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Divergence Method for Exponential Stability Study of Autonomous Dynamical Systems

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ABSTRACT The recent proposed divergence method is used for investigation of an exponential stability of autonomous dynamical systems. New necessary conditions are derived by using Divergence theorem. Taking into account the relation between gradient and divergence of the vector field, novel sufficient conditions are obtained. These sufficient conditions are applied for design the state-feedback control laws ensuring exponential stability of the closed-loop system. Examples illustrate the efficiency of the proposed methods.

INDEX TERMS Autonomous system, sufficient condition, divergence theorem, control, divergence of the vector field, exponential stability.

I. INTRODUCTION

The paper is devoted to the development of divergence method for stability study of dynamical systems. Interest of this method is associated with its relationship to many problems in physics and mechanics, in particular to processes that are described by the continuity equation [1]–[4].

First results of stability study of dynamical systems by using a divergence and flow of the vector field were proposed in [5]–[8] for second order dynamical systems. These results were improved in [9]–[13] for arbitrary order systems. An overview of these and others corresponding methods is described in more detail in [14]–[17]. Also, in [14]–[17] new necessary and sufficient stability conditions were proposed. Differently from [9]–[13] methods [14]–[17] allow one to extend the class of investigated systems. Additionally, in [14]–[17] the relation between necessary and sufficient conditions as well as continuity equation in electromagnetism, fluid dynamics, energy and heat, probability distributions, and quantum mechanics is established. However, all of these methods are proposed for investigation of stability or asymptotic stability, but not exponential one.

There are many tasks where stability and asymptotic stability is not enough. For example, the exponentially stable systems are more robust under parametric uncertainty and external disturbances, than stable or asymptotically stable

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systems [18], [19]. Also, if continuous system is an exponentially stable, then it remains stable under discrete control [20]. Currently, tasks with discrete controls are widespread, e.g., networked control [21]–[23], security control [24], event-triggered control [25], [26], sampled-data control [27], etc.

Therefore, the main contribution of the present paper is a generalization of results [14]–[17] to solution of the following problems:

- i) design new necessary and sufficient exponential stability conditions;
- ii) design new method of control law design ensuring exponential stability of the closed-loop systems.

The paper is organized as follows. Section II describes problem formulation. Section III proposes necessary and sufficient exponential stability conditions. Section IV describes application of the proposed results to linear systems. Section V proposes methods for designing the state feedback control laws. All proposed theoretical results are illustrated by numerical examples. Finally, Section VI collects some conclusions.

Notations. In the paper the superscript T stands for matrix or vector transposition; \mathbb{R}^n denotes the n dimensional Euclidean space with the vector norm $|\cdot|$; $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices; $\nabla\{W(x)\} = \left[\frac{\partial W}{\partial x_1}, \dots, \frac{\partial W}{\partial x_n} \right]^T$ is a gradient of the scalar function $W(x)$; $\nabla \cdot \{h(x)\} = \frac{\partial h_1}{\partial x_1} + \dots + \frac{\partial h_n}{\partial x_n}$ is a divergence of the vector field $h(x) = [h_1(x), \dots, h_n(x)]^T$;

$int\{\bar{\Upsilon}\}$ is the set of interior points in $\bar{\Upsilon}$. We mean that the equilibrium point is stable if it is Lyapunov stable [28].

II. PROBLEM FORMULATION

In the paper we consider autonomous systems. An autonomous systems (or time-invariant systems) are described by autonomous differential equations which do not explicitly depend on time, i.e.

$$\dot{x}(t) = f(x(t)). \quad (1)$$

Here $x = [x_1, \dots, x_n]^T$ is the state, $f = [f_1, \dots, f_n]^T : D \rightarrow \mathbb{R}^n$ is the continuously differentiable function in $D \subset \mathbb{R}^n$. The set D contains the origin and $f(0) = 0$. Assume that the domain of attraction D_A of the equilibrium point $x = 0$ coincides with the domain D . However, all obtained results is valid if $D_A \subset D$ or $D_A = \mathbb{R}^n$.

It is required to design new exponential stability conditions in the divergence form. The proposed results will be obtained by using resent methods [14]–[17].

III. STABILITY CONDITIONS

In [14]–[17] new necessary asymptotic stability conditions were obtained. Now, we generalize these results for getting new necessary exponential stability conditions.

Denote by $\Upsilon = \{x \in D : \mu(x) = C, C > 0\}$ and $V = \Upsilon \cup int\{\Upsilon\}$, as well as denote by $\bar{\Upsilon} = \{x \in D : \mu^{-1}(x) = C, C > 0\}$ and $\bar{V} = \bar{\Upsilon} \cup int\{\bar{\Upsilon}\}$.

Theorem 1: If the equilibrium point $x = 0$ of (1) is an exponentially stable, then there exists twice differentiable function $\mu(x) > 0$ in $x \in D \setminus \{0\}$, $\mu(0) \geq 0$ and $|\nabla\{\mu(x)\}| \neq 0$ for any $x \in D \setminus \{0\}$ such that one of the following conditions holds:

$$\begin{aligned} (i_{T1}) \quad & \int_V \nabla \cdot \left\{ |\nabla\{\mu(x)\}|f(x) + \alpha \frac{\mu(x)}{|\nabla\{\mu(x)\}|} \nabla\{\mu(x)\} \right\} dV \leq 0, \\ (ii_{T1}) \quad & \int_{\bar{V}} \nabla \cdot \left\{ |\nabla\{\mu^{-1}(x)\}|f(x) - \alpha \frac{\mu^{-1}(x)}{|\nabla\{\mu^{-1}(x)\}|} \nabla\{\mu^{-1}(x)\} \right\} \\ & d\bar{V} \geq 0, \end{aligned}$$

where $\alpha > 0$.

Proof 1: Since the equilibrium point $x = 0$ of (1) is an exponentially stable, then, according to [28, Corollary 3.4] and taking $\mu(x)$ as Lyapunov function, the following inequality holds

$$\dot{\mu}(x) \leq -\alpha\mu(x). \quad (2)$$

Considering $\dot{\mu}(x) = \nabla\{\mu(x)\}f(x)$, integrate (2) as follows

$$\oint_{\Upsilon} [\nabla\mu(x)f(x) + \alpha\mu(x)] d\Upsilon \leq 0. \quad (3)$$

By using Divergence theorem, rewrite (3) in the form (i_{T1}) .

Considering $\nabla\{\mu^{-1}(x)\} = -\mu^{-2}(x)\nabla\{\mu(x)\}$, rewrite (2) as follows

$$\nabla\{\mu^{-1}(x)\}f(x) \geq \alpha\mu^{-1}(x). \quad (4)$$

Integrating (4), we have

$$\oint_{\bar{\Upsilon}} \left[\nabla\mu^{-1}(x)f(x) - \alpha\mu^{-1}(x) \right] d\bar{\Upsilon} \geq 0. \quad (5)$$

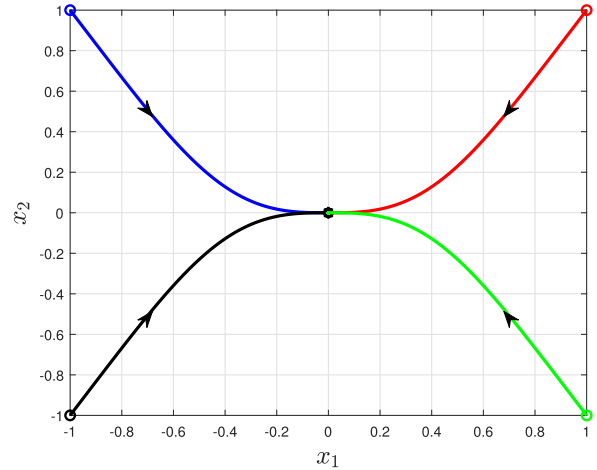


FIGURE 1. Phase portrait of (6).

By using Divergence theorem, rewrite (5) in the form (ii_{T1}) . Theorem 1 is proved.

Remark 1: The conditions (i_{T1}) and (ii_{T1}) are presented in the integral forms. However, it can be rewritten in the corresponding differential forms if

$$\begin{aligned} (i_{R1}) \quad & \nabla \cdot \left\{ |\nabla\{\mu(x)\}|f(x) + \alpha \frac{\mu(x)}{|\nabla\{\mu(x)\}|} \nabla\{\mu(x)\} \right\} \leq 0, \\ (ii_{R1}) \quad & \nabla \cdot \left\{ |\nabla\{\mu^{-1}(x)\}|f(x) - \alpha \frac{\mu^{-1}(x)}{|\nabla\{\mu^{-1}(x)\}|} \nabla\{\mu^{-1}(x)\} \right\} \geq 0. \end{aligned}$$

Illustrate the result of Theorem 1 in the following example.

Example 1: Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_1x_2^2 - x_1^3, \\ \dot{x}_2 &= -2x_2 - x_2x_1^2 - x_2^3, \end{aligned} \quad (6)$$

with the equilibrium point $(0, 0)$. The phase portrait of (6) is shown in Fig. 1. Here and below, phase trajectory has its own colour for appropriate initial point.

Investigate the equilibrium point of (6). Introduce

$$\mu(x) = |x|^{2\beta}, \quad (7)$$

where β is a positive integer. Calculate the following auxiliary expressions:

$$\begin{aligned} \nabla\{\mu(x)\} &= 2\beta(x_1^2 + x_2^2)^{\beta-1}x, \\ \nabla\{\mu^{-1}(x)\} &= -2\beta(x_1^2 + x_2^2)^{-\beta-1}x, \\ |\nabla\{\mu(x)\}| &= 2\beta(x_1^2 + x_2^2)^{\beta-0.5}, \\ |\nabla\{\mu^{-1}(x)\}| &= 2\beta(x_1^2 + x_2^2)^{-\beta-0.5}, \\ \frac{\mu(x)}{|\nabla\{\mu(x)\}|} \nabla\{\mu(x)\} &= (x_1^2 + x_2^2)^{\beta-0.5}x, \\ \frac{\mu^{-1}(x)}{|\nabla\{\mu^{-1}(x)\}|} \nabla\{\mu^{-1}(x)\} &= -(x_1^2 + x_2^2)^{-\beta-0.5}x, \\ \nabla \cdot \{|\nabla\{\mu(x)\}|f(x)\} &\leq -2\beta(x_1^2 + x_2^2)^{\beta-0.5} \\ &\quad \times [2\beta + 2 + (2\beta + 3)(x_1^2 + x_2^2)], \\ \nabla \cdot \{|\nabla\{\mu^{-1}(x)\}|f(x)\} &\geq 2\beta(x_1^2 + x_2^2)^{-\beta-0.5} \\ &\quad \times [2\beta - 2 + (2\beta - 3)(x_1^2 + x_2^2)], \\ \nabla \cdot \left\{ \frac{\mu(x)}{|\nabla\{\mu(x)\}|} \nabla\{\mu(x)\} \right\} &= (2\beta + 1)(x_1^2 + x_2^2)^{\beta-0.5}, \end{aligned}$$

$$\begin{aligned} & \nabla \cdot \left\{ \frac{\mu^{-1}(x)}{|\nabla\{\mu^{-1}(x)\}|} \nabla\{\mu^{-1}(x)\} \right\} \\ &= (2\beta - 1)(x_1^2 + x_2^2)^{\beta-0.5}. \end{aligned}$$

Using the last relations, verify the conditions (i_{T1}) and (ii_{T1}) :

$$\begin{aligned} & \nabla \cdot \left\{ |\nabla\{\mu(x)\}|f(x) + \alpha \frac{\mu(x)}{|\nabla\{\mu(x)\}|} \nabla\{\mu(x)\} \right\} \\ & \leq -(x_1^2 + x_2^2)^{\beta-0.5} [4\beta^2 + 4\beta - 2\alpha\beta - \alpha \\ & \quad + 2\beta(2\beta + 3)(x_1^2 + x_2^2)] < 0 \quad \text{for } \alpha < \frac{4\beta(\beta + 1)}{2\beta + 1}, \end{aligned}$$

$$\begin{aligned} & \nabla \cdot \left\{ |\nabla\{\mu^{-1}(x)\}|f(x) - \alpha \frac{\mu^{-1}(x)}{|\nabla\{\mu^{-1}(x)\}|} \nabla\{\mu^{-1}(x)\} \right\} \\ & \geq (x_1^2 + x_2^2)^{\beta-0.5} [4\beta^2 - 4\beta - 2\alpha\beta + \alpha + 2\beta(2\beta - 3) \\ & \quad \times (x_1^2 + x_2^2)] > 0 \quad \text{for } \beta \geq 2 \text{ and } \alpha < \frac{4\beta(\beta - 1)}{2\beta - 1}. \end{aligned}$$

As a result, the equilibrium point $x = 0$ is an exponentially stable.

In the next theorem we will introduce new auxiliary function $\xi(x)$ that allows one to extend the class of investigated systems in comparison with Theorem 1.

Theorem 2: If the equilibrium point $x = 0$ of (1) is an exponentially stable, then there exists twice differentiable function $\mu(x) > 0$ and differentiable function $\xi(x) > 0$ in $x \in D \setminus \{0\}$, $\mu(0) \geq 0$ and $|\nabla\{\mu(x)\}| \neq 0$ for any $x \in D \setminus \{0\}$ such that one of the following conditions holds:

$$\begin{aligned} (i_{T2}) \quad & \int_V \nabla \cdot \left\{ \xi(x)f(x) + \alpha \frac{\mu(x)\xi(x)}{|\nabla\{\mu(x)\}|^2} \nabla\{\mu(x)\} \right\} dV \leq 0, \\ (ii_{T2}) \quad & \int_{\bar{V}} \nabla \cdot \left\{ \xi^{-1}(x)f(x) - \alpha \frac{\mu^{-1}(x)\xi^{-1}(x)}{|\nabla\{\mu^{-1}(x)\}|^2} \nabla\{\mu^{-1}(x)\} \right\} \\ & d\bar{V} \geq 0, \end{aligned}$$

where $\alpha > 0$.

In particular, choosing $\xi(x) = |\nabla\{\mu(x)\}|$, we have the condition of Theorem 1.

Remark 2: The conditions (i_{T2}) and (ii_{T2}) can be rewritten in the corresponding differential forms if

$$\begin{aligned} (i_{R2}) \quad & \nabla \cdot \left\{ |\nabla\{\mu(x)\}|f(x) + \alpha \frac{\mu(x)}{|\nabla\{\mu(x)\}|} \nabla\{\mu(x)\} \right\} \leq 0, \\ (ii_{R2}) \quad & \nabla \cdot \left\{ |\nabla\{\mu^{-1}(x)\}|f(x) - \alpha \frac{\mu^{-1}(x)}{|\nabla\{\mu^{-1}(x)\}|} \nabla\{\mu^{-1}(x)\} \right\} \geq 0. \end{aligned}$$

Let us show in the following example, that Theorems 1 and 2 ensure only necessary conditions.

Example 2: Choose $\xi(x) = 1$ and $\mu(x) = |x|^{2\beta}$. Thus, $\nabla\{\mu(x)\} = 2\beta|x|^{\beta-2}x$, $|\nabla\{\mu(x)\}| = 2\beta|x|^{2\beta-1}$, and $\frac{\mu(x)\xi(x)}{|\nabla\{\mu(x)\}|^2} \nabla\{\mu(x)\} = \frac{n}{2\beta}$. Taking into account condition (i_{R2}) , we have

$$\nabla \cdot \{f(x)\} \leq -\frac{n\alpha}{2\beta} < 0.$$

Note, that the necessary stability condition [5]–[10] looks like $\nabla \cdot \{f(x)\} < 0$.

The next theorem generalises the sufficient stability condition [14]–[17] to exponential one.

Theorem 3: Let $\mu(x)$ be a positive definite continuously differentiable function in D , $\alpha > 0$ and $x = 0$ be an equilibrium point of the system (1). If one of the following inequalities

$$\begin{aligned} (i_{T2}) \quad & \nabla \cdot \{\mu(x)f(x)\} \leq -\mu(x)[\alpha - \nabla \cdot \{f(x)\}], \\ (ii_{T2}) \quad & \nabla \cdot \{\mu^{-1}(x)f(x)\} \geq \alpha\mu^{-1}(x) \text{ and } \nabla \cdot \{f(x)\} \leq 0, \\ (iii_{T2}) \quad & \nabla \cdot \{\mu(x)f(x)\} \leq -\alpha\mu(x) \text{ and } \nabla \cdot \{\mu^{-1}(x)f(x)\} \geq \\ & \alpha\mu^{-1}(x), \end{aligned}$$

holds for $x \in D \setminus \{0\}$, then $x = 0$ is an exponentially stable.

Proof 2: Consider each case (i_{T2}) – (iii_{T2}) separately. Rewrite inequality (i_{T2}) as follows

$$\nabla \cdot \{\mu(x)f(x)\} - \nabla \cdot \{f(x)\}\mu(x) \leq -\alpha\mu(x). \quad (8)$$

Taking into account well-known relation [3]

$$\nabla\{\mu(x)\}^T f(x) = \nabla \cdot \{\mu(x)f(x)\} - \nabla \cdot \{f(x)\}\mu(x), \quad (9)$$

represent the left-hand side of (8) in the form (2), i.e., $x = 0$ is an exponentially stable.

From (ii_{T2}) we have

$$\nabla \cdot \{\mu^{-1}(x)f(x)\} - \nabla \cdot \{f(x)\}\mu^{-1}(x) \geq \alpha\mu^{-1}(x). \quad (10)$$

Taking into account the first expression in (9), rewrite (10) as follows

$$\nabla\{\mu^{-1}(x)\}f(x) \geq \alpha\mu^{-1}(x).$$

Considering $\nabla\{\mu^{-1}(x)\}f(x) = -\mu^{-2}\nabla\{\mu(x)\}f(x)$, we get (2).

From (iii_{T2}) , one gets

$$\nabla \cdot \{\mu(x)f(x)\} - \mu^2(x) \nabla \cdot \{\mu^{-1}(x)f(x)\} \leq -2\alpha\mu(x)$$

or in the form

$$\begin{aligned} & \nabla \cdot \{\mu(x)f(x)\} - \nabla \cdot \{f(x)\}\mu(x) \\ & \quad - \mu^2(x) \nabla \cdot \{\mu^{-1}(x)f(x)\} + \nabla \cdot \{f(x)\}\mu(x) \\ & \leq -2\alpha\mu(x). \end{aligned} \quad (11)$$

Taking into account $\nabla\{\mu(x)\}^T f(x) = -\mu^2(x)\nabla \cdot \{\mu^{-1}(x)f(x)\} + \nabla \cdot \{f(x)\}\mu(x)$ derived from (9), we get (2) from (11). Theorem 3 is proved.

Example 3: Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 - x_1x_2^2 - x_1^3, \\ \dot{x}_2 &= -x_1 - x_2 - x_2x_1^2 - x_2^3, \end{aligned} \quad (12)$$

with the equilibrium point $(0, 0)$. The phase portrait of (12) is shown in Fig. 2.

Investigate the equilibrium point of (12). According to Theorem 3, introduce $\mu(x)$ in the form (7). Calculate the following auxiliary expressions:

$$\nabla \cdot \{f(x)\} = -2 - 4x_1^2 - 4x_2^2, \quad (13)$$

$$\begin{aligned} \nabla \cdot \{\mu(x)f(x)\} &= -2\beta(x_1^2 + x_2^2)^{\beta-1} [x_1^2 + x_2^2 \\ & \quad + 2x_1^2x_2^2 + x_1^4 + x_2^4] \\ & \quad - 2(x_1^2 + x_2^2)^\beta (1 + 2x_1^2 + 2x_2^2), \end{aligned} \quad (14)$$

$$\begin{aligned} \nabla \cdot \{\mu^{-1}(x)f(x)\} &= (x_1^2 + x_2^2)^{-\beta} [-2 + 2\beta \\ & \quad + (2\beta - 4)(x_1^2 + x_2^2)]. \end{aligned} \quad (15)$$

Taking into account (14), verify the condition (i_{T2})

$$\nabla \cdot \{\mu(x)f(x)\} \leq -|x|^{2\beta} [2\beta - (-2 - 4x_1^2 - 4x_2^2)].$$

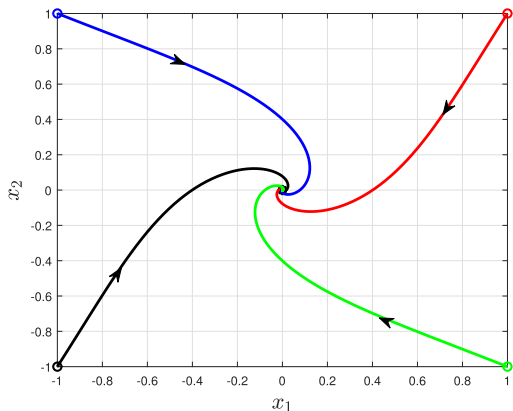


FIGURE 2. Phase portrait of (12).

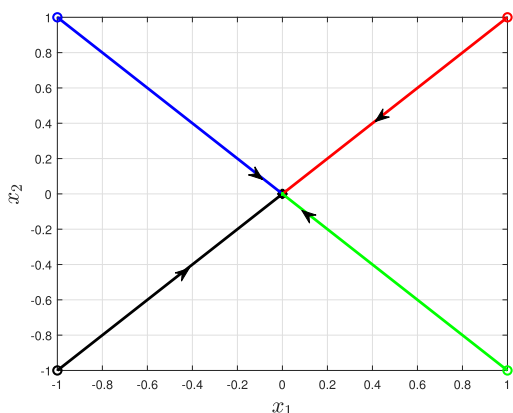


FIGURE 3. Phase portrait of (17).

Considering (13) and (15), verify the condition (ii_{T2})

$$\nabla \cdot \{f(x)\} < 0,$$

$$\nabla \cdot \{\mu^{-1}(x)f(x)\} \geq |x|^{-2\beta}[-2 + 2\beta] \text{ for } \beta \geq 2. \quad (16)$$

Using (14), one gets $\nabla \cdot \{\mu(x)f(x)\} \leq -2(\beta + 1)(x_1^2 + x_2^2)^\beta$.

Considering (16), the condition (iii_{T2}) holds.

Example 4: Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + 0.5 \sin(x_2), \\ \dot{x}_2 &= -x_2 + 0.5 \sin(x_1), \end{aligned} \quad (17)$$

with the equilibrium point $(0, 0)$. The phase portrait of (17) is shown in Fig. 3.

Introduce $\mu(x)$ as in (7). Calculate the following expressions

$$\nabla \cdot \{f(x)\} = -2, \quad (18)$$

$$\begin{aligned} \nabla \cdot \{\mu(x)f(x)\} &= -2\beta(x_1^2 + x_2^2)^{\beta-1} \\ &\quad \times [x_1 \sin(x_1) + x_2 \sin(x_2)] \\ &\quad - 2(\beta + 1)(x_1^2 + x_2^2)^\beta, \end{aligned} \quad (19)$$

$$\begin{aligned} \nabla \cdot \{\mu^{-1}(x)f(x)\} &= 2(\beta - 1)(x_1^2 + x_2^2)^{-\beta} \\ &\quad - \beta(x_1^2 + x_2^2)^{-\beta-1} \\ &\quad \times (x_1 \sin(x_1) + x_2 \sin(x_2)). \end{aligned} \quad (20)$$

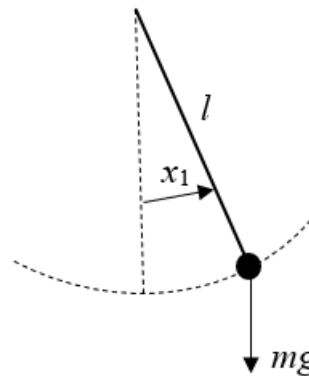


FIGURE 4. The pendulum.

Taking into account (19), verify the condition (i_{T2})

$$\begin{aligned} \nabla \cdot \{\mu(x)f(x)\} &\leq \beta(x_1^2 + x_2^2)^{\beta-1} [x_1^2 + x_2^2] \\ &\quad - 2(\beta + 1)(x_1^2 + x_2^2)^\beta = -(\beta + 2)(x_1^2 + x_2^2)^\beta. \end{aligned} \quad (21)$$

Considering (18) and (20), verify the condition (ii_{T2})

$$\begin{aligned} \nabla \cdot \{\mu^{-1}(x)f(x)\} &\geq 2(\beta - 1)(x_1^2 + x_2^2)^{-\beta} \\ &\quad - 0.5\beta(x_1^2 + x_2^2)^{-\beta-1}(x_1^2 + x_2^2) \\ &= (1.5\beta - 2)(x_1^2 + x_2^2)^{-\beta} \geq 0 \text{ for } \beta \geq 2. \end{aligned} \quad (22)$$

Using (21) and (22), the condition (iii_{T2}) holds.

Example 5: Consider the pendulum (see Fig. 4 [28]) which is described by the following equation

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{g}{l} \sin(x_1) - \frac{k}{m}x_2, \end{aligned} \quad (23)$$

where x_1 is the deflection angle of the rod from the vertical axis, x_2 is the angular velocity, g is the acceleration of gravity, l is the rod length, k is the friction coefficient, m is the cargo mass. Let us investigate the equilibrium point $(0, 0)$. The phase portrait of (23) in the vicinity of $(0, 0)$ is shown in Fig. 5 for $\frac{g}{l} = 1$ and $\frac{k}{m} = 1$.

Denote by $a = \frac{g}{l}$ and $b = \frac{k}{m}$. Introduce $\mu(x)$ as follows

$$\mu(x) = a[1 - \cos(x_1)] + 0.5x_2^2.$$

Calculate the following expressions:

$$\nabla \cdot \{f(x)\} = -b, \quad (24)$$

$$\begin{aligned} \nabla \cdot \{\mu(x)f(x)\} &= -b(x_2^2 + a[1 - \cos(x_1)] \\ &\quad + 0.5x_2^2), \end{aligned} \quad (25)$$

$$\nabla \cdot \{\mu^{-1}(x)f(x)\} = \frac{b(0.5x_2^2 - a[1 - \cos(x_1)])}{(a[1 - \cos(x_1)] + 0.5x_2^2)^2}. \quad (26)$$

Taking into account (25), verify the condition (i_{T2})

$$\nabla \cdot \{\mu(x)f(x)\} \leq -b[a[1 - \cos(x_1)] + 0.5x_2^2].$$

According with (i_{T2}) , we have $\alpha = 0$. Following [14]–[17], the equilibrium point $(0, 0)$ is an asymptotically stable.

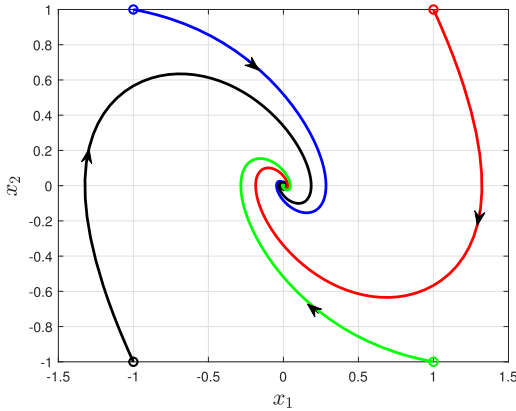


FIGURE 5. Phase portrait of (23) in the vicinity of the point (0, 0).

Considering (24) and (26), verify condition (ii_{T2})

$$\nabla \cdot \{\mu^{-1}(x)f(x)\} \geq \frac{b}{a[1 - \cos(x_1)] + 0.5x_2^2}.$$

According with (ii_{T2}), one gets $\alpha = b$. Therefore, the equilibrium point (0, 0) is an exponentially stable. Also, following to (iii_{T2}), the equilibrium point (0, 0) is an exponentially stable for $\alpha = b$.

Consider Lyapunov function $V = a[1 - \cos(x_1)] + 0.5x_2^2$ in the form of total system energy [28]. Taking time derivative of V along the trajectories of (23), one gets $\dot{V} = -abx_2^2$ [28]. Thus, only asymptotic stability of the equilibrium point (0, 0) can be established by using Lyapunov method for the same V and μ .

IV. STABILITY OF LINEAR SYSTEMS

Consider the linear system

$$\dot{x} = Ax, \tag{27}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$. Introduce $\mu(x) = (x^T Q x)^\beta$ with $Q = Q^T > 0$ and $\beta \geq 1$. Calculate the following relations

$$\begin{aligned} & \nabla \cdot \{\mu(x)f(x)\} \\ &= \beta(x^T Q x)^{\beta-1} x^T [QA + A^T Q + \frac{\text{trace}(A)}{\beta} Q] x, \\ & \nabla \cdot \{\mu^{-1}(x)f(x)\} \\ &= -\beta(x^T Q x)^{-\beta-1} \\ & \quad \times x^T \left[QA + A^T Q - \frac{\text{trace}(A)}{\beta} Q \right] x. \end{aligned} \tag{28}$$

According to condition (i_{T2}) and (28), the linear system (27) is an exponentially stable if $\nabla \cdot \{\mu(x)f(x)\} + \mu(x)[\alpha - \nabla \cdot \{f(x)\}] = \beta(x^T Q x)^{\beta-1} x^T [QA + A^T Q + \frac{\alpha}{\beta} Q] x \leq 0$ or

$$QA + A^T Q + \frac{\alpha}{\beta} Q \leq 0. \tag{29}$$

The inequality (29) is the well-known linear matrix inequality for exponential stability of linear system (27) [29].

According to condition (ii_{T2}) and (28), the linear system is exponential stable if $\nabla \cdot \{\mu^{-1}(x)f(x)\} - \alpha\mu^{-1}(x) = \beta(x^T Q x)^{\beta-1} x^T [QA + A^T Q + \frac{\alpha}{\beta} Q] x \geq 0$ or

$$\begin{aligned} QA + A^T Q + \frac{\alpha - \text{trace}(A)}{\beta} Q &\leq 0, \\ \text{trace}(A) &\leq 0. \end{aligned} \tag{30}$$

The result (30) is similar to (29).

According to condition (iii_{T2}) and (28), the linear system is exponential stable if $\nabla \cdot \{\mu(x)f(x)\} + \alpha\mu(x) = \beta(x^T Q x)^{\beta-1} x^T [QA + A^T Q + \frac{2\alpha}{\beta} Q] x \leq 0$ or

$$\begin{aligned} QA + A^T Q + \frac{2\alpha}{\beta} Q &\leq 0, \\ QA + A^T Q + \frac{\alpha - \text{trace}(A)}{\beta} Q &\leq 0, \\ \text{trace}(A) &\leq 0. \end{aligned} \tag{31}$$

The results (31) are similar to (29).

V. CONTROL LAW DESIGN

Consider a dynamical system in the form

$$\dot{x} = \psi(x, u), \tag{32}$$

where $x \in D \subset \mathbb{R}^n$, $u(x) \in \mathbb{R}^m$ is the control signal, $\psi(x, u) = \xi(x) + g(x)u(x)$, the functions $\xi(x)$, $g(x)$ and $u(x)$ are continuously differentiable in D , $\psi(0, u(0)) = 0$ and the system (32) is stabilizable in D .

Theorem 4: Let $\mu(x)$ be a positive definite continuously differentiable function in $x \in D$. The closed-loop system is an exponentially stable if the control law $u(x)$ is chosen such that one of the following conditions

$$\begin{aligned} (i_{T3}) \quad & \nabla \cdot \{\mu(x)\psi(x, u)\} \leq -\mu(x)[\alpha - \nabla \cdot \{\psi(x, u)\}]; \\ (ii_{T3}) \quad & \nabla \cdot \{\mu^{-1}(x)\psi(x, u)\} \geq \alpha\mu^{-1}(x) \quad \text{and} \\ & \nabla \cdot \{\psi(x, u)\} \leq 0; \\ (iii_{T3}) \quad & \nabla \cdot \{\mu(x)\psi(x, u)\} \leq -\alpha\mu(x) \quad \text{and} \\ & \nabla \cdot \{\mu^{-1}(x)\psi(x, u)\} \geq \alpha\mu^{-1}(x), \end{aligned}$$

holds for any $x \in D \setminus \{0\}$, where $\alpha > 0$.

Since the system (32) is stabilizable in D , then the proof of Theorem 4 is similar to the proof of Theorem 3 (denoting by $f(x) = \psi(x, u(x))$).

Example 6: Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 - x_1^3 - x_1 x_2^2, \\ \dot{x}_2 &= u. \end{aligned} \tag{33}$$

The phase trajectories under $u = 0$ are presented in Fig. 7. It is required to design the control law ensuring the exponential stability of the closed-loop system.

Choose $\mu(x)$ in the form (7). Find the following relations

$$\begin{aligned} & \nabla \cdot \{\mu(x)\psi(x, u)\} \\ &= 2\beta(x_1^2 + x_2^2)^{\beta-1} \\ & \quad \times (-x_1^2 + x_1 x_2 - x_1^4 - x_1^2 x_2^2 + x_2 u) \\ & \quad \times (x_1^2 + x_2^2)^\beta (-1 - 3x_1^2 - x_2^2 + \frac{\partial u}{\partial x_2}), \end{aligned}$$

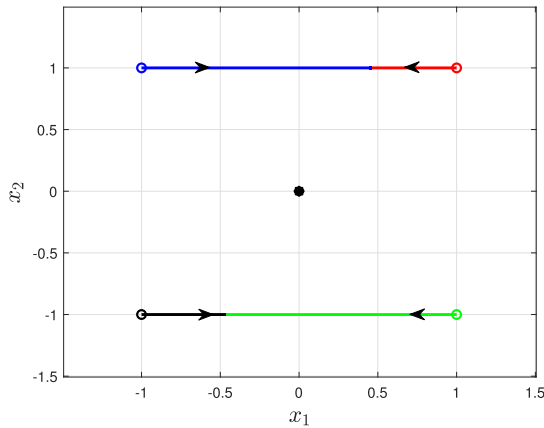


FIGURE 6. Phase portrait of (33).

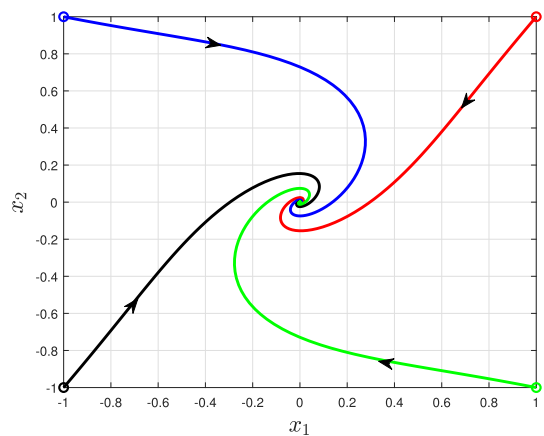


FIGURE 7. Phase portrait of the closed-loop system.

$$\begin{aligned} \nabla \cdot \{\mu^{-1}(x)\psi(x, u)\} &= \beta(x_1^2 + x_2^2)^{-\beta-1} \\ &\times [(x_1^2 + x_2^2)(-1 - 3x_1^2 - x_2^2 + \frac{\partial u}{\partial x_2}) \\ &+ 2\beta(x_1^2 - x_1x_2 + x_1^4 + x_1^2x_2^2 - x_2u)], \end{aligned} \quad (34)$$

Introducing

$$u = -x_1 - x_2 - x_2x_1^2 - x_2^3,$$

rewrite (34) as follows

$$\begin{aligned} \nabla \cdot \{\mu(x)\psi(x, u)\} &= -(x_1^2 + x_2^2)^\beta \\ &\times (2\beta + 2\beta(x_1^2 + x_2^2)) - (2 - 4x_1^2 - 4x_2^2) \\ &\leq -(x_1^2 + x_2^2)^\beta \\ &\times (2\beta - (-2 - 4x_1^2 - 4x_2^2)), \\ \nabla \cdot \{\mu^{-1}(x)\psi(x, u)\} &= (x_1^2 + x_2^2)^{-\beta} \\ &\times 2[\beta - 1 + (\beta - 2)x_1^2 + (\beta - 2)x_2^2] \\ &\geq 2[\beta - 1](x_1^2 + x_2^2)^{-\beta} \text{ for } \beta \geq 3. \end{aligned} \quad (35)$$

Thus, the conditions (iT_3) and (iiT_3) hold. Additionally estimating

$$\begin{aligned} \nabla \cdot \{\mu(x)\psi(x, u)\} &\leq -(x_1^2 + x_2^2)^\beta (2\beta - (-2 - 4x_1^2 - 4x_2^2)) \\ &\leq -2\beta(x_1^2 + x_2^2)^\beta, \end{aligned}$$

we have fulfilment of the condition $(iiiT_3)$.

VI. CONCLUSION

In this paper the recent divergence method [14]–[17] is used for investigation of an exponential stability of autonomous dynamical systems. The property of exponential stability makes it possible to increase system robustness to certain types of perturbations and uncertainties. New necessary and sufficient exponential stability conditions are derived from [14]–[17]. Examples illustrate the efficiency of the proposed method by the simulations.

The advantage of the proposed method is that the formulated necessary and sufficient conditions have the form of a continuity equation, which is widely used in mechanics and physics, as well as in some industrial tasks [30]. The difficulties are in the selection of the density function (auxiliary function $\mu(x)$) such that the obtained conditions hold. This problem is equivalent to the problem of finding the Lyapunov function.

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