 + ||\eta||_0^2\right) \le (\lambda(\eta, \eta) + A^{-1}\xi, \xi), \quad \forall\, \xi \in \boldsymbol{V}, \ \eta \in W. \quad (59)$$

Let

$$\tau_1^n = y^n(u_h^n) - Y^n(u_h^n), \quad \sigma_1^n = \boldsymbol{p}^n(u_h^n) - P^n(u_h^n), \quad (60)$$
$$\tau_2^n = z^n(u_h^n) - Z^n(u_h^n), \quad \sigma_2^n = \boldsymbol{q}^n(u_h^n) - Q^n(u_h^n). \quad (61)$$

The estimates for $\tau_1^n, \tau_2^n, \sigma_1^n$, and σ_2^n have been given in [17] as follows.

Lemma 3.1: For $t \in J$ and for fully small and independent h, there exists $C>0$,

$$||\tau_1^n||_0 + ||\tau_1^n||_{0,\infty} + ||\sigma_1^n||_0 \le Ch, \quad (62)$$
$$||\tau_2^n||_0 + ||\tau_2^n||_{0,\infty} + ||\sigma_2^n||_0 \le Ch, \quad (63)$$
$$||\mathrm{div}\sigma_1^n||_0 + ||\mathrm{div}\sigma_2^n||_0 \le Ch. \quad (64)$$

The estimates for $\tau_{1t}^n, \tau_{2t}^n, \sigma_{1t}^n$, and σ_{2t}^n have been given in [17] as follows.

Lemma 3.2: For $t \in J$ and for fully small and independent h, there exists $C>0$,

$$||\tau_{1t}^n||_0 + ||\tau_{1t}^n||_{0,\infty} + ||\sigma_{1t}^n||_0 \le Ch, \quad (65)$$
$$||\tau_{2t}^n||_0 + ||\tau_{2t}^n||_{0,\infty} + ||\sigma_{2t}^n||_0 \le Ch, \quad (66)$$
$$||\mathrm{div}\sigma_{2t}^n||_0 + ||\mathrm{div}\sigma_{1t}^n||_0 \le Ch. \quad (67)$$

By means of Lemmas 3.1-3.2 and Gronwall's Lemma (see, e.g., [41]), we can get the errors estimates.

Theorem 3.1: There exists $C > 0$, which is independent of h, such that

$$|||y(u_h^n) - y_h|||_{L^\infty(J;L^2(\Omega))} + |||\boldsymbol{p}(u_h^n) - \boldsymbol{p}_h|||_{L^\infty(J;H(\mathrm{div}))} \le C(h + \Delta t), \quad (68)$$
$$|||z(u_h^n) - z_h|||_{L^\infty(J;L^2(\Omega))} + |||\boldsymbol{q}(u_h^n) - \boldsymbol{q}_h|||_{L^\infty(J;H(\mathrm{div}))} \le C(h + \Delta t). \quad (69)$$

Proof: Let $\eta_1^n = Y^n(u_h^n) - y_h^n$, $\eta_2^n = P^n(u_h^n) - \boldsymbol{p}_h^n$, *where* $n = 1, 2, \cdots, N$. *We obtain that*

$$(y_t^n(u_h^n), \omega) + (div\boldsymbol{p}^n(u_h^n), \omega) + (\phi(y^n(u_h^n)), \omega) = (f + u_h^n, \omega), \quad (70)$$

for any ω *belongs to* W.

We can get following equation (71) by the elliptic mixed projection

$$(Y_t^n(u_h^n), \omega_h) + (divP^n(u_h^n), \omega_h) = (y_t^n(u_h^n), \omega_h) + (div\boldsymbol{p}^n(u_h^n), \omega_h) - (\tau_{1t}^n, \omega_h) + \lambda(\tau_1^n, \omega_h), \quad (71)$$

for any $\omega_h \in W_h$. By combining (70) and (71), we have

$$(Y_t^n(u_h^n), \omega_h) + (divP^n(u_h^n), \omega_h) = (f + u_h^n, \omega_h) - (\phi(y^n(u_h^n)), \omega_h) - (\tau_{1t}^n, \omega_h) + \lambda(\tau_1^n, \omega_h),$$

Next, from (39) and (72), we derive that

$$(Y_t^n(u_h^n) - d_t y_h^n, \omega_h) + (div\eta_2^n, \omega_h) + \lambda(\eta_1^n, \omega_h) = (I_h\phi(y_h^n) - \phi(y^n(u_h^n)), \omega_h) - (\tau_{1t}^n, \omega_h) - \lambda(y_h^n - y^n(u_h^n), \omega_h), \tag{72}$$

It follows from substituting (38) into (55),

$$(A^{-1}(P^n(u_h^n) - \boldsymbol{p}_h^n), v_h) - (Y^n(u_h^n) - y_h^n, divv_h) = 0, \tag{73}$$

for any v_h belongs to V_h.

Put $\omega_h = \eta_1^n$ into (72) and put $v_h = \eta_2^n$ into (73), then we get

$$(d_t\eta_1^n, \eta_1^n) + (A^{-1}\eta_2^n, \eta_2^n) + \lambda(\eta_1^n, \eta_1^n) = (d_tY^n(u_h^n) - Y_t^n(u_h^n), \eta_1^n) + (I_h\phi(y_h^n) - \phi(y^n(u_h^n)), \eta_1^n) - (\tau_{1t}^n, \eta_1^n) - \lambda(y_h^n - y^n(u_h^n), \eta_1^n). \tag{74}$$

We shall estimate the left side of (74).

We apply the coercivity property (59) and get

$$(A^{-1}\eta_2^n, \eta_2^n) + \lambda(\eta_1^n, \eta_1^n) \geq C(||\eta_1^n||_0^2 + ||\eta_2^n||_0^2). \tag{75}$$

Note that

$$(d_t\eta_1^n, \eta_1^n) = \frac{1}{2\Delta t}\left(1, (\eta_1^n)^2 - (\eta_1^{n-1})^2\right) + \frac{1}{2\Delta t}\left(1, (\eta_1^n - \eta_1^{n-1})^2\right). \tag{76}$$

Taking $R^n = \int_0^{\eta_1^n} sds$, we find

$$R^n - R^{n-1} = \int_{\eta_1^{n-1}}^{\eta_1^n} sds. \tag{77}$$

Note that (76) can be transformed into

$$\frac{R^n - R^{n-1}}{\Delta t} \leq \frac{R^n - R^{n-1}}{\Delta t} + \frac{1}{2\Delta t}\left(1, (\eta_1^n - \eta_1^{n-1})^2\right) = (d_t\eta_1^n, \eta_1^n). \tag{78}$$

We use the Lemma 3.2 to derive the estimation of the right side of (74)

$$\left|(d_tY^n(u_h^n) - Y_t^n(u_h^n), \eta_1^n)\right| \leq CE_n^2 + C||\eta_1^n||_0^2, \tag{79}$$

$$\left|(\tau_{1t}^n, \eta_1^n)\right| \leq Ch^2 + C||\eta_1^n||_0^2, \tag{80}$$

where

$$E_n^2 = \left(\int_{t^{n-1}}^{t^n}\left\|\frac{\partial^2 Y}{\partial t^2}(\cdot, s)\right\| ds\right)^2 \leq C(\Delta t)^2. \tag{81}$$

Using Lemma 3.1, $\phi(\cdot) \in W^{2,\infty}(-R, R)$ and the triangle inequality, we have

$$\begin{aligned}
&(I_h\phi(y_h^n) - \phi(y^n(u_h^n)), \eta_1^n) \\
&= (I_h\phi(y_h^n) - \phi(y_h^n) + \phi(y_h^n) - \phi(y^n(u_h^n)), \eta_1^n) \\
&= (I_h\phi(y_h^n) - \phi(y_h^n), \eta_1^n) + (\phi(y_h^n) - \phi(y^n(u_h^n)), \eta_1^n) \\
&\leq Ch||\eta_1^n||_0 \cdot ||\phi(y_h^n)||_1 + C||\eta_1^n||_0 \cdot ||y_h^n - y^n(u_h^n)||_0 \\
&\leq Ch||\eta_1^n||_0 + C||\eta_1^n||_0 \cdot (||\tau_1^n||_0 + ||\eta_1^n||_0) \\
&\leq C||\eta_1^n||_0^2 + Ch^2,
\end{aligned} \tag{82}$$

and

$$\begin{aligned}
(y_h^n - y^n(u_h^n), \eta_1^n) &\leq ||y_h^n - y^n(u_h^n)||_0 \cdot ||\eta_1^n||_0 \\
&\leq (||\tau_1^n||_0 + ||\eta_1^n||_0) \cdot ||\eta_1^n||_0 \\
&\leq C||\eta_1^n||_0^2 + Ch^2.
\end{aligned} \tag{83}$$

Then we multiply each estimates of the terms in (74)-(83) by $\triangle t$ and sum on n, we get

$$R^l + C\sum_{n=1}^{l}\left(||\eta_1^n||_0^2 + ||\eta_2^n||_0^2\right)\Delta t \leq C\left(h^2 + (\Delta t)^2\right)\Delta t + C\sum_{n=1}^{l}||\eta_1^n||_0^2\Delta t, \tag{84}$$

for $2 \leq l \leq N$ and $\eta_1^0 = 0$ has been used. By virtue of the Gronwall's Lemma, we verify that

$$||\eta_1^l||_0^2 + \sum_{n=1}^{l}\left(||\eta_1^n||_0^2 + ||\eta_2^n||_0^2\right) \leq C\left(h^2 + (\Delta t)^2\right), \tag{85}$$

or equivalently,

$$||\eta_1^l||_0^2 \leq C\left(h^2 + (\Delta t)^2\right), \quad \text{for any } l \leq N. \tag{86}$$

So we get

$$|||Y(u_h^n) - y_h|||_{L^\infty(L^2(\Omega))} \leq C(h + \Delta t). \tag{87}$$

According to the trigonometric inequality of Lemma 3.1, we have

$$|||y(u_h^n) - y_h|||_{L^\infty(L^2(\Omega))} \leq C(h + \Delta t). \tag{88}$$

Then, by defining the test function as $d_t\eta_1^n$ and taking in (72), we have

$$(d_t\eta_1^n, d_t\eta_1^n) + (div\eta_2^n, d_t\eta_1^n) + \lambda(\eta_1^n, d_t\eta_1^n) = (d_tY^n(u_h^n) - Y_t^n(u_h^n), d_t\eta_1^n) + (I_h\phi(y_h^n) - \phi(y^n(u_h^n)), d_t\eta_1^n) - (\tau_{1t}^n, d_t\eta_1^n) - \lambda(y_h^n - y^n(u_h^n), d_t\eta_1^n), \quad n = 1, 2, \cdots, N. \tag{89}$$

Combining (38) and (55), we obtain

$$(A^{-1}\eta_2^n, v_h) - (\eta_1^n, divv_h) = 0. \tag{90}$$

By taking different time into (90), and let $v_h = \eta_2^n$ as test function, we get

$$(A^{-1}d_t\eta_2^n, \eta_2^n) - (d_t\eta_1^n, div\eta_2^n) = 0, \tag{91}$$

for $n = 1, 2, \cdots, N$.

Substituting (91) into (89) we get

$$(d_t\eta_1^n, d_t\eta_1^n) + (A^{-1}d_t\eta_2^n, \eta_2^n) + \lambda(\eta_1^n, d_t\eta_1^n)$$

$$= (d_t Y^n(u_h^n) - Y_t^n(u_h^n), d_t\eta_1^n) + (I_h\phi(y_h^n) - \phi(y^n(u_h^n)), d_t\eta_1^n) - (\tau_{1t}^n, d_t\eta_1^n) - \lambda(y_h^n - y^n(u_h^n), d_t\eta_1^n), \tag{92}$$

for $n = 1, 2, \cdots, N$.

From the inequality (78), we have

$$C\frac{(\eta_1^n)^2 - (\eta_1^{n-1})^2}{2\triangle t} \leq \lambda(\eta_1^n, d_t\eta_1^n), \tag{93}$$

$$C\frac{(\eta_2^n)^2 - (\eta_2^{n-1})^2}{2\triangle t} \leq (A^{-1}d_t\eta_2^n, \eta_2^n). \tag{94}$$

Now we limit each term of (92) on the right side

$$\begin{aligned}
&(I_h\phi(y_h^n) - \phi(y^n(u_h^n)), d_t\eta_1^n)\\
&= (I_h\phi(y_h^n) - \phi(y_h^n) + \phi(y_h^n) - \phi(y^n(u_h^n)), d_t\eta_1^n)\\
&= (I_h\phi(y_h^n) - \phi(y_h^n), d_t\eta_1^n) + (\phi(y_h^n) - \phi(y^n(u_h^n)), d_t\eta_1^n)\\
&\leq Ch||\phi(y_h^n)||_1 \cdot ||d_t\eta_1^n||_0 + C||y_h^n - y^n(u_h^n)||_0 \cdot ||d_t\eta_1^n||_0\\
&\leq C(h + \triangle t)||d_t\eta_1^n||_0\\
&\leq C(h + \triangle t)^2 + \delta||d_t\eta_1^n||_0^2,
\end{aligned} \tag{95}$$

and

$$\begin{aligned}
(y_h^n - y^n(u_h^n), d_t\eta_1^n) &\leq ||y_h^n - y^n(u_h^n)||_0 \cdot ||d_t\eta_1^n||_0\\
&\leq C(h + \triangle t)||d_t\eta_1^n||_0\\
&\leq C(h + \triangle t)^2 + \delta||d_t\eta_1^n||_0^2,
\end{aligned} \tag{96}$$

where $\delta > 0$ *is fully small. And we can get the following inequalities*

$$(d_t Y^n(u_h^n) - Y_t^n(u_h^n), d_t\eta_1^n) \leq CE_n^2 + \delta||d_t\eta_1^n||_0^2, \tag{97}$$

$$(\tau_{1t}^n, d_t\eta_1^n) \leq Ch^2 + \delta||d_t\eta_1^n||_0^2, \tag{98}$$

where E_n^2 *is defined in (81).*

Next, from applying the bounds for each term of the sum in both sides, we have

$$\begin{aligned}
&\sum_{n=1}^{l}||d_t\eta_1^n||_0^2\triangle t + \frac{||\eta_1^l||_0^2 + ||\eta_2^l||_0^2}{2}\\
&\leq C\left(h^2 + (\triangle t)^2\right)\triangle t + C\delta\sum_{n=1}^{l}||d_t\eta_1^n||_0^2\triangle t\\
&\quad + C\left(||\eta_1^1||_0^2 + ||\eta_2^1||_0^2\right).
\end{aligned} \tag{99}$$

From the Gronwall's Lemma, we have

$$\begin{aligned}
&\sum_{n=1}^{l}||d_t\eta_1^n||_0^2 + ||\eta_1^l||_0^2 + ||\eta_2^l||_0^2\\
&\leq C\left(h^2 + (\triangle t)^2\right) + C\left(||\eta_1^1||_0^2 + ||\eta_2^1||_0^2\right).
\end{aligned} \tag{100}$$

Now, we get some bounds on η_1^1 *and* η_2^1. *By using (85), we obtain*

$$||\eta_1^1||_0 + ||\eta_2^1||_0 \leq C(h + \triangle t). \tag{101}$$

A combination of (100) and (101) yields

$$\sum_{n=1}^{l}||d_t\eta_1^n||_0^2 + ||\eta_1^l||_0^2 + ||\eta_2^l||_0^2 \leq C\left(h^2 + (\triangle t)^2\right), \tag{102}$$

or equivalently,

$$||\eta_2^l||_0^2 \leq C\left(h^2 + (\triangle t)^2\right), \tag{103}$$

where $l \leq N$.

Then, according to the triangle inequality, Lemma 3.1 and (102), we get

$$|||\boldsymbol{p}(u_h^n) - \boldsymbol{p}_h|||_{L^\infty(J;L^2(\Omega))} \leq C(h + \triangle t). \tag{104}$$

From (39) and taking $w_h = div\eta_2^n$ *as a test function, we get*

$$\begin{aligned}
&(div(\boldsymbol{p}^n(u_h^n) - \boldsymbol{p}_h^n), div\eta_2^n)\\
&= -(y_t^n(u_h^n) - d_t y_h^n, div\eta_2^n) - (\phi(y^n(u_h^n)) - I_h\phi(y_h^n), div\eta_2^n).
\end{aligned} \tag{105}$$

Thanks to (56) and (105), we obtain

$$\begin{aligned}
(div\eta_2^n, div\eta_2^n) &= -(y_t^n(u_h^n) - d_t y_h^n, div\eta_2^n)\\
&\quad - (\phi(y^n(u_h^n)) - I_h\phi(y_h^n), div\eta_2^n)\\
&\quad + \lambda(y^n(u_h^n) - Y^n(u_h^n), div\eta_2^n)\\
&\leq C\left(h^2 + (\triangle t)^2\right) + \delta||div\eta_2^n||_0^2.
\end{aligned} \tag{106}$$

So, we have

$$||div\eta_2^n||_0 \leq C(h + \triangle t). \tag{107}$$

It is easy to see the following inequality according to the triangle inequality, Lemma 3.1 and (107)

$$|||\boldsymbol{p}(u_h^n) - \boldsymbol{p}_h|||_{L^\infty(H(\mathrm{div}))} \leq C(h + \triangle t). \tag{108}$$

Hence we have proved (68). The process of proving (69) can also refer to the above steps, which is omitted here. □

Let some intermediate errors as follows

$$\epsilon_1^n = \boldsymbol{p}^n - \boldsymbol{p}^n(u_h^n), \quad \varepsilon_1^n = y^n - y^n(u_h^n), \tag{109}$$

$$\epsilon_2^n = \boldsymbol{q}^n - \boldsymbol{q}^n(u_h^n), \quad \varepsilon_2^n = z^n - z^n(u_h^n). \tag{110}$$

From (11)-(16), we obtain

$$(A^{-1}\epsilon_1^n, \nu) - (\varepsilon_1^n, \mathrm{div}\nu) = 0, \tag{111}$$

$$(y_t^n - d_t y^n(u_h^n), \omega) + (\mathrm{div}\epsilon_1^n, \omega) + (\tilde{\phi}'(y^n)\varepsilon_1^n, \omega) = (u^n - u_h^n, \omega), \tag{112}$$

for $n = 1, 2, \cdots, N$. And

$$(A^{-1}\epsilon_2^{n-1}, \nu) - (\varepsilon_2^{n-1}, \mathrm{div}\nu) = -(\epsilon_1^n, \nu), \tag{113}$$

$$\begin{aligned}
&-(z_t^n - d_t z^n(u_h^n), \omega) + (\mathrm{div}\epsilon_2^{n-1}, \omega) + (\phi'(y^n)\varepsilon_2^{n-1}, \omega)\\
&\quad + (\tilde{\phi}''(y^n)r_1^n z^{n-1}(u_h^n), \omega) = (\varepsilon_1^n, \omega),
\end{aligned} \tag{114}$$

for $n = N, N-1, \cdots, 1$, where $\nu_h \in \boldsymbol{V}_h$, $\omega_h \in W_h$.

Theorem 3.2: There exists $C > 0$, that is independent of h and $\triangle t$, such that

$$|||y - y(u_h^n)|||_{L^\infty(L^2(\Omega))} + |||\boldsymbol{p} - \boldsymbol{p}(u_h^n)|||_{L^\infty(J;H(\mathrm{div}))}$$

$$\leq C(h + \Delta t + |||u - u_h|||_{L^2(L^2(\Omega))}), \quad (115)$$

$$|||z - z(u_h^n)|||_{L^\infty(L^2(\Omega))} + |||\boldsymbol{q} - \boldsymbol{q}(u_h^n)|||_{L^\infty(H(\mathrm{div}))}$$
$$\leq C(h + \Delta t + |||u - u_h|||_{L^2(L^2(\Omega))}). \quad (116)$$

Proof: Define the test functions as $v = \epsilon_1^n$ and $\omega = \varepsilon_1^n$ for $n = 1, 2, \cdots, N$, then from equations (111) and (112), we get

$$(A^{-1}\epsilon_1^n, \epsilon_1^n) + (\tilde{\phi}'(y^n)\varepsilon_1^n, \varepsilon_1^n)$$
$$= (u^n - u_h^n, \varepsilon_1^n) - (y_t^n - d_t y^n(u_h^n), \varepsilon_1^n).$$

Combined (79) with the δ-Cauchy inequality, we have

$$||\epsilon_1^n||_0^2 + ||\varepsilon_1^n||_0^2$$
$$\leq C\left(h^2 + (\Delta t)^2 + ||u^n - u_h^n||_0^2\right) + \delta||\varepsilon_1^n||_0^2, \quad (117)$$

for any fully small $\delta > 0$. And we get

$$||\epsilon_1^n||_0 + ||\varepsilon_1^n||_0 \leq C(h + \Delta t + ||u^n - u_h^n||_0). \quad (118)$$

Let $w = div\epsilon_1^n$ as a test function in (112), then we get

$$||div\epsilon_1^n||_0^2 = (u^n - u_h^n, div\epsilon_1^n) - (y_t^n - d_t y^n(u_h^n), div\epsilon_1^n)$$
$$- (\tilde{\phi}'(y_h^n)\varepsilon_1^n, div\epsilon_1^n)$$
$$\leq C||y_t^n - d_t y^n(u_h^n)||_0^2 + C||u^n - u_h^n||_0^2$$
$$+ C||\varepsilon_1^n||_0^2 + \delta||div\epsilon_1^n||_0^2. \quad (119)$$

Moreover, using the estimation (118), we have

$$||div\epsilon_1^n||_0 \leq C(h + \Delta t + ||u^n - u_h^n||_0). \quad (120)$$

Similarly, define the test functions $v = \epsilon_2^{n-1}$ and $\omega = \varepsilon_2^{n-1}$ for $n = N, N-1, \cdots, 1$, then from equations (113) and (114), we get

$$(A^{-1}\epsilon_2^{n-1}, \epsilon_2^{n-1}) + (\phi'(y^n)\varepsilon_2^{n-1}, \varepsilon_2^{n-1})$$
$$= (\varepsilon_1^n, \varepsilon_2^{n-1}) + (z_t^n - d_t z^n(u_h^n), \varepsilon_2^{n-1})$$
$$- (\tilde{\phi}''(y^n)z^{n-1}(u_h^n)\varepsilon_1^n, \varepsilon_2^{n-1}).$$

Then, we use (79) and the δ-Cauchy inequality to obtain

$$||\epsilon_2^{n-1}||_0^2 + ||\varepsilon_2^{n-1}||_0^2$$
$$\leq C\left((\Delta t)^2 + h^2 + ||u^n - u_h^n||_0^2\right) + \delta||\varepsilon_2^{n-1}||_0^2, \quad (121)$$

or equivalently,

$$||\epsilon_2^{n-1}||_0 + ||\varepsilon_2^{n-1}||_0 \leq C(h + \Delta t + ||u^n - u_h^n||_0). \quad (122)$$

Using the δ-Cauchy inequality and taking $\omega = div\epsilon_2^{n-1}$ as a test function in (114), we have

$$||div\epsilon_2^{n-1}||_0^2$$
$$= (\varepsilon_1^n, div\epsilon_2^{n-1}) - (\phi'(y^n)\varepsilon_2^{n-1}, div\epsilon_2^{n-1})$$
$$+ (z_t^n - d_t z^n(u_h^n), div\epsilon_2^{n-1}) - (\tilde{\phi}''(y^n)z^{n-1}(u_h^n)\varepsilon_1^n, div\epsilon_2^{n-1})$$
$$\leq C||z_t^n - d_t z^n(u_h^n)||_0^2 + C||\varepsilon_1^n||_0^2$$
$$+ C||\varepsilon_2^{n-1}||_0^2 + \delta||div\epsilon_2^{n-1}||_0^2. \quad (123)$$

By using the estimations (118) and (122), we confirm that

$$||div\epsilon_2^{n-1}||_0 \leq C(h + \Delta t + ||u^n - u_h^n||_0). \quad (124)$$

The above process proves Theorem 3.2. □

IV. A PRIORI ERROR ESTIMATES

We present a detailed approach to the approximation problem of semi-linear parabolic optimal control for mixed finite element systems with fully discrete interpolation coefficients estimated in this section. Define $S(\cdot)$ as a G-differential uniform convex functional for $K \to \mathbb{R}$, which is subjected to the following equations

$$S(u^n) = \frac{1}{2}||\boldsymbol{p}^n - \boldsymbol{p}_d||^2 + \frac{1}{2}||y^n - y_d||^2 + \frac{\alpha}{2}||u^n||^2,$$
$$S(u_h^n) = \frac{1}{2}||\boldsymbol{p}^n(u_h^n) - \boldsymbol{p}_d||^2 + \frac{1}{2}||y^n(u_h^n) - y_d||^2 + \frac{\alpha}{2}||u_h^n||^2.$$

for $S(\cdot)$ is uniform convex near the solution u (see, e.g., [27]).

We are going to prove the following equations

$$(S'(u^n), v) = (z^n + \alpha u^n, v),$$
$$(S'(u_h^n), v) = (z^n(u_h^n) + \alpha u_h^n, v),$$

where $(\boldsymbol{p}^n(u_h^n), y^n(u_h^n), \boldsymbol{q}^{n-1}(u_h^n), z^{n-1}(u_h^n))$ is the solution of (47)-(52) with $\tilde{u} = u_h^n$. In many applications, $S(\cdot)$ is uniform convex near the solution u (see, e.g., [27]). Then there is a $c > 0$, independent of h, such that

$$\int_0^T (S'(u) - S'(v), u - v) \geq c|||u - v|||_{L^2(J;L^2(\Omega))}^2, \quad (125)$$

for $v \in U$, and for some fully small $\varepsilon > 0$, $\|u - v\|_{L^2(0,T;L^2(\Omega))} \leq \varepsilon$. The convexity of $S(\cdot)$ is closely related to the second order sufficient conditions of the optimal control problem, which is assumed in many studies on numerical methods of the problem. Then we get the following results by using Theorem 3.2.

Theorem 4.1: Let $(\boldsymbol{p}^n, y^n, \boldsymbol{q}^{n-1}, z^{n-1}, u^n)$ belong to $(\boldsymbol{V} \times W)^2 \times G$, and $(\boldsymbol{p}_h^n, y_h^n, \boldsymbol{q}_h^{n-1}, z_h^{n-1}, u_h^n)$ belong to $(\boldsymbol{V}_h \times W_h)^2 \times G_h$, which is the solutions of (11)-(17) and (38)-(44), respectively. Suppose that $z^n + \alpha u^n \in H^1(\Omega)$, we obtain

$$|||u - u_h|||_{L^2(J;L^2(\Omega))} \leq C(h + \Delta t), \quad (126)$$

$$|||y - y_h|||_{L^\infty(J;L^2(\Omega))} + |||\boldsymbol{p} - \boldsymbol{p}_h|||_{L^\infty(J;H(\mathrm{div}))} \leq C(h + \Delta t), \quad (127)$$

$$|||z - z_h|||_{L^\infty(J;L^2(\Omega))} + |||\boldsymbol{q} - \boldsymbol{q}_h|||_{L^\infty(J;H(\mathrm{div}))} \leq C(h + \Delta t). \quad (128)$$

Proof: Firstly, by putting $\widehat{u} = u_h^n$ into (17), and putting $\widehat{u}_h = Q_h u^n$ into (44), we get

$$\int_0^T (z^n + \alpha u^n, u_h^n - u^n) \geq 0, \quad (129)$$

and

$$\int_0^T (z_h^n + \alpha u_h^n, Q_h u^n - u_h^n) \geq 0. \quad (130)$$

It follows (129), (130), and $Q_h u^n - u_h^n = Q_h u^n - u^n + u^n - u_h^n$ that we have

$$\int_0^T (z_h^n + \alpha u_h^n - z^n - \alpha u^n, u^n - u_h^n)$$
$$+ \int_0^T (z_h^n + \alpha u_h^n, Q_h u^n - u^n) \geq 0. \quad (131)$$

We get the following derivations by using the uniform convexity of $S(\cdot)$ and (131),

$$\begin{aligned}
&c|||u-u_h|||^2_{L^2(J;L^2(\Omega))}\\
&\leq \int_0^T (S'(u^n), u^n-u_h^n) - \int_0^T (S'(u_h^n), u^n-u_h^n)\\
&= \int_0^T (z^n+\alpha u^n, u^n-u_h^n) - \int_0^T (z^n(u_h^n)+\alpha u_h^n, u^n-u_h^n)\\
&\leq \int_0^T (z_h^n - z^n(u_h^n), u^n-u_h^n) + \int_0^T (\alpha(u_h^n-u^n), Q_h u^n-u^n)\\
&\quad + \int_0^T (z_h^n-z^n(u_h^n), Q_h u^n-u^n) + \int_0^T (z^n+\alpha u^n, Q_h u^n-u^n)\\
&\quad + \int_0^T (z^n(u_h^n)-z^n, Q_h u^n-u^n). \qquad (132)
\end{aligned}$$

Secondly, we use the δ-Caunchy inequality and Theorem 3.1 to estimate the right side of (132). For any $\delta > 0$,

$$\begin{aligned}
&\int_0^T (z_h^n - z^n(u_h^n), u^n-u_h^n)\\
&\quad \leq |||z_h - z(u_h^n)|||_{L^2(J;L^2(\Omega))} \cdot |||u-u_h|||_{L^2(J;L^2(\Omega))}\\
&\quad \leq \delta|||u-u_h|||^2_{L^2(J;L^2(\Omega))} + C(h+\Delta t)^2. \qquad (133)
\end{aligned}$$

It is easy to see by using the δ-Caunchy inequality and (46),

$$\int_0^T (\alpha(u_h^n-u^n), Q_h u^n-u^n) \leq Ch^2+\delta|||u-u_h|||^2_{L^2(J;L^2(\Omega))}. \qquad (134)$$

From (46) we get the projection of Q_h approximation property, so we have

$$\int_0^T (z_h^n - z^n(u_h^n), Q_h u^n - u^n) \leq C(h+\Delta t)^2. \qquad (135)$$

Considering Theorem 3.2, the property (46), and the δ-Cauchy inequality, we have

$$\int_0^T (z^n(u_h^n)-z^n, Q_h u^n-u^n) \leq Ch^2+\delta|||u-u_h|||^2_{L^2(J;L^2(\Omega))}. \qquad (136)$$

So,

$$||\omega||_{-k,\Omega} = \sup_{\varrho\in 0, \varrho\neq 0} \frac{|(\omega,\varrho)|}{||\varrho||_{k,\Omega}}, \qquad (137)$$

and combined with (46), we have

$$\begin{aligned}
&\int_0^T (z^n+\alpha u^n, Q_h u^n-u^n)\\
&\quad \leq C||z^n+\alpha u^n||_{1,\Omega}\cdot||Q_h u^n-u^n||_{-1,\Omega} \leq Ch^2. \qquad (138)
\end{aligned}$$

Substituting (133)-(138) into (132), it is clear that

$$|||u-u_h|||_{L^2(J;L^2(\Omega))} \leq C(h+\Delta t). \qquad (139)$$

Therefore we have proved (126).

Finally, considering Theorem 3.1, Theorem 3.2, (139), the triangle inequality, and by taking the state and the co-state variables in our estimate problems, we have that

$$|||y-y_h|||_{L^\infty(J;L^2(\Omega))}+|||\boldsymbol{p}-\boldsymbol{p}_h|||_{L^\infty(J;H(\mathrm{div}))} \leq C(h+\Delta t), \qquad (140)$$

TABLE 1. The errors estimates for the control and the state variables.

h	Errors				
	$\lvert\lvert\lvert u-u_h\rvert\rvert\rvert$	$\lvert\lvert\lvert \boldsymbol{p}-\boldsymbol{p}_h\rvert\rvert\rvert$	$\lvert\lvert\lvert y-y_h\rvert\rvert\rvert$	$\lvert\lvert\lvert \boldsymbol{q}-\boldsymbol{q}_h\rvert\rvert\rvert$	$\lvert\lvert\lvert z-z_h\rvert\rvert\rvert$
1/16	5.64972E-02	2.03498E-01	3.64610E-02	5.91284E-01	7.01598E-02
1/32	2.84163E-02	1.01359E-01	1.81396E-02	2.95479E-01	3.50779E-02
1/64	1.36850E-02	5.05545E-02	9.06922E-03	1.47969E-01	1.75401E-02
1/128	7.01364E-03	2.52946E-02	4.53492E-03	7.39516E-02	8.76945E-03

and

$$|||z-z_h|||_{L^\infty(J;L^2(\Omega))}+|||\boldsymbol{q}-\boldsymbol{q}_h|||_{L^\infty(J;H(\mathrm{div}))} \leq C(h+\Delta t). \qquad (141)$$

By using inequalities (140) and (141), we obtain (127) and (128). □

V. NUMERICAL EXAMPLE

We are going to give the theoretical results by an example in this section, to verify the prior error estimates in the state, the co-state and the control. OCPs are handled with the code developed in the freely available AFEPACK, see for details in [31].

We define the OCPs as follows

$$\min_{u(t)\in K\subset U} \left\{ \frac{1}{2}\int_0^T \left(||\boldsymbol{p}-\boldsymbol{p}_d||^2 + ||y-y_d||^2 + ||u||^2 \right) dt \right\},$$

$$\begin{aligned}
&y_t + \mathrm{div}\boldsymbol{p} + y^5 = u + f, \quad x\in\Omega,\\
&y(x,t) = 0, \quad x\in\partial\Omega,\\
&\mathrm{div}\boldsymbol{q} + 5y^4 z - z_t = y - y_d, \quad x\in\Omega,\\
&z(x,t) = 0, \quad x\in\partial\Omega,
\end{aligned}$$

for $\boldsymbol{p} = -\nabla y$, $\boldsymbol{q} = -\nabla z - \boldsymbol{p} + \boldsymbol{p}_d$, $y(x,0) = 0$, and $z(x,T) = 0$.

Let $\Omega = [0,1]\times[0,1]$, and $T = 1$. We compute the convergence order via an equation: order $\simeq \frac{\log(E_i/E_{i+1})}{\log(h_i/h_{i+1})}$, where i means the spatial partition, E_i is the approximations on L^∞-norm for the state and the co-state, L^2 norm is used for control approximation.

The testing data is given as follows

$$\begin{aligned}
u &= \max(1.0-z, 0),\\
y &= \sin 2\pi x_1 \sin 2\pi x_2 \sin \pi t,\\
z &= 2\sin 2\pi x_1 \sin 2\pi x_2 \sin \pi t,\\
f &= y_t + \mathrm{div}\boldsymbol{p} + y^5 - u,\\
y_d &= z_t + y - 5y^4 z - \mathrm{div}\boldsymbol{q},\\
\boldsymbol{p} &= \begin{pmatrix} 2\pi\cos 2\pi x_1 \sin 2\pi x_2 \sin \pi t\\ 2\pi\cos 2\pi x_2 \sin 2\pi x_1 \sin \pi t \end{pmatrix},\\
\boldsymbol{q} &= \frac{3}{2}\boldsymbol{p}_d = 3\begin{pmatrix} 2\pi\cos 2\pi x_1 \sin 2\pi x_2 \sin \pi t\\ 2\pi\cos 2\pi x_2 \sin 2\pi x_1 \sin \pi t \end{pmatrix}.
\end{aligned}$$

In the test, we define the state and the control variables for $\Delta t = h$ on the same mesh partition. We study the convergence of the solution and its order in this case. Table 1. shows the deviations $|||\boldsymbol{p}-\boldsymbol{p}_h|||_{L^\infty(J;H(\mathrm{div}))}$, $|||u-u_h|||_{L^2(J;L^2(\Omega))}$, $|||\boldsymbol{q}-\boldsymbol{q}_h|||_{L^\infty(J;H(\mathrm{div}))}$, $|||y-y_h|||_{L^\infty(J;L^2(\Omega))}$, and $|||z-z_h|||_{L^\infty(J;L^2(\Omega))}$ with $h = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$, respectively. Fig. 1 shows the order of convergence through the slope, where dofs means the degree of freedoms.

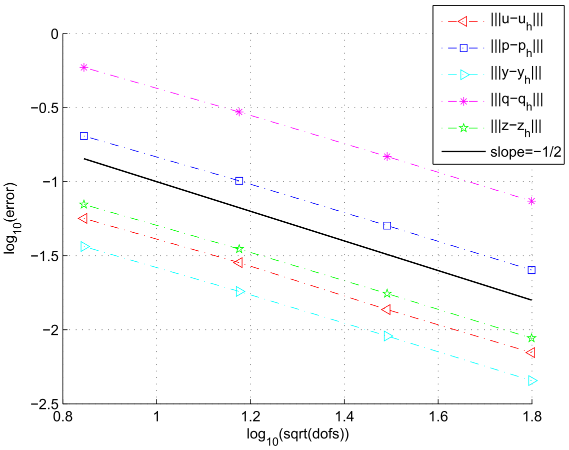

FIGURE 1. Convergence orders of $u - u_h, p - p_h, y - y_h, q - q_h$ and $z - z_h$ in different norms.

From this numerical example, we find that the order of convergence is $\mathcal{O}(h + \triangle t)$, which confirms our theoretical results for prior error estimates.

VI. CONCLUSION AND FUTURE WORK

We explored the fully discrete interpolation coefficients MFEMs for the OCPs governed by semi-linear parabolic equations. We use the interpolation operator $I_h(\phi(y_h))$ to compute the nonlinear term $\phi(y_h)$, and interpolation coefficients mixed finite elements is used to the space discretization of the state variable, while the discretization of time is based on difference methods. By applying MFEMs, we gave the control and coupled approximations with priori error estimates.

The priori error estimates for semi-linear parabolic problems by fully discrete interpolation coefficients MFEMs may be new.

In the future, it is interesting to apply interpolation coefficients for nonlinear hyperbolic problems by MFEMs. Meanwhile, we may consider superconvergence of nonlinear parabolic OCPs.

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