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Delta Modulator Based Quantised State-Feedback Control of Networked Linear Systems

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ABSTRACT This study proposes the design of quantised state-feedback controller using a Delta-Modulator (Δ-M) for linear networked systems. This modulator can be considered as one type of sliding mode quantiser (SMQ) and offers several advantages such as lower design complexity, lower cost, and less noisy. The stability conditions of both the continuous-time and the discrete-time Δ-M based quantised control systems are derived. The bounds of the switching function, which ensure that the steady state behaviour of the system is *periodic*, are derived. The effectiveness of the Δ-M based quantised control system is investigated using a ZigBee protocol based experimental communication network with inherent imperfections associated with a real-time network. It is shown that the designed quantised state-feedback controller could achieve the desired performance.

INDEX TERMS Delta modulator, state-feedback control, switched control.

I. INTRODUCTION

During the past few years, sliding mode techniques have been popular in control applications such as controller design [1], [2], observer design [3] and sliding mode quantiser (SMQs) [4], [5] due to their robustness and simplicity in converting input signals to switching signals which are described by a sequence of binary values. Different types of SMQs can be used to generate these binary values such as delta modulator (Δ-M), sigma-delta modulator ($\Sigma\Delta$-M), delta-sigma modulator ($\Delta\Sigma$-M) [6]–[8], pulse-width-modulator (PWM) based sliding mode [9], where these are used in many control applications [10]–[18].

However, the switching components in SMQs add more complexities to the system. If these quantisers are to be used effectively in control applications, they should satisfy both the stability and equivalence conditions [9]. In [5], [9] detailed investigations on the stability of SMQs in continuous-time (CT) systems have been carried out where it has been found out that the SMQs can be used in high-speed switching. However, the gain of the switching function affects the stability of the closed loop system. Under the assumption of ideal sliding mode and high switching frequency, the SMQs are equivalent to the inputs. This ensures the equivalence conditions [17], [19].

Most of the modern controllers are often implemented in the discrete-time domain where discretisation would introduce delays into switching components. In real-time, the effects of discretisation and the factors which affect the implementation under high switching frequency (for SMQs) have been reported in [20]–[22]. However, the results presented in [20]–[22] do not explicitly address the issues related to equivalence conditions. Therefore this has been the primary motivation of this study where the performance of SMQs designed using equivalence conditions is investigated. Amongst various quantisers, this study focuses on using Delta-modulators (Δ-M) as a quantiser. This modulator is extensively being used in power converters [6], [17] and in applications where bandwidth utilisation of the communication channel is necessary. Some further advantages of, the Δ-M include its simplicity in implementation, requirements of fewer hardware resources, lower complexity, cost-effective operations, lower noise, and many others [23]–[26].

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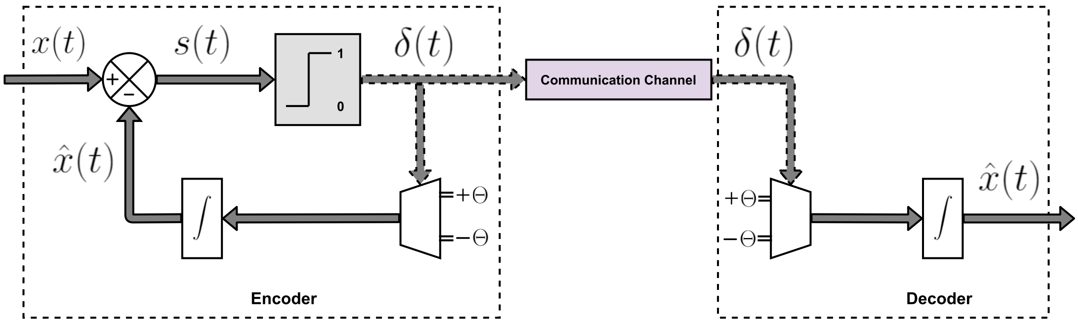


FIGURE 1. **Continuous-time delta-modulator (Δ-M).**

In this study, a Δ-M based quantised state-feedback controller is designed for multi-input-multi-output (MIMO) systems; both in CT and discrete-time (DT) domains. The main contributions of this paper are highlighted as follows:

1) Established the stability conditions for both CT and DT Δ-M based quantised state-feedback systems.
2) Derived sufficient conditions for the existence of periodic orbits (or *zig-zag* behaviour) of the switching function and states.
3) Determined the bound of the switching function which results in periodic orbits.
4) Validated the theoretical findings using a real ZigBee protocol based communication system.

The rest of the paper is organised as follows. Section-II and section-III discuss, CT-Δ-M and DT-Δ-M respectively and the stability conditions are derived. The effectiveness of the proposed control strategy and the theoretical findings are validated using two simulation examples in section-IV followed by conclusions on section-V.

For the rest of the paper, $\|x\|$ and $\varrho(A) \triangleq \max_{1\le i\le n} |\lambda_i|$ respectively denote the standard Euclidean norm of the vector $x \in R^n$ and the spectral radius of the matrix $A \in R^{n\times n}$.

II. CONTINUOUS-TIME SWITCHED CONTROL SYSTEM

A. REVIEW OF CONTINUOUS-TIME DELTA MODULATOR

The block diagram of a MIMO CT Δ-M is shown in Figure. 1.

It consists of two integrators, one on the feedback path and the other in the decoder side. Input of the quantiser is the difference between the output of the integrator on the feedback path and the input signal. This is a static quantiser and the output $\delta(t)$ is a single-bit signal which is transmitted through a communication channel before received by the decoder. Under ideal conditions, the output of the decoder $\hat{x}(t)$ is equivalent to the input signal $\bar{x}(t)$ [7], [10], [19].

The MIMO CT Δ-M is described as:

$$s(t) = x\,(t) - \hat{x}\,(t) \tag{1a}$$
$$\dot{\hat{x}}(t) = \Theta\,\text{sgn}(s(t)) \tag{1b}$$

where $x \in \mathbb{R}^n$, $\hat{x} \in \mathbb{R}^n$, $s \in \mathbb{R}^n$ and $\Theta \in \mathbb{R}^{n\times n}$ respectively denote the input signal, the quantised signal, the switching signal and the gain of the 2-level quantiser. Further, $\text{sgn}(s(t)) \in \left[\{-1, 1\}, \cdots, \{-1, 1\}\right]^T \in \mathbb{R}^n$, where,

$$\text{sgn}(s_i(t)) = \begin{cases} +1, & \text{if } s_i(t) \ge 0; \\ -1, & \text{if } s_i(t) < 0; \end{cases} \quad i = (1, 2, \cdots, n) \tag{2}$$

The communication channel between the encoder and the decoder carries a single-bit n-dimensional signal which is expressed as:

$$\delta(t) = \frac{1}{2}\left[1_n + \text{sgn}(s(t))\right] \tag{3}$$

where $\delta(t) \in \left[\{0, 1\}, \cdots, \{0, 1\}\right]^T \in \mathbb{R}^n$.

The quantiser gain Θ is often selected as a positive definite diagonal matrix for design simplicity. This is expressed as:

$$\Theta = \text{diag}\{\theta_{ii}\} \quad \forall i = (1, 2, \cdots, n). \tag{4}$$

For proper operation of the MIMO CT Δ-M, the quantisation gain Θ must be same in both the encoder and the decoder [10], [19]. In [27], it has been shown that the existence of the sliding mode (i.e. $s(t)^T\,\dot{s}(t) \le 0$) is ensured provided $\|\dot{x}(t)\|_\infty \le \hat{\theta}; \hat{\theta} = \min(\theta_{ii})$.

B. CONTINUOUS-TIME QUANTISED STATE-FEEDBACK CONTROL SYSTEM

The present study focuses on stabilisation problem of linear time invariant (LTI) systems via state feedback which is described by:

$$\dot{\bar{x}}(t) = A\,\bar{x}(t) + B\,u(t) \tag{5a}$$
$$u(t) = -K\,\bar{x}(t) \tag{5b}$$

where $\bar{x} \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ respectively denote the system states and the control signal. The pair (A, B) is assumed to be controllable and the closed loop system is expressed as:

$$\dot{\bar{x}}(t) = A_{cl}\,\bar{x}(t) \tag{6}$$

where $A_{cl} = A - BK$.

The feedback control gain K is designed such that the poles of the closed-loop system A_{cl} are at desired locations. The solution of (6) gives:

$$\bar{x}(t) = e^{A_{cl}t}\,\bar{x}(0) \tag{7}$$

where $\bar{x}(0)$ is the initial conditions of $\bar{x}(t)$. It is assumed that

$$\left\| e^{A_{cl}t} \right\| \leq c\, e^{-\lambda t} \tag{8}$$

The solution of (7) is therefore bounded and satisfies the condition:

$$\|\bar{x}(t)\| \leq c\, e^{-\lambda t}\, \|\bar{x}(0)\| \tag{9}$$

Using (1) and (5), the quantised state-feedback control system can be described as:

$$\dot{x}(t) = A\, x(t) + B\, \hat{u}(t) \tag{10a}$$
$$\dot{\hat{x}}(t) = \Theta\, \text{sgn}(s(t)) \tag{10b}$$
$$\hat{u}(t) = -K\, \hat{x}(t) \tag{10c}$$
$$s(t) = x(t) - \hat{x}(t) \tag{10d}$$

where $s(t)$ denotes the quantisation error. For the sake of convenience let us re-write (10) as:

$$\dot{x}(t) = A_{cl}\, x(t) + BK\, s(t) \tag{11a}$$
$$\dot{s}(t) = A_{cl}\, x(t) + BK\, s(t) - \Theta\, \text{sgn}(s(t)). \tag{11b}$$

The objective is to $\lim_{t\to\infty} s(t) = 0_n$ in the equivalent sliding mode.

Proposition 1: Assume $x(0)$ *is known and* (8) *holds. If there exists a quantiser gain* $\Theta = \text{diag}\{\theta_{ii}\}$ *such that*

$$c\, \varrho(A_{cl})\, e^{-\lambda t}\, \|x(0)\| < \hat{\theta}, \tag{12}$$

then the quantised state feedback control systems (11)*, is stable and converges to the system* (5)*.*

Proof: From (11),

$$\begin{aligned} s(t)^T\, \dot{s}(t) &= s^T(t)\, (A_{cl}\, x(t) + BK\, s(t) - \Theta\, \text{sgn}(s(t))) \\ &\leq s^T(t) A_{cl}\, x(t) + s^T(t)\, BK\, s(t) - \hat{\theta}\, \|s(t)\|_1 \\ &\leq \varrho(A_{cl})\, \|s(t)\|\, \|x(t)\| + \|BK\|\, \|s(t))\|^2 \\ &\quad -\hat{\theta}\, \|s(t)\|_1 \\ &\leq \|s(t)\|\, \{\varrho(A_{cl})\, \|x(t)\| + \|BK\|\, \|s(t))\| - \hat{\theta}\} \end{aligned} \tag{13}$$

Note that since, $\varrho(A) \triangleq \max\{|\lambda| : \lambda \in \text{spec(A)}\}$ and $\|s(t)\|_2 \leq \|s(t)\|_1$, the solution for (11) is given by:

$$x(t) = e^{A_{cl}t}\, \bar{x}(0) + \int_0^t BK\, e^{A_{cl}(t-\tau)}\, s(\tau) d\tau \tag{14}$$

where

$$\int_0^t s(\tau) d\tau = \left[\int_0^t s_0(\tau) d\tau \int_0^t s_1(\tau) d\tau \cdots \int_0^t s_n(\tau) d\tau \right]^T.$$

Considering (8), the bound of the solution can be found and is given by:

$$\begin{aligned} \|x(t)\| &\leq c\, e^{-\lambda t}\, \|x(0)\| + c \int_0^t \|BK\|\, e^{-\lambda(t-\tau)}\, s(\tau) d\tau \\ &\leq c\, e^{-\lambda t}\, \|x(0)\| + \frac{c\, \|BK\|}{\lambda} \sup_{0<\tau<t} \|s(t)\| \end{aligned} \tag{15}$$

Since $\sup_{0<\tau<t} \|s(t)\| \leq \|s(t)\|$, manipulating (13) and (15) gives,

$$c\, \varrho(A_{cl})\, e^{-\lambda t}\, \|x(0)\| + \left(1 + \frac{c}{\lambda}\right) \|BK\|\, \|s(t)\| \leq \hat{\theta}) \tag{16}$$

Thus, $s(t)^T\, \dot{s}(t) \leq 0$ and the sliding mode is ensured.

Consider the case of ideal sliding mode and assume that $\hat{x}(0) = x(0)$ (i.e. $s(0) = 0$). Then the switching function (11b) will initiate an equivalent sliding mode and remain there indefinitely (i.e. $s(t) = 0$) [7]. Therefore, if the condition (12) is true, then using (11b) and (13), it can be proved that $s(t)^T\, \dot{s}(t) \leq 0$. ■

Remark 1: If $\|s(0)\| \geq 0$, then the switching function (11b) requires finite-time (also called *reaching-time*), to change its mode from the *reaching-mode* to *equivalent-mode*. However, this finite-time may destabilise the entire closed loop system. Hence, in this paper, initial values of the system is assumed to be known and $\hat{x}(0)$ is selected such that $\|s(0)\| = 0$.

Remark 2: Higher values of feedback control gain K will make the degree of stability of the system higher. However, it forces θ_{ii} for all $i = (1, 2, \ldots, n)$ to be chosen high, which results in the chattering phenomenon.

Remark 3: In Proposition 1, an ideal sliding mode is assumed where infinite sampling switch is considered. As a result of the imperfection of switch elements, the switching manifold can be described as a boundary layer. The width of the boundary layer is defined as:

$$\|s(t)\| < \epsilon(h) \tag{17}$$

where h denotes the sampling time. It is worth to note that, $\lim_{h\to 0} \epsilon(h) = 0$. By choosing the optimal quantiser gain $\boldsymbol{\Theta}$ and reducing sampling frequency of the switching; the boundary layer (17) can be minimised, undesired oscillations (*chattering phenomena*) can be attenuated and the system can be stabilised.

III. DISCRETE-TIME SWITCHED CONTROL SYSTEMS

A. EULER DISCRETISED SYSTEM DESCRIPTION

The discrete-time equivalent of the CT system (5), using Euler discretisation is expressed as:

$$\bar{x}(k+1) = \Phi\, \bar{x}(k) + \Gamma\, u(k) \tag{18a}$$
$$u(k) = -K\, \bar{x}(k) \tag{18b}$$

where $\Phi = I_n + hA = \{\phi_{ij}\}$ *and* $\Gamma = hB$. Note that $\bar{x}(k)$, $u(k)$ denote respectively $\bar{x}(kh)$ and $u(kh)\; \forall\, k \in [kh, (k+1)h]$, where h is the sampling period.

Simplifying (18) gives,

$$\bar{x}(k+1) = \Phi_{cl}\, \bar{x}(k) \tag{19}$$

where

$$\Phi_{cl} = I_n + h\, A_{cl} \tag{20}$$

The sampling time $h \in \mathbb{R}_+$ is selected in the range [28]:

$$0 < h < h_{max} = \frac{2}{\varrho\,(A_{cl})} \tag{21}$$

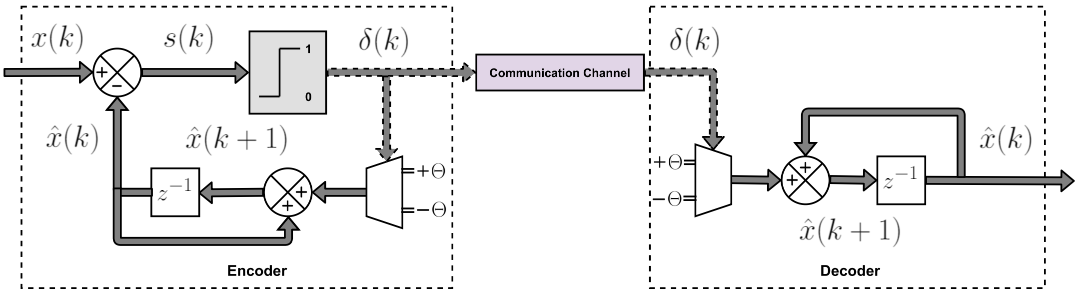


FIGURE 2. Discrete-time delta-modulator (Δ-M).

This will ensure that the eigenvalues of Φ_{cl} are at desired locations. Substituting (20) in (19) gives:

$$\Delta\bar{x}(k) = hA_{cl}\,\bar{x}(k) \tag{22}$$

where $\Delta\bar{x}(k) = \bar{x}(k+1) - \bar{x}(k)$.

Re-write (22) as:

$$\Delta\bar{x}(k) = hA_{cl}\,(\Phi_{cl})^k\,\bar{x}(0) \tag{23}$$

where $\bar{x}(k) = (\Phi_{cl})^k\,\bar{x}(0)$.

If the feedback gain matrix K is selected such that, $\|\Phi_{cl}\| < 1$, the it can be shown that

$$\|\Delta\bar{x}(k)\| < \ldots < \|\Delta\bar{x}(1)\| < \|\Delta\bar{x}(0)\|\,. \tag{24}$$

This proves,

$$\|\Delta\bar{x}(k)\| < h\,\varrho\,(A_{cl})\,\|\bar{x}(0)\| \tag{25}$$

B. DISCRETE-TIME DELTA MODULATOR

The block diagram of a MIMO discrete-time Δ-M is shown in Figure. 2. From Figure. 2, the dynamics of the discrete-time Δ-M can be expressed as:

$$\hat{x}(k+1) = \hat{x}(k) + \Theta\,\mathrm{sgn}(s(k)) \tag{26a}$$

$$s(k) = x(k) - \hat{x}(k) \tag{26b}$$

where Θ is a positive diagonal matrix i.e., $\Theta = \mathrm{diag}\{\theta_{ii}\}\forall i = (1, 2, \cdots, n)$.

Further $\mathrm{sgn}(s(k)) \in \left[\{-1,1\},\cdots,\{-1,1\}\right]^T \in \mathbb{R}^n$, where,

$$\mathrm{sgn}(s_i(k)) = \begin{cases} +1, & \text{if } s_i(k) \geq 0; \\ -1, & \text{if } s_i(k) < 0; \end{cases} \quad i = (1, 2, \cdots, n) \tag{27}$$

The communication channel between the encoder and the decoder carries a single-bit n-dimensional signal which is expressed as:

$$\delta(k) = \frac{1}{2}\left[1_n + \mathrm{sgn}(s(k))\right]. \tag{28}$$

This implies that $\delta(k) \in \left[\{0,1\},\,\cdots,\{0,1\}\right]^T \in \mathbb{R}^n$.

For the proper operation of the MIMO DT Δ-M, the quantisation gain Θ must be same in both the encoder and the decoder [10], [19]. In [27], it has been shown that the existence of the sliding mode (i.e. $|s(k+1)| \leq |s(k)|$) is ensured provided $\|\Delta x(k)\|_\infty \leq \hat{\theta})\,\forall\,\Delta x(k) = x(k+1) - x(k)$.

C. DISCRETE-TIME QUANTISED STATE-FEEDBACK CONTROL SYSTEM

Using (18) and (26), the quantised state-feedback control system is described as:

$$x(k+1) = \Phi\,x(k) + \Gamma\,\hat{u}(k) \tag{29a}$$

$$\hat{x}(k+1) = \hat{x}(k) + \Theta\,\mathrm{sgn}(s(k)) \tag{29b}$$

$$\hat{u}(k) = -K\,\hat{x}(k) \tag{29c}$$

$$s(k) = x(k) - \hat{x}(k) \tag{29d}$$

Substituting (29c) and (29d) into (29a) gives:

$$x(k+1) = \Phi_{cl}x(k) + \Gamma K\,s(k) \tag{30}$$

Re-write (30) as:

$$1\Delta x(k) = hA_{cl}\,x(k) + \Gamma K\,s(k) \tag{31}$$

Using (20), (29) and (31), the dynamics of the switching function $s(k)$ can be described as:

$$\begin{aligned} s(k+1) &= x(k+1) - \hat{x}(k+1) \\ &= \Phi_{cl}x(k) + \Gamma K\,s(k) - \hat{x}(k) - \Theta\,\mathrm{sgn}(s(k)) \\ &= [\Phi_{cl} - I_n]\,x(k) + [\Gamma K + I_n]\,s(k) \\ &\quad -\Theta\,\mathrm{sgn}(s(k)) \\ &= \Delta x(k) + s(k) - \Theta\,\mathrm{sgn}(s(k)) \end{aligned} \tag{32}$$

Definition 1: If there exists $\delta > 0$ and $M > 0$ such that $|f(h)| < M\,|g(h)|$ for $h < \delta$, then the order of $f(h)$ is equal $g(h)$ provided $h \to 0$. This is denoted by $f(h) = \mathcal{O}\,(g(h))$.

Lemma 1: Let $s_i(k)$ be defined as in (32). If $|s(k+1)| < |s(k)|$ and

$$\sup_{n\geq 0} |\Delta x_i(k)| < \theta_{ii}(h) < \infty \tag{33}$$

then,

$$|s_i(\infty)| \leq 2\theta_{ii}(h) < \varepsilon \tag{34}$$

for $i = 1, \cdots, n$ where ε is a positive constant.

Proof: Consider two cases: when $s_i(k) > 0$ and $s_i(k) \leq 0$ From (32), when $s_i(k) > 0$,

$$\begin{aligned} s_i(k+1) &= \Delta x_i(k) + s_i(k) - \theta_{ii} \\ &< s_i(k) < 2\theta_{ii} \end{aligned} \tag{35}$$

Correspondingly, when $s_i(k) \leq 0$,

$$\begin{aligned} s_i(k+1) &= \Delta x_i(k) + s_i(k) + \theta_{ii} \\ &> s_i(k) > 2\theta_{ii} \end{aligned} \tag{36}$$

From the above two cases $|s_i(k)|$ is bounded such that $|s_i(k)| < 2\theta_{ii}$. Further, as $k \to \infty$, $s_i(k)$ will be bounded as:

$$|s_i(\infty)| \leq 2\theta_{ii} < \varepsilon \tag{37}$$

Using Definition 1, (31) can be expressed as:

$$\Delta x_i(k) = \alpha_i\, x(k) + \beta_i\, s(n) \tag{38}$$

where α_i and β_i respectively denote the i^{th} row of hA_{cl} and $\varGamma K$. If both α_i and β_i can be expressed as $\alpha_i = \mathcal{O}(h)$ and $\beta_i = \mathcal{O}(h)$, (33) can be satisfied with arbitrary small h. This also means that θ_{ii} is a function of h (i.e. $\theta_{ii}(h)$) and $\theta_{ii} \to 0$ as $h \to 0$. ■

Remark 4: For an arbitrarily small value of h, $\Delta x(k)$ tends to its counterpart of the nominal system $\Delta\bar{x}(k)$ in (22). Considering (25), this also implies that $\|\Delta x(k)\| \leq h\,\varrho\,(A_{cl})\,\|\bar{x}(0)\| \leq \theta_{ii}(h)$.

Proposition 2: If the condition in Lemma 1 is true, then (30) *is bounded as,*

$$\frac{\sqrt{n}\theta_{ii}\,\|\varGamma K\|}{1 - \|\varPhi_{cl}\|}. \tag{39}$$

Proof: From (30):

$$\begin{aligned} \|x(k+1)\| &= \|\varPhi_{cl}\, x(k) + \varGamma K\, s(k)\| \\ &\leq \|\varPhi_{cl}\|\,\|x(k)\| + \|\varGamma K\|\,\|s(k)\| \end{aligned} \tag{40}$$

If $I_n - \varPhi_{cl}$ is non-singular, iterating (40) κ times gives,

$$\begin{aligned} \|x(\kappa)\| &\leq \|\varPhi_{cl}\|^{\kappa}\,\|x(0)\| + \|\varGamma K\| \sum_{i=0}^{\kappa-1} \|\varPhi_{cl}\|^{i}\,\|s(\kappa - i - 1)\| \\ &\leq \|\varPhi_{cl}\|^{\kappa}\,\|x(0)\| + \sqrt{n}\theta_{ii}\,\|\varGamma K\| \sum_{i=0}^{\kappa-1} \|\varPhi_{cl}\|^{i} \\ &\leq \|\varPhi_{cl}\|^{\kappa}\,\|x(0)\| + \frac{\sqrt{n}\theta_{ii}\,\|\varGamma K\|\,(1 - \|\varPhi_{cl}\|^{\kappa})}{1 - \|\varPhi_{cl}\|} \end{aligned} \tag{41}$$

When $\kappa \to \infty$, it can be shown that $\|x(\infty)\| \leq \frac{\sqrt{n}\theta_{ii}\|\varGamma K\|}{1-\|\varPhi_{cl}\|}$ which completes the proof. ■

D. BOUNDARY LAYER OF SLIDING MODE AND PERIODICITY

In the following, we will estimate the boundary width of sliding mode which is associated to the quantization error.

Lemma 2 [29]: Let $x(k)$, f_s respectively denote the discretised samples of $x(t)$ and the over-sampling frequency of Δ-M which satisfies

$$\frac{f_s}{2\mathcal{B}_{\mathcal{X}}} > 2^{\alpha}. \tag{42}$$

where $\mathcal{X}$, $\mathcal{B}_{\mathcal{X}}$ respectively denote the upper bound of x and the bandwidth of the closed-loop system and α is a positive number. Then,

$$\|\Delta x(k)\| \leq \frac{\pi}{2^{\alpha}}\mathcal{X} \tag{43}$$

The bound of $s(k)$ in the equivalent-based sliding mode, is presented in the following [19], [30].

Lemma 3: If the diagonal elements of Θ is selected such that

$$\hat{\theta} = \min(\theta_{ii}) \geq \frac{\pi}{2^{\alpha}}\mathcal{X} \geq \|\Delta x(k)\| \tag{44}$$

and the initial conditions satisfy $s(0) = 0$, then the system (26) exhibits quasi-sliding motion and the switching function $s(k)$ remain inside the boundary layer Ω $\left(\|s(k)\| \leq \Omega = \frac{(\|\Delta x(k)\| + \sqrt{n}\theta_{ii})^2}{2(\theta_{ii} - n\|\Delta x(k)\|)}\right)$ indefinitely.

Proof: Consider the dynamics of $s(k)$ in (32). Let us choose a positive-definite, DT candidate Lyapunov function,

$$\mathcal{V}(k) = s^T(k)s(k) \tag{45}$$

Using (43), the first difference $\Delta\mathcal{V}(k)$ of the Lyapunov function $\mathcal{V}(k)$ can be written as:

$$\begin{aligned} \Delta\mathcal{V}(k) &= s^T(k+1)s(k+1) - s^T(k)s(k) \\ &= \left[\Delta x(k) + s(k) - \Theta\,\mathrm{sgn}(s(k))\right]^T \\ &\quad \times \left[\Delta x(k) + s(k) - \Theta\,\mathrm{sgn}(s(k))\right] - s^T(k)s(k) \\ &= \Delta x^T(k)\Delta x(k) + \Delta x^T(k)s(k) \\ &\quad - \Delta x^T(k)\Theta\,\mathrm{sgn}(s(k)) + s^T(k)\Delta x(k) \\ &\quad + s^T(k)s(k) - s^T(k)\Theta\,\mathrm{sgn}(s(k)) \\ &\quad - \Theta\,\mathrm{sgn}(s^T(k))x(k) - \Theta\,\mathrm{sgn}(s^T(k))s(k) \\ &\quad + \Theta\,\mathrm{sgn}(s^T(k)\Theta\,\mathrm{sgn}(s(k)) - s^T(k)s(k) \\ &= \|\Delta x(k)\|^2 + 2\sum_{i=1}^{n} \Delta x_i(k)s_i(k) \\ &\quad - 2\sum_{i=1}^{n} \Delta x_i(k)\theta_{ii}\mathrm{sgn}(s_i(k)) \\ &\quad - 2\sum_{i=1}^{n} |s_i(k)|\theta_{ii} + \sum_{i=1}^{n} \theta_{ii}^2 \\ &\leq \|\Delta x(k)\|^2 + 2\|\Delta x(k)\|_1\|s(k)\|_1 + n\theta_{ii}^2 \\ &\quad - 2\left(\sum_{i=1}^{n} \Delta x_i(k)\theta_{ii}\mathrm{sgn}(s_i(k)) + \sum_{i=1}^{n} |s_i(k)|\theta_{ii}\right) \\ &\leq \|\Delta x(k)\|^2 + 2\|\Delta x(k)\|_1\|s(k)\|_1 + n\theta_{ii}^2 \\ &\quad + 2\sum_{i=1}^{n} |\Delta x_i(k)|\theta_{ii} - 2\sum_{i=1}^{n} |s_i(k)|\theta_{ii} \\ &\leq \|\Delta x(k)\|^2 + 2\|\Delta x(k)\|_1\|s(k)\|_1 + n\theta_{ii}^2 \\ &\quad + 2\theta_{ii}\|\Delta x(k)\|_1 - 2\theta_{ii}\|s(k)\|_1 \\ &\leq \|\Delta x(k)\|^2 + 2\sqrt{n}\|\Delta x(k)\|\hat{\theta} + \hat{\theta}^2 \\ &\quad + 2\left(n\|\Delta x(k)\| - \hat{\theta}\right)\|s(k)\| \end{aligned} \tag{46}$$

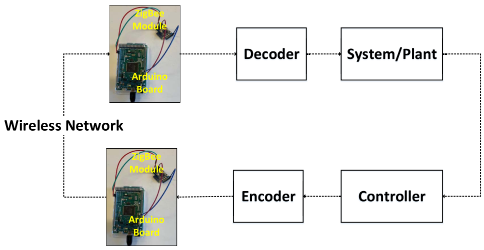


FIGURE 3. Networked control using ZigBee protocol based communication network.

where $\hat{\theta}$ satisfies the dominance condition with respect to $\Delta x(k)$. Solving $\Delta \mathcal{V}(k)$ with respect to $s(k)$, suggests that $s(k)$ is bounded such that $\|s(k)\| \leq \Omega$, where,

$$\Delta \mathcal{V}(k) \leq 0 \implies$$
$$\Omega = \frac{\|\Delta x(k)\|^2 + 2\sqrt{n}\|\Delta x(k)\|\hat{\theta} + \hat{\theta}^2}{2\left(\hat{\theta} - n\|\Delta x(k)\|\right)}$$
$$\Omega = \frac{\left(\|\Delta x(k)\| + \sqrt{n}\hat{\theta}\right)^2}{2\left(\hat{\theta} - n\|\Delta x(k)\|\right)} \tag{47}$$

■

Remark 5: When the switching function $s(k)$ is bounded (i.e., $\|s(k)\| \leq \Omega$), the trajectory of the switching function will exhibit periodic behaviour [30]. The period is dependent on the choice of sampling time h and quantisation gain Θ.

Remark 6: Let us re-write (29a) as:

$$x(k+1) = \Phi_{cl}\, x(k) + \Psi s(k) \tag{48}$$

where $\Psi = \Gamma K$. When $\varrho(\Phi_{cl}) < 1$ and $s(k)$ exhibits periodic behaviour, then $x(k)$ and $\hat{x}(k)$ also exhibit similar behaviour and their periods will be similar [13].

IV. SIMULATION RESULTS

The effectiveness of the proposed Δ-M based quantised state-feedback controller design is demonstrated in a practical networked environment considering examples of both discrete and continuous time systems. In this study, the wireless communication channel is implemented using two Arduino boards and two Zigbee modules as hardware in loop (HIL). The modulated signal is transmitted to the Zigbee module 2 in Arduino board 2 from the Zigbee module 1 in Arduino board 1. This board is connected to the computer and with the plant, the controller, and the quantiser. The Zigbee module 2 in Arduino board 2 acts as a hop device in another computer which transmits the signal back to the Zigbee module 1 in Arduino board 1 which then transmits the signal into the demodulator (see figure 3.)

A. EXAMPLE 1

Consider a continuous-time linear system with three number of inputs described by:

$$\dot{x}(t) = A\, x(t) + B\, u(t) \tag{49}$$

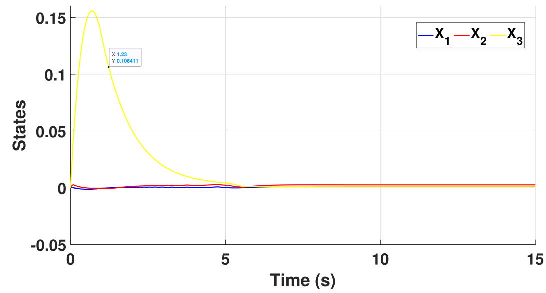


FIGURE 4. Behaviour of the states for the Example 1 for the continuous-time system.

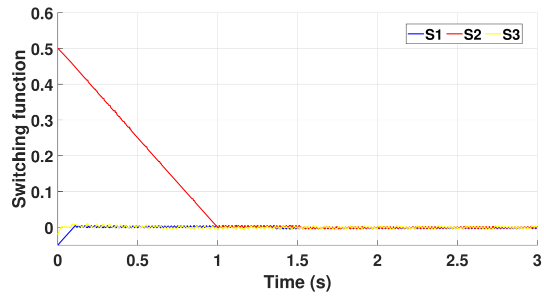


FIGURE 5. Behaviour of the switching function for the Example 1 for the continuous-time system.

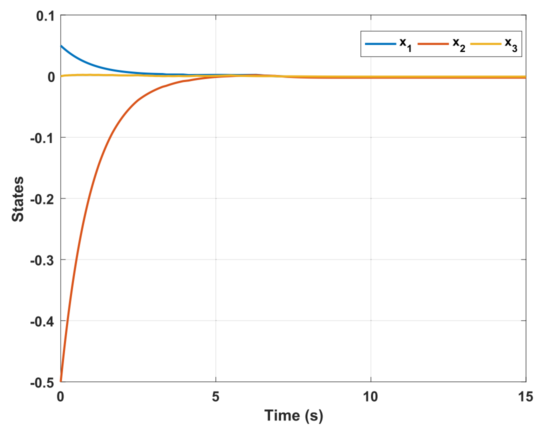
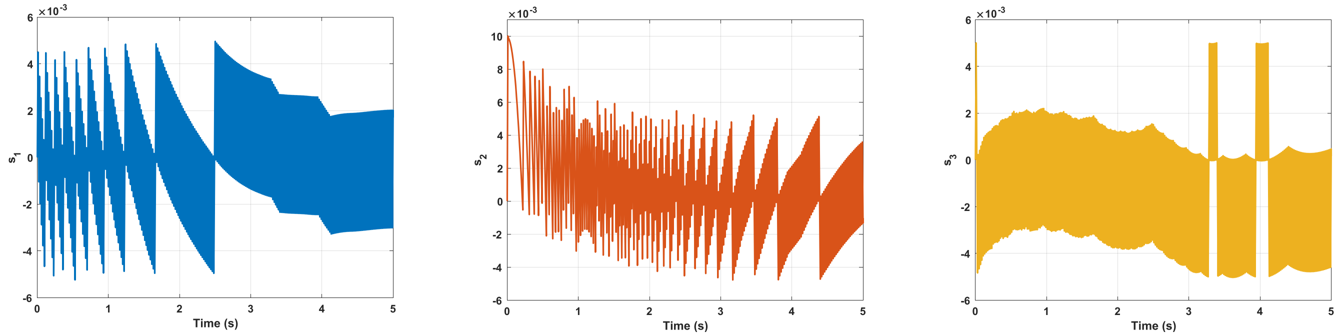


FIGURE 6. Behaviour of states over time for the Example 1 for the discrete-time system.

where A and B are given by,

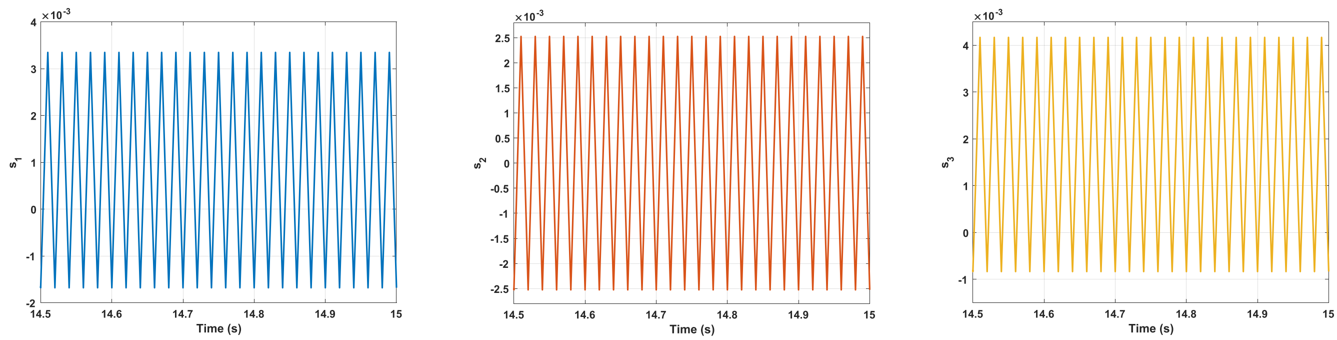
$$A = \begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & -1 \\ -1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The controller and the quantisation gain of the Δ-M based quantiser are designed following the method described in section-II for the continuous-time system.The initial conditions x_0 of the the plant is taken as $x_0 = [0.05; -0.5; 0]$, and the poles and the feedback gain K is calculated by placing the closed loop poles at $x_{poles,cl} = -1, -1, -1$. The performance of the controller in stabilising the states are shown in Figure 4 and the behaviour of the switching function is shown in Figure 5.



(a) s_1 (b) s_2 (c) s_3

FIGURE 7. Behaviour of the switching function for the example 1 for the discrete-time system.



(a) s_1 (b) s_2 (c) s_3

FIGURE 8. Zig-zag (periodic) behaviour of the switching function for the example 1 for the discrete-time system.

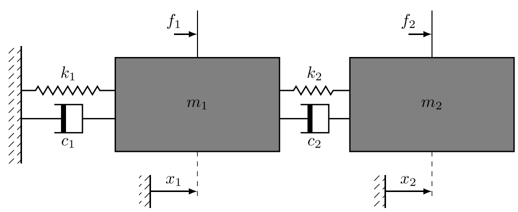


FIGURE 9. Simulated example 2: Mechanical damper system.

After designing the controller for continuous time systems, in the next phase, the Δ-M based quantised controller is designed for the discrete-time system. The discrete equivalent of the continuous-time system described in (49), at a sampling rate of $h = 0.01$ seconds, is given by:

$$x(k+1) = \Phi x(k) + \Gamma u(k) \tag{50}$$

where

$$\Phi = \begin{bmatrix} 0.9900 & 0 & -0.0100 \\ -0.0100 & 0.9900 & -0.0100 \\ -0.0100 & 0.0100 & 0.9900 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 0.0100 & 0 & 0 \\ 0 & 0.0100 & 0 \\ 0 & 0 & 0.0100 \end{bmatrix}$$

The controller and the gain of the Δ-M based quantiser are designed following the approaches described in section-II and

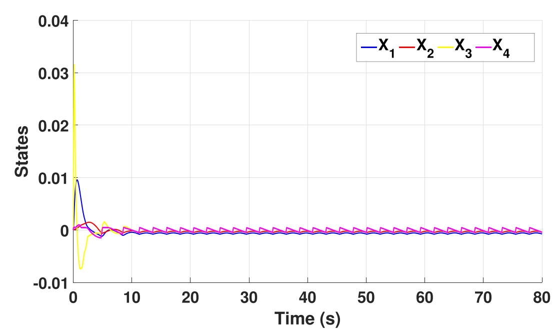


FIGURE 10. Behaviour of the states for the Example 2 for the continuous-time system.

section-III. The initial conditions and x_0 and feedback gain K are same as the continues-time system. The quantisation gain Θ is given by:

$$\Theta = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.05 \end{bmatrix}$$

From Figure. 6, it is observed that all the states converge to zero within finite time. Further, the switching function starts inside the region Ω, as can be seen in Figure. 7, and stays there indefinitely with a period of 2 (see Figure. 8). It is worth to emphasis that although the period of the

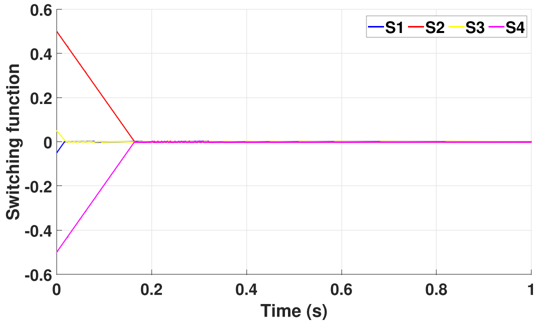


FIGURE 11. Behaviour of the switching function for the Example 2 for the continuous-time system.

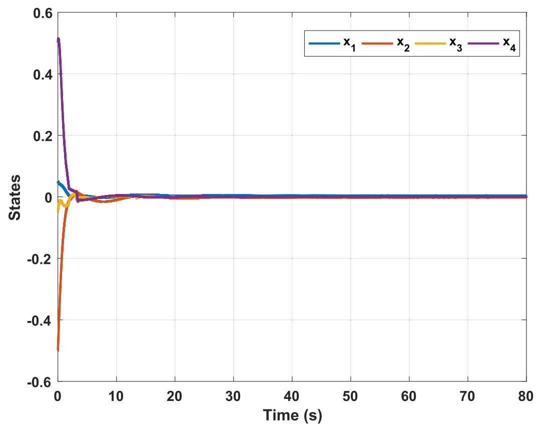


FIGURE 12. Behaviour of the states for the Example 2 for the discrete-time system.

switching function depends on the choice of the quantisation gain and the sampling time, this remains always within the region Ω.

B. EXAMPLE 2

The effectiveness of the proposed controller is further shown considering an example of a mechanical system shown in Figure. 9. The continuous-time dynamics of this system is described as [31]:

$$\dot{x}(t) = A\,x(t) + B\,u(t) \tag{51}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} & -\dfrac{c_1+c_2}{m_1} & \dfrac{c_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2}{m_2} & \dfrac{c_2}{m_2} & -\dfrac{c_2}{m_2} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{1}{m_1} & 0 \\ 0 & \dfrac{1}{m_2} \end{bmatrix}$$

The values of various parameters of the system, used in the simulation, are:$k_1 = 1$; $k_2 = 1$; $m_1 = 1$; $m_2 = 2$; $c_1 = 1$; $c_2 = 1$. The design of controller and the quantisation gain follows same procedure as used in example 1.

The closed loop poles are placed at $-2+1i, -2-1i, -2+0.5i, -2-0.5i$ and the initial condition x_0 are taken as $[0.05; -0.5; -0.05; 0.5]$. The controller and the gain of the Δ-M based quantiser are designed following the method described in section-II for the continuous-time system. The performance of the controller in stabilising the states are shown in Figure 10 and the behaviour of the switching function is shown in Figure 11.

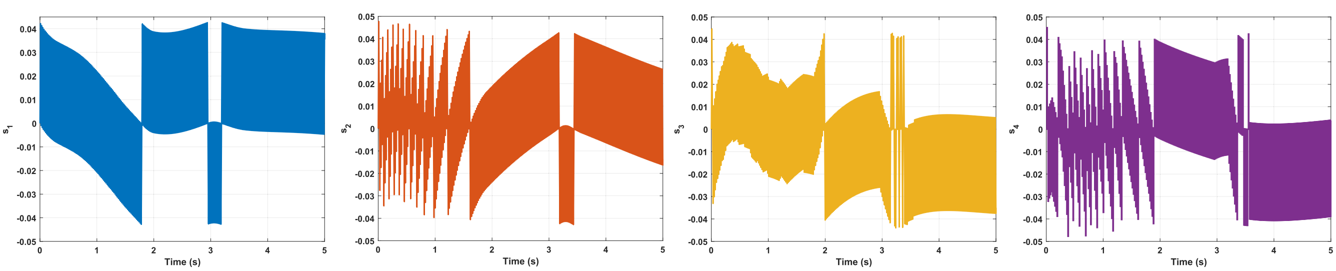
After designing the controller for continuous time systems, in the next phase, the Δ-M based quantised controller is designed for the discrete-time system. The discrete equivalent of the continuous-time system described in (51) at a sampling rate of $h = 0.01$ seconds is given by:

$$x(k+1) = \Phi x(k) + \Gamma u(k) \tag{52}$$

where

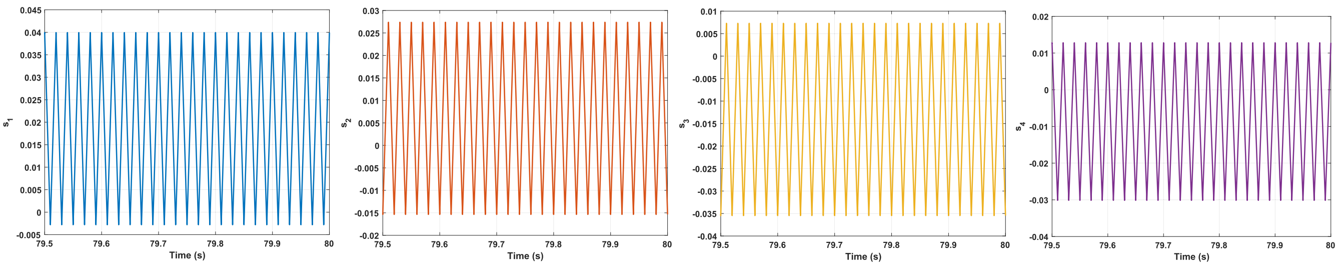
$$\Phi = \begin{bmatrix} 1.0000 & 0 & 0.0100 & 0 \\ 0 & 1.0000 & 0 & 0.0100 \\ -0.0200 & 0.0100 & 0.9900 & 0.0100 \\ 0.0050 & -0.0050 & 0.0050 & 0.9950 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.0100 & 0 \\ 0 & 0.0050 \end{bmatrix}$$



(a) s_1 (b) s_2 (c) s_3 (d) s_4

FIGURE 13. Behaviour of the switching function for the Example 2 for the discrete-time system.



(a) s_1 (b) s_2 (c) s_3 (d) s_4

FIGURE 14. Zig-zag (periodic) behaviour of the switching function for the example 2 for the discrete-time system.

The quantisation gain matrix Θ is given by:

$$\Theta = \begin{bmatrix} 0.0427 & 0 & 0 & 0 \\ 0 & 0.0427 & 0 & 0 \\ 0 & 0 & 0.0427 & 0 \\ 0 & 0 & 0 & 0.0427 \end{bmatrix}$$

From Figure. 12, it is obvious that all the states converge to zero is within finite time. Similar to example 1, the switching function starts inside the region Ω, and remain there indefinitely with a period 2 (see Figure. 13 and Figure. 14).

From both examples, it can be seen that the performance of the quantised feedback controller is satisfactory. Although this study mainly focuses on quantisation related issues, the behaviour of the designed Δ-M based quantised control system is investigated under many networked constraints such as random packet delay, transmission delay and packet losses in the practical Zigbee protocol based networked control system. From the results, it is observed that the proposed one-bit quantised controller could maintain the stability of the overall system.

V. CONCLUSION

In this study, a delta modulator based single-bit quantised state-feedback controller is designed for linear networked control systems. The stability conditions Δ-M is derived for both in CT and DT domains. It is confirmed that the stability of delta Modulator is heavily dependent on the quantiser gain and on some properties of the input signals. The bound of the switching function is derived such that *periodic* behaviour in the steady-state is ensured. The effectiveness of the designed delta modulator based single-bit quantised control approach is illustrated using a practical ZigBee protocol based networked control system which inherent many network imperfections like packet losses, transmission delays, and bit-rate constraints. The results of the experimental simulations were carried out using two examples and the results of the simulations confirm the theoretical findings.

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