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The $MAP/PH/N/\infty$ Queueing-Inventory System With Demands From a Random Environment

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ABSTRACT This study investigates a multi-server queueing-inventory system in a random environment. The system has an unlimited waiting space and an inventory capacity of S units. The customer arrives at the system according to the Markovian arrival process. Whenever all the servers are busy, an arriving customer either goes to an unlimited waiting space or leaves the system. And if there is a free server with positive inventory, then the arriving customer gets service immediately. Inventories are filled under the instantaneous replenishment policy. We consider that the service time follows a phase-type distribution, and we assess the joint probability distribution of the inventory level and the number of customers in the steady-state case. On top of that, we extract the sojourn time distributions of arbitrary customers using the Laplace-Stieltjes transform. Finally, a few numerical examples are provided to illustrate our mathematical model.

INDEX TERMS Infinite queue, Markovian arrival process, multi-server, phase-type distribution, random environment.

I. INTRODUCTION

The queueing-inventory model's applications are transportation systems, grocery shops, computer network systems, etc. Many researchers have worked in the queueing-inventory system for the past few decades. Sigman and Levi [18] and Melikov and Molchanov [12] put forward a discourse on the queueing-inventory system. Nowadays, the Markovian arrival process (MAP) is one of the most challenging models in the queueing-inventory system, which plays a significant part in the queueing-inventory system.

Neuts [14] was inaugurated as the MAP, many input flows are considered in the MAP class, such as stationary Poisson (M), Erlangian (E_k), hyper-Markovian (HM), phase-type (PH), and the Markov Modulated Poisson Process (MMPP). Generally speaking, the MAP is correlated, so it's ideal for modelling correlated and burst traffic in modern telecommunication networks. Day by day, the number of researchers attracted to MAP in the queueing-inventory system is increasing.

Chakravarthy [2] explored a single server queueing model with the customers arriving with MAP compliant with the FCFS rule. The customers may request an optional secondary service from the same server where they have received service immediately or wait until a predetermined threshold is reached by the number of customers waiting for such services. Krishnamoorthy *et al.* [10] examined a single server queue with several services. The system provides multiple services for each customer, one of which is required and the other undesired. The customers arrive under MAP, and in both undesired and required phases, the service time has a PH distribution. Finally,

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they analyse the stochastic decomposition of the system state. Some literatures for MAP are Suganya *et al.* [21], Nair and Jose [13], Punalal and Babu [15], Seokjun Lee *et al.* [16], and Valentina Klimenok *et al.* [20]. Chakravarthy and Khizer Hayat [1] addressed the two vendors in the queueing-inventory model and who's accountable for inventory replenishment with the (s, S) policy. Also, they consider the demands under MAP, lead time depending on the vendor, and service time according to PH distribution.

Random environment: The present state of the Markovian random process with a finite state space determines the system's behaviour. The procedure of this process under a fixed state is known as a "random environment."

The random environment is a very prevalent phenomenon that occurs in business administration. Chesoong Kim *et al.* [7] considered a multi-server retrial queueing model with the Batch Markovian Arrival Process, and service time follows the PH-type. The system has a random environment. Finally, they calculated the ergodicity condition and stable algorithms.

Chesoong Kim *et al.* [6] and [8] are a few references to literature on systems that operate in a random environment. Jeganathan *et al.* [22], Alexander Dudin *et al.* [4], and Chesoong Kim *et al.* [9] are some multi-server pieces of literature.

Jeganathan [5] considered a single server retrial inventory model with server interruptions, multiple vacations, and demands from a finite population of N sources. Stock products have exponential lifetimes, and the ordering policy $(0, S)$ is used. The total expected cost rate, as well as a variety of system performance measures, are determined. Dhanya Shajin *et al.* [17] considered an $M/M/1$ queue, in which the inventory has a common lifetime and is exponentially distributed. When the common lifetime is reached or the inventory level drops to zero for the first time in a cycle, whichever one first occurs a replenishment order is placed to restore the inventory level to S (zero lead time). Customers arrive at a Poisson process with an exponentially distributed service time. Before the realization of a common lifetime, reservations and cancellations of sold items were allowed, and they derived long-run performance measures and a revenue function.

The impulsion of our work comes from those results in the above survey. To the best of our knowledge, there is no research into demands from a random environment with a queueing-inventory model. In this article, we consider the arrival process as MAP and the service process as a phase-type distribution under the fixed state of a random environment. Whenever the states of a random environment are changed, the parameters of the arrival and service flow are changed as well. Furthermore, to avoid losing customers due to a lack of inventory, we use an instantaneous replenishment policy.

The perspective of this work is as follows: The model representation of the queueing - inventory model with the random environment, multi-server, and MAP is portrayed in section II. Section III portrayed the analysis of the model. The joint probability distribution of the number of customers in the unlimited waiting hall and the stock stage is studied in section IV. The stability condition is shown in subsection IV-A and we calculate the steady-state probability vector in section IV-B. We acquired some important system peculiarities in section V, and we constructed the cost function in subsection V-A. We describe the distribution of sojourn time for an arbitrary customer in section VI. Some numerical examples are provided in section VII and, finally, we give a conclusion in section VIII.

A. NOTATIONS AND ABBREVIATIONS

- $[A]_{ij}$: The element of submatrix at (i, j) the position of A
- 0 : Zero matrix
- e : A column vector of 1's appropriate dimension
- $A \otimes B$: Kronecker product of matrices A and B
- $A \oplus B$: Kronecker sum of matrices A and B
- MAP : Markovian Arrival Process
- PH : Phase-type
- LST : Laplace-Stieltjes Transform
- FCFS : First Come First Serve
- $\delta_{ij} = \begin{cases} 1, & \text{j} = i \\ 0 & otherwise. \end{cases}$
- $\bar{\delta}_{ij} = 1 - \delta_{ij}$

II. MODEL FORMULATION

This article deals with a multi-server (N) queueing-inventory system. The system has a maximum inventory capacity of S units and the demands appear from a random environment with R various modes, all of which need a single unit of the item. The behaviour of the system depends on the state of the random environment. The random environment is applied by the stochastic process $\{r_t, \ t \geq 0\}$, which is an irreducible continuous-time Markov chain with the state space $\{1, 2, \ldots, R\}$ and the infinitesimal generator H of the random environment. The customers arrive to the system follows the MAP and it is represented by $(D_0^{(r)}, D_1^{(r)})$, here $D_0^{(r)}$ and $D_1^{(r)}$ are of size $(M_1 \times M_1)$ under the fixed state of random environment $r (\in \{1, 2, \ldots, R\})$. The behaviour of the MAP under the fixed state r is completely characterized by the above matrices. The underlying Markov process $X_4(t)$ of the MAP has the generator, $D^{(r)} = (D_0^{(r)} + D_1^{(r)})$. Here,

$$[D_0^{(r)}]_{ij} = \begin{cases} \lambda_i^{(r)} & \text{j} = i \\ \lambda_i^{(r)} p_{ij}^{(r)} & otherwise \end{cases} \text{ and } [D_1^{(r)}]_{ij} = \lambda_i^{(r)} q_{ij}^{(r)}.$$

where,

- $\lambda_i^{(r)}$ = The mean rate of the underlying Markov chain stays in the state $i \in 1, 2, \ldots, M_1$ for an exponential interval of time.
- $p_{ij}^{(r)}$ = The probability of the underlying Markov chain changes state transition from state i to state j without generation of customers $(i, j \in 1, 2, \ldots, M_1)$.

- $q_{ij}^{(r)}$ = The probability of the underlying Markov chain state transition from state i to state j $(i, j \in 1, 2, \ldots, M_1)$ happens a customer arrives.
- $[D_0^{(r)}]_{ii}$ = Determined in such a way that $(D_0^{(r)} + D_1^{(r)})\mathbf{e} = \mathbf{0}$.

The stationary row vector $\eta^{(r)}$ of length M_1 is unique solution to the system $\eta^{(r)}D^{(r)} = \mathbf{0}$ and $\eta^{(r)}\mathbf{e} = 1$ under the fixed state r. The average arrival rate $\lambda^{(r)} = \eta^{(r)}D_1^{(r)}\mathbf{e}$ under fixed state r.

During a customer's arrival epoch, if there is a free server with positive inventory, the arriving customer gets service immediately. On the other hand, if all servers are busy, the system give an offer to the arriving customer. The customer may go to unlimited waiting space with probability $p^{(r)}$ or exit the system with probability $q^{(r)}(=1 - p^{(r)})$ under fixed state r. The stock list is stuffing according to instantaneous replenishment principle. The squared coefficient of variation $c_{var}^{(r)}$ of inter-arrival times of customers under the fixed state r is given as

$$c_{var}^{(r)} = 2\lambda^{(r)}\eta^{(r)}(-D_0^{(r)})^{-1}\mathbf{e} - 1 \tag{1}$$

The correlation coefficient $c_{cor}^{(r)}$ of inter-arrival times of customers under the fixed state r is given (see Chakravarthy [3]), as

$$c_{cor}^{(r)} = \frac{\lambda^{(r)}\eta^{(r)}(-D_0^{(r)})^{-1}D_1^{(r)}(-D_0^{(r)})^{-1}\mathbf{e} - 1}{2\lambda^{(r)}\eta^{(r)}(-D_0^{(r)})^{-1}\mathbf{e} - 1} \tag{2}$$

The service time of a customer has PH distribution with an irreducible representation $(\alpha^{(r)}, T^{(r)})$ under the fixed state r of order M_2. This service time can be interpreted as the time until the underlying Markov process $\{\mathbf{X}_5(t), t \geq 0\}$ with finite state space $1, 2, \ldots, M_2, M_2 + 1$ reaches the single absorbing state $M_2 + 1$, conditioned on the fact that the initial state of this process is selected from among the states $1, 2, \ldots, M_2$ according to initial probability vector $\alpha^{(r)} = (\alpha_1^{(r)}, \alpha_2^{(r)}, \ldots, \alpha_{M_2}^{(r)})$. The transition rates into the absorbing state are given as $\mathbf{T}_0^{(r)} = -T^{(r)}\mathbf{e}$. The mean service time of customer calculated as $\mu^{(r)} = \alpha^{(r)}(-T^{(r)})^{-1}\mathbf{e}$.

In this process, we can track the number of servers in each phase and it is denoted by $K_n = \begin{pmatrix} n + M_2 - 1 \\ M_2 - 1 \end{pmatrix}$, where $n (\in 0, 1, 2, \ldots, N)$ identical phase type servers and M_2 phases becomes eminently tractable.

The squared coefficient of variation of service times is given by

$$c_{service,var}^{(r)} = \frac{2\alpha^{(r)}(-T^{(r)})^{-2}\mathbf{e}}{(\alpha^{(r)}(-T^{(r)})^{-1}\mathbf{e})^2} - 1 \tag{3}$$

III. ANALYSIS

In this section, we construct the transition rate matrix on the queueing-inventory model. The Markov process of the form $\vartheta_t = \{(X_1(t), X_2(t), X_3(t), X_4(t), \mathbf{X}_5(t)), t \geq 0\}$, with state space $E = (n, r, i, m_1, x_5^{(1)}, x_5^{(2)}, \ldots, x_5^{(M_2)})$, where

$$n = 0, 1, 2, \ldots; \quad r = 1, \ldots, R; \ i = 1, \ldots, S;$$
$$m_1 = 1, 2, \ldots M_1, \quad x_5^{(m_2)} = 0, 1, \ldots, min\{n, N\}$$
$$m_2 = 1, 2, \ldots, M_2.$$

Here,

$X_1(t)$ – The number of customers in the system at time t.
$X_2(t)$ – The state of random environment at time t.
$X_3(t)$ – The number of items in the inventory at time t.
$X_4(t)$ – Phase of the customers arrival process at time t.
$\mathbf{X}_5(t) = \{X_5^{(1)}(t), X_5^{(2)}(t), \ldots, X_5^{(M_2)}(t)\}, \ t \geq 0,$
$X_5^{(m_2)}(t)$ – The number of servers at phase m_2 of service, which lies between 0 and $min\{n, N\}$ at time t.

where, $\sum_{m_2=1}^{M_2} X_5^{(m_2)}(t) = min\{n, N\}$.

Theorem 1: The infinitesimal generator U has the following block-tridiagonal structure U, as shown at the bottom of the next page. Nonzero blocks $A_{n_1 n_2}, n_1, n_2 \geq 0$, are of the following form:

- $A_{n(n+1)} = \text{diag}\{(B_n)_r, r = 1, 2, \ldots, R\}, \ n = 0, 1, 2, \ldots,$
  For $n = 0, 1, \ldots, N - 1; r = 1, 2, \ldots, R$
  $$(B_n)_r = I_S \otimes (D_1^{(r)} \otimes P_n(\alpha^{(r)}))$$
  For $n = N, N + 1, \ldots; \ r = 1, 2, \ldots, R.$
  $$(B_n)_r = I_S \otimes (p^{(r)}D_1^{(r)} \otimes I_{K_N}).$$
- $A_{n(n-1)} = \text{diag}\{(C_n)_r, r = 1, 2, \ldots, R\}, \ n = 1, 2, 3, \ldots,$
  For $n = 1, 2, \ldots, N; \ r = 1, 2, \ldots, R$
  $$(C_n)_r = G \otimes (I_{M_1} \otimes L_{N-n}(N, Q^{(r)})),$$
  $$Q^{(r)} = \begin{pmatrix} 0 & \mathbf{0} \\ T_0^{(r)} & T^{(r)} \end{pmatrix}, \quad r = 1, 2, \ldots, R$$
  For $n = N + 1, N + 2, \ldots; r = 1, 2, \ldots, R$
  $$(C_n)_r = G \otimes I_{M_1} \otimes L_0(N, Q^{(r)})P_{N-1}(\alpha^{(r)}),$$
  $$[G]_{ij} = \begin{cases} 1, & j = i - 1, \ i = 2, \ldots, S \\ 1, & j = S, \ i = 1 \\ 0, & otherwise \end{cases}$$
- $A_{nn} = \text{diag}\{I_S \otimes F_2, r = 1, 2, \ldots, R\} + \Delta_n + H \otimes I_{l_2}, \ n = 0, 1, 2, \ldots, N - 1$, where
  $$F_2 = D_0^{(r)} \oplus A_n(N, T^{(r)}),$$
  $$\Delta_n = -\text{diag}\{I_{l_1} \otimes \text{diag}\{A_n(N, T^{(r)})\mathbf{e} + L_{N-n}(N, Q^{(r)})\mathbf{e}\}, r = 1, 2, \ldots, R\},$$
  $$n = 1, 2, \ldots, N, \quad \Delta_0 = 0.$$
  $A_{nn} = \text{diag}\{I_S \otimes F_3, r = 1, 2, \ldots, R\} + \Delta_N + H \otimes I_{l_3}$, $n = N, N + 1, N + 2, \ldots,$ where
  $$F_3 = (D_0^{(r)} + q^{(r)}D_1^{(r)}) \oplus A_N(N, T^{(r)}),$$
  $$l_1 = SM_1, \quad l_2 = SM_1K_n, \ l_3 = SM_1K_N$$

Proof: The demonstration is carried out by examining all potential transitions of the Markov process $\{\vartheta_t, t \geq 0\}$ in an infinitely small time period and putting these rates into block matrix form.

Here,

- $P_n(\alpha^{(r)})$ - Transition probabilities of the process $\{\mathbf{X}_5(t), t \geq 0\}$, at the epoch of starting the new service given that n servers are busy at this epoch under fixed state r.
- $L_{N-n}(N, Q^{(r)})$ - Transitions of this process at the service completion epoch given that n servers are busy at this epoch under fixed state r.
- $A_n(N, T^{(r)})$ - Transitions of this process, which do not lead to the service completion given that n servers are busy under fixed state r.
- Δ_n - The total intensity of leaving the corresponding states of this process, given that n servers are busy under fixed state r.

where $\mathbf{X}_5(t) = \{X_5^{(1)}(t), X_5^{(2)}(t), \ldots, X_5^{(M_2)}(t))\}, t \geq 0$.

The matrices $P_n(\alpha^{(r)}), L_{N-n}(N, Q^{(r)}), A_n(N, T^{(r)})$ are introduced in Ramasami [24] and Ramaswami and Lucantoni [11].

The output rate from the corresponding state is specified by the absolute value of each diagonal element of A_{nn} which is negative. The rates of the diagonal transition are represented by the diagonal elements of the matrices.

- For $n = 0$
$$-diag\{I_S \otimes D_0^{(r)} \oplus A_n(N, T^{(r)}), r = 1, 2, \ldots, R\} - H \otimes I_{l_2}.$$
- For $n = 1, 2, \ldots, N-1$
$$-diag\{I_S \otimes D_0^{(r)} \oplus A_n(N, T^{(r)}), r = 1, 2, \ldots, R\} + diag\{I_{l_1} \otimes diag\{A_n(N, T^{(r)})\mathbf{e} + L_{N-n}(N, Q^{(r)})\mathbf{e}\}, r = 1, 2, \ldots, R\} - H \otimes I_{l_2}$$
- $n = N, N+1, N+2, \ldots$
$$-diag\{I_S \otimes (D_0^{(r)} + q^{(r)} D_1^{(r)}) \oplus A_N(N, T^{(r)}), r = 1, 2, \ldots, R\} + \mathrm{diag}\{I_{l_1} \otimes \mathrm{diag}\{A_n(N, T^{(r)})\mathbf{e} + L_{N-n}(N, Q^{(r)})\mathbf{e}\}, r = 1, 2, \ldots, R\} - H \otimes I_{l_3}$$

Elements of blocks A_{nn+1}, $n \geq 0$ indicate the rates of the transitions that result in raising the number of customers in the system by one. The following are the transitions:

- When the number of available servers is less than N, MAP control process transitions are followed by the customer's arrival, and the server initiates the service for the next customer.
For $n = 0, 1, 2, \ldots, N-1$
$$diag\{I_S \otimes (D_1^{(r)} \otimes P_n(\alpha^{(r)})), r = 1, 2, \ldots, R\},$$
- When all the N servers are busy, the arriving customer choose to wait in the infinite size waiting hall. The rates of these occurrences are represented by
For $n = N, N+1, \ldots$
$$diag\{I_S \otimes (p^{(r)} D_1^{(r)} \otimes I_{K_N}), \ r = 1, 2, \ldots, R\},$$

The rates of transitions that result in a one-to-one reduction in the number of customers in the system, are specified by the elements of block A_{nn-1}, $n \geq 1$. The following are the transitions:

- Customer service is completed when the number of available servers is less than N, and the rates of such events are determined by matrix elements.
For $n = 1, 2, \ldots, N$
$$diag\{G \otimes (I_{M_1} \otimes L_{N-n}(N, Q^{(r)})), \ r = 1, 2, \ldots, R\},$$
where
$$[G]_{ij} = \begin{cases} 1, & j = i-1, \ i = 2, \ldots, S \\ 1, & j = i+S-1, \ i = 1 \\ 0, & otherwise \end{cases}$$
- Customer service is completed and initiates the service for the next customer when all N servers are busy and the rates of such events are determined by matrix elements.

$$U = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ \vdots \\ N \\ N+1 \\ N+2 \\ \vdots \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & \ldots & N-1 & N & N+1 & N+2 & \ldots \end{array} \\ \begin{pmatrix} A_{00} & A_{01} & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots \\ A_{10} & A_{11} & A_{12} & \ldots & 0 & 0 & 0 & 0 & \ldots \\ 0 & A_{21} & A_{22} & \ddots & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & A_{N(N-1)} & A_{NN} & A_{N(N+1)} & 0 & \ldots \\ 0 & 0 & 0 & \ldots & 0 & A_{(N+1)N} & A_{(N+1)(N+1)} & A_{N(N+1)} & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 & A_{(N+1)N} & A_{(N+1)(N+1)} & \ldots \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \end{array}$$

For $n = N+1, N+2, \ldots$

$$diag\{G \otimes (I_{M_1} \otimes L_0(N, Q^{(r)})P_{N-1}(\alpha^{(r)})), \quad r = 1, 2, \ldots, R\}.$$

IV. JOINT PROBABILITY DISTRIBUTION UNDER STEADY-STATE

In this section, we calculate the steady-state probability vector and ergodicity criterion. Let us consider $\mathcal{A} = \hat{A}_1 + \hat{A}_0 + \hat{A}_2$, where $\hat{A}_1 = A_{n(n-1)}$, $\hat{A}_0 = A_{nn}$, and $\hat{A}_2 = A_{n(n+1)}, n \geq N+1$.

Let $\mathfrak{Y}$ be the steady-state probability vector of the matrix $\mathcal{A}$ such that $\mathfrak{Y}\mathcal{A} = \mathbf{0}$, $\mathfrak{Y}\mathbf{e} = 1$ yields

$$\mathfrak{Y} = (\mathfrak{Y}_1, \mathfrak{Y}_2, \ldots, \mathfrak{Y}_R),$$

where

$$\mathfrak{Y}_r = (\mathfrak{Y}_{(r,1)}, \mathfrak{Y}_{(r,2)}, \ldots, \mathfrak{Y}_{(r,S)}), \quad 1 \leq r \leq R.$$

A. STABILITY CONDITION

We give the condition for stability through theorem on the queueing-inventory system.

Theorem 2: The queueing-inventory system under study is stable if and only if

$$\sum_{r=1}^{R}\sum_{i=1}^{S}\mathfrak{Y}_{(r,s)}(p^{(r)}D_1^{(r)} \otimes I_{K_N})\mathbf{e} < \sum_{r=1}^{R}\sum_{i=1}^{S}\mathfrak{Y}_{(r,s)}(I_{M_1} \otimes L_0(N, Q^{(r)})P_{N-1}(\alpha^{(r)})\mathbf{e} \quad (4)$$

Proof: From the standard results of Neuts [14] on the positive recurrence of U we have

$$\mathfrak{Y}\hat{A}_2\mathbf{e} < \mathfrak{Y}\hat{A}_1\mathbf{e}$$

and by applying the structure of the matrices $\hat{A}_2$ and $\hat{A}_1$ and $\mathfrak{Y}$ the declared results follows.

It can be seen from the structure of the rate matrix U and from the theorem (2), that the Markov process $\{\vartheta_t,\ t \geq 1\}$ with the state space E is regular (refer Ross [19]).

B. STEADY-STATE PROBABILITY VECTOR

We derive the steady-state probability vector $\mathbf{x}$ and also calculate rate matrix $\mathcal{R}$. The steady-state probability vector $\mathbf{x}$ of the generator U such that $\mathbf{x}U = \mathbf{0},\ \mathbf{x}\mathbf{e} = 1$. Segregation for $\mathbf{x}$ yields

$$\mathbf{x} = (\mathbf{x}_0,\ \mathbf{x}_1,\ \mathbf{x}_2, \ldots),$$

where

- $\mathbf{x}_n = (\mathbf{x}_{(n,1)}, \mathbf{x}_{(n,2)}, \ldots, \mathbf{x}_{(n,R)}),\ n \geq 0$
- $\mathbf{x}_{(n,r)} = (\mathbf{x}_{(n,r,1)}, \mathbf{x}_{(n,r,2)}, \ldots, \mathbf{x}_{(n,r,S)}),\ n \geq 0,\ 1 \leq r \leq R$
- $\mathbf{x}_{(n,r,i)} = (\mathbf{x}_{(n,r,i,1)}, \mathbf{x}_{(n,r,i,2)}, \ldots, \mathbf{x}_{(n,r,i,M_1)}), n \geq 0,\ 1 \leq r \leq R,\ 1 \leq i \leq S$

The vectors $\mathbf{x}_n,\ 0 \leq n \leq N$ are obtained by solving the following system of equations:

$$\mathbf{x}_0A_{00} + \mathbf{x}_1A_{10} = \mathbf{0} \quad (5)$$

$$\mathbf{x}_{n-1}A_{(n-1)n} + \mathbf{x}_nA_{nn} + \mathbf{x}_{n+1}A_{(n+1)n} = \mathbf{0}, \quad 1 \leq n \leq N-1 \quad (6)$$

$$\mathbf{x}_{n-1}A_{(n-1)n} + \mathbf{x}_nA_{nn} + \mathbf{x}_{n+1}\hat{A}_1 = \mathbf{0}, \quad n = N \quad (7)$$

$$\mathbf{x}_{n-1}\hat{A}_2 + \mathbf{x}_n\hat{A}_0 + \mathbf{x}_{n+1}\hat{A}_1 = \mathbf{0}, \quad n \geq N+1 \quad (8)$$

The vectors $\mathbf{x}_n, n \geq N+1$, are obtained by the matrix geometric relation:

$$\mathbf{x}_n = \mathbf{x}_N\mathcal{R}^{n-N}, \quad n \geq N+1 \quad (9)$$

Insert (9) in (8), then the equation (8) becomes the quadratic equation:

$$\mathcal{R}^2\hat{A}_1 + \mathcal{R}\hat{A}_0 + \hat{A}_2 = \mathbf{0}, \quad (10)$$

where $\mathcal{R}$ is minimal non-negative solution, it is defined by $\mathcal{R}$, as shown at the bottom of the next page. The $\mathcal{R}$ matrix can be computed using the logarithmic reduction algorithm (refer Latouche and Ramaswami [23]).

Equation (7) can be written as

$$\mathbf{x}_{N-1}A_{(N-1)N} + \mathbf{x}_N(A_{NN} + \mathcal{R}\hat{A}_1) = \mathbf{0} \quad (11)$$

Equation (9) in normalizing condition as

$$\sum_{n=1}^{N-1}\mathbf{x_n}\mathbf{e} + \mathbf{x_N}(I - \mathcal{R})^{-1}\mathbf{e} = 1, \quad (12)$$

where

$$\mathbf{x}_n = \mathbf{x}_0\prod_{n_1=1}^{n}\mathcal{Z}_{n_1}, \quad 1 \leq n \leq N \quad (13)$$

with

$$\mathcal{Z}_{n_1} = \begin{cases} -A_{(n_1-1)n_1}(A_{n_1n_1} + \mathcal{Z}_{(n_1+1)n_1})^{-1}, & 1 \leq n_1 \leq N-1 \\ -A_{(N-1)N}(\hat{A}_0 + \mathcal{R}\hat{A}_1)^{-1}, & \mathrm{n}_1 = N \end{cases} \quad (14)$$

V. SOME IMPORTANT SYSTEM PECULIARITIES

In this section, we acquire some important peculiarities criteria of queueing-inventory system in the steady-state.

1) Expected number of customers in the system is

$$E_S = \sum_{n=1}^{N}n\mathbf{x}_n\mathbf{e} + N\mathbf{x_N}\mathcal{R}(I-\mathcal{R})^{-1}\mathbf{e} + \mathbf{x}_N\mathcal{R}(I-\mathcal{R})^{-2}\mathbf{e}.$$

2) Expected number of customers in the waiting hall is

$$E_{WH} = \mathbf{x}_N\mathcal{R}(I-\mathcal{R})^{-2}\mathbf{e}.$$

3) Expected number of busy servers is

$$E_{BS} = \sum_{n=1}^{N-1}n\mathbf{x}_n\mathbf{e} + \sum_{n=N}^{\infty}N\mathbf{x}_n\mathbf{e}.$$

4) Expected loss rate is

$$E_L = \sum_{n=N}^{\infty}\sum_{r=1}^{R}\sum_{i=1}^{S}\mathbf{x}_{(n,r,i)}(q^{(r)}D_1^{(r)} \otimes I_{K_N})\mathbf{e}.$$

5) Expected number of items in the inventory is

$$E_I = \sum_{n=0}^{\infty}\sum_{r=1}^{R}\sum_{i=1}^{S} i\mathbf{x}_{(n,r,i)}\mathbf{e}.$$

6) Expected reorder rate is

$$E_R = \sum_{n=1}^{N}\sum_{r=1}^{R}\mathbf{x}_{(n,r,1)}(I_{M_1}\otimes L_{N-n}(N,Q^{(r)}))\mathbf{e} + \sum_{n=N+1}^{\infty}\sum_{r=1}^{R}\mathbf{x}_{(n,r,1)}(I_{M_1}\otimes L_0(N,Q^{(r)}) \times P_{N-1}(\alpha^{(r)})\mathbf{e}.$$

7) The intensity of out flow of customer, which get the service in the system, is determined as

$$\lambda_{service} = \sum_{n=1}^{N}\mathbf{x}_n L_{N-n}\mathbf{e} + \sum_{n=N+1}^{\infty}\mathbf{x}_n \mathrm{diag}\{I_S\otimes I_{M_1} \otimes L_0(N,Q^{(r)}),\ r=1,2,\ldots,R\}\mathbf{e},$$

If all the N servers are busy, then the arriving customer has a choice to make whether to join the infinite size waiting hall or leave the system. The leaving customer is considered lost.

8) The loss probability of an arbitrary customers is

$$P_{loss} = 1-\frac{\lambda_{service}}{\lambda},$$

where the average total arrival rate λ is given as $\lambda = \eta\mathrm{diag}\{D_1^{(r)}, r = 1,2,\ldots,R\}\mathbf{e}$ and the vector η is the unique solution to the following system of equation: $\eta(H\otimes I_{M_1} + \mathrm{diag}\{D_0^{(r)}+D_1^{(r)}, r=1,2,\ldots,R\}) = \mathbf{0}$, $\eta\mathbf{e} = 1$.

A. CONSTRUCTION OF THE COST FUNCTION

The expected total cost function per unit time is given by

$$C(S,N) = c_{sc}E_R + c_hE_I + c_sE_S + c_bE_{BS} + c_lE_L,$$

where

c_{sc} : Setup cost per order.
c_h : The inventory carrying cost per unit time.
c_s : Waiting cost of a customer in the system per unit time.
c_b : The service cost for each server per unit time.
c_l : The loss cost of a customer per unit time.

VI. SOJOURN TIME ANALYSIS

In this section, we determine the sojourn time distribution for an arbitrary customer's in the system.

Theorem 3: The LST $g(s)$ of the distribution of an arbitrary customer's sojourn time in the system is

$$g(s) = P_{loss} + \frac{1}{\lambda}\{\sum_{n=0}^{N-1}\sum_{r=1}^{R}\sum_{i=1}^{S}\mathbf{x}(n,r,i)(D^{(r)}\otimes I_{K_n}) \times \mathbf{e}\alpha^{(r)}\mathbf{f}(s,r) + \sum_{n=N}^{\infty}\sum_{r=1}^{R}\sum_{i=1}^{S}(1-q^{(r)})\mathbf{x}(n,r,i) \times (D_1^{(r)}\mathbf{e}\otimes I_{K_N})\mathbf{w}(s,n-N,r)\}, \tag{15}$$

where $\mathbf{f}(s,r) = (f(s,r,1), f(s,r,2),\ldots,f(s,r,M_2))^T$, $r = 1,2,\ldots,R$

Proof: Let $g(s) = \int_0^{\infty} e^{-sx}d\mathbf{G}(x)$, $Re\, s > 0$, be the LST of an arbitrary customer's sojourn time distribution, where $\mathbf{G}(x)$ is distribution function of a system customer's sojourn time.

$g(s)$ denotes the probability that no catastrophe occurs during the customer's stay. The proof is based on the claim 1 and claim 2 as well as the rules of law of total probability, a probabilistic sense of the LSTs.

Claim 1: The formula $\mathbf{f}(s) = (-\mathfrak{T} - H\otimes I_{M_2} + sI)^{-1}\mathfrak{T}_o$ produces the vector $\mathbf{f}(s)$, where $\mathfrak{T} = \mathrm{diag}\{T^{(r)}, r = 1,2,\ldots,R\}$ *and* $\mathfrak{T}_o = ((\mathbf{T_0}^{(1)})^T,\ldots,(\mathbf{T_0}^{(R)})^T)^T$.

Proof of Claim 1: Let $f(s,r,m_2)$ be the probability that a catastrophe will not arrive during the rest of the customer's service time in the system, the position of the customer in the system is n, $n\geq 1$ To calculate the unknown vector $\mathbf{f}(s)$ using a probabilistic understanding of the LST and the law of total probability:

$$f(s,r,m_2) = (-(\mathfrak{T}^{(r)})_{m_2,m_2} - H_{r,r} + s)^{-1} \times ((\mathfrak{T}_0^{(r)})_{m_2} + \sum_{r'=1,r'\neq r}^{R} H_{r,r'}f(s,r',m_2) + \sum_{m_2'=1,m_2'\neq m_2}^{M_2}(\mathfrak{T}^{(r)})_{m_2,m_2'}f(s,r,m_2')),$$
$$r = 1,2,\ldots,R, \quad m_2 = 1,2,\ldots,M_2. \tag{16}$$

The system (16) can be rewritten with the help of $\mathfrak{T}$, $\mathfrak{T}_0$ in matrix form as

$$(\mathfrak{T} + H\otimes I_{M_2} - sI)\mathbf{f}(s) = -\mathfrak{T}_o, \tag{17}$$

$$\mathcal{R} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ R-1 \\ R \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & \ldots & R-1 & R \end{array} \\ \begin{pmatrix} \mathcal{R}_{(1,1)} & \mathcal{R}_{(1,2)} & \ldots & \mathcal{R}_{(1,R-1)} & \mathcal{R}_{(1,R)} \\ \mathcal{R}_{(2,1)} & \mathcal{R}_{(2,2)} & \ldots & \mathcal{R}_{(2,R-1)} & \mathcal{R}_{(2,R)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{R}_{(R-1,1)} & \mathcal{R}_{(R-1,2)} & \ldots & \mathcal{R}_{(R-1,R-1)} & \mathcal{R}_{(R-1,R)} \\ \mathcal{R}_{(R,1)} & \mathcal{R}_{(R,2)} & \ldots & \mathcal{R}_{(R,R-1)} & (\mathcal{R}_{(R,R)} \end{pmatrix} \end{array}$$

because the matrix $\mathfrak{T} + H \otimes I_{M_2}$ represents a subgenerator, the matrix $(\mathfrak{T}+H\otimes I_{M_2}-sI)^{-1}$ exists, the assertion of lemma follows instantly.

Claim 2: The vector $\mathbf{w}(s, n)$ can be calculated using the recursion shown below:

$$\mathbf{w}(s,1) = (-\mathfrak{A}-H \otimes I_{K_N} + sI)^{-1} \times (\mathfrak{L}(I_R \otimes \mathbf{e}_{K_N})\alpha \mathbf{f}(s)), \tag{18}$$

$$\mathbf{w}(s,n+1) = (-\mathfrak{A}-H \otimes I_{K_N} + sI)^{-1} \times (\mathfrak{L}\mathbf{w}(s,n)), \quad n \geq 1, \tag{19}$$

where $\mathfrak{A} = \operatorname{diag}\{A_N(N, T^{(r)}) - \operatorname{diag}\{A_N(N, T^{(r)})\mathbf{e} + L_0(N, Q^{(r)})\mathbf{e}\}, r = 1,\dots,R\}$, $\alpha = \operatorname{diag}\{\alpha^{(r)}, r = 1,2,\dots,R\}$, $\mathbf{f}(s) = (\mathbf{f}(s,1), \mathbf{f}(s,2),\dots,\mathbf{f}(s,R))^T$, $\mathbf{w}(s,n) = (\mathbf{w}^T(s,n,1), \mathbf{w}^T(s,n,2),\dots,\mathbf{w}^T(s,n,R))^T, n \geq 1.$ and $\mathfrak{L} = \operatorname{diag}\{L_0(N, Q^{(r)})P_{N-1}(\alpha^{(r)})\}, r = 1,\dots,R\}$

Proof of Claim 2: Let $w(s, n, r, i, m_1, x_5^{(1)}, x_5^{(2)}, \dots, x_5^{(M_2)}), n \geq 1, r = 1,2,\dots,R, m_1 = 1,2,\dots,M_1, x_5^{(m_2)} = 0,1,\dots,min\{n,N\}, m_2 = 1,2,\dots,M_2$ be the probability that a catastrophe will not arrive during the rest of the customer's sojourn time in the system.

Enumerate the probabilities $w(s, n, r, i, m_1, x_5^{(1)}, x_5^{(2)}, \dots, x_5^{(M_2)})$ in the lexicographic order of the components $x_5^{(1)}, x_5^{(2)}, \dots, x_5^{(M_2)}$ and build column vectors $\mathbf{w}(s, n, r)$ from these probabilities.

To calculate the unknown vector $\mathbf{w}(s,n,r), n \geq 1.r = 1,2,\dots,R$ using a probabilistic understanding of the LST and the law of total probability:

$$\begin{aligned}\mathbf{w}(s,n,r) &= ((s-H_{r,r})I_{K_N} - A^{(r)})^{-1} \times (\bar{\delta}_{n1}L_0(N, Q^{(r)}) \\ &\times \mathbf{e}\alpha^{(r)}\mathbf{f}(s,r) + (1-\bar{\delta}_{n1}L^{(r)} + \mathbf{w}(s,n-1,r) \\ &+ \sum_{r'=1, r'\neq r}^{R} H_{r,r'}\mathbf{w}(s,n,r')),\end{aligned} \tag{20}$$

the system (20) can be rewritten in the form:

$$\begin{aligned}&(\mathfrak{A} - sI + H \otimes I_{T_N})\mathbf{w}(s,n) + \bar{\delta}_{n1}\mathfrak{L}\mathbf{e}\alpha\mathbf{f}(s) \\ &\quad + (1-\bar{\delta}_{n1})(\mathfrak{L}\mathbf{w}(s,n-1)) = \mathbf{0}^T, \quad n \geq 1,\end{aligned} \tag{21}$$

let us present the column vector $\mathbf{w}(s, n), n \geq 1$ in I-A. The matrix $\mathfrak{A} + H \otimes I_{K_N}$ represents a subgenerator, the matrix $(\mathfrak{A} + H \otimes I_{K_N} - sI)^{-1}$ exists, the assertion of lemma follows instantly.

Corollary 1:

1) The average sojourn time $\mathbf{G}_{soj}$ of an arbitrary customer is

$$\begin{aligned}\mathbf{G_{soj}} &= -g'(s)|_{s=0} \\ &= -\frac{1}{\lambda}\Big(\sum_{n=0}^{N-1}\sum_{r=1}^{R}\sum_{i=1}^{S}\mathbf{x}(n,r,i)(D_1^{(r)} \otimes I_{K_n}) \\ &\times \mathbf{e}\alpha^{(r)}\frac{\partial \mathbf{f}(s,r)}{\partial s}|_{s=0} + \sum_{n=N}^{\infty}\sum_{r=1}^{R}\sum_{i=1}^{S}(1-q^{(r)})x \\ &\times (n,r,i)(D_1^{(r)}\mathbf{e} \otimes I_{K_N})\frac{\partial \mathbf{w}(s,n-N,r)}{\partial s}|_{s=0}).\end{aligned} \tag{22}$$

Here the vectors

$$\frac{\partial \mathbf{w}(s,n,r)}{\partial s}|_{s=0}, \quad n \geq 1, \; r = 1,2,\dots,R.$$

As sub-vectors of the vectors, there are computed.

$$\frac{\partial \mathbf{w}(s,1)}{\partial s}|_{s=0} = (\mathfrak{A} + H \otimes I_{K_N})^{-1} \times (\mathbf{e} - (\mathfrak{L}(I_R \otimes \mathbf{e_{K_N}})\alpha\frac{d\mathbf{f}(s)}{ds}|_{s=0})), \tag{23}$$

$$\frac{\partial \mathbf{w}(s,n+1)}{\partial s}|_{s=0} = (\mathfrak{A} + H \otimes I_{K_N})^{-1} \times (\mathbf{e} - (\mathfrak{L}\frac{\partial \mathbf{w}(s,n)}{ds}|_{s=0})), \quad n \geq 1, \tag{24}$$

and the values $\frac{\partial \mathbf{f}(s,r)}{\partial s}|_{s=0}$ are determined as the sub-vectors of the vectors

$$\frac{d\mathbf{f}(s)}{ds}|_{s=0} = (\mathfrak{T} + H \otimes I_{M_2})^{-1}\mathbf{e}. \tag{25}$$

2) The formula determines an arbitrary customer's average waiting time $\mathbf{G}_{wait}$.

$$\begin{aligned}\mathbf{G}_{wait} &= -\frac{1}{\lambda}\sum_{n=N}^{\infty}\sum_{r=1}^{R}\sum_{i=1}^{S}(1-q^{(r)}) \times \mathbf{x}(n,r,i) \\ &\times (D_1^{(r)}\mathbf{e} \otimes I_{K_N})\frac{\partial \mathbf{z}(s,n+1,r)}{\partial s}|_{s=0},\end{aligned} \tag{26}$$

where the values

$$\frac{\partial \mathbf{z}(s,n,r)}{\partial s}|_{s=0}, \quad n \geq 1, \; r = 1,2,\dots,R.$$

are calculated as the entries of the vector

$$\frac{\partial \mathbf{z}(s,1)}{\partial s}|_{s=0} = (\mathfrak{A} + H \otimes I_{K_N})^{-1}\mathbf{e}, \tag{27}$$

TABLE 1. The function of total cast rate with two variable C(S, N), when $R = 1$.

S\N	8	9	10	11	12
12	286.34851	276.30994	24.87577	259.28332	337.65203
13	**285.99396**	**276.01869**	**11.47978**	**259.12985**	335.44483
14	286.17320	276.27907	16.21048	259.45466	333.68530
15	286.79926	276.99503	19.18256	260.18759	332.34298
16	287.80950	278.09883	21.35109	261.27880	331.39141
17	289.15767	279.54137	23.96585	262.68988	330.80743
18	290.80882	281.28616	26.74627	264.39152	**330.57097**
19	292.73632	283.30557	29.65321	266.36068	330.66405

TABLE 2. When $R = 1$, the effect of E_L dependence of S and N.

N\S	11	12	13	14	15
6	105.4102	106.8240	108.3650	110.0125	111.8461
7	91.4032	93.5134	95.7241	97.9251	100.2652
8	72.0817	75.1982	78.2248	81.3520	84.4168
9	23.1252	25.8246	28.7132	31.7465	34.8465
10	25.3571	28.2783	31.2127	34.3168	37.5122

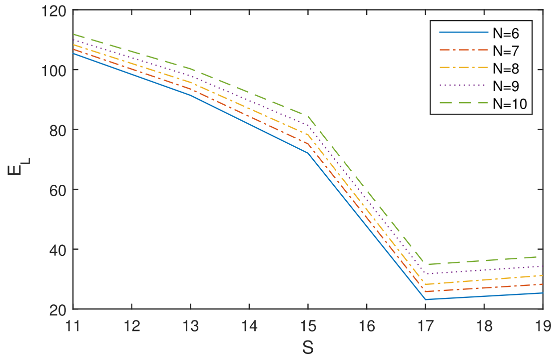


FIGURE 1. When $R = 1$, the effect of E_L dependence of S and N.

TABLE 3. When $R = 1$, the ramification of E_{WH} dependence of S and N.

N\S	13	14	15	16	17	18
6	0.0134	0.0135	0.0136	0.0137	0.0138	0.0139
8	0.0068	0.0069	0.0070	0.0071	0.0072	0.0073
10	0.0011	0.0011	0.0012	0.0012	0.0013	0.0013
12	0.0005	0.0005	0.0005	0.0005	0.0005	0.0006

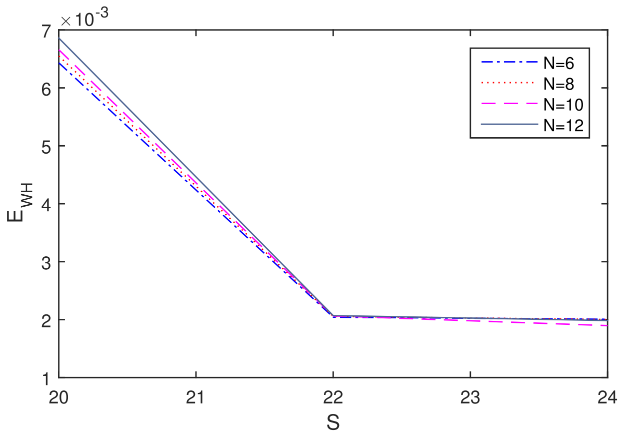


FIGURE 2. When $R = 1$, the effect of E_{WH} dependence of S and N.

$$\frac{\partial \mathbf{z}(s,n+1)}{\partial s}|_{s=0} = (\mathfrak{A} + H \otimes I_{K_N})^{-1} \times (\mathbf{e} - ((\mathfrak{L}\frac{\partial \mathbf{z}(s,n)}{\partial s}|_{s=0})), \quad n \geq 1. \tag{28}$$

VII. NUMERICAL ILLUSTRATION

We give a few descriptive numerical examples that expose the convexity of the expected cost rate and consider single arrival mode of the random environment ($R = 1$), the MAP for the appearance of demands are

1) **Hyper-exponential(HEX):**

$$D_0^{(1)} = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix},$$
$$D_1^{(1)} = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}$$

TABLE 4. Ramification of some system peculiarity measures with N and different $\lambda^{(1)}$.

N	E_I	E_{WH}	E_S	E_L
		$\lambda^{(1)} = 1$		
12	143.5432	147.8479	306.2228	54.6999
13	142.6272	121.2670	314.7580	53.7170
14	141.6442	32.9123	332.6682	16.7311
15	138.9366	32.0487	334.8701	15.5297
16	137.6537	31.1994	336.1817	12.8329
17	137.4375	30.5100	338.4123	11.9112
		$\lambda^{(1)} = 2$		
12	171.0240	226.7578	368.1008	122.8916
13	170.5714	194.3731	369.7239	113.0988
14	170.2567	181.0135	371.0218	103.6045
15	169.8805	170.1573	371.9408	94.7528
16	169.7999	161.0272	374.1648	83.5438
17	169.4996	151.0442	374.7167	82.3019
		$\lambda^{(1)} = 3$		
12	169.5829	195.7964	364.7214	110.4381
13	168.2833	175.1286	365.1517	92.5726
14	167.9590	176.0518	366.0438	85.0127
15	167.1550	157.1210	366.4506	80.5605
16	166.4634	132.3614	367.8949	75.9610
17	166.1882	75.3763	367.9251	71.0349

TABLE 5. When $R = 1$, the ramification of cost values c_s, c_b, and c_l corresponding to N.

c_s		c_b			c_b			c_b		
	c_l	0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3
0.01	0.1	27.9257	27.5725	27.2193	26.4231	26.0699	25.7166	24.9204	24.5672	24.2140
	0.4	87.0849	86.7317	86.3785	85.5822	85.2290	84.8758	84.0796	83.7264	83.3731
0.03	0.1	27.7620	27.4088	27.0555	26.2593	25.9061	25.5529	24.7567	24.4034	24.0502
	0.4	86.9212	86.5679	86.2147	85.4185	85.0653	84.7120	83.9158	83.5626	83.2094
0.05	0.1	27.5982	27.2450	26.8918	26.0956	25.7423	25.3891	24.5929	24.2397	23.8865
	0.4	86.7574	86.4042	86.0510	85.2547	84.9015	84.5483	83.7521	83.3988	83.0456
0.06	0.1	27.4345	27.0813	26.7280	25.9318	25.5786	25.2254	24.4292	24.0759	23.7227
	0.4	86.5936	86.2404	85.8872	85.0910	84.7378	84.3845	83.5883	83.2351	82.8819
0.07	0.1	27.2707	26.9175	26.5643	25.7681	25.4148	25.0616	24.2654	23.9122	23.5590
	0.4	86.4299	86.0767	85.7235	84.9272	84.5740	84.2208	83.4246	83.0713	82.7181

2) **Negative Correlation (NC):**

$$D_0^{(1)} = \begin{bmatrix} -2.73 & 2.73 & 0 \\ 0 & -2.73 & 0 \\ 0 & 0 & -7.3 \end{bmatrix},$$
$$D_1^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.0273 & 0 & 2.7027 \\ 7.227 & 0 & 0.073 \end{bmatrix}$$

3) **Positive Correlation (PC):**

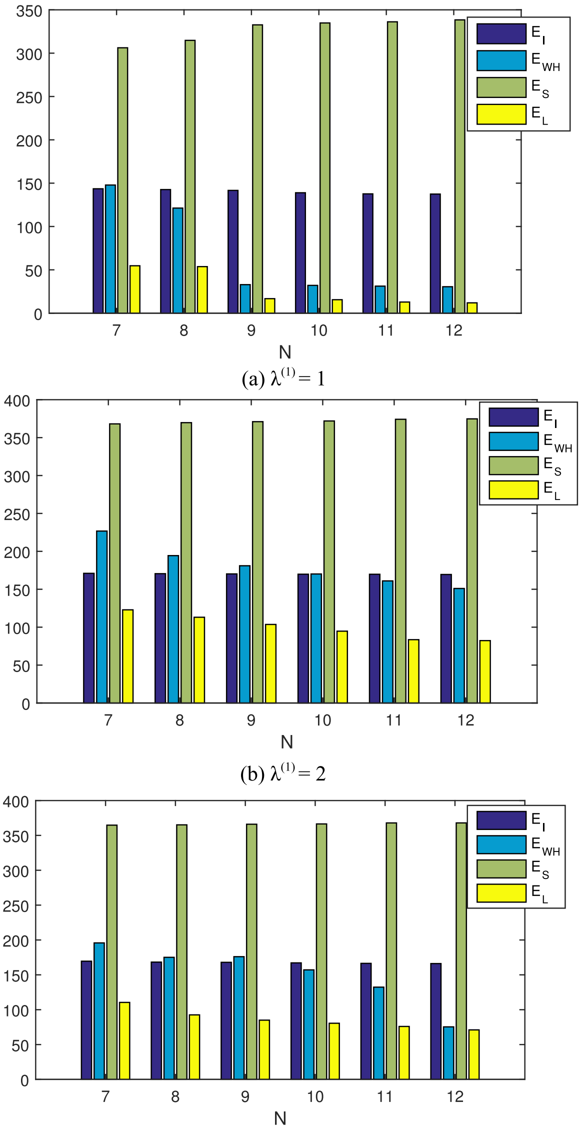
$$D_0^{(1)} = \begin{bmatrix} -2.73 & 2.73 & 0 \\ 0 & -2.73 & 0 \\ 0 & 0 & -7.3 \end{bmatrix},$$
$$D_1^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 2.7027 & 0 & 0.0273 \\ 0.073 & 0 & 7.227 \end{bmatrix}$$

The service time distribution characterized by

$$T_0^{(1)} = (0.9,\ 0.1), \quad T^{(1)} = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix}.$$

The demand process has negative(positive) correlated arrival with coefficient of variance,

$$c_{var}^{(1)} = 2\lambda^{(1)}\eta^{(1)}(-D_0^{(1)})^{-1}\mathbf{e} - 1 = \mathbf{1.2285}(\mathbf{1.2285})$$

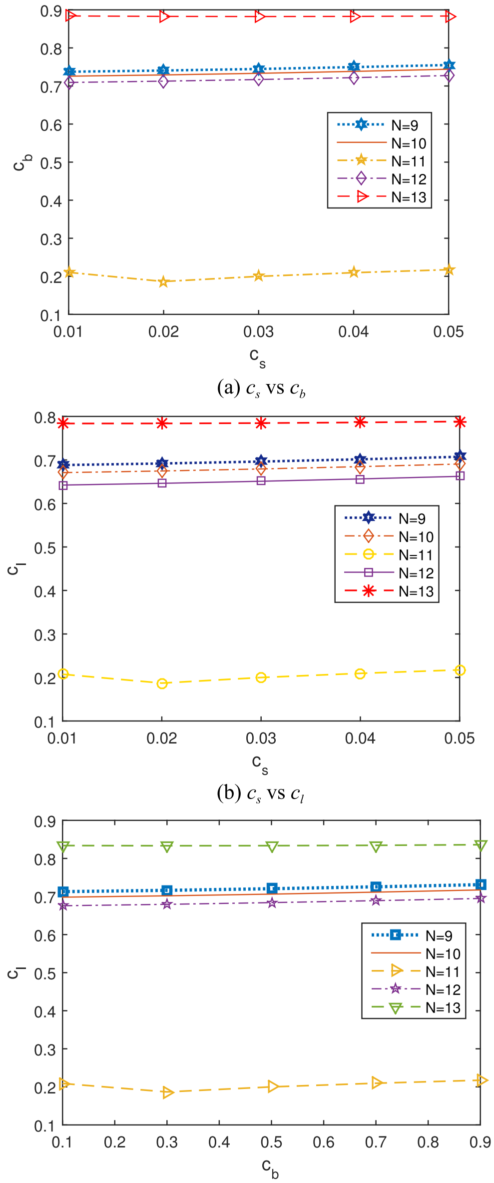


(a) $\lambda^{(1)} = 1$

(b) $\lambda^{(1)} = 2$

(c) $\lambda^{(1)} = 3$

FIGURE 3. Ramification some system peculiarity measures with N and different $\lambda^{(1)}$.



(a) c_s vs c_b

(b) c_s vs c_l

(c) c_b vs c_l

FIGURE 4. When $R = 1$, the ramification of cost values c_s, c_w, c_b, and c_l corresponding to N.

and coefficient of correlation,

$$c_{cor}^{(1)} = (\lambda^{(1)}\eta^{(1)})(-D_0^{(1)})^{-1}D_1^{(1)}(-D_0^{(1)})^{-1})\mathbf{e} - 1)/c_{var}^{(1)}$$
$$= -\mathbf{0.3742(0.3742)}$$

with arrival rate $\lambda^{(1)} = 2.2999$. The average service time is **0.1900** and the coefficient variance of this time is **5.0388**. Table 1 displayed the behaviour of the cost function of two variables $C(S, N)$, for the case of hyper-exponential distribution. The values are divulged **bold** in each column indicate the minimum cost rate whereas, the least cost rate is specified in each row by underlining the values. Thus a value (bold and underlined) spectacles the local minimum of the function $C(S, N)$. The optimal cost value $C(S, N)^* = \mathbf{11.47978}$ is achieve at $S^* = \mathbf{13}, N^* = \mathbf{10}$ with parameters $\lambda^{(1)} = 1$, $p^{(1)} = 0.7$, $q^{(1)} = 1-p^{(1)}$, $c_h = 0.15$, $c_s = 0.055$, $c_{sc} = 0.01$ $c_b = 0.12$, $c_l = 0.4$. Table 1 shows that the function $C(S, N)$ is convex.

TABLE 6. The function of total cast rate with two variable C(S, N), when $R = 2$.

S/N	7	8	9	10	11
19	2274.390	2161.060	2179.456	2199.120	2219.655
20	2267.583	**2160.591**	**2179.035**	**2198.717**	**2219.260**
21	2261.870	2160.639	2179.127	2198.825	2219.375
22	2257.124	2161.150	2179.678	2199.391	2219.947
23	2253.238	2162.082	2180.646	2200.373	2220.936
24	2250.125	2163.398	2181.997	2201.736	2222.304
25	2247.709	2165.069	2183.699	2203.450	2224.024
26	2245.927	2167.071	2185.730	2205.492	2226.071
27	2244.727	2169.382	2188.068	2207.840	2228.424
28	2244.061	2171.984	2190.696	2210.477	2231.065
29	**2243.891**	2174.862	2193.597	2213.388	2233.980
30	2244.181	2178.004	2196.761	2216.560	2237.156

TABLE 7. When $R = 2$, the effect of E_R.

N\S	11	12	13	14	15
6	167.6418	167.7740	167.8118	167.8473	167.8781
7	182.9274	183.0408	183.0808	183.1176	183.1496
8	198.2099	198.3078	198.3498	198.3878	198.4209
9	213.4897	213.5750	213.6189	213.6580	213.6921
10	228.7671	228.8422	228.8879	228.9281	228.9630

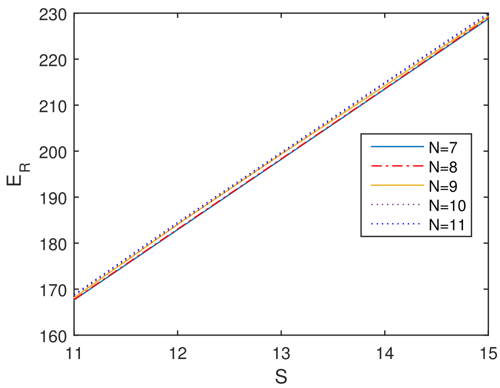


FIGURE 5. When $R = 2$, the effect of E_R.

Table 2 and figure 1 demonstrate E_L the dependence of S and N, whenever number of server is increase E_L is decrease but if take number of server is 10 in our corresponding to S, we got increase value and E_L is increase whenever number of item increase. Table 3 and figure 2 shows effects of E_{WH}(expected number of customers in the waiting hall) dependence of S and N, whenever N increase E_{WH} decrease and S increase E_{WH} also increase. In Table 4 displayed ramification of N and some system peculiarity measures with $\lambda^{(1)}$, whenever N increase E_S(expected number of customers in the system) also increases but E_I(expected inventory level), E_{WH} and E_L(expected lost) are decrease corresponding all $\lambda^{(1)}$ values. In particular $\lambda^{(1)} = 1$, $\lambda^{(1)} = 2$ and $\lambda^{(1)} = 3$, figure 3 (a), (b) and (c) are showed respectively. Table 5 and figure 4 demonstrate ramification of waiting cost of a customer in the system (c_s), the cost of busy server(c_b) and the lost cost of a customer(c_l) values. In figure 4 (a) and (b) are shows ramification of c_s with c_b and c_l respectively and (c) demonstrate c_b with c_l corresponding expected total cost rate are studied.

Consider two different state of arrival mode in the random environment ($R = 2$), customers are confined by the infinitesimal generator.

$$H = \begin{bmatrix} -0.10 & 0.10 \\ 0.01 & -0.01 \end{bmatrix}$$

The MAP for the appearance of demands $D_0^{(1)}$, $D_0^{(1)}$ are considered as we used in the case of $R = 1$.

1) **Hyper-exponential (HEX):**

$$D_0^{(2)} = \begin{pmatrix} -15 & 0 \\ 0 & -5 \end{pmatrix},$$

$$D_1^{(2)} = \begin{pmatrix} 13.5 & 1.5 \\ 4.5 & 0.5 \end{pmatrix}$$

TABLE 8. When $R = 2$, the effect of E_L dependence of S and N.

N\S	11	12	13	14	15
6	200.4897	232.5856	266.4074	300.9934	336.2915
7	198.0893	229.9232	263.5347	298.0012	333.2579
8	196.0839	227.7019	261.1556	295.5570	330.8299
9	194.3837	225.8215	259.1587	293.5380	328.8744
10	192.9244	224.2101	257.4637	291.8559	327.2955

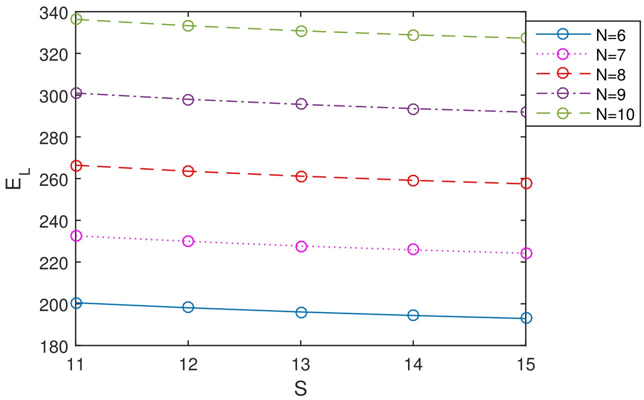


FIGURE 6. When $R = 2$, the effect of E_L.

TABLE 9. Ramification of some system peculiarity measures with different $\lambda^{(2)}$ and N.

N	E_I	E_S	E_{WH}	E_L
$\lambda^{(2)} = 11.5$				
4	65.9296	619.3078	167.7582	114.5873
7	66.0033	630.3700	167.7176	104.3482
8	66.9984	675.4985	167.6976	92.5821
9	67.3985	684.5671	166.6199	91.5856
$\lambda^{(2)} = 12$				
4	65.9480	639.5285	167.7814	118.3285
7	66.0019	645.3621	167.6727	112.8682
8	66.9988	653.1873	167.5942	105.1540
9	66.4988	668.5625	166.5148	98.2634
$\lambda^{(2)} = 12.5$				
4	65.9627	664.5263	167.7318	122.9536
7	66.0010	668.5209	167.6467	117.6701
8	66.9780	675.5831	167.5335	103.1413
9	66.9990	679.2662	166.5332	95.6841

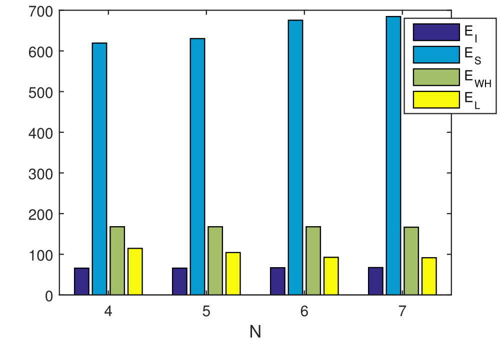
TABLE 10. When $R = 2$, the ramification of cost values c_s, c_b, and c_l.

c_s		c_b								
1.45	c_l	1.5	2.0	2.5	1.5	2.0	2.5	1.5	2.0	2.5
	0.01	1875.4205	1973.4625	2071.5044	2140.4020	2238.4439	2336.4858	2405.3834	2503.4254	2601.4673
	0.02	1877.4036	1975.4455	2073.4875	2142.3851	2240.4270	2338.4689	2407.3665	2505.4084	2603.4504
1.50	0.01	1911.1343	2009.1762	2107.2181	2176.1157	2274.1576	2372.1996	2441.0972	2539.1391	2637.1810
	0.02	1913.1173	2011.1593	2109.2012	2178.0988	2276.1407	2374.1827	2443.0803	2541.1222	2639.1641
1.55	0.01	1946.8480	2044.8899	2142.9319	2211.8295	2309.8714	2407.9133	2476.8109	2574.8528	2672.8948
	0.02	1948.8311	2046.8730	2144.9149	2213.8125	2311.8545	2409.8964	2478.7940	2576.8359	2674.8778
1.60	0.01	1982.5617	2080.6037	2178.6456	2247.5432	2345.5851	2443.6270	2512.5246	2610.5666	2708.6085
	0.02	1984.5448	2082.5867	2180.6287	2249.5263	2347.5682	2445.6101	2514.5077	2612.5497	2710.5916
1.65	0.01	2018.2755	2116.3174	2214.3593	2283.2569	2381.2989	2479.3408	2548.2384	2646.2803	2744.3222
	0.02	2020.2586	2118.3005	2216.3424	2285.2400	2383.2819	2481.3239	2550.2215	2648.2634	2746.3053

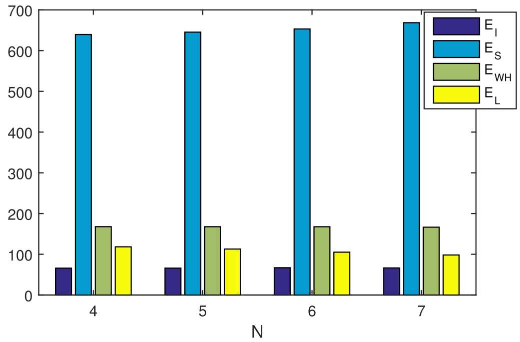
2) **Negative Correlation (NC):**

$$D_0^{(2)} = \begin{bmatrix} -2.35 & 2.35 & 0 \\ 0 & -2.35 & 0 \\ 0 & 0 & -3.5 \end{bmatrix},$$

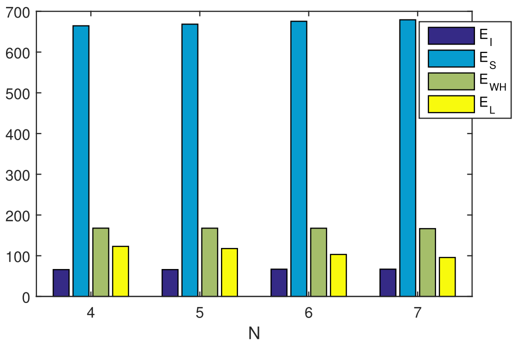
$$D_1^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 2.3265 & 0 & 0.0235 \\ 0.035 & 0 & 3.465 \end{bmatrix}$$



(a) $\lambda^{(2)} = 11.5$

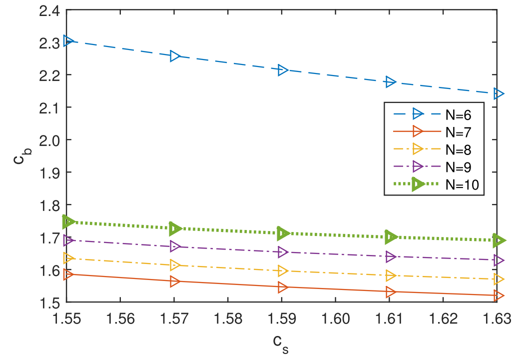


(b) $\lambda^{(2)} = 12$

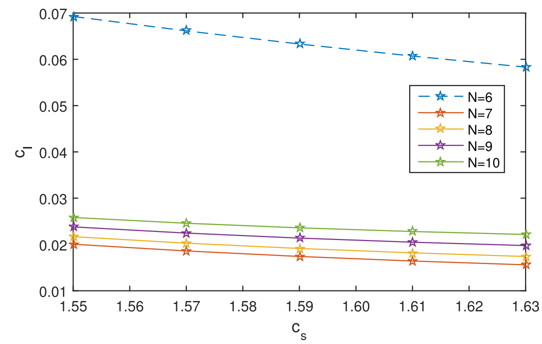


(c) $\lambda^{(2)} = 12.5$

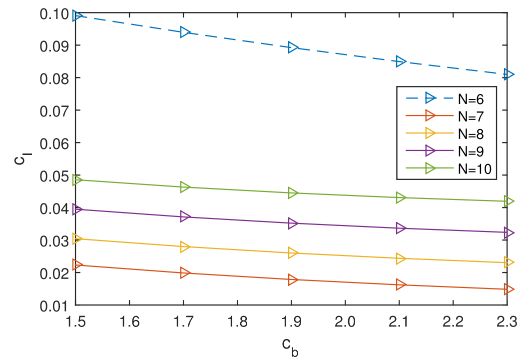
FIGURE 7. Ramification some system peculiarity measures with different $\lambda^{(2)}$ and N.



(a) c_s vs c_b



(b) c_s vs c_l



(c) c_b vs c_l

FIGURE 8. When $R = 2$, the ramification of cost values c_s, c_w, c_b, and c_l corresponding to N.

3) **Positive Correlation (PC):**

$$D_0^{(2)} = \begin{bmatrix} -2.35 & 2.35 & 0 \\ 0 & -2.35 & 0 \\ 0 & 0 & -3.5 \end{bmatrix},$$

$$D_1^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0.0235 & 0 & 2.3265 \\ 3.465 & 0 & 0.035 \end{bmatrix}.$$

When $R = 2$ the service time distribution:

$$T_0^{(2)} = (1,\ 0), \quad T^{(2)} = \begin{pmatrix} -0.3 & 0.3 \\ 0 & -0.3 \end{pmatrix}.$$

The demand process has negative(positive) correlated arrival with coefficient of variance

$$c_{var}^{(2)} = 2\lambda^{(2)}\eta^{(2)}(-D_0^{(2)})^{-1}\mathbf{e} - 1 = \mathbf{12.7420(12.7420)}$$

and coefficient of correlation

$$c_{cor}^{(2)} = (\lambda^{(2)}\eta^{(2)})(-D_0^{(2)})^{-1}D_1^{(2)}(-D_0^{(2)})^{-1})\mathbf{e} - 1)/c_{var}^{(2)} = \mathbf{-0.3440(0.3440)}$$

with arrival rate $\lambda^{(2)} = 12.5000$. The average service time is **6.6667** and the coefficient variance of this time is **0.5000**. Here, $r = 1$ demand martices as consider to $R = 1$ matrices. Table 6 displayed the behaviour of the cost function of two variables $C(S, N)$, for the case of hyper-exponential distribution. The values are divulged **bold** in each column indicate the minimum cost rate whereas, the least cost rate is specified in each row by underlining the values. Thus a value (bold and underlined) spectacles the local minimum of the function $C(S, N)$. The optimal cost value $C(S, N)^* = \mathbf{2160.591}$ is achieve at $S^* = \mathbf{20}$, $N^* = \mathbf{8}$ with parameters $\lambda^{(1)} = 1$, $p^{(1)} = 0.7$, $q^{(1)} = 1 - p^{(1)}$, $\lambda^{(2)} = 12.5$, $p^{(2)} = 0.7$, $q^{(2)} = 1 - p^{(2)}$, $c_h = 0.15$, $c_s = 1.55$, $c_{sc} = 0.37$, $c_b = 0.12$, $c_l = 0.01$. Table 6 shows that the function $C(S, N)$ is convex.

Table 7 and figure 5 demonstrate E_R the dependence of S and N with $R = 2$, whenever number of server and number of item are increases E_R also is increase. Table 8 and figure 6 shows effects of E_R(expected reorder) dependence of S and N with $R = 2$, whenever N increase E_R decrease and S increase E_R also increase. In Table 9 displayed ramification of N and some system peculiarity measures with $\lambda^{(2)}$ under second sate of random environment, whenever N increase E_S(expected number of customer in the system) and E_I(expected inventory level) are also increase but E_{WH}(expected number of customers in the waiting hall) and E_L(expected lost) are decrease corresponding all $\lambda^{(2)}$ values. In particular $\lambda^{(2)} = 11.5$, $\lambda^{(2)} = 12$ and $\lambda^{(2)} = 12.5$, figure 7 (a), (b) and (c) are showed respectively. Table 10 demonstrate ramification of waiting cost of a customer in the system (c_s), the cost of busy server(c_b) and the lost cost of a customer(c_l) values when $R = 2$. Figure 8 (a) and (b) are shows ramification of c_s with c_b and c_l respectively and (c) demonstrate c_b with c_l, corresponding expected total cost rate are studied.

VIII. CONCLUSION

In this suggested work, we discussed the queueing-inventory model with multi-server (N) and demands from R varieties of mode from a random environment. The customers arrival occurs according to the MAP. We obtained a steady-state vector with the help stability condition, and we derived the distribution of the sojourn time of an arbitrary customer in the system. We provided sensitive analysis in table 1 and table 6 to understand how the S and N changes cause expected cost rate $R = 1$ and $R = 2$, respectively. It is done only after obtaining the local optima, S^*, and N^*. The corresponding expected total cost rate is studied for $R = 2$.

This paper will be extended to a positive lead time and random environment-based on multiple types of service in the future.

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