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A Numerical Approach for Solving Nonlinear Optimal Control Problems Using the Hybrid Scheme of Fitness Dependent Optimizer and Bernstein Polynomials

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ABSTRACT In this paper, a heuristic scheme based on the hybridization of Bernstein Polynomials (BPs) and nature-inspired optimization techniques is presented to achieve the numerical solution of Nonlinear Optimal Control Problems (NOCPs) efficiently. The solution of NOCP is approximated by the linear combination of BPs with unknown coefficients. The unknown coefficients are estimated by transforming the NOCP into an error minimization problem and formulating the objective function. The Genetic Algorithm (GA) and Fitness Dependent Optimizer (FDO) are used for solving the objective function and obtaining the optimum values of the unknown coefficients. The findings and statistical results indicate the represented hybrid scheme offers encouraging results and outperforms the most recent and popular methods proposed in the literature, which ultimately validates the efficacy and productivity of the recommended approach. Furthermore, statistical analysis is incorporated to examine the reliability and stability of the suggested technique. Consequently, the remarkable difference is evident in simplicity, flexibility, and effectiveness compared to the other methods considered.

INDEX TERMS Optimal control problems, optimization problem, Bernstein polynomials, fitness dependent optimizer, genetic algorithm.

I. INTRODUCTION

Optimal Control Problems (OCPs) contain complex mathematical operations and have practical applications and importance in almost every field of science; i.e., engineering, economics, biomedicine, aircraft systems, robotics, etc. [1], [2]. Solving these OCPs numerically sometimes becomes very sophisticated, and hence obtaining optimal solutions for such problems could be quite tedious [3].

Due to the nonlinear and dynamical nature of OCPs, researchers have presented several numerical methods to determine the optimal solution for such problems and further improve already proposed methods [4]–[6]. To highlight a few of these available numerical techniques, Stryk and Bulirsch [7] provided a list of normally utilized direct and indirect techniques to find the numerical solution for OCPs and transformed various OCPs to some Nonlinear Programming (NLP) problems on successfully employing the parametrization or discretization techniques. Yıldız and Karasozen [8] applied various discretization methods on the distributed OCPs determined by the unsteady diffusion-convection reaction equation having no control constraints and supported their convergence rate noticed theoretically via the solution obtained. Liu *et al.* [9] investigated a class of OCPs by considering generally known nonlinear time-delay systems containing free terminal time. The authors applied the control parametrization scheme to generate a gradient-based nonlinear optimization technique

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and yielded an approximated solution. Teo *et al.* [10], Lee *et al.* [11] introduced the Control Parametrization Enhancing Technique (CPET) for constrained OCPs, and its convergence properties were elaborated via some numerical problems generating a computationally convenient and numerically accurate optimal solution. Vlassenbroeck [12], Vlassenbroeck and Dooren [13] adopted the numerical method governed by Chebyshev series expansion for solving nonlinear constrained OCPs and clarified the applicability of its theoretical and computational considerations by applying it to some approximation problems. Jaddu [14], [15] encountered some numerical techniques, for instance, quasi-linearization and state parameterization, solving constrained and unconstrained OCPs, like container crane problems and F8 fighter aircraft, via Chebyshev Polynomials and later modified these techniques to include NOCPs which enhanced the optimal solution quality and validated the analytical approach. Elnagar *et al.* [16] implemented Legendre Polynomials (LPs) for the discretization of various OCPs and approximation of control and state variables, whereas Edrisi-Tabri and Lakestani [17], Edrisi-Tabri *et al.* [18] obtained the approximated solution by implementing some linear B-spline functions on nonlinear constrained quadratic OCPs. Shienyu [19] presented a numerical technique for approximation of NOCPs using Block Pulse Functions (BPFs), whereas Mohan and Kar [20] executed BPFs and LPs, finding the optimal solution for NOCPs. Besides, Mashayekhi *et al.* [21] acquired the approximated solution of NOCPs by deploying BPFs and Bernoulli Polynomials. Kafash *et al.* [22] employed a numerical scheme for effectively solving the OCPs using Boubaker Polynomials Expansion Scheme (BPES), whereas Ouda *et al.* [23] proposed an indirect method for OCPs using BPES with the findings that the BPES technique is trustworthy. Dehghan [4] tailored an iterative numerical method based on Cardan Polynomials (CP), providing an approximated solution for OCPs with fast convergence rates. Cichella *et al.* [3] exploited BPs to approximate nonlinear constrained OCPs, and besides rigorous analysis, the authors validated the theoretical findings by applying the method to diverse optimization problems. Yousefi *et al.* [24] evaluated the approximate solution of different fractional OCPs using BPs.

Besides continuous improvements in the available numerical methods for dealing with OCPs, researchers have paved the way for nature-inspired optimization techniques to demonstrate their importance and efficacy in operating NOCPs in the engineering and applied sciences domain [25]. These nature-inspired optimization techniques are derivative-free, conceptually simple, do not require complex mathematical calculations, and prove their effectiveness by providing better solutions to complex real-world problems on a continuous-time grid as compared to other numerical techniques [26]. To name just a few nature-inspired optimization techniques, GA handled competently the three link hopping robot amid flight phase for optimally solving the posture control problem [26]–[28]. Particle Swarm Optimization (PSO) addressed subtly the bilocal OCP of the dc motor for rendering the desired solution [29]–[31]. Differential Evolution (DE) proved its capability by presenting the optimum solution for OCP concerned with the power flow in Microgrids (MGs) connected in a network [32], [33]. Ant Colony Optimization (ACO) dealt favorably with the complexities of the Area Traffic Control (ATC) problem and ultimately provided an optimal solution [34]. Artificial Bee Colony Optimization (ABCO) proved efficient in demonstrating a linear quadratic optimal controller design and achieving an optimum solution for a nonlinear inverted pendulum [35]–[38]. FDO dealt successfully with multi-source interconnected power system by implementing a productive Automatic Generation Control (AGC) [39]–[41]. Such algorithms are enough to unveil the comparatively better results and enhanced ability of nature-inspired optimization techniques to provide the global solution with a rapid convergence rate in comparison to the previously utilized techniques applied to various complex optimization problems.

Although nature-inspired optimization techniques have proved their efficiency at each level in reliably obtaining the solution to nonlinear problems, sometimes finding the optimum solution to a problem is not possible by using such techniques alone. In this case, a new approach known as the hybridization approach evolved for solving various real-world problems. This approach occasionally utilizes local search methods for the execution of some population-based optimization algorithms to provide better results and reduce processing time. These hybrid approaches further proved quite impressive in refining the solution obtained by nature-inspired optimization techniques solely. The excellent real-world problem-solving capability of hybrid nature-inspired optimization techniques can be analyzed through consideration of the aforementioned references in which SUN *et al.* [42] presented a Hybrid Improved GA (HIGA) approach and implemented it for solving OCPs related to chemical processes, e.g., Fed-Batch Bioreactor. Victoire and Jeyakumar [43] suggested a hybridized technique, i.e., PSO with Successive Quadratic Programming (PSO-SQP), to find the optimal solution for relevant problems, which was later amended by Modares and Sistani [25] via a new hybrid algorithm, i.e., Improved PSO combining SQP (IPSO-SQP), for further reinforcing previously attained solution. Nezhadhosein *et al.* [26] dealt excellently with several NOCPs, including the Chemical Reactor Problem (CRP), with the help of Modified Hybrid GA (MHGA) that integrates GA and SQP (GA-SQP), yielding the best solution. Nezhadhosein *et al.* [44] integrated DE and MHGA solving numerous NOCPs, including Temperature Control for Consecutive Reaction (TCCR) problem. Similarly, Xu *et al.* [45] recommended a novel hybrid scheme for Bayesian Optimization (BO) based on the hybridization of BO and PSO (BO-PSO), solving some complex optimization problems tremendously, for instance, Distributed Propulsion Configuration (DPC) aircraft having integrated flight/propulsion optimum control. Chiu *et al.* [46] proposed a hybrid approach of the Sine Cosine Algorithm (SCA) and FDO (SCA-FDO) and proved its efficiency by applying it to several optimization problems. Abbas *et al.* [47] suggested a hybrid technique combining Multi-Layer Perceptron (MLP) and FDO (FDO-MLP) to obtain the optimal values for the weights and biases of the Neural Network (NN) and minimize the Mean Square Error (MSE) of the optimization problem.

In this paper, a contemporary and promising nature-inspired optimization algorithm, namely FDO, is hybridized with BPs for numerically solving the NOCPs. In the proposed method, the solution of NOCPs is approximated using the linear combination of BPs with unknown coefficients. The given original system is transformed into an equivalent optimization problem by formulating the objective function. FDO and GA are employed to solve the optimization problems and yield the optimum values for the unknown coefficients, consequently providing the numerical solution for NOCPs. Moreover, to quantify the effectiveness of the suggested hybrid technique, the Absolute Error (AE) is minimized to a significant value by providing a better solution than previously developed approaches. Finally, to authenticate the efficacy and capability of the suggested scheme, a statistical analysis is conducted.

The content of this research work is divided into the following sections: Section 2 defines the basic generalized mathematical form of the OCPs. Section 3 introduces nature-inspired optimization techniques used in this work. Section 4 elaborates on the proposed methodology adopted and its hybridization scheme with BPs. Section 5 demonstrates the implementation of the proposed hybrid-based approach. In Section 6, statistical analysis for the proposed technique is presented, followed by the conclusion of this work in Section 7.

II. OPTIMAL CONTROL PROBLEMS

Generally, the OCP could be represented mathematically as the classical calculus of variation with the following crucial components: (i) A mathematical system, which is required to be controlled; (ii) A desired output for the relevant system; (iii) A set containing permissible input(s); (iv) A performance index/cost function, which assists in measuring the efficiency of the control process under consideration.

Optimal control is concerned with the process which is described by the problem of determining a control law, on the fixed time interval $[t_i, t_f]$, for a given system of nonlinear differential equations as follows:

$$u(t) = f(t, x(t), \dot{x}(t)) \tag{1}$$

The initial/boundary conditions are represented by a set that provides the value of the system state variable(s) x at the initial time t_i and final time t_f as given below:

$$x(t_i) = x_0, \quad x\left(t_f\right) = x_f \tag{2}$$

where $u(t)$ represents the mathematical model for a given dynamic system that requires the control processes and is generally demonstrated by a collection of first-order differential equations. Here $x(.) : [t_i, t_f] \to \mathbb{R}$ and $u(.) : [t_i, t_f] \to \mathbb{R}$ are the state variable and control variable, respectively, whereas f represents the continuously differentiable real-valued function. The assumption here is u is a piecewise continuous function during the time interval $t \in [t_i, t_f] \to \mathbb{R}$. Therefore, changes applied to u are directly proportional to the solution of the relevant differential equation(s). In addition, x_0 denotes some known vector of initial condition(s).

The performance index J, either minimized or maximized, is demonstrated mathematically in a scalar function and describes the desired specifications. By minimizing J of the OCP, the selection of the most feasible optimal solution is made easy. The performance index can be formulated as:

$$J = \int_{t_i}^{t_f} L(t, x(t), u(t))dt \tag{3}$$

where L is a scalar function, differentiable in all arguments, and generally a non-negative function, i.e., $L(t, 0, 0) = 0$. Additionally, if J of any optimization problem is represented mathematically, as demonstrated by (3), it could be referred to as the Lagrange problem.

Hence, the main goal of OCP is finding such a control u which transfers the system $u(t)$ from position $x(t_i) = x_0$ to $x\left(t_f\right) = x_f$ within the time frame $\left(t_f - t_i\right)$ and ultimately generates the optimum value for J [1], [4], [5]. Furthermore, interested readers may see [1]–[6] and references therein for a detailed description of the OCPs considered in this work, as shown in (1)–(3).

III. NATURE-INSPIRED OPTIMIZATION TECHNIQUES

This section briefly describes GA, a well-known evolutionary algorithm, and FDO, a recently introduced algorithm, used in this work.

A. GENETIC ALGORITHM

GA represents the heuristic optimization method established on concepts of natural selection, crossover, and mutation that uses a global search approach. The central idea behind this algorithm is the survival of the fittest theory, which permits only fit and competitive chromosome(s) to generate the next iteration. The working principle for GA comprises the population of individuals called chromosomes, where every chromosome contains a potential solution for a given problem. A randomly generated fitness-based value is allotted to each chromosome, representing the quality of the solution. The next step involves the selection and recombination processes (i.e., child chromosomes are produced by recombining the parent chromosomes), yielding the next generation based on the fittest individuals. This procedure is repeated over successive generations, evaluating the best solution using the genetic operator(s), namely selection, crossover, and mutation where necessary, until the algorithm reaches the stopping criteria. Further, the maximum number of generations accomplished, or the desired fitness level achieved, is considered the stopping criteria for GA [25]–[28].

The pseudocode for GA is presented in Algorithm 1 below:

Algorithm 1 Genetic Algorithm Pseudocode [27]

Start $t \to 0$
Generate Initial Population *Pi*
Evaluate Population *Pi*
while (termination criteria not satisfied)
Repeat
for $i = 1$ to size (population) **do**
SELECT *N* chromosomes
Find chromosome with the lowest fitness
Remove chromosome with the lowest fitness
CROSSOVER create new chromosome
Evaluate new chromosome
MUTATION apply (optional)
Evaluate mutated chromosomes
end for
end while

B. FITNESS DEPENDENT OPTIMIZER

FDO is a recently introduced heuristic algorithm applied to various real-world problems, e.g., aperiodic antenna array designs and frequency-modulated sound waves, achieving the optimal solution. FDO not only shares some features with PSO but also contains significant variabilities. It imitates the reproduction behavior of the bee swarms when they search for suitable hives. Further, the main idea behind this algorithm is derived from the method used by scout bees to select an appropriate hive from several potential ones. Moreover, in FDO, every scout bee looking for desired hives describes a possible solution. Therefore, choosing the best hive from a group of potential hives increases the likelihood of achieving the optimum solution [48].

FDO initiates the population of scout bees generated randomly within the search domain. The position and objective function of every scout bee are crucial for locating the hive. Eventually, the main goal of the scout bee is finding a better hive, i.e., a new solution, and if this purpose is served, the previously searched solution could be avoided. On the other hand, if the scout bee is unsuccessful in achieving a better solution, the prior solution could be adopted to modify its position [48], [49].

Scout bee is represented as follows:

$$S_b(b = 1, 2, \ldots, n) \tag{4}$$

The random search-based procedures of scout bees that initiate within the search domain depend on fitness weight fw and random walk methods. As evident from (5) below, the pace p with which the current scout changes position is imperative because the scout bee movement relies on it. The scout determines a new and improved solution by adding p [48], [49]:

$$S_{b,t+1} = S_{b,t} + p \tag{5}$$

where b describes the current scout bee, i.e., search agent, S denotes the artificial scout bee, and t refers to the current iteration. Here p represents the movement rate and direction of the artificial scout bee and counts on fw. Though, the direction of p entirely relies on some random operation. Hence, fw, which lies within the range of [0, 1], can be evaluated for minimization problems by (6) as follows [49]:

$$fw = \left| \frac{s^*_{b,tfitness}}{s_{b,tfitness}} \right| - wf \tag{6}$$

where $s_{b,tfitness}$ represents the optimum solution for the current scout bee, $s^*_{b,tfitness}$ exhibits the best solution determined by the scout bee thus far, and wf denotes the weight factor which has a value of 1 or 0.

FDO contains a random number R that lies within the scope of $[-1, 1]$. Although there are various ways to execute the random walk method, the Levy flight mechanism is employed since it possesses a favorable distribution curve and helps yield more stable movements. Henceforth, it is evident that FDO demands simple calculations for determining the objective function as it only requires R and fw [49].

The lower and upper bounds are used to find a global solution after a random initiation of the scout bee within the search domain. Now, fw is evaluated based on the following considerations [49].

If $fw = 1$ or $fw = 0$ or $s_{b,tfitness} = 0$ then p can be calculated as follows [49]:

$$p = s_{b,t} * R \tag{7}$$

Likewise, for $fw > 0$ and $fw < 1$, R can be executed based on the following conditions:

If $R < 0$ using (8) can help find p else if $R \geq 0$ then (9) could be used as mentioned below [48]–[51]:

$$p = distance_{best_{bee}} * fw * (-1) \tag{8}$$

$$p = distance_{best_{bee}} * fw \tag{9}$$

where $distance_{best_{bee}}$ is measured as:

$$distance_{best_{bee}} = s_{b,t} - s^*_{b,t} \tag{10}$$

The pseudocode for FDO is demonstrated in Algorithm 2 below:

IV. PROPOSED METHODOLOGY

This section presents the research methodology for the hybridization of Nature-Inspired Computing (NIC) techniques with BPs as follows:

To obtain the approximate numerical solution for OCPs considered in this work, e.g., (1)–(3), the approximate numeric solution of a given NOCP is assumed to be a linear combination of BPs with unknown parameters, as demonstrated in (11)–(13) below:

The following is an approximation of the state variable:

$$x(t) = \sum_{i=0}^{k} \alpha_i B_{i,k}(t) \tag{11}$$

The control variable is calculated as a function of the unknown parameters of the state variable(s) by using (1),

Algorithm 2 Fitness Dependent Optimizer Pseudocode [48]

scout bee population initialization $S_{b,t}$; ($b = 1, 2, 3,.., n$)
while iteration (t) limit not achieved **do**
for all scout bees $S_{b,t}$
  search best scout bee $s^*_{b,t}$
  execute random walk $R\epsilon[-1, 1]$
  % random number R
if $S_{b,tfitness} == 0$
  $fw = 0$ %fitness weight fw
else
use $fw = \left|\frac{S^*_{b,tfitness}}{S_{b,tfitness}}\right| - wf$ % weight factor wf
end if
 if $fw == 0$ OR $fw == 1$
  $p = (s_{b,t} * R)$ %pace p
 else
  if $R \geq 0$
   $p = (s_{b,t} - s^*_{b,t}) * fw$
 else
   $p = (s_{b,t} - s^*_{b,t}) * fw * (-1)$
 end if
 end if
 determine $S_{b,t+1} = S_{b,t} + p$
 if $S_{b,t+1fitness} < S_{b,tfitness}$
  accept $S_{b,t+1fitness}$
  save p
 else
  compute $S_{b,t+1} = S_{b,t} + p$ % p with previous value
 if $S_{b,t+1fitness} < S_{b,tfitness}$
  accept $S_{b,t+1fitness}$
  save p
 else
  keep current position
 end if
 end if
end for
end while

as formulated below:

$$u(t) = f(t, \sum_{i=0}^{k} \alpha_i B_{i,k}(t), \sum_{i=0}^{k} \alpha_i B_{i,k}^{\cdot}(t)) \tag{12}$$

The approximate values of $x(t)$ and $u(t)$ from (11) and (12), respectively, are substituted into (3), determining J as follows:

$$J = \int_{t_i}^{t_f} \begin{array}{l} L(t, \sum_{i=0}^{k} \alpha_i B_{i,k}(t), \\ f(t, \sum_{i=0}^{k} \alpha_i B_{i,k}(t), \sum_{i=0}^{k} \alpha_i B_{i,k}^{\cdot}(t)))dt \end{array} \tag{13}$$

where $(\alpha_0, \alpha_{1,...,}\alpha_k)$ refers to the unknown coefficients/ parameters to be determined, and k denotes the degree of BPs.

TABLE 1. Unknown parameters obtained by GA-BP and FDO-BP for Problem 01.

Parameters	*GA-BP*	*FDO-BP*
α_0	0.000000000	0.000000000
α_1	0.053182659	0.053182382
α_2	0.106360373	0.106364771
α_3	0.160826556	0.160813382
α_4	0.217777653	0.217794545
α_5	0.278649217	0.278637669
α_6	0.344795858	0.344799164
α_7	0.417935396	0.417935294
α_8	0.500000001	0.500000000

TABLE 2. Exact solution and approximate numerical values for state variable of Problem 01.

t	$x(t)$ *Exact*	$x(t)$ *GA-BP*	$x(t)$ *FDO-BP*	$x(t)$ *CP [4]*
0.0	0.000000000	0.000000000	0.000000000	0.000000000
0.1	0.042616852	0.042616666	0.042616852	0.042702960
0.2	0.085660227	0.085660290	0.085660227	0.085725159
0.3	0.129560919	0.129561224	0.129560919	0.129554968
0.4	0.174758300	0.174758449	0.174758300	0.174680761
0.5	0.221704721	0.221704679	0.221704721	0.221590909
0.6	0.270870037	0.270870103	0.270870037	0.270773784
0.7	0.322746312	0.322746558	0.322746312	0.322717759
0.8	0.377852740	0.377852840	0.377852740	0.377911205
0.9	0.436740845	0.436740703	0.436740845	0.436842495
1.0	0.500000000	0.500000001	0.500000000	0.500000000

TABLE 3. Exact solution and approximate numerical values for control variable of Problem 01.

t	$u(t)$ *Exact*	$u(t)$ *GA-BP*	$u(t)$ *FDO-BP*	$u(t)$ *CP [4]*
0.0	0.425459064	0.425461272	0.425459056	0.427061311
0.1	0.427588133	0.427587749	0.427588134	0.427811839
0.2	0.433996647	0.434000515	0.433996646	0.433446089
0.3	0.444748746	0.444749019	0.444748747	0.443964059
0.4	0.459952040	0.459949422	0.459952041	0.459365751
0.5	0.479758688	0.479758154	0.479758687	0.479651163
0.6	0.504366922	0.504369153	0.504366920	0.504820296
0.7	0.534023030	0.534023569	0.534023030	0.534873150
0.8	0.569023821	0.569020739	0.569023823	0.569809725
0.9	0.609719592	0.609719247	0.609719592	0.609630021
1.0	0.656517643	0.656516840	0.656517648	0.654334038

A. PROPOSED METHODOLOGY FOR HYBRIDIZATION OF NATURE-INSPIRED COMPUTATIONAL TECHNIQUE WITH BERNSTEIN POLYNOMIAL BASIS FUNCTION

This subsection specifically demonstrates the research methodology adopted in this work for the hybrid scheme of NIC techniques and BPs solving NOCPs as follows:

To deal with the OCPs considered in this paper, we assume that the approximate numerical solution for $u(t)$, as represented in (1), consists of a linear combination of BPs with $k = 8$ as mentioned below:

$$x(t) = \sum_{i=0}^{8} \alpha_i B_{i,8}(t) \tag{14}$$

$$u(t) = f(t, \sum_{i=0}^{8} \alpha_i B_{i,8}(t), \sum_{i=0}^{8} \alpha_i B_{i,8}^{\cdot}(t)) \tag{15}$$

TABLE 4. Absolute Error values for state variable of Problem 01.

t	AE GA-BP	AE FDO-BP	AE CP [4]
0.0	0.00E+00	0.00E+00	0.00E+00
0.1	1.86E-07	2.12E-11	8.61E-05
0.2	6.28E-08	1.06E-10	6.49E-05
0.3	3.05E-07	1.56E-10	5.95E-06
0.4	1.49E-07	1.61E-11	7.75E-05
0.5	4.17E-08	7.78E-11	1.14E-04
0.6	6.57E-08	1.11E-10	9.63E-05
0.7	2.46E-07	2.53E-10	2.86E-05
0.8	9.98E-08	1.08E-10	5.85E-05
0.9	1.43E-07	5.55E-12	1.02E-04
1.0	1.00E-09	0.00E+00	0.00E+00

TABLE 5. Absolute Error values for control variable of Problem 01.

t	AE GA-BP	AE FDO-BP	AE CP [4]
0.0	2.21E-06	8.12E-09	1.60E-03
0.1	3.83E-07	8.77E-10	2.24E-04
0.2	3.87E-06	1.61E-09	5.51E-04
0.3	2.73E-07	9.38E-10	7.85E-04
0.4	2.62E-06	1.79E-09	5.86E-04
0.5	5.34E-07	8.34E-10	1.08E-04
0.6	2.23E-06	2.35E-09	4.53E-04
0.7	5.38E-07	2.73E-11	8.50E-04
0.8	3.08E-06	2.24E-09	7.86E-04
0.9	3.46E-07	6.50E-10	8.96E-05
1.0	8.03E-07	5.25E-09	2.18E-03

TABLE 6. Exact and approximate numeric values for performance index of Problem 01.

performance index J	Solution
Exact	0.328258821
GA-BP	0.328258823
FDO-BP	0.328258821
CP [4]	0.328258837
Chebyshev Polynomials [1]	0.328258837
Mehne Method [52]	0.328476957

TABLE 7. Absolute Error values for performance index of Problem 01.

performance index J	Absolute Error
GA-BP	1.32E-09
FDO-BP	5.50E-11
CP [4]	1.54E-08
Chebyshev Polynomials [1]	1.56E-08
Mehne Method [52]	2.18E-04

An error minimization problem is evaluated by the transformation of NOCP, achieving the optimal values for unknown coefficients by utilizing such equations, like for

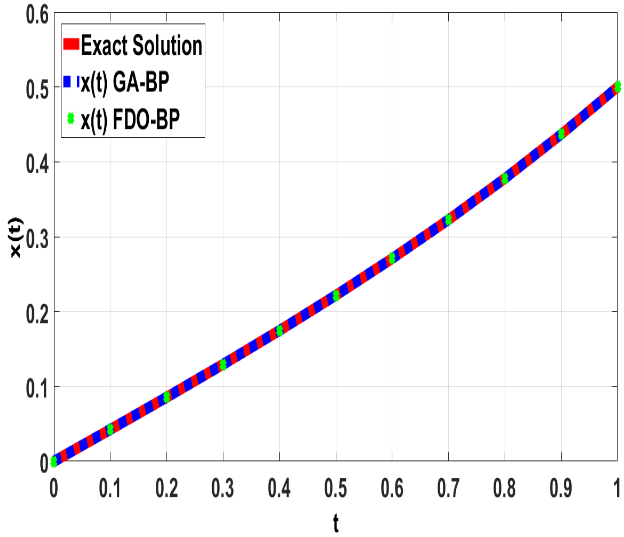

FIGURE 1. $x(t)$ approximation in comparison to the exact solution for Problem 01.

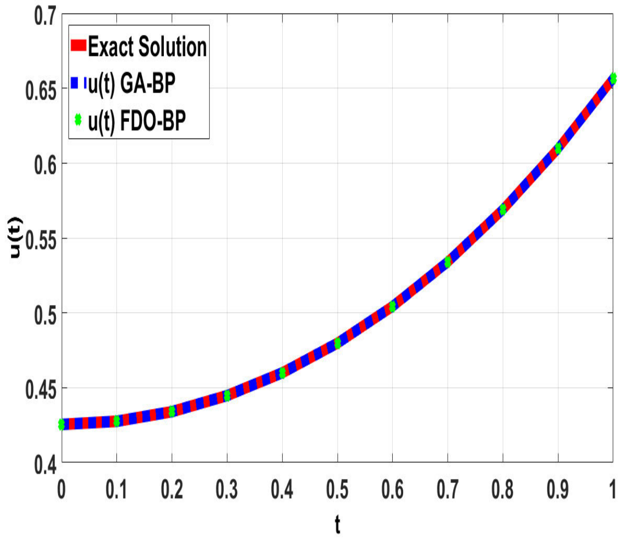

FIGURE 2. $u(t)$ approximation in comparison to the exact solution for Problem 01.

Problem 01, mentioned as follows:

$$\varepsilon_1 = \frac{1}{N+1}\sum_{i=0}^{N}(\dot{x}(t) - u(t))^2 \tag{16}$$

$$\varepsilon_2 = \frac{1}{2}((x(0) - 0)^2 + (x(1) - \frac{1}{2})^2) \tag{17}$$

$$\varepsilon_j = \varepsilon_1 + \varepsilon_2 \tag{18}$$

where N denotes the number of steps employed in the domain of [0, 1], ε_1 is the mean of the sum of square error, ε_2 is the mean of the sum of square error for given initial condition(s), and j represents the total number of iterations accomplished.

TABLE 8. Unknown parameters obtained by GA-BP and FDO-BP for Problem 02.

Parameters	*GA-BP*	*FDO-BP*
α_0	1.000000000	1.000000000
α_1	0.904800477	0.904800731
α_2	0.827458938	0.827458512
α_3	0.765705397	0.765707246
α_4	0.717876161	0.717874012
α_5	0.682769957	0.682771783
α_6	0.659627226	0.659626558
α_7	0.648054222	0.648054290
α_8	0.648051690	0.648054289

TABLE 9. Exact solution and approximate numerical values for state variable of Problem 02.

t	$x(t)$ *Exact*	$x(t)$ *GA-BP*	$x(t)$ *FDO-BP*	$x(t)$ *CP [4]*
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.928717757	0.928717665	0.928717751	0.929032661
0.2	0.866730433	0.866730281	0.866730431	0.866931521
0.3	0.813417638	0.813417460	0.813417640	0.813352820
0.4	0.768245801	0.768245649	0.768245800	0.767952796
0.5	0.730762826	0.730762707	0.730762824	0.730387689
0.6	0.700593571	0.700593441	0.700593574	0.700313738
0.7	0.677436092	0.677435881	0.677436100	0.677387182
0.8	0.661058620	0.661058177	0.661058626	0.661264260
0.9	0.651297246	0.651296154	0.651297251	0.651601211
1.0	0.648054274	0.648051690	0.648054289	0.648054274

TABLE 10. Exact solution and approximate numerical values for control variable of Problem 02.

t	$u(t)$ *Exact*	$u(t)$ *GA-BP*	$u(t)$ *FDO-BP*	$u(t)$ *CP [4]*
0.0	-1.26159415	-1.26159618	-1.26159414	-1.25515026
0.1	-1.12959742	-1.12959795	-1.12959744	-1.12928579
0.2	-1.00890609	-1.00890655	-1.00890602	-1.01129203
0.3	-0.89831222	-0.89831210	-0.89831223	-0.90099709
0.4	-0.79670897	-0.79670849	-0.79670900	-0.79822911
0.5	-0.70307945	-0.70307922	-0.70307942	-0.70281619
0.6	-0.61648658	-0.61648693	-0.61648651	-0.61458647
0.7	-0.53606372	-0.53606493	-0.53606371	-0.53336804
0.8	-0.46100596	-0.46100952	-0.46100602	-0.45898905
0.9	-0.39056211	-0.39057154	-0.39056204	-0.39127760
1.0	-0.32402713	-0.32404610	-0.32402715	-0.33006181

Further, using the heuristic computational algorithms, such as GA and FDO, the objective function ε_j is minimized to an optimum level. The optimal values for $(\alpha_0, \alpha_1, \ldots, \alpha_8)$, which provide minimal error, are utilized, and ultimately the approximate solution is achieved. A similar methodology is applied to different OCPs considered in this research work to attain the desired approximate numerical solution.

V. SIMULATION AND RESULTS

The proposed technique of Section 4 is applied to various NOCPs, using the hybrid scheme of nature-inspired optimization algorithms and BPs, to ensure that the suggested approach is valid and surpasses other existing methods. The simulations are computed using the MATLAB tool accurately. In addition, a comparative analysis between the proposed method and previously presented techniques is provided, certifying the effectiveness of the recommended approach.

TABLE 11. Absolute Error values for state variable of Problem 02.

t	*AE GA-BP*	*AE FDO-BP*	*AE CP [4]*
0.0	0.00E+00	5.47E-12	0.00E+00
0.1	9.23E-08	6.41E-09	3.15E-04
0.2	1.52E-07	2.48E-09	2.01E-04
0.3	1.78E-07	1.78E-09	6.48E-05
0.4	1.52E-07	1.25E-09	2.93E-04
0.5	1.19E-07	2.20E-09	3.75E-04
0.6	1.30E-07	3.43E-09	2.80E-04
0.7	2.11E-07	8.12E-09	4.89E-05
0.8	4.43E-07	5.82E-09	2.06E-04
0.9	1.09E-06	4.90E-09	3.04E-04
1.0	2.58E-06	1.50E-08	0.00E+00

TABLE 12. Absolute Error values for control variable of Problem 02.

t	*AE GA-BP*	*AE FDO-BP*	*AE CP [4]*
0.0	2.03E-06	7.06E-09	6.44E-03
0.1	5.23E-07	1.75E-08	3.12E-04
0.2	4.58E-07	6.66E-08	2.39E-03
0.3	1.21E-07	2.68E-09	2.68E-03
0.4	4.80E-07	3.45E-08	1.52E-03
0.5	2.27E-07	2.92E-08	2.63E-04
0.6	3.46E-07	7.12E-08	1.90E-03
0.7	1.21E-06	4.65E-09	2.70E-03
0.8	3.55E-06	5.67E-08	2.02E-03
0.9	9.43E-06	7.18E-08	7.15E-04
1.0	1.90E-05	1.57E-08	6.03E-03

TABLE 13. Exact and approximate numeric values for performance index of Problem 02.

performance index J	*Solution*
Exact	0.380797077978
GA-BP	0.380797077992
FDO-BP	0.380797077978
CP [4]	0.380797113192
Basis Polynomials [5]	0.380797078000
Polynomials Bases [53]	0.380813756200

TABLE 14. Absolute Error values for performance index of Problem 02.

performance index J	*Absolute Error*
GA-BP	1.45E-11
FDO-BP	2.00E-15
CP [4]	3.52E-08
Basis Polynomials [5]	1.00E-10
Polynomials Bases [53]	1.67E-05

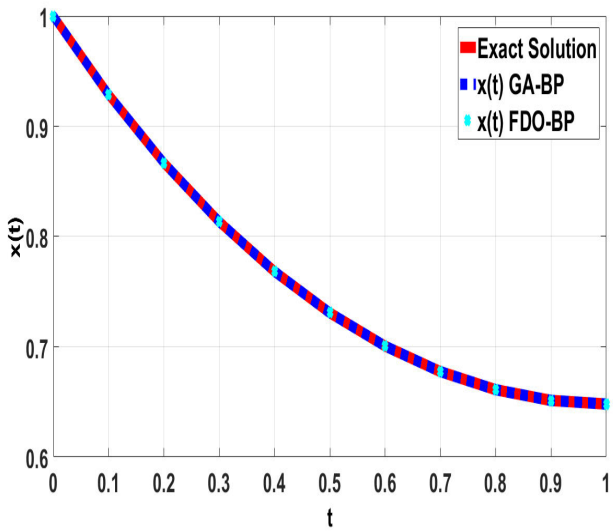

FIGURE 3. $x(t)$ approximation in comparison to the exact solution for Problem 02.

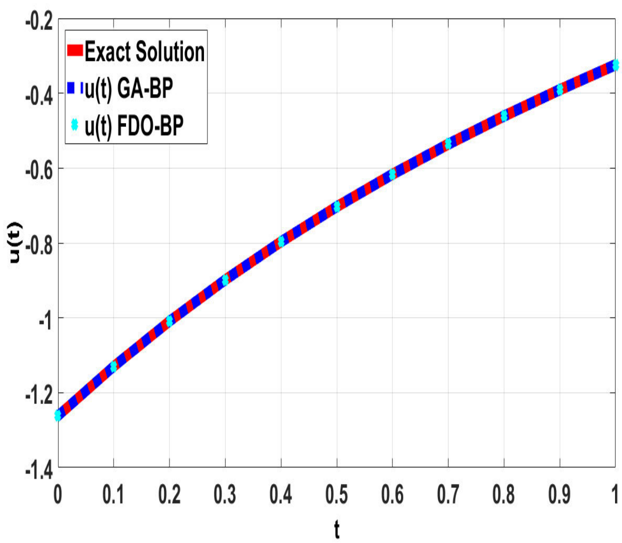

FIGURE 4. $u(t)$ approximation in comparison to the exact solution for Problem 02.

For the execution of GA, the parameter settings include population size as 1000 and generations as 900, along with the other ones. However, for FDO, the scout bee number is 20, and the maximum iterations are 1500. Further, $k = 8$ is chosen for BPs achieving the best values for unknown coefficients. Additionally, the parameter settings are the same for all the NOCPs considered in this paper.

A. PROBLEM 01

Consider the approximation of OCP as demonstrated below [4]:

$$J = \int_0^1 x^2(t) + u^2(t)\,dt \tag{19}$$

TABLE 15. Unknown parameters obtained by GA-BP and FDO-BP for Problem 03.

Parameters	*GA-BP*	*FDO-BP*
α_0	0.000000000	0.000000000
α_1	0.079015138	0.079015066
α_2	0.140172915	0.140173433
α_3	0.185355441	0.185353893
α_4	0.215845798	0.215849371
α_5	0.232436629	0.232433436
α_6	0.235444730	0.235446369
α_7	0.224761680	0.224761753
α_8	0.199780879	0.199788229

TABLE 16. Exact solution and approximate numerical values for state variable of Problem 03.

t	$x(t)$ *Exact*	$x(t)$ *GA-BP*	$x(t)$ *FDO-BP*
0.0	0.000000000	0.000000000	0.000000000
0.1	0.058313294	0.058313313	0.058313326
0.2	0.107201871	0.107201861	0.107201898
0.3	0.147155024	0.147154950	0.147155050
0.4	0.178572618	0.178572492	0.178572669
0.5	0.201769091	0.201768950	0.201769155
0.6	0.216976600	0.216976403	0.216976644
0.7	0.224347348	0.224346847	0.224347379
0.8	0.223955103	0.223953742	0.223955162
0.9	0.215795941	0.215792669	0.215796016
1.0	0.199788200	0.199780879	0.199788229

TABLE 17. Exact solution and approximate numerical values for control variable of Problem 03.

t	$u(t)$ *Exact*	$u(t)$ *GA-BP*	$u(t)$ *FDO-BP*
0.0	0.632120559	0.632121104	0.632120527
0.1	0.593430340	0.593430302	0.593430554
0.2	0.550671036	0.550670509	0.550670917
0.3	0.503414696	0.503413949	0.503414884
0.4	0.451188364	0.451187936	0.451188686
0.5	0.393469340	0.393469086	0.393469347
0.6	0.329679954	0.329678384	0.329679736
0.7	0.259181779	0.259176064	0.259181881
0.8	0.181269247	0.181255129	0.181269699
0.9	0.095162582	0.095132225	0.095162409
1.0	0.000000000	-6.55E-05	3.78983E-08

subject to:

$$u(t) = \dot{x}(t) \tag{20}$$

with respect to boundary conditions:

$$x(0) = 0, \quad x(1) = \frac{1}{2} \tag{21}$$

TABLE 18. Absolute Error values for state variable of Problem 03.

t	*AE GA-BP*	*AE FDO-BP*
0.0	0.00E+00	0.00E+00
0.1	1.86E-08	3.20E-08
0.2	1.01E-08	2.68E-08
0.3	7.44E-08	2.57E-08
0.4	1.26E-07	5.16E-08
0.5	1.40E-07	6.42E-08
0.6	1.97E-07	4.43E-08
0.7	5.01E-07	3.09E-08
0.8	1.36E-06	5.89E-08
0.9	3.27E-06	7.48E-08
1.0	7.32E-06	2.90E-08

TABLE 19. Absolute Error values for control variable of Problem 03.

t	*AE GA-BP*	*AE FDO-BP*
0.0	5.45E-07	3.19E-08
0.1	3.87E-08	2.13E-07
0.2	5.27E-07	1.19E-07
0.3	7.48E-07	1.88E-07
0.4	4.28E-07	3.22E-07
0.5	2.54E-07	6.43E-09
0.6	1.57E-06	2.18E-07
0.7	5.72E-06	1.02E-07
0.8	1.41E-05	4.52E-07
0.9	3.04E-05	1.73E-07
1.0	6.55E-05	3.79E-08

TABLE 20. Exact and approximate numeric values for performance index of Problem 03.

performance index J	*Solution*
Exact	0.084045620362
GA-BP	0.084045620224
FDO-BP	0.084045620362
Chebyshev Polynomials [1]	0.084045580400
Mehne Method [52]	0.084024961800

TABLE 21. Absolute Error values for performance index of Problem 03.

performance index J	*Absolute Error*
GA-BP	1.38E-10
FDO-BP	3.61E-14
Chebyshev Polynomials [1]	4.00E-08
Mehne Method [52]	2.07E-05

having the exact solution mentioned as follows:

$$x(t) = \frac{e}{e^2 - 1} \sinh(t), \quad u(t) = \frac{e}{e^2 - 1} \cosh(t) \quad (22)$$

Table 1 provides the optimum values of the unknown parameters obtained by applying the proposed hybrid technique, i.e., GA-BP and FDO-BP. The approximate numerical values for $x(t)$ and $u(t)$ are mentioned in Tables 2 and 3, respectively. Similarly, Tables 4 and 5 show the AE values for $x(t)$ and $u(t)$, respectively, at various points in t. Moreover, J obtained by the suggested scheme and previously developed approaches is shown in Table 6, and their AEs are provided in Table 7 for a comprehensive comparison. In addition, a comparison between the exact solution and approximate numerical values for $x(t)$ and $u(t)$ is demonstrated in Figures 1 and 2, respectively.

It is readily apparent from Tables 6 and 7 that the outcome of the proposed method for J is much closer to the actual analytical solution, i.e., AE for J is minimized sufficiently to an optimal value.Also, it is evident from Figures 1 and 2 that the exact and approximate solutions for $x(t)$ and $u(t)$,

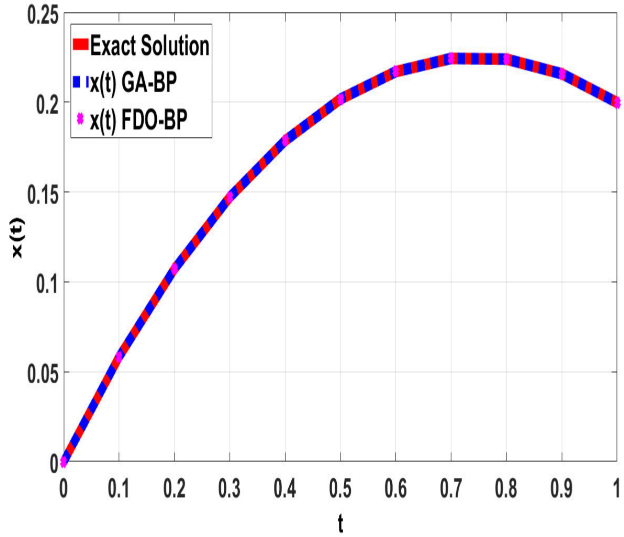

FIGURE 5. $x(t)$ approximation in comparison to the exact solution for Problem 03.

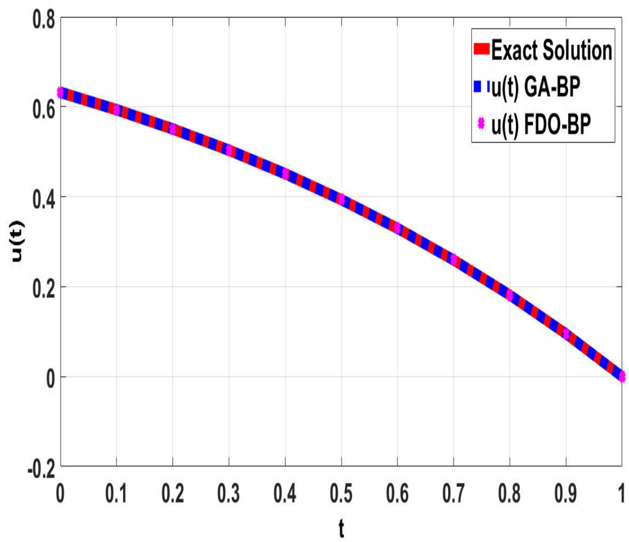

FIGURE 6. $u(t)$ approximation in comparison to the exact solution for Problem 03.

TABLE 22. Unknown parameters obtained by GA-BP and FDO-BP for Problem 04.

Parameters	*GA-BP*	*FDO-BP*
α_0	1.000000000	1.000000000
α_1	0.826772739	0.826772570
α_2	0.689257213	0.689260513
α_3	0.579219193	0.579209544
α_4	0.490751852	0.490764366
α_5	0.419603599	0.419595320
α_6	0.362530177	0.362532941
α_7	0.317215725	0.317215719
α_8	0.281965976	0.281969535

TABLE 23. Exact solution and approximate numerical values for state variable of Problem 04.

t	*x(t) Exact*	*x(t) GA-BP*	*x(t) FDO-BP*
0.0	1.000000000	1.000000000	1.000000000
0.1	0.870972416	0.870972277	0.870972438
0.2	0.759393333	0.759393311	0.759393308
0.3	0.663027446	0.663027520	0.663027358
0.4	0.579944224	0.579944161	0.579944144
0.5	0.508479231	0.508479010	0.508479182
0.6	0.447200783	0.447200556	0.447200732
0.7	0.394881267	0.394881005	0.394881211
0.8	0.350472548	0.350471860	0.350472540
0.9	0.313084970	0.313083288	0.313085009
1.0	0.281969535	0.281965976	0.281969535

TABLE 24. Exact solution and approximate numerical values for control variable of Problem 04.

t	*u(t) Exact*	*u(t) GA-BP*	*u(t) FDO-BP*
0.0	-0.385818596	-0.385818088	-0.385819440
0.1	-0.328060144	-0.328060678	-0.328060023
0.2	-0.276873838	-0.276871995	-0.276874673
0.3	-0.231234244	-0.231234480	-0.231234613
0.4	-0.190227048	-0.190229081	-0.190226793
0.5	-0.153030737	-0.153031798	-0.153030613
0.6	-0.118900146	-0.118899992	-0.118900351
0.7	-0.087151524	-0.087153551	-0.087151402
0.8	-0.057148839	-0.057156567	-0.057148150
0.9	-0.028291037	-0.028305707	-0.028291018
1.0	0.000000000	-3.20E-05	6.30E-08

respectively, are in close proximity to each other, which signifies optimal approximation.

B. PROBLEM 02

Find the optimal control for minimization of the OCP stated as follows [4]:

$$J=\int_0^1 \left(\frac{5}{8}x^2(t)+0.5x(t)\,u(t)+0.5u^2(t)\right)dt \tag{23}$$

TABLE 25. Absolute Error values for state variable of Problem 04.

t	*AE GA-BP*	*AE FDO-BP*
0.0	0.00E+00	0.00E+00
0.1	1.40E-07	2.17E-08
0.2	2.24E-08	2.54E-08
0.3	7.41E-08	8.79E-08
0.4	6.26E-08	7.99E-08
0.5	2.21E-07	4.89E-08
0.6	2.27E-07	5.10E-08
0.7	2.62E-07	5.60E-08
0.8	6.88E-07	7.61E-09
0.9	1.68E-06	3.91E-08
1.0	3.56E-06	3.62E-10

TABLE 26. Absolute Error values for control variable of Problem 04.

t	*AE GA-BP*	*AE FDO-BP*
0.0	5.08E-07	8.44E-07
0.1	5.34E-07	1.22E-07
0.2	1.84E-06	8.34E-07
0.3	2.36E-07	3.69E-07
0.4	2.03E-06	2.54E-07
0.5	1.06E-06	1.24E-07
0.6	1.54E-07	2.05E-07
0.7	2.03E-06	1.22E-07
0.8	7.73E-06	6.89E-07
0.9	1.47E-05	1.89E-08
1.0	3.20E-05	6.30E-08

TABLE 27. Exact and approximate numeric values for performance index of Problem 04.

performance index J	*Solution*
Exact	0.192909298093
GA-BP	0.192909298126
FDO-BP	0.192909298093
Basis Polynomials [5]	0.192909299000
Chebyshev Polynomials [1]	0.192909776000
Mehne Method [52]	0.193828723000

TABLE 28. Absolute Error values for performance index of Problem 04.

performance index J	*Absolute Error*
GA-BP	3.24E-11
FDO-BP	9.00E-14
Basis Polynomials [5]	9.07E-10
Chebyshev Polynomials [1]	4.78E-07
Mehne Method [52]	9.19E-04

subject to:

$$u(t)=\dot{x}(t)-0.5x(t) \tag{24}$$

with boundary conditions:

$$x(0)=1,\quad x(1)=0.64805 \tag{25}$$

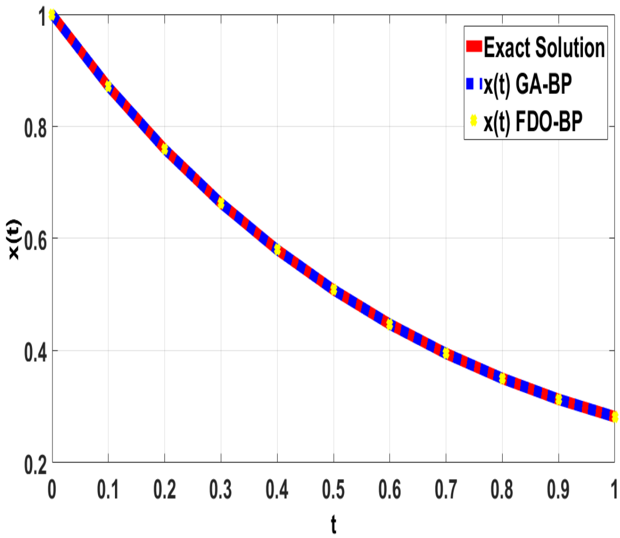

FIGURE 7. ***x*(*t*) approximation in comparison to the exact solution for Problem 04.**

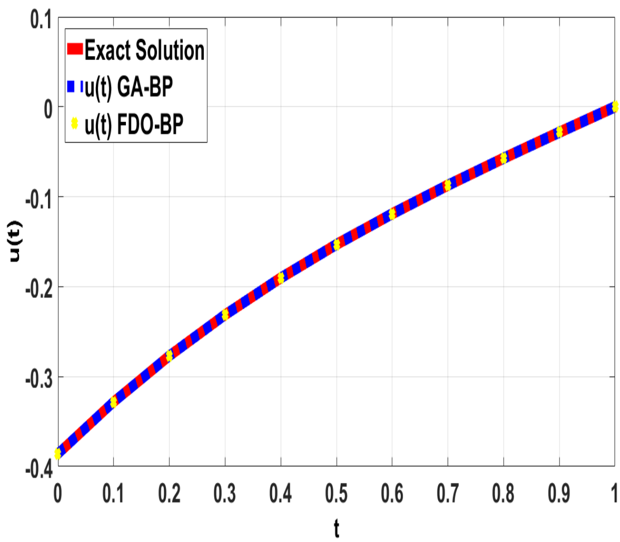

FIGURE 8. ***u*(*t*) approximation in comparison to the exact solution for Problem 04.**

containing analytical solution expressed as follows:

$$x(t) = \frac{\cosh(1-t)}{\cosh(1)}, u(t) = -\frac{((\tanh(1-t)+0.5)\cosh(1-t))}{\cosh(1)} \quad (26)$$

The optimal values of unknown coefficients attained by executing the proposed hybrid scheme are exhibited in Table 8. Besides, Tables 9 and 11 represent the approximate numeric values and AE for $x(t)$, respectively. Likewise, Tables 10 and 12 demonstrate the approximate solution and AE values for $u(t)$, respectively. Correspondingly, the numerical solution for J obtained by the recommended approach and some formerly developed methods is provided in Table 13, whereas their AE is presented in Table 14. Additionally, Figure 3 depicts the exact values and approximate results for $x(t)$, whereas Figure 4 compares the actual analytical values with the approximate solution for $u(t)$.

It is apparent from Tables 13 and 14 that the solution presented by the suggested technique is better than the previous methods, as the AE between the exact and approximate solutions is reduced to a considerable value.In addition, the results depicted in Figures 3 and 4 for $x(t)$ and $u(t)$, respectively, disclose that the presented hybrid-based approaches, i.e., FDO-BP and GA-BP, provide a better approximate solution.

C. PROBLEM 03

Consider the minimization of OCP expressed below [1]:

$$J = \int_0^1 (x(t) - \frac{1}{2}u^2(t))dt \quad (27)$$

subject to:

$$u(t) = \dot{x}(t) + x(t) \quad (28)$$

with respect to the boundary conditions:

$$x(0) = 0, \quad x(1) = \frac{1}{2}(1 - \frac{1}{e})^2 \quad (29)$$

the analytical results are achieved by using:

$$x(t) = 1 - \frac{1}{2}e^{t-1} + \left(\frac{1}{2e} - 1\right)e^{-t},$$
$$u(t) = 1 - e^{t-1} \quad (30)$$

Table 15 represents the optimum values for unknown parameters acquired by adopting the suggested hybrid approach. In the same way, Tables 16, 17, and 20 express a comparison between the analytical results and approximate numerical solutions for $x(t)$, $u(t)$, and J, respectively. Correspondingly, Tables 18, 19, and 21 exhibit the measure of AE for $x(t)$, $u(t)$, and J, respectively. Furthermore, Figures 5 and 6 compare the actual analytical values with the approximate numeric values for $x(t)$ and $u(t)$, accordingly.

The AE generated by the proposed hybrid approach is comparatively lower than the AE obtained from the formerly suggested techniques, as demonstrated in Tables 18, 19, and 21, indicating the effectiveness of the recommended hybrid scheme. Moreover, it is readily apparent from Figure 5 that the exact solution and the approximate result for $x(t)$, achieved by utilizing the proposed technique, are in good alignment with each other, showing optimal approximation. Likewise, Figure 6 reveals that the analytical and approximate results for $u(t)$ are in close proximity to each other, eventually testifying to the importance of FDO-BP and GA-BP.

D. PROBLEM 04

The following OCP is concerned with [5]:

$$J = \frac{1}{2}\int_0^1 x^2(t) + u^2(t)\,dt \quad (31)$$

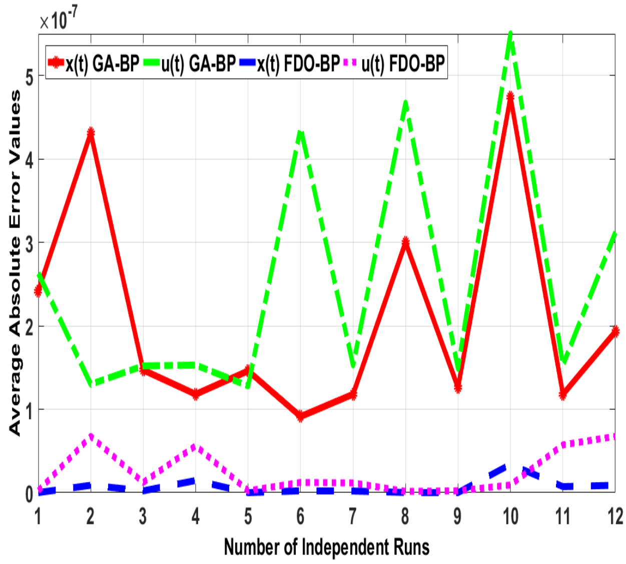

FIGURE 9. Average absolute error represented graphically for $x(t)$ and $u(t)$ of Problem 01.

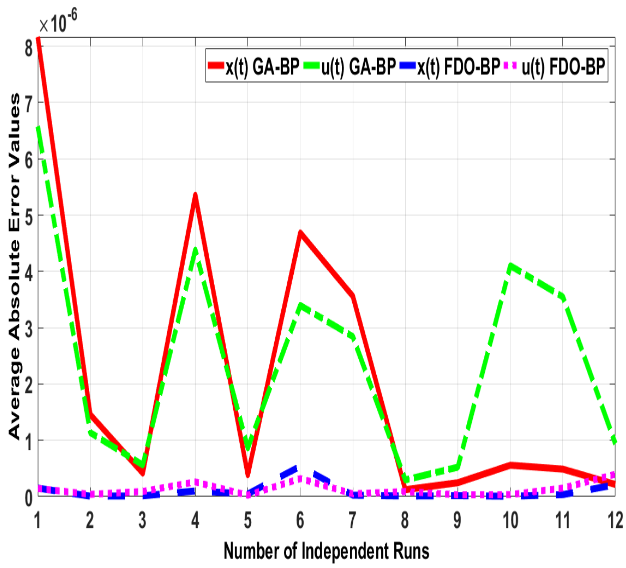

FIGURE 10. Average absolute error represented graphically for $x(t)$ and $u(t)$ of Problem 02.

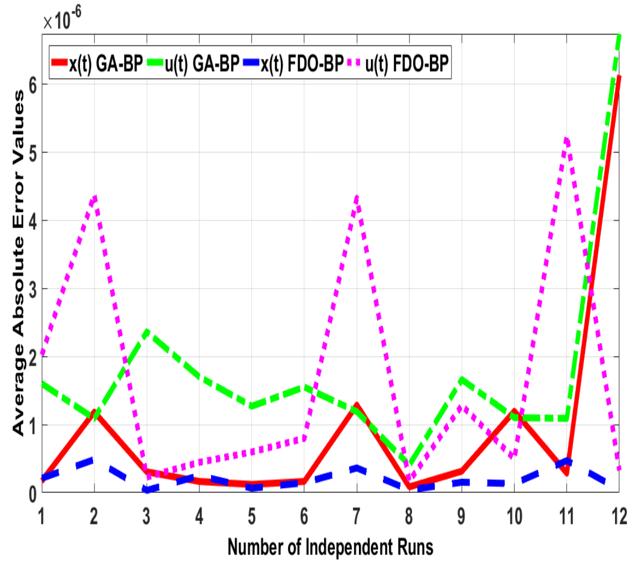

FIGURE 11. Average absolute error represented graphically for $x(t)$ and $u(t)$ of Problem 03.

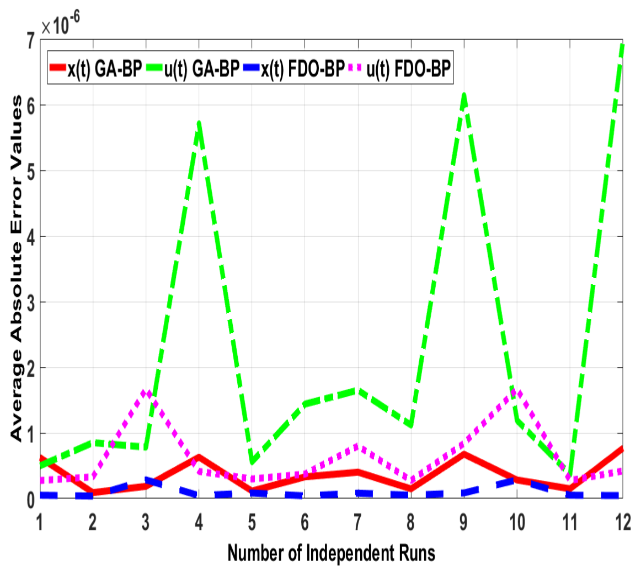

FIGURE 12. Average absolute error represented graphically for $x(t)$ and $u(t)$ of Problem 04.

subject to:

$$u(t) = \dot{x}(t) + x(t) \tag{32}$$

with boundary condition:

$$x(0) = 1 \tag{33}$$

The analytical solution is:

$$x(t) = \cosh\left(\sqrt{2}\,t\right) + \delta \sinh\left(\sqrt{2}\,t\right), \quad u(t) = \left(1+\sqrt{2}\delta\right)\cosh\left(\sqrt{2}\,t\right) + (\sqrt{2}+\delta)\sinh(\sqrt{2}\,t) \tag{34}$$

where

$$\delta = -\frac{\cosh\left(\sqrt{2}\right) + \sqrt{2}\sinh\left(\sqrt{2}\right)}{\sqrt{2}\cosh\left(\sqrt{2}\right) + \sinh\left(\sqrt{2}\right)} \tag{35}$$

Table 22 shows the unknown coefficients generated by successfully operating the proposed hybrid technique. Besides, Tables 23, 24, and 27 compare the exact and approximate solutions for $x(t)$, $u(t)$, and J, respectively, highlighting the excellence of FDO-BP and GA-BP. Furthermore, Tables 25, 26, and 28 make evident the evaluated AE values for $x(t)$, $u(t)$, and J, respectively. In addition, Figures 7 and 8

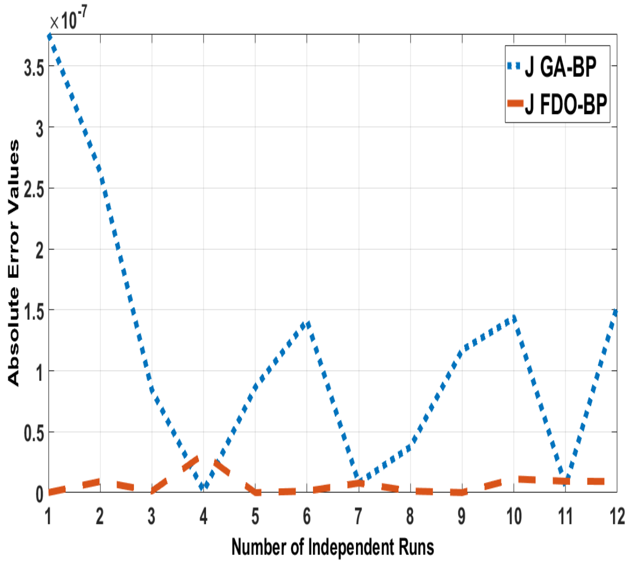

FIGURE 13. Graphical representation of Absolute Error for performance index of Problem 01.

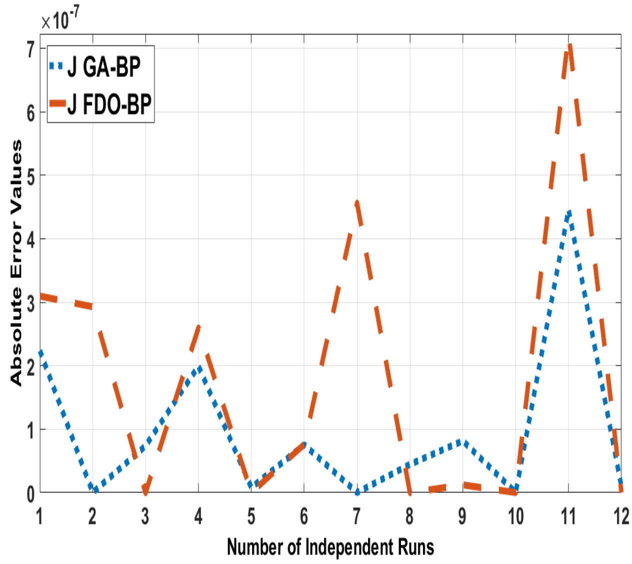

FIGURE 15. Graphical representation of Absolute Error for performance index of Problem 03.

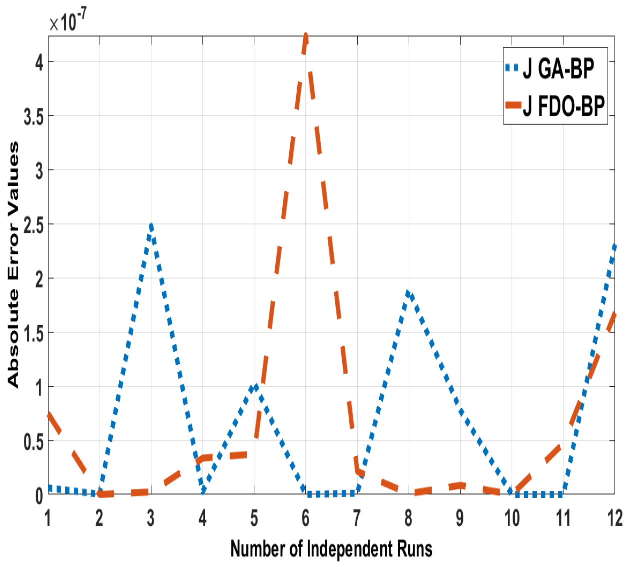

FIGURE 14. Graphical representation of Absolute Error for performance index of Problem 02.

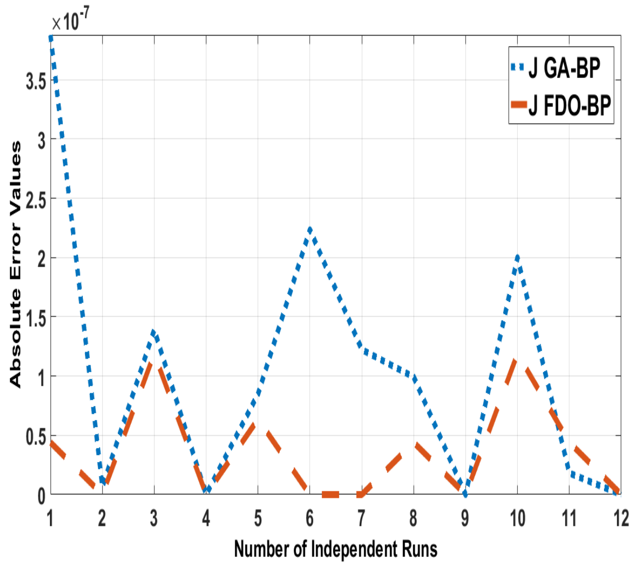

FIGURE 16. Graphical representation of Absolute Error for performance index of Problem 04.

depict the analytical and approximate values for $x(t)$ and $u(t)$, accordingly.

As mentioned in Tables 25, 26, and 28, the AE is reduced sufficiently, which discloses the significance of the proposed technique. Additionally, the attained approximate results and the actual analytical solution, plotted in Figures 7 and 8 for $x(t)$ and $u(t)$, respectively, reveal that the FDO-BP and GA-BP hybrid techniques render an optimum outcome.

VI. STATISTICAL ANALYSIS FOR PROPOSED TECHNIQUE

This section describes the statistical analysis implemented on all the OCPs mentioned above, i.e., Problems 01–04, to ascertain that the proposed technique is reliable, stable, and efficient. For this purpose, at least 12 independent runs are executed for the presented hybrid scheme, i.e., GA-BP and FDO-BP, keeping all the parameter settings unchanged, as defined earlier. The graphical representation of the average absolute error for $x(t)$ and $u(t)$ of all the concerned OCPs is provided in Figures 9–12.

It could be observed from Figures 9–12 that the average absolute error values for $x(t)$ and $u(t)$ are minimized adequately by FDO-BP and GA-BP for 12 independent runs, which verifies the stability, reliability, and efficiency of the presented hybrid-based approach.

TABLE 29. Statistical analysis for Optimal Control Problems.

Problem	*Technique*	*state variable* $x(t)$				*control variable* $u(t)$				*performance index J*			
		MIN	MAX	MEAN	SD	MIN	MAX	MEAN	SD	MIN	MAX	MEAN	SD
Problem 01 [4]	GA-BP	9.13E-08	4.74E-07	2.09E-07	1.29E-07	1.28E-07	5.50E-07	2.54E-07	1.52E-07	1.32E-09	3.76E-07	1.18E-07	1.12E-07
	FDO-BP	7.77E-11	3.39E-08	6.71E-09	9.76E-09	1.51E-09	6.75E-08	2.52E-08	2.78E-08	5.50E-11	3.15E-08	6.88E-09	8.92E-09
Problem 02 [4]	GA-BP	1.22E-07	8.17E-06	2.14E-06	2.67E-06	2.96E-07	6.58E-06	2.43E-06	2.00E-06	1.45E-11	2.48E-07	7.15E-08	9.81E-08
	FDO-BP	4.67E-09	5.38E-07	9.60E-08	1.54E-07	2.31E-08	3.97E-07	1.36E-07	1.25E-07	2.00E-15	4.23E-07	6.80E-08	1.22E-07
Problem 03 [1]	GA-BP	8.52E-08	6.13E-06	9.51E-07	1.69E-06	4.06E-07	6.73E-06	1.81E-06	1.62E-06	1.38E-10	4.46E-07	9.69E-08	1.33E-07
	FDO-BP	3.47E-08	4.86E-07	2.00E-07	1.62E-07	1.69E-07	5.24E-06	1.69E-06	1.86E-06	3.61E-14	7.23E-07	1.77E-07	2.35E-07
Problem 04 [5]	GA-BP	9.09E-08	7.68E-07	3.67E-07	2.47E-07	3.71E-07	7.00E-06	2.28E-06	2.46E-06	3.24E-11	3.88E-07	1.07E-07	1.19E-07
	FDO-BP	3.80E-08	2.85E-07	9.56E-08	9.01E-08	2.75E-07	1.67E-06	6.37E-07	5.20E-07	9.00E-14	1.19E-07	3.61E-08	4.52E-08

The graphical representation of the AEs for J of all the concerned OCPs is provided in Figures 13–16 for at least 12 independent runs of FDO-BP and GA-BP.

The numerical results illustrated in Figures 13–16 for Problems 01–04 suggest that FDO-BP and GA-BP provide optimum values of AE for J, which ultimately authenticates the stability and efficacy of the presented scheme.

The parameters investigated for the sake of statistical analysis of all the concerned NOCPs are as follows: the minimum (MIN), the maximum (MAX), the MEAN, and the Standard Deviation (SD). Further, the MIN and MAX parameters help detect the best and worst values, respectively. Similarly, the MEAN and SD values assist in determining the central tendency and measuring the degree of variation in the solution obtained by the proposed hybrid technique, verifying its reliability and robustness. The results of the statistical analysis are demonstrated in Table 29.

It could be observed from Table 29, for Problems 01–04, the MEAN value of $x(t)$, $u(t)$, and J are approximately 10^{-06} to 10^{-09}, 10^{-06} to 10^{-08}, and 10^{-07} to 10^{-09}, respectively. Likewise, for Problems 01–04, the SD value of $x(t)$, $u(t)$, and J ranges from 10^{-06} to 10^{-09}, 10^{-06} to 10^{-08}, and 10^{-07} to 10^{-09}, respectively. Consequently, it could be perceived from Table 29 that the MEAN and SD values are close to each other, which indicates a low deviation from the solution and ultimately proves the supremacy of the presented technique in terms of stability and efficiency [54]–[56].

VII. CONCLUSION

This research work presents the variant of the nature-inspired optimization technique that utilizes the strengths of GA-BP and FDO-BP as an alternative to deal with the NOCPs optimally. The suggested approach determines the quality of the solutions by considering various NOCPs. Experimental solutions achieved led to determining that the proposed hybrid scheme is more trustworthy in figuring out superior quality solutions with rational computational iteration than the other numerical techniques reported in the literature. The recommended technique tends to optimize the OCP systems by minimizing the AEs generated by $x(t)$, $u(t)$, and J to a competitive level than the other numerical methods, e.g., AE for J of Problems 01–04 is 5.50E-11, 2.00E-15, 3.61E-14, and 9.00E-14, respectively. The statistical analysis supported our findings that the suggested scheme is appropriate in optimizing nonlinear systems by improving the convergence rates and the number of objective function evaluations being key parameters in systems optimization.

In the future, we intend to employ the proposed method for solving other such OCPs, including Continuous Stirred Tank Chemical Reactor (CSTCR), Free Floating Robot (FFR), and Van Der Pol (VDP) oscillator. A hybrid scheme of FDO and GA with local techniques, such as Interior Point Algorithm (IPA) and Active Set Algorithm (ASA), shall be attempted. Additionally, different basis functions, e.g., Boubaker Polynomials, could be used with FDO and other optimization algorithms for solving real-world OCPs.

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