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An Advanced Study on the Bonferroni Mean Operators for Managing Cubic Intuitionistic Complex Fuzzy Soft Settings and Their Applications in Decision Making

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ABSTRACT The main influence of this analysis is to initiate the fundamental theory of cubic intuitionistic complex fuzzy soft (CICFS) information which is a very informative and effective tool for handling ambiguity and complications in reality. Further, in the consideration of CICFS information, we utilized the important algebraic laws, score value, and accuracy value and try to determine some rules for finding the relation between any CICFS numbers. Moreover, by using the presented information, we diagnosed the CICFS Bonferroni mean (CICFSBM) operator, CICFS weighted Bonferroni mean (CICFSWBM) operator and evaluated their important results, and described their important properties (Idempotency, Monotonicity, and Boundedness). Finally, to evaluate deficiencies with the help of invented operators, we established a MADM (''multi-attribute decision-making'') tool under the availability of CICFSBM, CICFSWBM operators. Finally, we described the supremacy and reliability of the diagnosed work with the help of comparative analysis and also explained their graphical representation to enhance the worth of the established approaches.

INDEX TERMS Bonferroni mean operators, cubic intuitionistic complex fuzzy soft settings, decision-making techniques.

I. INTRODUCTION

Clustering analysis is a feasible technique tool for any institution that needs to discover discrete collections of intellectuals, sales transfers, or other sorts of behaviors and things. Similarly, pattern recognition, medical diagnosis, image segmentation, networking systems, decision-making, and MADM techniques also played a very essential role in the environment of classical set theory. A lot of scholars have utilized the above-discussed applications in the field of fuzzy set (FS) theory, which was diagnosed by Zadeh [1] in 1965, where the classical type of set is normally called crisp set. The FS theory can be required in a massive space of domain in which information is inaccurate and vague, such as bioinformatics. FS theory has been represented to be an important technique to explain situations in which the information is vague or unreliable. FS theory easily handles such sort of situations by attributing a grade to which many terms belong to a set. We know that the theory of FS developed has represented useful circumstances in various fields of analysis. The theory of FS is well-known because it manages vague and unreliable information which cannot be addressed by crisp. But in various places (available information in the form of yes or no), the theory of FS has not worked effectively. To evaluate the above satiation perfectly, the theory of intuitionistic FS (IFS) is a very meaningful technique that described information in the form of truth and falsity grades, proposed by Atanassov [2]. The IFS has shown valuable applications in various fields, for instance, aggregation operators [3], [4], decision-making [5], [6], hybrid aggregation operators [7], [8], and different types of measures [9].

Complexity and inconsistency are involved in every field of life and due to this reason, experts/individuals faced a lot of difficulty during the decision-making process. For instance, if two people talking about the final match of PSL (''Pakistan supper league''). The favorite team of person-A is

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MS ("Multan Sultan") and the favorite team of person-B is LQ ("Lahore Qalandar"). The main opinion of person-A is of the form: if MS wins the toss and decided to bat first and they give a target between 180 to 200, they have a 100% chance to win the match. Similarly, person-B gave their opinion in the form if LQ wins the toss and decided to bat first and they give a target between 170 to 210, they have a 100% chance to win the match. Focused on the target values which are not particular but necessary lie between 180 to 200 or 170 to 210, such sort of problem is very difficult for IFS to evaluate. For this, the theory of interval-valued IFS (IVIFS) is very valuable for managing such sort of complicated information, diagnosed by Atanassov [10]. The theory of IVIFS includes two different grades, called truth and falsity grades in the form of intervals. The above-cited problem can easily depict by using the IVIFS. Many applications have been illustrated here, for instance, aggregation operators [11], decision-making [12], hybrid aggregation operators [13], [14], and different types of measures [15], [16].

The theory of complex FS (CFS) is very valuable and massive feasible for handling such sort of information that cannot be evaluated from FS, diagnosed by Ramot *et al.* [17]. The mathematical framework of CFS has included one grade, called truth grade whose real and imaginary parts are represented by the amplitude and phase term whose values belong to the unit interval. Further, some well-known theories based on CFS have been done, for example, cross-entropy measures for CFS and their applications [18]. CFSs have a lot of applications in the field of different analyses, but in various places (available information in the form of yes or no in the shape of complex numbers), then the theory of CFS has not worked effectively. To evaluate the above satiation perfectly, the theory of complex IFS (CIFS) is a very meaningful technique that described information in the form of truth and falsity grades in the shape of polar coordinates, proposed by Alkouri and Salleh [19]. Further, the theory of complex interval-valued IFS (CIVIFS) was diagnosed by Garg and Rani [20]. The theory of CIFS and CIVIFS has gotten a huge number of values from distinct scholars, for instance, another view of complex intuitionistic fuzzy soft sets was invented by Ali *et al.* [21], analysis of cyber-security and cyber-crimes in oil and gas using CIF relation was diagnosed by Jan *et al.* [22] and the theory idea of group theory in the region of CIFS was utilized by Gulzar *et al.* [23].

The main theory of soft set (SS) [24], invented in 1999, contains a function defined from a set of parameters to a unit interval. SS theory has a lot of implementations in the region of different theories, for instance, the theory of generalized SS was diagnosed by Majumdar and Samanta [25], generalized intuitionistic fuzzy SS, diagnosed by Agarwal *et al.* [26], group decision-making technique based on generalized intuitionistic fuzzy SS was invented by Garg and Arora [27]. Thirunavukarasu *et al.* [28] firstly employed the theory of SS in the environment of CFS. Further, Quek *et al.* [29] invented some algebraic structures based on complex intuitionistic fuzzy SS and their application. The main theory of the Bonferroni mean (BM) operator was diagnosed by Bonferroni [30] in 1950. Further, Xu and Yager [31] diagnosed the intuitionistic fuzzy BM operators. Xia *et al.* [32] presented the generalized intuitionistic fuzzy BM operators. Garg and Rani [33] explored the generalized BM operators for CIFSs. The theory of generalized BM operators for CIFS was also diagnosed by Garg and Rani [34]. BM operators for cubic IFSs were diagnosed by Kaur and Garg [35]. Garg and Arora [36] invented the BM operators for intuitionistic fuzzy soft sets. From the above-cited information, we noticed that the theory of the CICFS setting is not proposed yet. The theory of CICFS information deals with that information that cannot deal with FSs, IFSs, cubic IFSs, CIFSs, and cubic IFSs. It is clear that the cubic IFS can deal with that information which includes one-dimension information at a time but failed to manage that information that contained two-dimension information in a singleton set. Here, we tried to explain the importance of two-dimension information involved in genuine life dilemmas, for instance, a person wants to purchase a new branded car based on the following features, called name of the car and fuel consumption of the car, look here the name of the car represented the amplitude term and the fuel consumption is represented the phase term. We noticed that such kind of theory cannot be handled from traditionally cubic IFS. For managing such sort of problematic issues, the theory of cubic intuitionistic complex fuzzy information is more effective because it can deal with two-dimension information very easily. The real part "amplitude part" represented the name of the car and the imaginary part "phase term (periodicity term)" expressed the fuel consumption of the car. Further, the theory of cubic IFS is the special case of the diagnosed cubic intuitionistic complex fuzzy soft sets. The main influence of this theory is analyzed below.

1) To utilize the fundamental theory of CICFS information is diagnosed which is a very informative and effective tool for handling ambiguity and complications in reality.
2) To diagnose the important algebraic laws, score value, and accuracy value and try to determine some rules for finding the relation between any CICFS numbers.
3) To diagnose the CICFSBM operator, CICFSWBM operator and evaluated their important results, and described their important properties (Idempotency, Monotonicity, and Boundedness).
4) To evaluate deficiencies with the help of invented operators, we established a MADM tool under the availability of CICFSBM, and CICFSWBM operators.
5) To describe the supremacy and reliability of the diagnosed work with the help of comparative analysis and also explained their graphical representation is to enhance the worth of the established approaches. The summary of the proposed work is given in the shape of Figure 1.

The main key factor of this scenario is invented here, in section 2, we revised the definition of CIFS and their

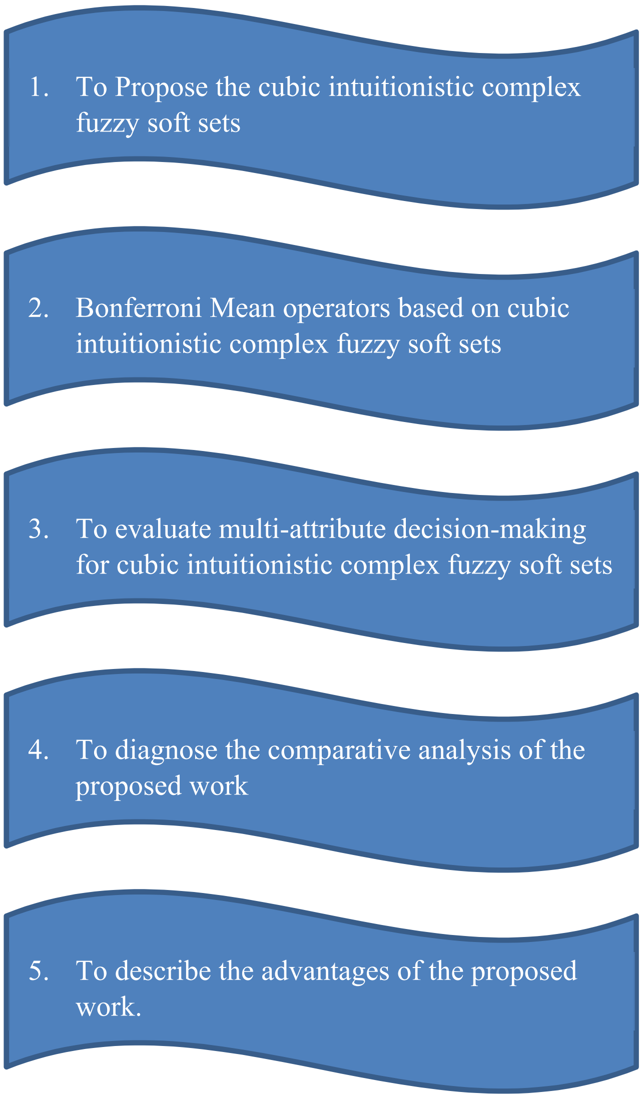

FIGURE 1. Geometrical shape of the proposed work.

algebraic, and similarly, we revised the theory of CIVIFS with various algebraic laws. Finally, we revised the theory of SS which will be helpful to the proposed theories. In section 3, the fundamental theory of CICFS information is diagnosed which is a very informative and effective tool for handling ambiguity and complications in reality. Further, in the consideration of CICFS information, we utilized the important algebraic laws, score value, and accuracy value and try to determine some rules for finding the relation between any CICFS numbers. In section 4, using the presented information, we diagnosed the CICFSBM operator, CICFSWBM operator and evaluated their important results, and described their important properties (Idempotency, Monotonicity, and Boundedness). In section 5, to evaluate deficiencies with the help of invented operators, we established a MADM tool under the availability of CICFSBM, CICFSWBM operators. Finally, we described the supremacy and reliability of the diagnosed work with the help of comparative analysis and also explained their graphical representation is to enhance the worth of the established approaches. In section 6, we concluded various remarks.

II. PRELIMINARIES

The key analysis of this theory is to revise the definition of CIFS and their algebraic and similarly, we revised the theory of CIVIFS with various algebraic laws. Finally, we revised the theory of SS which will be helpful to the proposed theories. CFSs have a lot of applications in the field of different analyses, but in various places (available information in the form of yes or no in the shape of complex numbers), then the theory of CFS has not worked effectively. To evaluate the above satiation perfectly, the theory of CIFS is a very meaningful technique that described information in the form of truth and falsity grades in the shape of polar coordinates.

Definition 1 [19]: The theory of the CIF set $\mathbb{I}_{IF}$ based on a universal set $\mathbb{U}_{UN}$ is deliberated by:

$$\mathbb{I}_{IF} = \left\{\left(\mu_{\mathbb{I}_{IF}}(x), \eta_{\mathbb{I}_{IF}}(x)\right) : x \in \mathbb{U}_{UN}\right\} \tag{1}$$

In Eq. (1), the term $\mu_{\mathbb{I}_{IF}}(x) = \mu_{\mathbb{I}_R}(x)\, e^{i2\pi\left(\mu_{\mathbb{I}_I}(x)\right)}$ and $\eta_{\mathbb{I}_{IF}}(x) = \eta_{\mathbb{I}_R}(x)\, e^{i2\pi\left(\eta_{\mathbb{I}_I}(x)\right)}$ are represented the well-known truth and falsity grades with techniques: $0 \leq \mu_{\mathbb{I}_R}(x) + \eta_{\mathbb{I}_R}(x) \leq 1$ and $0 \leq \mu_{\mathbb{I}_I}(x) + \eta_{\mathbb{I}_I}(x) \leq 1$. For more diagnoses, the refusal grade is represented by: $\xi_{\mathbb{I}_{IF}} = \xi_{\mathbb{I}_R}(x)\, e^{i2\pi\left(\xi_{\mathbb{I}_I}(x)\right)} = \left(1 - \left(\mu_{\mathbb{I}_R}(x) + \eta_{\mathbb{I}_R}(x)\right)\right) e^{i2\pi\left(1-\left(\mu_{\mathbb{I}_I}(x)+\eta_{\mathbb{I}_I}(x)\right)\right)}$. Finally, throughout this manuscript, the CIF numbers (CIFNs) are represented by: $\mathbb{I}_{IF-\ell} = \left(\mu_{\mathbb{I}_{R-\ell}} e^{i2\pi\left(\mu_{\mathbb{I}_{I-\ell}}\right)}, \eta_{\mathbb{I}_{R-\ell}} e^{i2\pi\left(\eta_{\mathbb{I}_{I-\ell}}\right)}\right), \ell = 1, 2, \ldots, \mathfrak{n}$. After investigating the theory of CIFSs, it is a very serious issue, how we are doing addition and multiplication by using any two CIFSs because, by following traditional addition and multiplication, we failed. For managing the above dilemmas, we review some operational laws proposed with the help of algebraic t-norm and t-conorm.

Definition 2 [19]: The theory of CIFNs $\mathbb{I}_{IF-\ell} = \left(\mu_{\mathbb{I}_{R-\ell}} e^{i2\pi\left(\mu_{\mathbb{I}_{I-\ell}}\right)}, \eta_{\mathbb{I}_{R-\ell}} e^{i2\pi\left(\eta_{\mathbb{I}_{I-\ell}}\right)}\right), \ell = 1, 2$ based on a universal set $\mathbb{U}_{UN}$, we have

$$\mathbb{I}_{IF-1} \oplus \mathbb{I}_{IF-2} = \begin{pmatrix} \left(\mu_{\mathbb{I}_{R-1}} + \mu_{\mathbb{I}_{R-2}} - \mu_{\mathbb{I}_{R-1}}\mu_{\mathbb{I}_{R-2}}\right) \\ e^{i2\pi\left(\mu_{\mathbb{I}_{I-1}}+\mu_{\mathbb{I}_{I-2}}-\mu_{\mathbb{I}_{I-1}}\mu_{\mathbb{I}_{I-2}}\right)}, \\ \left(\eta_{\mathbb{I}_{R-1}}\eta_{\mathbb{I}_{R-2}}\right) e^{i2\pi\left(\eta_{\mathbb{I}_{I-1}}\eta_{\mathbb{I}_{I-2}}\right)} \end{pmatrix} \tag{2}$$

$$\mathbb{I}_{IF-1} \otimes \mathbb{I}_{IF-2} = \begin{pmatrix} \left(\mu_{\mathbb{I}_{R-1}}\mu_{\mathbb{I}_{R-2}}\right) e^{i2\pi\left(\mu_{\mathbb{I}_{I-1}}\mu_{\mathbb{I}_{I-2}}\right)}, \\ \left(\eta_{\mathbb{I}_{R-1}} + \eta_{\mathbb{I}_{R-2}} - \eta_{\mathbb{I}_{R-1}}\eta_{\mathbb{I}_{R-2}}\right) \\ e^{i2\pi\left(\eta_{\mathbb{I}_{I-1}}+\eta_{\mathbb{I}_{I-2}}-\eta_{\mathbb{I}_{I-1}}\eta_{\mathbb{I}_{I-2}}\right)} \end{pmatrix} \tag{3}$$

$$\varphi\mathbb{I}_{IF-1} = \begin{pmatrix} 1 - \left(1 - \mu_{\mathbb{I}_{R-1}}\right)^{\varphi} e^{i2\pi\left(1-\left(1-\mu_{\mathbb{I}_{I-1}}\right)^{\varphi}\right)}, \\ \eta_{\mathbb{I}_{R-1}}^{\varphi} e^{i2\pi\left(\eta_{\mathbb{I}_{I-1}}^{\varphi}\right)} \end{pmatrix} \tag{4}$$

$$\mathbb{I}_{IF-1}^{\varphi}=\begin{pmatrix}\mu_{\mathbb{I}_{R-1}}^{\varphi}e^{i2\pi\left(\mu_{\mathbb{I}_{I-1}}^{\varphi}\right)},\\ 1-\left(1-\eta_{\mathbb{I}_{R-1}}\right)^{\varphi}e^{i2\pi\left(1-\left(1-\eta_{\mathbb{I}_{I-1}}\right)^{\varphi}\right)}\end{pmatrix}\tag{5}$$

Still, many authors have faced a serious problem, how we are finding the relation between any two CIFNs. For this, we review the mathematical shape of two different ideas, explained below.

Definition 3 [19]: The theory of CIFNs $\mathbb{I}_{IF-\ell}=\left(\mu_{\mathbb{I}_{R-\ell}}e^{i2\pi\left(\mu_{\mathbb{I}_{I-\ell}}\right)},\eta_{\mathbb{I}_{R-\ell}}e^{i2\pi\left(\eta_{\mathbb{I}_{I-\ell}}\right)}\right)$, $\ell=1$ based on a universal set $\mathbb{U}_{UN}$. Then the function

$$\mathbb{S}^{SV}\left(\mathbb{I}_{IF-1}\right)=\frac{1}{2}\left(\mu_{\mathbb{I}_{R-1}}-\eta_{\mathbb{I}_{R-1}}+\mu_{\mathbb{I}_{I-1}}-\eta_{\mathbb{I}_{I-1}}\right),\quad \mathbb{S}^{SV}\left(\mathbb{I}_{IF-1}\right)\in[-1,1]\tag{6}$$

Called score value (SV) and the function

$$\mathbb{H}^{SV}\left(\mathbb{I}_{IF-1}\right)=\frac{1}{2}\left(\mu_{\mathbb{I}_{R-1}}+\eta_{\mathbb{I}_{R-1}}+\mu_{\mathbb{I}_{I-1}}+\eta_{\mathbb{I}_{I-1}}\right),\quad \mathbb{S}^{SV}\left(\mathbb{I}_{IF-1}\right)\in[-1,1]\tag{7}$$

Called accuracy value (AV). Further, based on Eq. (6) and Eq. (7), we deliberated various rules: when $\mathbb{S}^{SV}\left(\mathbb{I}_{IF-1}\right)>\mathbb{S}^{SV}\left(\mathbb{I}_{IF-2}\right)$, then we assume that $\mathbb{I}_{IF-1}>\mathbb{I}_{IF-2}$, similarly, if $\mathbb{S}^{SV}\left(\mathbb{I}_{IF-1}\right)=\mathbb{S}^{SV}\left(\mathbb{I}_{IF-2}\right)$, we use Eq. (7), if $\mathbb{H}^{SV}\left(\mathbb{I}_{IF-1}\right)>\mathbb{H}^{SV}\left(\mathbb{I}_{IF-2}\right)$, then we assume that $\mathbb{I}_{IF-1}>\mathbb{I}_{IF-2}$, if $\mathbb{H}^{SV}\left(\mathbb{I}_{IF-1}\right)=\mathbb{H}^{SV}\left(\mathbb{I}_{IF-2}\right)$, then we assume that $\mathbb{I}_{IF-1}=\mathbb{I}_{IF-2}$. CIFSs have a lot of applications in the field of different analyses, but in various places (available information in the form of yes or no in the shape of interval-valued complex numbers), then the theory of CIFS has not worked effectively. To evaluate the above satiation perfectly, the theory of CIVIFS is a very meaningful technique that described information in the form of truth and falsity grades in the shape of polar coordinates, whose real and unreal parts are developed in the shape of interval-valued.

Definition 4 [20]: The theory of the CIVIF set $\mathbb{I}_{CIF}$ based on a universal set $\mathbb{U}_{UN}$ is deliberated by:

$$\mathbb{I}_{CIF}=\left\{\left(\mu_{\mathbb{I}_{CIF}}(x),\eta_{\mathbb{I}_{CIF}}(x)\right):x\in\mathbb{U}_{UN}\right\}\tag{8}$$

In Eq. (1), the term $\mu_{\mathbb{I}_{CIF}}(x)=\left[\mu_{\mathbb{I}_R}^{-}(x),\mu_{\mathbb{I}_R}^{+}(x)\right]e^{i2\pi\left(\left[\mu_{\mathbb{I}_I}^{-}(x),\mu_{\mathbb{I}_I}^{+}(x)\right]\right)}$ and $\eta_{\mathbb{I}_{CIF}}(x)=\left[\eta_{\mathbb{I}_R}^{-}(x),\eta_{\mathbb{I}_R}^{+}(x)\right]e^{i2\pi\left(\left[\eta_{\mathbb{I}_I}^{-}(x),\eta_{\mathbb{I}_I}^{+}(x)\right]\right)}$ are represented the well-known truth and falsity grades in the shape of complex numbers with techniques: $0\le\mu_{\mathbb{I}_R}^{+}(x)+\eta_{\mathbb{I}_R}^{+}(x)\le1$ and $0\le\mu_{\mathbb{I}_I}^{+}(x)+\eta_{\mathbb{I}_I}^{+}(x)\le1$. For more diagnoses, the refusal grade is represented by:

$$\begin{aligned}\xi_{\mathbb{I}_{CIF}}&=\left[\xi_{\mathbb{I}_R}^{-}(x),\xi_{\mathbb{I}_R}^{+}(x)\right]e^{i2\pi\left(\left[\xi_{\mathbb{I}_I}^{-}(x),\xi_{\mathbb{I}_I}^{+}(x)\right]\right)}\\&=\begin{bmatrix}\left(1-\left(\mu_{\mathbb{I}_R}^{-}(x)+\eta_{\mathbb{I}_R}^{-}(x)\right)\right),\\\left(1-\left(\mu_{\mathbb{I}_R}^{+}(x)+\eta_{\mathbb{I}_R}^{+}(x)\right)\right)\end{bmatrix}\\&\quad\times e^{i2\pi\begin{bmatrix}\left(1-\left(\mu_{\mathbb{I}_I}^{-}(x)+\eta_{\mathbb{I}_I}^{-}(x)\right)\right),\\\left(1-\left(\mu_{\mathbb{I}_I}^{+}(x)+\eta_{\mathbb{I}_I}^{+}(x)\right)\right)\end{bmatrix}}.\end{aligned}$$

Finally, throughout this manuscript, the CIVIF numbers (CIVIFNs) are represented by:

$$\mathbb{I}_{CIF-\ell}=\begin{pmatrix}\left[\mu_{\mathbb{I}_{R-\ell}}^{-},\mu_{\mathbb{I}_{R-\ell}}^{+}\right]e^{i2\pi\left(\left[\mu_{\mathbb{I}_{I-\ell}}^{-},\mu_{\mathbb{I}_{I-\ell}}^{+}\right]\right)},\\\left[\eta_{\mathbb{I}_{R-\ell}}^{-},\eta_{\mathbb{I}_{R-\ell}}^{+}\right]e^{i2\pi\left(\left[\eta_{\mathbb{I}_{I-\ell}}^{-},\eta_{\mathbb{I}_{I-\ell}}^{+}\right]\right)}\end{pmatrix},\quad \ell=1,2,\ldots,\mathfrak{n}.$$

After investigating the theory of CIVIFSs, it is a very serious issue, how we are doing addition and multiplication by using any two CIVIFSs, because, by following traditional addition and multiplication, we failed. For managing the above dilemmas, we review some operational laws proposed with the help of algebraic t-norm and t-conorm.

Definition 5 [20]: The theory of CIVIFNs

$$\mathbb{I}_{CIF-\ell}=\begin{pmatrix}\left[\mu_{\mathbb{I}_{R-\ell}}^{-},\mu_{\mathbb{I}_{R-\ell}}^{+}\right]e^{i2\pi\left(\left[\mu_{\mathbb{I}_{I-\ell}}^{-},\mu_{\mathbb{I}_{I-\ell}}^{+}\right]\right)},\\\left[\eta_{\mathbb{I}_{R-\ell}}^{-},\eta_{\mathbb{I}_{R-\ell}}^{+}\right]e^{i2\pi\left(\left[\eta_{\mathbb{I}_{I-\ell}}^{-},\eta_{\mathbb{I}_{I-\ell}}^{+}\right]\right)}\end{pmatrix},\quad \ell=1,2$$

based on a universal set $\mathbb{U}_{UN}$, we have

$$\mathbb{I}_{CIF-1}\oplus\mathbb{I}_{CIF-2}=\begin{pmatrix}\begin{bmatrix}\left(\mu_{\mathbb{I}_{R-1}}^{-}+\mu_{\mathbb{I}_{R-2}}^{-}-\mu_{\mathbb{I}_{R-1}}^{-}\mu_{\mathbb{I}_{R-2}}^{-}\right),\\\left(\mu_{\mathbb{I}_{R-1}}^{+}+\mu_{\mathbb{I}_{R-2}}^{+}-\mu_{\mathbb{I}_{R-1}}^{+}\mu_{\mathbb{I}_{R-2}}^{+}\right)\end{bmatrix}\\ e^{i2\pi\begin{bmatrix}\left(\mu_{\mathbb{I}_{I-1}}^{-}+\mu_{\mathbb{I}_{I-2}}^{-}-\mu_{\mathbb{I}_{I-1}}^{-}\mu_{\mathbb{I}_{I-2}}^{-}\right),\\\left(\mu_{\mathbb{I}_{I-1}}^{+}+\mu_{\mathbb{I}_{I-2}}^{+}-\mu_{\mathbb{I}_{I-1}}^{+}\mu_{\mathbb{I}_{I-2}}^{+}\right)\end{bmatrix}},\\\left[\left(\eta_{\mathbb{I}_{R-1}}^{-}\eta_{\mathbb{I}_{R-2}}^{-}\right),\left(\eta_{\mathbb{I}_{R-1}}^{+}\eta_{\mathbb{I}_{R-2}}^{+}\right)\right]\\ e^{i2\pi\left[\left(\eta_{\mathbb{I}_{I-1}}^{-}\eta_{\mathbb{I}_{I-2}}^{-}\right),\left(\eta_{\mathbb{I}_{I-1}}^{+}\eta_{\mathbb{I}_{I-2}}^{+}\right)\right]}\end{pmatrix}\tag{9}$$

$$\mathbb{I}_{CIF-1}\otimes\mathbb{I}_{CIF-2}=\begin{pmatrix}\left[\left(\mu_{\mathbb{I}_{R-1}}^{-}\mu_{\mathbb{I}_{R-2}}^{-}\right),\left(\mu_{\mathbb{I}_{R-1}}^{+}\mu_{\mathbb{I}_{R-2}}^{+}\right)\right]\\ e^{i2\pi\left[\left(\mu_{\mathbb{I}_{I-1}}^{-}\mu_{\mathbb{I}_{I-2}}^{-}\right),\left(\mu_{\mathbb{I}_{I-1}}^{+}\mu_{\mathbb{I}_{I-2}}^{+}\right)\right]},\\\begin{bmatrix}\left(\eta_{\mathbb{I}_{R-1}}^{-}+\eta_{\mathbb{I}_{R-2}}^{-}-\eta_{\mathbb{I}_{R-1}}^{-}\eta_{\mathbb{I}_{R-2}}^{-}\right),\\\left(\eta_{\mathbb{I}_{R-1}}^{+}+\eta_{\mathbb{I}_{R-2}}^{+}-\eta_{\mathbb{I}_{R-1}}^{+}\eta_{\mathbb{I}_{R-2}}^{+}\right)\end{bmatrix}\\ e^{i2\pi\left[\left(\eta_{\mathbb{I}_{I-1}}^{-}+\eta_{\mathbb{I}_{I-2}}^{-}-\eta_{\mathbb{I}_{I-1}}^{-}\eta_{\mathbb{I}_{I-2}}^{-}\right),\left(\eta_{\mathbb{I}_{I-1}}^{+}+\eta_{\mathbb{I}_{I-2}}^{+}-\eta_{\mathbb{I}_{I-1}}^{+}\eta_{\mathbb{I}_{I-2}}^{+}\right)\right]}\end{pmatrix}\tag{10}$$

$$\varphi\mathbb{I}_{CIF-1}=\begin{pmatrix}\left[1-\left(1-\mu_{\mathbb{I}_{R-1}}^{-}\right)^{\varphi},1-\left(1-\mu_{\mathbb{I}_{R-1}}^{+}\right)^{\varphi}\right]\\ e^{i2\pi\left(\left[1-\left(1-\mu_{\mathbb{I}_{I-1}}^{-}\right)^{\varphi},1-\left(1-\mu_{\mathbb{I}_{I-1}}^{+}\right)^{\varphi}\right]\right)},\\\left[\eta^{-\varphi}_{\mathbb{I}_{R-1}},\eta^{+\varphi}_{\mathbb{I}_{R-1}}\right]e^{i2\pi\left(\left[\eta^{-\varphi}_{\mathbb{I}_{I-1}},\eta^{+\varphi}_{\mathbb{I}_{I-1}}\right]\right)}\end{pmatrix}\tag{11}$$

$$\mathbb{I}^{\varphi}_{CIF-1} = \begin{pmatrix} \left[\mu^{-\varphi}_{\mathbb{I}_{R-1}}, \mu^{+\varphi}_{\mathbb{I}_{R-1}}\right] e^{i2\pi\left(\left[\mu^{-\varphi}_{\mathbb{I}_{I-1}}, \mu^{+\varphi}_{\mathbb{I}_{I-1}}\right]\right)}, \\ \left[1-\left(1-\eta^{-}_{\mathbb{I}_{R-1}}\right)^{\varphi}, 1-\left(1-\eta^{+}_{\mathbb{I}_{R-1}}\right)^{\varphi}\right] \\ e^{i2\pi\left(\left[1-\left(1-\eta^{-}_{\mathbb{I}_{I-1}}\right)^{\varphi}, 1-\left(1-\eta^{+}_{\mathbb{I}_{I-1}}\right)^{\varphi}\right]\right)} \end{pmatrix} \quad (12)$$

Still, many authors have faced a serious problem, how we are finding the relation between any two CIVIFNs. For this, we review the mathematical shape of two different ideas, explained below.

Definition 6 [20]: The theory of CIVIFNs $\mathbb{I}_{CIF-\ell} = \left(\left[\mu^{-}_{\mathbb{I}_{R-\ell}}, \mu^{+}_{\mathbb{I}_{R-\ell}}\right] e^{i2\pi\left(\left[\mu^{-}_{\mathbb{I}_{I-\ell}}, \mu^{+}_{\mathbb{I}_{I-\ell}}\right]\right)}, \left[\eta^{-}_{\mathbb{I}_{R-\ell}}, \eta^{+}_{\mathbb{I}_{R-\ell}}\right] e^{i2\pi\left(\left[\eta^{-}_{\mathbb{I}_{I-\ell}}, \eta^{+}_{\mathbb{I}_{I-\ell}}\right]\right)}\right), \ell = 1$ based on a universal set $\mathbb{U}_{UN}$. Then the function

$$\mathbb{S}^{SV}\left(\mathbb{I}_{CIF-1}\right) = \frac{1}{4}\left(\mu^{-}_{\mathbb{I}_{R-1}} - \eta^{-}_{\mathbb{I}_{R-1}} + \mu^{-}_{\mathbb{I}_{I-1}} - \eta^{-}_{\mathbb{I}_{I-1}} + \mu^{+}_{\mathbb{I}_{R-1}} - \eta^{+}_{\mathbb{I}_{R-1}} + \mu^{+}_{\mathbb{I}_{I-1}} - \eta^{+}_{\mathbb{I}_{I-1}}\right), \quad \mathbb{S}^{SV}\left(\mathbb{I}_{IF-1}\right) \in [-1, 1] \quad (13)$$

Called SV and the function

$$\mathbb{H}^{SV}\left(\mathbb{I}_{CIF-1}\right) = \frac{1}{4}\left(\mu^{-}_{\mathbb{I}_{R-1}} + \eta^{-}_{\mathbb{I}_{R-1}} + \mu^{-}_{\mathbb{I}_{I-1}} + \eta^{-}_{\mathbb{I}_{I-1}} + \mu^{+}_{\mathbb{I}_{R-1}} + \eta^{+}_{\mathbb{I}_{R-1}} + \mu^{+}_{\mathbb{I}_{I-1}} + \eta^{+}_{\mathbb{I}_{I-1}}\right), \quad \mathbb{S}^{SV}\left(\mathbb{I}_{IF-1}\right) \in [-1, 1] \quad (14)$$

Called AV. Further, based on Eq. (13) and Eq. (14), we deliberated various rules: when $\mathbb{S}^{SV}\left(\mathbb{I}_{CIF-1}\right) > \mathbb{S}^{SV}\left(\mathbb{I}_{CIF-2}\right)$, then we assume that $\mathbb{I}_{CIF-1} > \mathbb{I}_{CIF-2}$, similarly, if $\mathbb{S}^{SV}\left(\mathbb{I}_{CIF-1}\right) = \mathbb{S}^{SV}\left(\mathbb{I}_{CIF-2}\right)$, we use Eq. (13), if $\mathbb{H}^{SV}\left(\mathbb{I}_{CIF-1}\right) > \mathbb{H}^{SV}\left(\mathbb{I}_{CIF-2}\right)$, then we assume that $\mathbb{I}_{CIF-1} > \mathbb{I}_{CIF-2}$, if $\mathbb{H}^{SV}\left(\mathbb{I}_{CIF-1}\right) = \mathbb{H}^{SV}\left(\mathbb{I}_{CIF-2}\right)$, then we assume that $\mathbb{I}_{CIF-1} = \mathbb{I}_{CIF-2}$.

Definition 7 [24]: Assume $\mathbb{E}_P$ be the set of parameters, the duplet $\left(\mathbb{F}_F, \mathbb{E}_P\right)$, represented the SS based on a universal set $\mathbb{U}_{UN}$, where $\mathbb{F}_F : \mathbb{E}_P \rightarrow P\left(\mathbb{U}_{UN}\right)$, where $P\left(\mathbb{U}_{UN}\right)$, representing the collection of all possible subsets of a universal set $\mathbb{U}_{UN}$.

III. CICFS SETTINGS

In this analysis, the fundamental theory of CICFS information is diagnosed which is a very informative and effective tool for handling ambiguity and complications in reality. Further, in the consideration of CICFS information, we utilized the important algebraic laws, score value, and accuracy value and try to determine some rules for finding the relation between any CICFS numbers.

Definition 8: The theory of the CICFS set is a duplet $\left(\mathbb{F}_F, \mathbb{E}_P\right)$ based on a universal set $\mathbb{U}_{UN}$ iff $\mathbb{F}_F : \mathbb{E}_P \rightarrow CICF\left(\mathbb{U}_{UN}\right)$, where $CICF\left(\mathbb{U}_{UN}\right)$, representing the collection of all possible cubic intuitionistic complex fuzzy subsets of a universal set $\mathbb{U}_{UN}$ is deliberated by:

$$\mathbb{F}_{e_k}\left(x_\ell\right) = \left\{\begin{pmatrix} x_\ell, \left(\mu_{\mathbb{I}_{CIF-k}}\left(x_\ell\right), \eta_{\mathbb{I}_{CIF-k}}\left(x_\ell\right)\right), \\ \left(\mu_{\mathbb{I}_{IF-k}}\left(x_\ell\right), \eta_{\mathbb{I}_{IF-k}}\left(x_\ell\right)\right) \end{pmatrix} : x \in \mathbb{U}_{UN}\right\} \quad (15)$$

In Eq. (15), the term $\mu_{\mathbb{I}_{CIF-k}}\left(x_\ell\right) = \left[\mu^{-}_{\mathbb{I}_{R-k}}\left(x_\ell\right), \mu^{+}_{\mathbb{I}_{R-k}}\left(x_\ell\right)\right] e^{i2\pi\left(\left[\mu^{-}_{\mathbb{I}_{I-k}}\left(x_\ell\right), \mu^{+}_{\mathbb{I}_{I-k}}\left(x_\ell\right)\right]\right)}$, $\mu_{\mathbb{I}_{IF-k}}\left(x_\ell\right) = \mu_{\mathbb{I}_{R-k}}\left(x_\ell\right) e^{i2\pi\left(\mu_{\mathbb{I}_{I-k}}\left(x_\ell\right)\right)}$ and $\eta_{\mathbb{I}_{CIF-k}}\left(x_\ell\right) = \left[\eta^{-}_{\mathbb{I}_{R-k}}\left(x_\ell\right), \eta^{+}_{\mathbb{I}_{R-k}}\left(x_\ell\right)\right] e^{i2\pi\left(\left[\eta^{-}_{\mathbb{I}_{I-k}}\left(x_\ell\right), \eta^{+}_{\mathbb{I}_{I-k}}\left(x_\ell\right)\right]\right)}$, $\eta_{\mathbb{I}_{IF-k}}\left(x_\ell\right) = \eta_{\mathbb{I}_{R-k}}\left(x_\ell\right) e^{i2\pi\left(\eta_{\mathbb{I}_{I-k}}\left(x_\ell\right)\right)}$ are represented the well-known truth and falsity grades in the shape of complex numbers with techniques: $0 \leq \mu^{+}_{\mathbb{I}_{R-k}}\left(x_\ell\right) + \eta^{+}_{\mathbb{I}_{R-k}}\left(x_\ell\right) \leq 1, 0 \leq \mu_{\mathbb{I}_{R-k}}\left(x_\ell\right) + \eta_{\mathbb{I}_{R-k}}\left(x_\ell\right) \leq 1$ and $0 \leq \mu^{+}_{\mathbb{I}_{I-k}}\left(x_\ell\right) + \eta^{+}_{\mathbb{I}_{I-k}}\left(x_\ell\right) \leq 1, 0 \leq \mu_{\mathbb{I}_{I-k}}\left(x_\ell\right) + \eta_{\mathbb{I}_{I-k}}\left(x_\ell\right) \leq 1$. For more diagnoses, the refusal grade is represented by:

$$\begin{aligned} \xi_{\mathbb{I}_{CIF-k}}\left(x_\ell\right) &= \left[\xi^{-}_{\mathbb{I}_{R-k}}\left(x_\ell\right), \xi^{+}_{\mathbb{I}_{R-k}}\left(x_\ell\right)\right] e^{i2\pi\left(\left[\xi^{-}_{\mathbb{I}_{I-k}}\left(x_\ell\right), \xi^{+}_{\mathbb{I}_{I-k}}\left(x_\ell\right)\right]\right)} \\ &= \begin{bmatrix} \left(1-\left(\mu^{-}_{\mathbb{I}_{R-k}}\left(x_\ell\right) + \eta^{-}_{\mathbb{I}_{R-k}}\left(x_\ell\right)\right)\right), \\ \left(1-\left(\mu^{+}_{\mathbb{I}_{R-k}}\left(x_\ell\right) + \eta^{+}_{\mathbb{I}_{R-k}}\left(x_\ell\right)\right)\right) \end{bmatrix} \\ &\quad \times e^{i2\pi\begin{bmatrix} \left(1-\left(\mu^{-}_{\mathbb{I}_{I-k}}\left(x_\ell\right) + \eta^{-}_{\mathbb{I}_{I-k}}\left(x_\ell\right)\right)\right), \\ \left(1-\left(\mu^{+}_{\mathbb{I}_{I-k}}\left(x_\ell\right) + \eta^{+}_{\mathbb{I}_{I-k}}\left(x_\ell\right)\right)\right) \end{bmatrix}}, \end{aligned}$$

$$\begin{aligned} \xi_{\mathbb{I}_{IF-k}}\left(x_\ell\right) &= \xi_{\mathbb{I}_{R-k}}\left(x_\ell\right) e^{i2\pi\left(\xi_{\mathbb{I}_{I-k}}\left(x_\ell\right)\right)} \\ &= \left(1-\left(\mu_{\mathbb{I}_{R-k}}\left(x_\ell\right) + \eta_{\mathbb{I}_{R-k}}\left(x_\ell\right)\right)\right) \\ &\quad \times e^{i2\pi\left(1-\left(\mu_{\mathbb{I}_{I-k}}\left(x_\ell\right) + \eta_{\mathbb{I}_{I-k}}\left(x_\ell\right)\right)\right)}. \end{aligned}$$

Finally, throughout this manuscript, the CICFS numbers (CICFSNs) are represented by:

$$\mathbb{I}_{\ell k} = \left(\begin{pmatrix} \left[\mu^{-}_{\mathbb{I}_{R-\ell k}}, \mu^{+}_{\mathbb{I}_{R-\ell k}}\right] e^{i2\pi\left(\left[\mu^{-}_{\mathbb{I}_{I-\ell k}}, \mu^{+}_{\mathbb{I}_{I-\ell k}}\right]\right)}, \\ \left[\eta^{-}_{\mathbb{I}_{R-\ell k}}, \eta^{+}_{\mathbb{I}_{R-\ell k}}\right] e^{i2\pi\left(\left[\eta^{-}_{\mathbb{I}_{I-\ell k}}, \eta^{+}_{\mathbb{I}_{I-\ell k}}\right]\right)} \end{pmatrix}, \left(\mu_{\mathbb{I}_{R-\ell k}} e^{i2\pi\left(\mu_{\mathbb{I}_{I-\ell k}}\right)}, \eta_{\mathbb{I}_{R-\ell k}} e^{i2\pi\left(\eta_{\mathbb{I}_{I-\ell k}}\right)}\right)\right),$$

$$\ell = 1, 2, \ldots, \mathfrak{n}.$$

Definition 9: The theory of CICFSNs

$$\mathbb{I}_{1k} = \left(\begin{pmatrix} \left[\mu^{-}_{\mathbb{I}_{R-1k}}, \mu^{+}_{\mathbb{I}_{R-1k}}\right] e^{i2\pi\left(\left[\mu^{-}_{\mathbb{I}_{I-1k}}, \mu^{+}_{\mathbb{I}_{I-1k}}\right]\right)}, \\ \left[\eta^{-}_{\mathbb{I}_{R-1k}}, \eta^{+}_{\mathbb{I}_{R-1k}}\right] e^{i2\pi\left(\left[\eta^{-}_{\mathbb{I}_{I-1k}}, \eta^{+}_{\mathbb{I}_{I-1k}}\right]\right)} \end{pmatrix}, \left(\mu_{\mathbb{I}_{R-1k}} e^{i2\pi\left(\mu_{\mathbb{I}_{I-1k}}\right)}, \eta_{\mathbb{I}_{R-1k}} e^{i2\pi\left(\eta_{\mathbb{I}_{I-1k}}\right)}\right)\right),$$

$$k = 1, 2$$

based on a universal set $\mathbb{U}_{UN}$, we have

$$\mathbb{I}_{11} \oplus \mathbb{I}_{12}$$

$$= \left(\begin{pmatrix} \begin{bmatrix} \left(\mu^-_{\mathbb{I}_{R-11}} + \mu^-_{\mathbb{I}_{R-12}} - \mu^-_{\mathbb{I}_{R-11}}\mu^-_{\mathbb{I}_{R-12}}\right), \\ \left(\mu^+_{\mathbb{I}_{R-11}} + \mu^+_{\mathbb{I}_{R-12}} - \mu^+_{\mathbb{I}_{R-11}}\mu^+_{\mathbb{I}_{R-12}}\right) \end{bmatrix} \\ e^{i2\pi\begin{bmatrix} \left(\mu^-_{\mathbb{I}_{I-11}} + \mu^-_{\mathbb{I}_{I-12}} - \mu^-_{\mathbb{I}_{I-11}}\mu^-_{\mathbb{I}_{I-12}}\right), \\ \left(\mu^+_{\mathbb{I}_{I-11}} + \mu^+_{\mathbb{I}_{I-12}} - \mu^+_{\mathbb{I}_{I-11}}\mu^+_{\mathbb{I}_{I-12}}\right) \end{bmatrix}}, \\ \left[\left(\eta^-_{\mathbb{I}_{R-11}}\eta^-_{\mathbb{I}_{R-12}}\right), \left(\eta^+_{\mathbb{I}_{R-11}}\eta^+_{\mathbb{I}_{R-12}}\right)\right] \\ e^{i2\pi\left[\left(\eta^-_{\mathbb{I}_{I-11}}\eta^-_{\mathbb{I}_{I-12}}\right), \left(\eta^+_{\mathbb{I}_{I-11}}\eta^+_{\mathbb{I}_{I-12}}\right)\right]} \end{pmatrix}, \\ \begin{pmatrix} \left(\mu_{\mathbb{I}_{R-11}} + \mu_{\mathbb{I}_{R-12}} - \mu_{\mathbb{I}_{R-11}}\mu_{\mathbb{I}_{R-12}}\right) \\ e^{i2\pi\left(\mu_{\mathbb{I}_{I-11}} + \mu_{\mathbb{I}_{I-12}} - \mu_{\mathbb{I}_{I-11}}\mu_{\mathbb{I}_{I-12}}\right)}, \\ \left(\eta_{\mathbb{I}_{R-11}}\eta_{\mathbb{I}_{R-12}}\right) e^{i2\pi\left(\eta_{\mathbb{I}_{I-11}}\eta_{\mathbb{I}_{I-12}}\right)} \end{pmatrix}\right) \tag{16}$$

Before discussing the proof of Eq. (16), we revise the idea of t-norm and t-conorm. A function $T : X \times X \to [0, 1]$, called triangular norm (T-norm), if it satisfied the commutative, associative, monotonicity, and neutral element "1" such that $T(x, 1) = x$. Example, $T(x, y) = xy$, represented the T-norm. similarly, a function $S : X \times X \to [0, 1]$, called triangular conorm (T-conorm), if it satisfied the commutative, associative, monotonicity, and neutral element "0" such that $T(x, 0) = x$. Example, $S(x, y) = x + y - xy$, represented the T-conorm. In Eq. (16), we have involved two norms, called t-norm $\left(\eta^-_{\mathbb{I}_{R-11}}\eta^-_{\mathbb{I}_{R-12}}\right)$, $\left(\eta^+_{\mathbb{I}_{R-11}}\eta^+_{\mathbb{I}_{R-12}}\right)$, $\left(\eta^-_{\mathbb{I}_{I-11}}\eta^-_{\mathbb{I}_{I-12}}\right)$, $\left(\eta^+_{\mathbb{I}_{I-11}}\eta^+_{\mathbb{I}_{I-12}}\right)$, $\left(\eta_{\mathbb{I}_{R-11}}\eta_{\mathbb{I}_{R-12}}\right)$, $\left(\eta_{\mathbb{I}_{I-11}}\eta_{\mathbb{I}_{I-12}}\right) \in [0, 1]$ and t-conorm

$$\begin{aligned} &\left(\mu^-_{\mathbb{I}_{R-11}} + \mu^-_{\mathbb{I}_{R-12}} - \mu^-_{\mathbb{I}_{R-11}}\mu^-_{\mathbb{I}_{R-12}}\right), \\ &\quad\left(\mu^+_{\mathbb{I}_{R-11}} + \mu^+_{\mathbb{I}_{R-12}} - \mu^+_{\mathbb{I}_{R-11}}\mu^+_{\mathbb{I}_{R-12}}\right), \\ &\quad\left(\mu^-_{\mathbb{I}_{I-11}} + \mu^-_{\mathbb{I}_{I-12}} - \mu^-_{\mathbb{I}_{I-11}}\mu^-_{\mathbb{I}_{I-12}}\right), \\ &\quad\left(\mu^+_{\mathbb{I}_{I-11}} + \mu^+_{\mathbb{I}_{I-12}} - \mu^+_{\mathbb{I}_{I-11}}\mu^+_{\mathbb{I}_{I-12}}\right), \\ &\quad\left(\mu_{\mathbb{I}_{R-11}} + \mu_{\mathbb{I}_{R-12}} - \mu_{\mathbb{I}_{R-11}}\mu_{\mathbb{I}_{R-12}}\right), \\ &\quad\left(\mu_{\mathbb{I}_{I-11}} + \mu_{\mathbb{I}_{I-12}} - \mu_{\mathbb{I}_{I-11}}\mu_{\mathbb{I}_{I-12}}\right) \in [0, 1], \end{aligned}$$

and obviously, the resultant value should be belonging to the unit interval. Hence, $\mathbb{I}_{11} \oplus \mathbb{I}_{12} \in [0, 1]$.

$$\mathbb{I}_{11} \otimes \mathbb{I}_{12} = \left(\begin{pmatrix} \left[\left(\mu^-_{\mathbb{I}_{R-11}}\mu^-_{\mathbb{I}_{R-12}}\right), \left(\mu^+_{\mathbb{I}_{R-11}}\mu^+_{\mathbb{I}_{R-12}}\right)\right] \\ e^{i2\pi\left[\left(\mu^-_{\mathbb{I}_{I-11}}\mu^-_{\mathbb{I}_{I-12}}\right), \left(\mu^+_{\mathbb{I}_{I-11}}\mu^+_{\mathbb{I}_{I-12}}\right)\right]} \\ \begin{bmatrix} \left(\eta^-_{\mathbb{I}_{R-11}} + \eta^-_{\mathbb{I}_{R-12}} - \eta^-_{\mathbb{I}_{R-11}}\eta^-_{\mathbb{I}_{R-12}}\right), \\ \left(\eta^+_{\mathbb{I}_{R-11}} + \eta^+_{\mathbb{I}_{R-12}} - \eta^+_{\mathbb{I}_{R-11}}\eta^+_{\mathbb{I}_{R-12}}\right) \end{bmatrix}, \\ e^{i2\pi\begin{bmatrix} \left(\eta^-_{\mathbb{I}_{I-11}} + \eta^-_{\mathbb{I}_{I-12}} - \eta^-_{\mathbb{I}_{I-11}}\eta^-_{\mathbb{I}_{I-12}}\right), \\ \left(\eta^+_{\mathbb{I}_{I-11}} + \eta^+_{\mathbb{I}_{I-12}} - \eta^+_{\mathbb{I}_{I-11}}\eta^+_{\mathbb{I}_{I-12}}\right) \end{bmatrix}} \end{pmatrix}, \\ \begin{pmatrix} \left(\mu_{\mathbb{I}_{R-11}}\mu_{\mathbb{I}_{R-12}}\right) e^{i2\pi\left(\mu_{\mathbb{I}_{I-11}}\mu_{\mathbb{I}_{I-12}}\right)}, \\ \left(\eta_{\mathbb{I}_{R-11}} + \eta_{\mathbb{I}_{R-12}} - \eta_{\mathbb{I}_{R-11}}\eta_{\mathbb{I}_{R-12}}\right) \\ e^{i2\pi\left(\eta_{\mathbb{I}_{I-11}} + \eta_{\mathbb{I}_{I-12}} - \eta_{\mathbb{I}_{I-11}}\eta_{\mathbb{I}_{I-12}}\right)} \end{pmatrix}\right) \tag{17}$$

$$\varphi\mathbb{I}_{11} = \left(\begin{pmatrix} \left[1 - \left(1 - \mu^-_{\mathbb{I}_{R-11}}\right)^\varphi, 1 - \left(1 - \mu^+_{\mathbb{I}_{R-11}}\right)^\varphi\right] \\ e^{i2\pi\left(\left[1 - \left(1 - \mu^-_{\mathbb{I}_{I-11}}\right)^\varphi, 1 - \left(1 - \mu^+_{\mathbb{I}_{I-11}}\right)^\varphi\right]\right)}, \\ \left[\eta^{-\varphi}_{\mathbb{I}_{R-11}}, \eta^{+\varphi}_{\mathbb{I}_{R-11}}\right] e^{i2\pi\left(\left[\eta^{-\varphi}_{\mathbb{I}_{I-11}}, \eta^{+\varphi}_{\mathbb{I}_{I-11}}\right]\right)} \end{pmatrix}, \\ \begin{pmatrix} 1 - \left(1 - \mu_{\mathbb{I}_{R-11}}\right)^\varphi e^{i2\pi\left(1 - \left(1 - \mu_{\mathbb{I}_{I-11}}\right)^\varphi\right)}, \\ \eta^\varphi_{\mathbb{I}_{R-11}} e^{i2\pi\left(\eta^\varphi_{\mathbb{I}_{I-11}}\right)} \end{pmatrix}\right) \tag{18}$$

$$\mathbb{I}^\varphi_{11} = \left(\begin{pmatrix} \left[\mu^{-\varphi}_{\mathbb{I}_{R-11}}, \mu^{+\varphi}_{\mathbb{I}_{R-11}}\right] e^{i2\pi\left(\left[\mu^{-\varphi}_{\mathbb{I}_{I-11}}, \mu^{+\varphi}_{\mathbb{I}_{I-11}}\right]\right)}, \\ \left[1 - \left(1 - \eta^-_{\mathbb{I}_{R-11}}\right)^\varphi, 1 - \left(1 - \eta^+_{\mathbb{I}_{R-11}}\right)^\varphi\right] \\ e^{i2\pi\left(\left[1 - \left(1 - \eta^-_{\mathbb{I}_{I-11}}\right)^\varphi, 1 - \left(1 - \eta^+_{\mathbb{I}_{I-11}}\right)^\varphi\right]\right)} \end{pmatrix}, \\ \begin{pmatrix} \mu^\varphi_{\mathbb{I}_{R-11}} e^{i2\pi\left(\mu^\varphi_{\mathbb{I}_{I-11}}\right)}, \\ 1 - \left(1 - \eta_{\mathbb{I}_{R-11}}\right)^\varphi e^{i2\pi\left(1 - \left(1 - \eta_{\mathbb{I}_{I-11}}\right)^\varphi\right)} \end{pmatrix}\right) \tag{19}$$

Definition 10: The theory of CICFSNs

$$\mathbb{I}_{1k} = \left(\begin{pmatrix} \left[\mu^-_{\mathbb{I}_{R-1k}}, \mu^+_{\mathbb{I}_{R-1k}}\right] e^{i2\pi\left(\left[\mu^-_{\mathbb{I}_{I-1k}}, \mu^+_{\mathbb{I}_{I-1k}}\right]\right)}, \\ \left[\eta^-_{\mathbb{I}_{R-1k}}, \eta^+_{\mathbb{I}_{R-1k}}\right] e^{i2\pi\left(\left[\eta^-_{\mathbb{I}_{I-1k}}, \eta^+_{\mathbb{I}_{I-1k}}\right]\right)} \end{pmatrix}, \\ \left(\mu_{\mathbb{I}_{R-1k}} e^{i2\pi\left(\mu_{\mathbb{I}_{I-1k}}\right)}, \eta_{\mathbb{I}_{R-1k}} e^{i2\pi\left(\eta_{\mathbb{I}_{I-1k}}\right)}\right)\right), k = 1$$

based on a universal set $\mathbb{U}_{UN}$. Then the function

$$\begin{aligned} \mathbb{S}^{SV}(\mathbb{I}_{11}) = \frac{1}{2}\Bigg(&\frac{1}{4}\Big(\mu^-_{\mathbb{I}_{R-11}} - \eta^-_{\mathbb{I}_{R-11}} + \mu^-_{\mathbb{I}_{I-11}} - \eta^-_{\mathbb{I}_{I-11}} \\ &+ \mu^+_{\mathbb{I}_{R-11}} - \eta^+_{\mathbb{I}_{R-11}} + \mu^+_{\mathbb{I}_{I-11}} - \eta^+_{\mathbb{I}_{I-11}}\Big) \\ &+ \frac{1}{2}\left(\mu_{\mathbb{I}_{R-11}} - \eta_{\mathbb{I}_{R-11}} + \mu_{\mathbb{I}_{I-11}} - \eta_{\mathbb{I}_{I-11}}\right)\Bigg), \\ &\mathbb{S}^{SV}(\mathbb{I}_{11}) \in [-1, 1] \end{aligned} \tag{20}$$

Called SV and the function

$$\begin{aligned} \mathbb{H}^{SV}(\mathbb{I}_{11}) = \frac{1}{2}\Bigg(&\frac{1}{4}\Big(\mu^-_{\mathbb{I}_{R-11}} + \eta^-_{\mathbb{I}_{R-11}} + \mu^-_{\mathbb{I}_{I-11}} + \eta^-_{\mathbb{I}_{I-11}} \\ &+ \mu^+_{\mathbb{I}_{R-11}} + \eta^+_{\mathbb{I}_{R-11}} + \mu^+_{\mathbb{I}_{I-11}} + \eta^+_{\mathbb{I}_{I-11}}\Big) \\ &+ \frac{1}{2}\left(\mu_{\mathbb{I}_{R-11}} + \eta_{\mathbb{I}_{R-11}} + \mu_{\mathbb{I}_{I-11}} + \eta_{\mathbb{I}_{I-11}}\right)\Bigg), \\ &\mathbb{S}^{SV}(\mathbb{I}_{IF-1}) \in [-1, 1] \end{aligned} \tag{21}$$

Called AV. Further, based on Eq. (20) and Eq. (21), we deliberated various rules: when $\mathbb{S}^{SV}(\mathbb{I}_{11}) > \mathbb{S}^{SV}(\mathbb{I}_{12})$, then we assume that $\mathbb{I}_{11} > \mathbb{I}_{12}$, similarly, if $\mathbb{S}^{SV}(\mathbb{I}_{11}) = \mathbb{S}^{SV}(\mathbb{I}_{12})$, we use Eq. (21), if $\mathbb{H}^{SV}(\mathbb{I}_{11}) > \mathbb{H}^{SV}(\mathbb{I}_{12})$, then we assume that $\mathbb{I}_{11} > \mathbb{I}_{12}$, if $\mathbb{H}^{SV}(\mathbb{I}_{11}) = \mathbb{H}^{SV}(\mathbb{I}_{12})$, then we assume that $\mathbb{I}_{CIF-1} = \mathbb{I}_{CIF-2}$.

IV. BONFERRONI MEAN OPERATORS USING CICFS INFORMATION

In this analysis, using the presented information, we diagnosed the CICFSBM operator, CICFSWBM operator and evaluated their important results, and described their important properties (Idempotency, Monotonicity, and Boundedness).

Definition 11: The theory of the CICFSBM (CICFSBM) operator based on the collection of CICFSNs

$$\mathbb{I}_{\ell k}=\left(\left(\begin{array}{c}\left[\mu^{-}_{\mathbb{I}_{R-\ell k}},\mu^{+}_{\mathbb{I}_{R-\ell k}}\right]e^{i2\pi\left(\left[\mu^{-}_{\mathbb{I}_{I-\ell k}},\mu^{+}_{\mathbb{I}_{I-\ell k}}\right]\right)},\\ \left[\eta^{-}_{\mathbb{I}_{R-\ell k}},\eta^{+}_{\mathbb{I}_{R-\ell k}}\right]e^{i2\pi\left(\left[\eta^{-}_{\mathbb{I}_{I-\ell k}},\eta^{+}_{\mathbb{I}_{I-\ell k}}\right]\right)}\end{array}\right),\left(\mu_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\mu_{\mathbb{I}_{I-\ell k}}\right)},\eta_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\eta_{\mathbb{I}_{I-\ell k}}\right)}\right)\right),$$

deliberated by:

$$CICFSBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11},\mathbb{I}_{12},\ldots,\mathbb{I}_{\mathbb{nm}}\right)=\left(\frac{1}{\mathbb{mn}\left(\mathbb{m}-1\right)\left(\mathbb{n}-1\right)}\oplus^{\mathbb{m}}_{\substack{k,j=1\\k\neq j}}\oplus^{\mathbb{n}}_{\substack{\ell,t=1\\ \ell\neq t}}\left(\mathbb{I}^{\mathcal{U}}_{\ell k}\otimes\mathbb{I}^{\mathcal{V}}_{jt}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\tag{22}$$

where $\mathcal{U},\mathcal{V}\geq 0$.

Theorem 1: The main theory of the CICFSBM operator is initiated in the shape of Eq. (23), by using Eq. (22) and Def. (9), we Eq. (23), as shown at the bottom of page 8, and Eq. (I), as shown at the bottom of page 9.

Proof: To prove the Eq. (23), we needed to follow the mathematical induction technique, such that

Step 1: When $\mathbb{n}=2$, then

$$CICFSBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11},\mathbb{I}_{12},\ldots,\mathbb{I}_{2\mathbb{m}}\right)=\left(\frac{1}{2\mathbb{m}\left(\mathbb{m}-1\right)}\oplus^{\mathbb{m}}_{\substack{k,j=1\\k\neq j}}\oplus^{2}_{\substack{\ell,t=1\\ \ell\neq t}}\left(\mathbb{I}^{\mathcal{U}}_{\ell k}\otimes\mathbb{I}^{\mathcal{V}}_{jt}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}$$

then,

$$\oplus^{2}_{\substack{\ell,t=1\\ \ell\neq t}}\left(\mathbb{I}^{\mathcal{U}}_{\ell k}\otimes\mathbb{I}^{\mathcal{V}}_{jt}\right)=\left(\mathbb{I}^{\mathcal{U}}_{1k}\otimes\mathbb{I}^{\mathcal{V}}_{2t}\right)\oplus\left(\mathbb{I}^{\mathcal{U}}_{2k}\otimes\mathbb{I}^{\mathcal{V}}_{1t}\right)$$

$$=\left(\left(\begin{array}{c}\left[\mu^{-\ \mathcal{U}}_{\mathbb{I}_{R-1k}}\mu^{-\ \mathcal{V}}_{\mathbb{I}_{R-2t}},\mu^{+\ \mathcal{U}}_{\mathbb{I}_{R-1k}}\mu^{+\ \mathcal{V}}_{\mathbb{I}_{R-2t}}\right]\\ e^{i2\pi\left(\left[\mu^{-\ \mathcal{U}}_{\mathbb{I}_{I-1k}}\mu^{-\ \mathcal{V}}_{\mathbb{I}_{I-2t}},\mu^{+\ \mathcal{U}}_{\mathbb{I}_{I-1k}}\mu^{+\ \mathcal{V}}_{\mathbb{I}_{I-2t}}\right]\right)},\\ \left[\begin{array}{c}1-\left(1-\eta^{-}_{\mathbb{I}_{R-1k}}\right)^{\mathcal{U}}\left(1-\eta^{-}_{\mathbb{I}_{R-2t}}\right)^{\mathcal{V}},\\ 1-\left(1-\eta^{+}_{\mathbb{I}_{R-1k}}\right)^{\mathcal{U}}\left(1-\eta^{+}_{\mathbb{I}_{R-2t}}\right)^{\mathcal{V}}\end{array}\right]\\ e^{i2\pi\left(\left[\begin{array}{c}1-\left(1-\eta^{-}_{\mathbb{I}_{I-1k}}\right)^{\mathcal{U}}\left(1-\eta^{-}_{\mathbb{I}_{I-2t}}\right)^{\mathcal{V}},\\ 1-\left(1-\eta^{+}_{\mathbb{I}_{I-1k}}\right)^{\mathcal{U}}\left(1-\eta^{+}_{\mathbb{I}_{I-2t}}\right)^{\mathcal{V}}\end{array}\right]\right)}\end{array}\right),\left(\begin{array}{c}\mu^{\mathcal{U}}_{\mathbb{I}_{R-1k}}\mu^{\mathcal{V}}_{\mathbb{I}_{R-2t}}e^{i2\pi\left(\mu^{\mathcal{U}}_{\mathbb{I}_{I-1k}}\mu^{\mathcal{V}}_{\mathbb{I}_{I-2t}}\right)},\\ \left(1-\left(1-\eta_{\mathbb{I}_{R-1k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-2t}}\right)^{\mathcal{V}}\right)\\ e^{i2\pi\left(1-\left(1-\eta_{\mathbb{I}_{I-1k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-2t}}\right)^{\mathcal{V}}\right)}\end{array}\right)\right)$$

$$\oplus\left(\left(\begin{array}{c}\left[\mu^{-\ \mathcal{U}}_{\mathbb{I}_{R-2k}}\mu^{-\ \mathcal{V}}_{\mathbb{I}_{R-1t}},\mu^{+\ \mathcal{U}}_{\mathbb{I}_{R-2k}}\mu^{+\ \mathcal{V}}_{\mathbb{I}_{R-1t}}\right]\\ e^{i2\pi\left(\left[\mu^{-\ \mathcal{U}}_{\mathbb{I}_{I-2k}}\mu^{-\ \mathcal{V}}_{\mathbb{I}_{I-1t}},\mu^{+\ \mathcal{U}}_{\mathbb{I}_{I-2k}}\mu^{+\ \mathcal{V}}_{\mathbb{I}_{I-1t}}\right]\right)},\\ \left[\begin{array}{c}1-\left(1-\eta^{-}_{\mathbb{I}_{R-2k}}\right)^{\mathcal{U}}\left(1-\eta^{-}_{\mathbb{I}_{R-1t}}\right)^{\mathcal{V}},\\ 1-\left(1-\eta^{+}_{\mathbb{I}_{R-2k}}\right)^{\mathcal{U}}\left(1-\eta^{+}_{\mathbb{I}_{R-1t}}\right)^{\mathcal{V}}\end{array}\right]\\ e^{i2\pi\left(\left[\begin{array}{c}1-\left(1-\eta^{-}_{\mathbb{I}_{I-2k}}\right)^{\mathcal{U}}\left(1-\eta^{-}_{\mathbb{I}_{I-1t}}\right)^{\mathcal{V}},\\ 1-\left(1-\eta^{+}_{\mathbb{I}_{I-2k}}\right)^{\mathcal{U}}\left(1-\eta^{+}_{\mathbb{I}_{I1t}}\right)^{\mathcal{V}}\end{array}\right]\right)}\end{array}\right),\left(\begin{array}{c}\mu^{\mathcal{U}}_{\mathbb{I}_{R-2k}}\mu^{\mathcal{V}}_{\mathbb{I}_{R-1t}}e^{i2\pi\left(\mu^{\mathcal{U}}_{\mathbb{I}_{I-2k}}\mu^{\mathcal{V}}_{\mathbb{I}_{I-1t}}\right)},\\ \left(1-\left(1-\eta_{\mathbb{I}_{R-2k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-1t}}\right)^{\mathcal{V}}\right)\\ e^{i2\pi\left(1-\left(1-\eta_{\mathbb{I}_{I-2k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-1t}}\right)^{\mathcal{V}}\right)}\end{array}\right)\right)$$

Hence, we have Eq. (II), as shown at the top of page 10. Similarly, when $\mathbb{m}=2$, then we have Eq. (III), as shown at the top of page 11.

The obtained theory holds for $\mathbb{m}=2$ and $\mathbb{n}=2$.

Step 2: When we fixed the value of $\mathbb{m}=s_1$, $\mathbb{n}=s_2+1$ and $\mathbb{m}=s_1+1$, $\mathbb{n}=s_2$, then for any $\mathbb{m}=s_1+1$, $\mathbb{n}=s_2+1$, we have

$$\begin{aligned}&\oplus^{s_1+1}_{\substack{k,j=1\\k\neq j}}\oplus^{s_2+1}_{\substack{\ell,t=1\\ \ell\neq t}}\left(\mathbb{I}^{\mathcal{U}}_{\ell k}\otimes\mathbb{I}^{\mathcal{V}}_{jt}\right)\\&=\left(\oplus^{s_1}_{\substack{k,j=1\\k\neq j}}\oplus^{s_2+1}_{\substack{\ell,t=1\\ \ell\neq t}}\left(\mathbb{I}^{\mathcal{U}}_{\ell k}\otimes\mathbb{I}^{\mathcal{V}}_{jt}\right)\right)\\&\quad\oplus\left(\oplus^{s_1}_{k=1}\oplus^{s_2+1}_{\substack{\ell,t=1\\ \ell\neq t}}\left(\mathbb{I}^{\mathcal{U}}_{\ell k}\otimes\mathbb{I}^{\mathcal{V}}_{j(s_1+1)}\right)\right)\\&\quad\oplus\left(\oplus^{s_1}_{k=1}\oplus^{s_2+1}_{\substack{\ell,t=1\\ \ell\neq t}}\left(\mathbb{I}^{\mathcal{U}}_{\ell(s_1+1)}\otimes\mathbb{I}^{\mathcal{V}}_{jt}\right)\right)\end{aligned}$$

First, to prove the above analysis, we have Eq. (IV), as shown at the top of page 12.

Step 2a: When $s_2=1$, then we have Eq. (V), as shown at the top of page 13.

Step 2b: When $s_2=z$, then we have Eq. (VI), as shown at the top of page 14.

Similarly, for $s_2=z+1$, we have Eq. (VII), as shown at the top of pages 15–17.

The result holds for $s_2=z+1$, further, we have Eq. (VIII), as shown at the top of page 18.

Similarly, we can prove Eq. (IX), as shown at the top of page 19.

Then, by using the above all analysis, we get Eq. (X), as shown at the top of pages 20 and 21.

Hence, the result is proved.

Additionally, using the information in Eq. (23), we illustrated various properties.

Property 1 (Idempotency): If $\mathbb{I}_{\ell k}=\mathbb{I}=\left(\left(\left[\mu^{-}_{\mathbb{I}_R},\mu^{+}_{\mathbb{I}_R}\right]e^{i2\pi\left(\left[\mu^{-}_{\mathbb{I}_I},\mu^{+}_{\mathbb{I}_I}\right]\right)},\left[\eta^{-}_{\mathbb{I}_R},\eta^{+}_{\mathbb{I}_R}\right]e^{i2\pi\left(\left[\eta^{-}_{\mathbb{I}_I},\eta^{+}_{\mathbb{I}_I}\right]\right)}\right),\left(\mu_{\mathbb{I}_R}e^{i2\pi\left(\mu_{\mathbb{I}_I}\right)},\eta_{\mathbb{I}_R}e^{i2\pi\left(\eta_{\mathbb{I}_I}\right)}\right)\right)$, deliberated by:

$$CICFSBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11}, \mathbb{I}_{12}, \ldots, \mathbb{I}_{\mathbb{n}\mathbb{m}}\right) = \mathbb{I} \tag{24}$$

Proof: Considered the $\mathbb{I}_{\ell k} = \mathbb{I} = \left(\begin{array}{c}\left(\left[\mu^{-}_{\mathbb{I}_R}, \mu^{+}_{\mathbb{I}_R}\right] e^{i2\pi\left(\left[\mu^{-}_{\mathbb{I}_I}, \mu^{+}_{\mathbb{I}_I}\right]\right)}, \left[\eta^{-}_{\mathbb{I}_R}, \eta^{+}_{\mathbb{I}_R}\right] e^{i2\pi\left(\left[\eta^{-}_{\mathbb{I}_I}, \eta^{+}_{\mathbb{I}_I}\right]\right)}\right), \\ \left(\mu_{\mathbb{I}_R} e^{i2\pi\left(\mu_{\mathbb{I}_I}\right)}, \eta_{\mathbb{I}_R} e^{i2\pi\left(\eta_{\mathbb{I}_I}\right)}\right)\end{array}\right)$, then using Eq. (23), we have Eq. (XI), as shown at the bottom of pages 22–24.

Hence, the result is proved.

Property 2 (Boundedness): If

$$\mathbb{I}^{+} = \left(\begin{array}{c}\left(\begin{array}{c}\left[\max\limits_{k}\max\limits_{\ell}\mu^{-}_{\mathbb{I}_{R-\ell k}}, \max\limits_{k}\max\limits_{\ell}\mu^{+}_{\mathbb{I}_{R-\ell k}}\right] e^{i2\pi\left(\left[\max\limits_{k}\max\limits_{\ell}\mu^{-}_{\mathbb{I}_{I-\ell k}}, \max\limits_{k}\max\limits_{\ell}\mu^{+}_{\mathbb{I}_{I-\ell k}}\right]\right)}, \\ \left[\min\limits_{k}\min\limits_{\ell}\eta^{-}_{\mathbb{I}_{R-\ell k}}, \min\limits_{k}\min\limits_{\ell}\eta^{+}_{\mathbb{I}_{R-\ell k}}\right] e^{i2\pi\left(\left[\min\limits_{k}\min\limits_{\ell}\eta^{-}_{\mathbb{I}_{I-\ell k}}, \min\limits_{k}\min\limits_{\ell}\eta^{+}_{\mathbb{I}_{I-\ell k}}\right]\right)}\end{array}\right), \\ \left(\begin{array}{c}\max\limits_{k}\max\limits_{\ell}\mu_{\mathbb{I}_{R-\ell k}} e^{i2\pi\left(\max\limits_{k}\max\limits_{\ell}\mu_{\mathbb{I}_{I-\ell k}}\right)}, \\ \min\limits_{k}\min\limits_{\ell}\eta_{\mathbb{I}_{R-\ell k}} e^{i2\pi\left(\min\limits_{k}\min\limits_{\ell}\eta_{\mathbb{I}_{I-\ell k}}\right)}\end{array}\right)\end{array}\right)$$

and

$$CICFSBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11}, \mathbb{I}_{12}, \ldots, \mathbb{I}_{\mathbb{n}\mathbb{m}}\right) = (A, B)$$

$$A = \left(\begin{array}{c}\left[\begin{array}{c}\left(1-\left(\prod\limits_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod\limits_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-{\mu^{-}_{\mathbb{I}_{R-\ell k}}}^{\mathcal{U}}{\mu^{-}_{\mathbb{I}_{R-jt}}}^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}, \\ \left(1-\left(\prod\limits_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod\limits_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-{\mu^{+}_{\mathbb{I}_{R-\ell k}}}^{\mathcal{U}}{\mu^{+}_{\mathbb{I}_{R-jt}}}^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\end{array}\right] e^{i2\pi\left[\begin{array}{c}\left(1-\left(\prod\limits_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod\limits_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-{\mu^{-}_{\mathbb{I}_{I-\ell k}}}^{\mathcal{U}}{\mu^{-}_{\mathbb{I}_{I-jt}}}^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}, \\ \left(1-\left(\prod\limits_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod\limits_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-{\mu^{+}_{\mathbb{I}_{I-\ell k}}}^{\mathcal{U}}{\mu^{+}_{\mathbb{I}_{I-jt}}}^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\end{array}\right]}, \\ \left[\begin{array}{c}1-\left(1-\prod\limits_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod\limits_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^{-}_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^{-}_{\mathbb{I}_{R-jt}}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}, \\ 1-\left(1-\prod\limits_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod\limits_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^{+}_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^{+}_{\mathbb{I}_{R-jt}}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\end{array}\right] e^{i2\pi\left[\begin{array}{c}1-\left(1-\prod\limits_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod\limits_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^{-}_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^{-}_{\mathbb{I}_{I-jt}}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}, \\ 1-\left(1-\prod\limits_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod\limits_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^{+}_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^{+}_{\mathbb{I}_{I-jt}}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\end{array}\right]}\end{array}\right) \tag{23}$$

$$\mathbb{I}^- = \left(\begin{array}{l}\left(\begin{array}{l}\left[\min_k \min_\ell \mu^-_{\mathbb{I}_{R-\ell k}}, \min_k \min_\ell \mu^+_{\mathbb{I}_{R-\ell k}}\right] e^{i2\pi\left(\left[\min_k \min_\ell \mu^-_{\mathbb{I}_{I-\ell k}}, \min_k \min_\ell \mu^+_{\mathbb{I}_{I-\ell k}}\right]\right)}, \\ \left[\max_k \max_\ell \eta^-_{\mathbb{I}_{R-\ell k}}, \max_k \max_\ell \eta^+_{\mathbb{I}_{R-\ell k}}\right] e^{i2\pi\left(\left[\max_k \max_\ell \eta^-_{\mathbb{I}_{I-\ell k}}, \max_k \max_\ell \eta^+_{\mathbb{I}_{I-\ell k}}\right]\right)}\end{array}\right), \\ \left(\begin{array}{l}\min_k \min_\ell \mu_{\mathbb{I}_{R-\ell k}} e^{i2\pi\left(\min_k \min_\ell \mu_{\mathbb{I}_{I-\ell k}}\right)}, \\ \max_k \max_\ell \eta_{\mathbb{I}_{R-\ell k}} e^{i2\pi\left(\max_k \max_\ell \eta_{\mathbb{I}_{I-\ell k}}\right)}\end{array}\right)\end{array}\right),$$

deliberated by:

$$\mathbb{I}^+ \leq CICFSBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11}, \mathbb{I}_{12}, \ldots, \mathbb{I}_{\mathbb{n}\mathbb{m}}\right) \leq \mathbb{I}^- \qquad (25)$$

Proof: Assume that

$$\mathbb{I}^+ = \left(\begin{array}{l}\left(\begin{array}{l}\left[\max_k \max_\ell \mu^-_{\mathbb{I}_{R-\ell k}}, \max_k \max_\ell \mu^+_{\mathbb{I}_{R-\ell k}}\right] e^{i2\pi\left(\left[\max_k \max_\ell \mu^-_{\mathbb{I}_{I-\ell k}}, \max_k \max_\ell \mu^+_{\mathbb{I}_{I-\ell k}}\right]\right)}, \\ \left[\min_k \min_\ell \eta^-_{\mathbb{I}_{R-\ell k}}, \min_k \min_\ell \eta^+_{\mathbb{I}_{R-\ell k}}\right] e^{i2\pi\left(\left[\min_k \min_\ell \eta^-_{\mathbb{I}_{I-\ell k}}, \min_k \min_\ell \eta^+_{\mathbb{I}_{I-\ell k}}\right]\right)}\end{array}\right), \\ \left(\begin{array}{l}\max_k \max_\ell \mu_{\mathbb{I}_{R-\ell k}} e^{i2\pi\left(\max_k \max_\ell \mu_{\mathbb{I}_{I-\ell k}}\right)}, \\ \min_k \min_\ell \eta_{\mathbb{I}_{R-\ell k}} e^{i2\pi\left(\min_k \min_\ell \eta_{\mathbb{I}_{I-\ell k}}\right)}\end{array}\right)\end{array}\right) \text{ and}$$

$$\mathbb{I}^- = \left(\begin{array}{l}\left(\begin{array}{l}\left[\min_k \min_\ell \mu^-_{\mathbb{I}_{R-\ell k}}, \min_k \min_\ell \mu^+_{\mathbb{I}_{R-\ell k}}\right] e^{i2\pi\left(\left[\min_k \min_\ell \mu^-_{\mathbb{I}_{I-\ell k}}, \min_k \min_\ell \mu^+_{\mathbb{I}_{I-\ell k}}\right]\right)}, \\ \left[\max_k \max_\ell \eta^-_{\mathbb{I}_{R-\ell k}}, \max_k \max_\ell \eta^+_{\mathbb{I}_{R-\ell k}}\right] e^{i2\pi\left(\left[\max_k \max_\ell \eta^-_{\mathbb{I}_{I-\ell k}}, \max_k \max_\ell \eta^+_{\mathbb{I}_{I-\ell k}}\right]\right)}\end{array}\right), \\ \left(\begin{array}{l}\min_k \min_\ell \mu_{\mathbb{I}_{R-\ell k}} e^{i2\pi\left(\min_k \min_\ell \mu_{\mathbb{I}_{I-\ell k}}\right)}, \\ \max_k \max_\ell \eta_{\mathbb{I}_{R-\ell k}} e^{i2\pi\left(\max_k \max_\ell \eta_{\mathbb{I}_{I-\ell k}}\right)}\end{array}\right)\end{array}\right), \text{ with}$$

$\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)} = \sigma$ then

$$\begin{aligned}
&\min_k \min_\ell \mu^+_{\mathbb{I}_{R-\ell k}} \\
&\leq \mu^+_{\mathbb{I}_{R-\ell k}} \leq \max_k \max_\ell \mu^+_{\mathbb{I}_{R-\ell k}} \\
\Longleftrightarrow &\left(\min_k \min_\ell \mu^+_{\mathbb{I}_{R-\ell k}} \mu^+_{\mathbb{I}_{R-jt}}\right)^{\mathcal{U}+\mathcal{V}} \\
&\leq {\mu^+_{\mathbb{I}_{R-\ell k}}}^{\mathcal{U}} {\mu^+_{\mathbb{I}_{R-jt}}}^{\mathcal{V}} \\
&\leq \left(\max_k \max_\ell \mu^+_{\mathbb{I}_{R-\ell k}} \mu^+_{\mathbb{I}_{R-jt}}\right)^{\mathcal{U}+\mathcal{V}} \\
\Longleftrightarrow &\ 1 - \left(\min_k \min_\ell \mu^+_{\mathbb{I}_{R-\ell k}} \mu^+_{\mathbb{I}_{R-jt}}\right)^{\mathcal{U}+\mathcal{V}} \\
&\leq 1 - {\mu^+_{\mathbb{I}_{R-\ell k}}}^{\mathcal{U}} {\mu^+_{\mathbb{I}_{R-jt}}}^{\mathcal{V}}
\end{aligned}$$

$$B = \left(\begin{array}{l}\left(1 - \left(\prod_{\substack{k,j=1 \\ k\neq j}}^{\mathbb{m}} \prod_{\substack{\ell,t=1 \\ \ell\neq t}}^{\mathbb{n}} \left(1 - \mu^{\mathcal{U}}_{\mathbb{I}_{R-\ell k}} \mu^{\mathcal{V}}_{\mathbb{I}_{R-jt}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} e^{i2\pi\left(1 - \left(\prod_{\substack{k,j=1 \\ k\neq j}}^{\mathbb{m}} \prod_{\substack{\ell,t=1 \\ \ell\neq t}}^{\mathbb{n}} \left(1 - \mu^{\mathcal{U}}_{\mathbb{I}_{I-\ell k}} \mu^{\mathcal{V}}_{\mathbb{I}_{I-jt}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}}, \\ \left(1 - \left(1 - \prod_{\substack{k,j=1 \\ k\neq j}}^{\mathbb{m}} \prod_{\substack{\ell,t=1 \\ \ell\neq t}}^{\mathbb{n}} \left(1 - \left(1-\eta_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}} \left(1-\eta_{\mathbb{I}_{R-jt}}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right) e^{i2\pi\left(1 - \left(1 - \prod_{\substack{k,j=1 \\ k\neq j}}^{\mathbb{m}} \prod_{\substack{\ell,t=1 \\ \ell\neq t}}^{\mathbb{n}} \left(1 - \left(1-\eta_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}} \left(1-\eta_{\mathbb{I}_{I-jt}}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right)}\end{array}\right) \qquad \text{(I)}$$

$$\left(\frac{1}{2\mathbb{m}\left(\mathbb{m}-1\right)}\oplus_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\oplus_{\substack{\ell,t=1\\\ell\neq t}}^{2}\left(\mathbb{I}_{\ell k}^{\mathcal{U}}\otimes\mathbb{I}_{jt}^{\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}$$

$$=\left(\begin{array}{l}\left(\begin{array}{l}\left[\begin{array}{l}\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{2}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-jt}}^{-}{}^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{m}(\mathbb{m}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{2}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-jt}}^{+}{}^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{m}(\mathbb{m}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\end{array}\right]\\e^{i2\pi\left[\begin{array}{l}\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{2}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-jt}}^{-}{}^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{m}(\mathbb{m}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{2}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-jt}}^{+}{}^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{m}(\mathbb{m}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\end{array}\right]},\\\left[\begin{array}{l}1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{2}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-jt}}^{-}\right)^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{m}(\mathbb{m}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{2}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-jt}}^{+}\right)^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{m}(\mathbb{m}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\end{array}\right]\\e^{i2\pi\left[\begin{array}{l}1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{2}\left(\left(\genfrac{}{}{0pt}{}{1-}{\eta_{\mathbb{I}_{I-\ell k}}^{-}}\right)^{\genfrac{}{}{0pt}{}{1-}{\mathcal{U}}}\left(\genfrac{}{}{0pt}{}{1-}{\eta_{\mathbb{I}_{I-jt}}^{-}}\right)^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{m}(\mathbb{m}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{2}\left(\left(\genfrac{}{}{0pt}{}{1-}{\eta_{\mathbb{I}_{I-\ell k}}^{+}}\right)^{\genfrac{}{}{0pt}{}{1-}{\mathcal{U}}}\left(\genfrac{}{}{0pt}{}{1-}{\eta_{\mathbb{I}_{I-jt}}^{+}}\right)^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{m}(\mathbb{m}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\end{array}\right]}\end{array}\right),\\\left(\begin{array}{l}\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{2}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{\mathcal{U}}\mu_{\mathbb{I}_{R-jt}}^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{m}(\mathbb{m}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\\e^{i2\pi\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{2}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{\mathcal{U}}\mu_{\mathbb{I}_{I-jt}}^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{m}(\mathbb{m}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}},\\\left(1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{2}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-jt}}\right)^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{m}(\mathbb{m}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\\e^{i2\pi\left(1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{2}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-jt}}\right)^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{m}(\mathbb{m}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}}\end{array}\right)\end{array}\right)\quad\text{(II)}$$

$$
\begin{aligned}
&CICFSBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11},\mathbb{I}_{12},\ldots,\mathbb{I}_{\mathbb{n}2}\right)\\
&=\left(\frac{1}{2\mathbb{n}\left(\mathbb{n}-1\right)}\oplus_{\substack{k,j=1\\k\neq j}}^{2}\oplus_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(\mathbb{I}_{\ell k}^{\mathcal{U}}\otimes\mathbb{I}_{jt}^{\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\\
&=\left(\begin{array}{c}
\left(\begin{array}{c}
\left[\begin{array}{c}
\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{2}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{-\ \mathcal{U}}\mu_{\mathbb{I}_{R-jt}}^{-\ \mathcal{V}}\right)^{\frac{1}{2\mathbb{n}(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\
\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{2}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{+\ \mathcal{U}}\mu_{\mathbb{I}_{R-jt}}^{+\ \mathcal{V}}\right)^{\frac{1}{2\mathbb{n}(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}
\end{array}\right]
e^{i2\pi\left[\begin{array}{c}
\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{2}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{-\ \mathcal{U}}\mu_{\mathbb{I}_{I-jt}}^{-\ \mathcal{V}}\right)^{\frac{1}{2\mathbb{n}(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\
\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{2}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{+\ \mathcal{U}}\mu_{\mathbb{I}_{I-jt}}^{+\ \mathcal{V}}\right)^{\frac{1}{2\mathbb{n}(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}
\end{array}\right]},\\
\left[\begin{array}{c}
1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{2}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-jt}}^{-}\right)^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{n}(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\
1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{2}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-jt}}^{+}\right)^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{n}(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}
\end{array}\right]
e^{i2\pi\left[\begin{array}{c}
1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{2}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-jt}}^{-}\right)^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{n}(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\
1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{2}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-jt}}^{+}\right)^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{n}(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}
\end{array}\right]}
\end{array}\right),\\
\left(\begin{array}{c}
\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{2}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{\mathcal{U}}\mu_{\mathbb{I}_{R-jt}}^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{n}(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}
e^{i2\pi\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{2}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{\mathcal{U}}\mu_{\mathbb{I}_{I-jt}}^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{n}(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}},\\
\left(\begin{array}{c}
\left(1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{2}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-jt}}\right)^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{n}(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right)\\
e^{i2\pi\left(1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{2}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-jt}}\right)^{\mathcal{V}}\right)^{\frac{1}{2\mathbb{n}(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right)}
\end{array}\right)
\end{array}\right)
\end{array}\right)
\end{aligned}
\quad \text{(III)}
$$

$$
\oplus_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(\mathbb{I}_{\ell k}^{\mathcal{U}}\otimes\mathbb{I}_{j(s_1+1)}^{\mathcal{V}}\right)=
\left(\begin{array}{c}
\left(\begin{array}{c}
\left[\begin{array}{c}
1-\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{-}{}^{\mathcal{V}}\right)\right),\\
1-\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{+}{}^{\mathcal{V}}\right)\right)
\end{array}\right]^{\frac{1}{\mathcal{U}+\mathcal{V}}}
e^{i2\pi\left[\begin{array}{c}
\left(1-\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{-}{}^{\mathcal{V}}\right)\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\
\left(1-\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{+}{}^{\mathcal{V}}\right)\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}
\end{array}\right]},\\
\left[\begin{array}{c}
\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}^{-}\right)^{\mathcal{V}}\right),\\
\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}^{+}\right)^{\mathcal{V}}\right)
\end{array}\right]
e^{i2\pi\left[\begin{array}{c}
\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}^{-}\right)^{\mathcal{V}}\right)\\
,\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}^{+}\right)^{\mathcal{V}}\right)
\end{array}\right]}
\end{array}\right),\\
\left(\begin{array}{c}
1-\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{\mathcal{V}}\right)
e^{i2\pi\left(1-\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{\mathcal{V}}\right)\right)},\\
\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}\right)^{\mathcal{V}}\right)\right)
e^{i2\pi\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}\right)^{\mathcal{V}}\right)\right)}
\end{array}\right)
\end{array}\right)
\quad \text{(IV)}
$$

$$
\begin{aligned}
&\oplus_{\substack{\ell,t=1\\ \ell\neq t}}^{2}\left(\mathbb{I}_{\ell k}^{\mathcal{U}}\otimes\mathbb{I}_{j(s_1+1)}^{\mathcal{V}}\right)\\
&=\left(\begin{pmatrix}
\begin{bmatrix}
1-\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{2}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{-}{}^{\mathcal{V}}\right)\right),\\
1-\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{2}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{+}{}^{\mathcal{V}}\right)\right)
\end{bmatrix}
e^{i2\pi\begin{bmatrix}
\left(1-\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{2}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{-}{}^{\mathcal{V}}\right)\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\
\left(1-\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{2}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{+}{}^{\mathcal{V}}\right)\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}
\end{bmatrix}},\\
\begin{bmatrix}
\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{2}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}^{-}\right)^{\mathcal{V}}\right),\\
\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{2}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}^{+}\right)^{\mathcal{V}}\right)
\end{bmatrix}
e^{i2\pi\begin{bmatrix}
\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{2}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}^{-}\right)^{\mathcal{V}}\right)\\
,\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{2}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}^{+}\right)^{\mathcal{V}}\right)
\end{bmatrix}}
\end{pmatrix},\right.\\
&\left.\qquad\begin{pmatrix}
1-\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{2}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{\mathcal{V}}\right)
e^{i2\pi\left(1-\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{2}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{\mathcal{V}}\right)\right)},\\
\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{2}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}\right)^{\mathcal{V}}\right)\right)
e^{i2\pi\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{2}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}\right)^{\mathcal{V}}\right)\right)}
\end{pmatrix}\right)
\end{aligned}
\tag{V}
$$

$$
\begin{aligned}
&\oplus_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(\mathbb{I}_{\ell k}^{\mathcal{U}}\otimes\mathbb{I}_{j(s_1+1)}^{\mathcal{V}}\right)\\
&=\left(\begin{pmatrix}
\left[\begin{array}{c}
1-\left(\prod\limits_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{-}{}^{\mathcal{V}}\right)\right),\\
1-\left(\prod\limits_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{+}{}^{\mathcal{V}}\right)\right)
\end{array}\right]
e^{i2\pi\left[\begin{array}{c}
\left(1-\left(\prod\limits_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{-}{}^{\mathcal{V}}\right)\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\
\left(1-\left(\prod\limits_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{+}{}^{\mathcal{V}}\right)\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}
\end{array}\right]},\\
\left[\begin{array}{c}
\prod\limits_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}^{-}\right)^{\mathcal{V}}\right),\\
\prod\limits_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}^{+}\right)^{\mathcal{V}}\right)
\end{array}\right]
e^{i2\pi\left[\begin{array}{c}
\prod\limits_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}^{-}\right)^{\mathcal{V}}\right),\\
\prod\limits_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}^{+}\right)^{\mathcal{V}}\right)
\end{array}\right]}
\end{pmatrix},\right.\\
&\qquad\left.\begin{pmatrix}
1-\prod\limits_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{\mathcal{V}}\right)
e^{i2\pi\left(1-\prod\limits_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{\mathcal{V}}\right)\right)},\\
\left(\prod\limits_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}\right)^{\mathcal{V}}\right)\right)
e^{i2\pi\left(\prod\limits_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}\right)^{\mathcal{V}}\right)\right)}
\end{pmatrix}\right)
\end{aligned}
\tag{VI}
$$

$$\oplus_{\substack{\ell,t=1\\ \ell\neq t}}^{z+2}\left(\mathbb{I}_{\ell k}^{\mathcal{U}}\otimes\mathbb{I}_{j(s_1+1)}^{\mathcal{V}}\right)=\left(\oplus_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(\mathbb{I}_{\ell k}^{\mathcal{U}}\otimes\mathbb{I}_{j(s_1+1)}^{\mathcal{V}}\right)\right)\oplus\left(\oplus_{\ell=1}^{z+1}\left(\mathbb{I}_{\ell(z+2)}^{\mathcal{U}}\otimes\mathbb{I}_{j(s_1+1)}^{\mathcal{V}}\right)\right)\oplus\left(\oplus_{\ell=1}^{z+1}\left(\mathbb{I}_{\ell k}^{\mathcal{U}}\otimes\mathbb{I}_{(z+2)(s_1+1)}^{\mathcal{V}}\right)\right)$$

$$=\left(\begin{pmatrix}\left[\begin{array}{c}1-\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{-}{}^{\mathcal{V}}\right)\right),\\ 1-\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{+}{}^{\mathcal{V}}\right)\right)\end{array}\right] e^{i2\pi\left[\begin{array}{c}\left(1-\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{-}{}^{\mathcal{V}}\right)\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\ \left(1-\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{+}{}^{\mathcal{V}}\right)\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\end{array}\right]},\\ \left[\begin{array}{c}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}^{-}\right)^{\mathcal{V}}\right),\\ \prod_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}^{+}\right)^{\mathcal{V}}\right)\end{array}\right] e^{i2\pi\left[\begin{array}{c}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}^{-}\right)^{\mathcal{V}}\right),\\ \prod_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}^{+}\right)^{\mathcal{V}}\right)\end{array}\right]}\end{pmatrix},\begin{pmatrix}1-\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{\mathcal{V}}\right) e^{i2\pi\left(1-\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{z+2}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{\mathcal{V}}\right)\right)},\\ \left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{z+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}\right)^{\mathcal{V}}\right)\right) e^{i2\pi\left(\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{z+2}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}\right)^{\mathcal{V}}\right)\right)}\end{pmatrix}\right)$$

$$\left(\left(\begin{array}{c}\left[\begin{array}{c}1-\left(\prod_{\ell=1}^{z+1}\left(1-\mu^{-}_{\mathbb{I}_{R-\ell(z+2)}}{}^{\mathcal{U}}\mu^{-}_{\mathbb{I}_{R-j(s_1+1)}}{}^{\mathcal{V}}\right)\right),\\ 1-\left(\prod_{\ell=1}^{z+1}\left(1-\mu^{+}_{\mathbb{I}_{R-\ell(z+2)}}{}^{\mathcal{U}}\mu^{+}_{\mathbb{I}_{R-j(s_1+1)}}{}^{\mathcal{V}}\right)\right)\end{array}\right]\\ e^{i2\pi\left[\begin{array}{c}\left(1-\left(\prod_{\ell=1}^{z+1}\left(1-\mu^{-}_{\mathbb{I}_{I-\ell(z+2)}}{}^{\mathcal{U}}\mu^{-}_{\mathbb{I}_{I-j(s_1+1)}}{}^{\mathcal{V}}\right)\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\ \left(1-\left(\prod_{\ell=1}^{z+1}\left(1-\mu^{+}_{\mathbb{I}_{I-\ell(z+2)}}{}^{\mathcal{U}}\mu^{+}_{\mathbb{I}_{I-j(s_1+1)}}{}^{\mathcal{V}}\right)\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\end{array}\right]},\\ \left[\begin{array}{c}\prod_{\ell=1}^{z+1}\left(1-\left(1-\eta^{-}_{\mathbb{I}_{R-\ell(z+2)}}\right)^{\mathcal{U}}\left(1-\eta^{-}_{\mathbb{I}_{R-j(s_1+1)}}\right)^{\mathcal{V}}\right),\\ \prod_{\ell=1}^{z+1}\left(1-\left(1-\eta^{+}_{\mathbb{I}_{R-\ell(z+2)}}\right)^{\mathcal{U}}\left(1-\eta^{+}_{\mathbb{I}_{R-j(s_1+1)}}\right)^{\mathcal{V}}\right)\end{array}\right]\\ e^{i2\pi\left[\begin{array}{c}\prod_{\ell=1}^{z+1}\left(1-\left(1-\eta^{-}_{\mathbb{I}_{I-\ell(z+2)}}\right)^{\mathcal{U}}\left(1-\eta^{-}_{\mathbb{I}_{I-j(s_1+1)}}\right)^{\mathcal{V}}\right),\\ \prod_{\ell=1}^{z+1}\left(1-\left(1-\eta^{+}_{\mathbb{I}_{I-\ell(z+2)}}\right)^{\mathcal{U}}\left(1-\eta^{+}_{\mathbb{I}_{I-j(s_1+1)}}\right)^{\mathcal{V}}\right)\end{array}\right]}\end{array}\right),\right.$$

$$\left.\oplus\left(\begin{array}{c}1-\prod_{\ell=1}^{z+1}\left(1-\mu^{\mathcal{U}}_{\mathbb{I}_{R-\ell(z+2)}}\mu^{\mathcal{V}}_{\mathbb{I}_{R-j(s_1+1)}}\right)\\ e^{i2\pi\left(1-\prod_{\ell=1}^{z+2}\left(1-\mu^{\mathcal{U}}_{\mathbb{I}_{I-\ell(z+2)}}\mu^{\mathcal{V}}_{\mathbb{I}_{I-j(s_1+1)}}\right)\right)},\\ \left(\prod_{\ell=1}^{z+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell(z+2)}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}\right)^{\mathcal{V}}\right)\right)\\ e^{i2\pi\left(\prod_{\ell=1}^{z+2}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell(z+2)}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}\right)^{\mathcal{V}}\right)\right)}\end{array}\right)\right)$$

$$\leq 1-\left(\max_k\max_\ell \mu^{+}_{\mathbb{I}_{R-\ell k}}\mu^{+}_{\mathbb{I}_{R-jt}}\right)^{\mathcal{U}+\mathcal{V}}$$

$$\Longleftrightarrow 1-\left(\min_k\min_\ell \mu^{+}_{\mathbb{I}_{R-\ell k}}\mu^{+}_{\mathbb{I}_{R-jt}}\right)^{\mathcal{U}+\mathcal{V}}$$

$$\leq \prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\mu^{+}_{\mathbb{I}_{R-\ell k}}{}^{\mathcal{U}}\mu^{+}_{\mathbb{I}_{R-jt}}{}^{\mathcal{V}}\right)^{\sigma}$$

$$\leq 1-\left(\max_k\max_\ell \mu^{+}_{\mathbb{I}_{R-\ell k}}\mu^{+}_{\mathbb{I}_{R-jt}}\right)^{\mathcal{U}+\mathcal{V}}$$

$$\Longleftrightarrow \min_k\min_\ell \mu^{+}_{\mathbb{I}_{R-\ell k}}\mu^{+}_{\mathbb{I}_{R-jt}}$$

$$\leq \left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\mu^{+}_{\mathbb{I}_{R-\ell k}}{}^{\mathcal{U}}\mu^{+}_{\mathbb{I}_{R-jt}}{}^{\mathcal{V}}\right)^{\sigma}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}$$

$$\leq \max_k\max_\ell \mu^{+}_{\mathbb{I}_{R-\ell k}}\mu^{+}_{\mathbb{I}_{R-jt}}$$

Similarly, we get

$$\min_k\min_\ell \mu^{-}_{\mathbb{I}_{R-\ell k}}\mu^{-}_{\mathbb{I}_{R-jt}}$$

$$\leq \left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\mu^{-}_{\mathbb{I}_{R-\ell k}}{}^{\mathcal{U}}\mu^{-}_{\mathbb{I}_{R-jt}}{}^{\mathcal{V}}\right)^{\sigma}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}$$

$$\leq \max_k\max_\ell \mu^{-}_{\mathbb{I}_{R-\ell k}}\mu^{-}_{\mathbb{I}_{R-jt}}$$

$$\min_k\min_\ell \mu^{+}_{\mathbb{I}_{I-\ell k}}\mu^{+}_{\mathbb{I}_{I-jt}}$$

$$\leq \left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\mu^{+}_{\mathbb{I}_{I-\ell k}}{}^{\mathcal{U}}\mu^{+}_{\mathbb{I}_{I-jt}}{}^{\mathcal{V}}\right)^{\sigma}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}$$

$$\leq \max_k\max_\ell \mu^{+}_{\mathbb{I}_{I-\ell k}}\mu^{+}_{\mathbb{I}_{I-jt}}$$

$$\min_k\min_\ell \mu^{-}_{\mathbb{I}_{I-\ell k}}\mu^{-}_{\mathbb{I}_{I-jt}}$$

$$\oplus\left(\left(\begin{array}{c}\left[\begin{array}{c}1-\left(\prod_{\ell=1}^{z+1}\left(1-\mu^{-}_{\mathbb{I}_{R-\ell k}}{}^{\mathcal{U}}\mu^{-}_{\mathbb{I}_{R-(z+2)(s_1+1)}}{}^{\mathcal{V}}\right)\right),\\ 1-\left(\prod_{\ell=1}^{z+1}\left(1-\mu^{+}_{\mathbb{I}_{R-\ell k}}{}^{\mathcal{U}}\mu^{+}_{\mathbb{I}_{R-(z+2)(s_1+1)}}{}^{\mathcal{V}}\right)\right)\end{array}\right] \\ e^{i2\pi\left[\begin{array}{c}\left(1-\left(\prod_{\ell=1}^{z+1}\left(1-\mu^{-}_{\mathbb{I}_{I-\ell k}}{}^{\mathcal{U}}\mu^{-}_{\mathbb{I}_{I-(z+2)(s_1+1)}}{}^{\mathcal{V}}\right)\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\ \left(1-\left(\prod_{\ell=1}^{z+1}\left(1-\mu^{+}_{\mathbb{I}_{I-\ell k}}{}^{\mathcal{U}}\mu^{+}_{\mathbb{I}_{I-(z+2)(s_1+1)}}{}^{\mathcal{V}}\right)\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\end{array}\right]}, \\ \left[\begin{array}{c}\prod_{\ell=1}^{z+1}\left(1-\left(1-\eta^{-}_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^{-}_{\mathbb{I}_{R-(z+2)(s_1+1)}}\right)^{\mathcal{V}}\right),\\ \prod_{\ell=1}^{z+1}\left(1-\left(1-\eta^{+}_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^{+}_{\mathbb{I}_{R-(z+2)(s_1+1)}}\right)^{\mathcal{V}}\right)\end{array}\right] \\ e^{i2\pi\left[\begin{array}{c}\prod_{\ell=1}^{z+1}\left(1-\left(1-\eta^{-}_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^{-}_{\mathbb{I}_{I-(z+2)(s_1+1)}}\right)^{\mathcal{V}}\right),\\ \prod_{\ell=1}^{z+1}\left(1-\left(1-\eta^{+}_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^{+}_{\mathbb{I}_{I-(z+2)(s_1+1)}}\right)^{\mathcal{V}}\right)\end{array}\right]}\end{array}\right), \left(\begin{array}{c}1-\prod_{\ell=1}^{z+1}\left(1-\mu^{\mathcal{U}}_{\mathbb{I}_{R-\ell k}}\mu^{\mathcal{V}}_{\mathbb{I}_{R-(z+2)(s_1+1)}}\right) \\ e^{i2\pi\left(1-\prod_{\ell=1}^{z+2}\left(1-\mu^{\mathcal{U}}_{\mathbb{I}_{I-\ell k}}\mu^{\mathcal{V}}_{\mathbb{I}_{I-(z+2)(s_1+1)}}\right)\right)}, \\ \left(\prod_{\ell=1}^{z+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-(z+2)(s_1+1)}}\right)^{\mathcal{V}}\right)\right) \\ e^{i2\pi\left(\prod_{\ell=1}^{z+2}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-(z+2)(s_1+1)}}\right)^{\mathcal{V}}\right)\right)}\end{array}\right)\right) \tag{VII}$$

$$\leq\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\mu^{-}_{\mathbb{I}_{I-\ell k}}{}^{\mathcal{U}}\mu^{-}_{\mathbb{I}_{I-jt}}{}^{\mathcal{V}}\right)^{\sigma}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}$$

$$\leq \max_k\max_\ell \mu^{-}_{\mathbb{I}_{I-\ell k}}\mu^{-}_{\mathbb{I}_{I-jt}}$$

Further,

$$\min_k\min_\ell \eta^{+}_{\mathbb{I}_{R-\ell k}} \leq \eta^{+}_{\mathbb{I}_{R-\ell k}}$$

$$\leq \max_k\max_\ell \eta^{+}_{\mathbb{I}_{R-\ell k}}$$

$$\Longleftrightarrow \left(1-\min_k\min_\ell \eta^{+}_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}+\mathcal{V}}$$

$$\leq \left(1-\eta^{+}_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^{+}_{\mathbb{I}_{R-jt}}\right)^{\mathcal{V}}$$

$$\leq \left(1-\max_k\max_\ell \mu^{+}_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}+\mathcal{V}}$$

$$\Longleftrightarrow \left(1-\min_k\min_\ell \eta^{+}_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}+\mathcal{V}}$$

$$\leq \prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^{+}_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^{+}_{\mathbb{I}_{R-jt}}\right)^{\mathcal{V}}\right)^{\sigma}$$

$$\leq \left(1-\max_k\max_\ell \mu^{+}_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}+\mathcal{V}}$$

$$\Longleftrightarrow \left(1-\min_k\min_\ell \eta^{+}_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}+\mathcal{V}}$$

$$\leq 1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^{+}_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^{+}_{\mathbb{I}_{R-jt}}\right)^{\mathcal{V}}\right)^{\sigma}$$

$$\leq \left(1-\max_k\max_\ell \mu^{+}_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}+\mathcal{V}}$$

$$\oplus_{k=1}^{s_1}\oplus_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(\mathbb{I}_{\ell k}^{\mathcal{U}}\otimes\mathbb{I}_{j(s_1+1)}^{\mathcal{V}}\right)$$

$$= \left(\begin{array}{c}\left(\begin{array}{c}\left[\begin{array}{c}\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{-}{}^{\mathcal{V}}\right)\right)\right),\\ \left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{+}{}^{\mathcal{V}}\right)\right)\right)\end{array}\right] e^{i2\pi\left[\begin{array}{c}\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{-}{}^{\mathcal{V}}\right)\right)\right),\\ \left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{+}{}^{\mathcal{V}}\right)\right)\right)\end{array}\right]},\\ \left[\begin{array}{c}\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}^{-}\right)^{\mathcal{V}}\right),\\ \prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}^{+}\right)^{\mathcal{V}}\right)\end{array}\right] e^{i2\pi\left[\begin{array}{c}\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}^{-}\right)^{\mathcal{V}}\right),\\ \prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}^{+}\right)^{\mathcal{V}}\right)\end{array}\right]}\end{array}\right),\\ \left(\begin{array}{c}\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}{}^{\mathcal{V}}\right)\right)\right) e^{i2\pi\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}{}^{\mathcal{V}}\right)\right)\right)},\\ \prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}\right)^{\mathcal{V}}\right) e^{i2\pi\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}\right)^{\mathcal{V}}\right)\right)}\end{array}\right)\end{array}\right) \tag{VIII}$$

$$
\begin{aligned}
&\oplus_{k=1}^{s_1}\oplus_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(\mathbb{I}_{\ell(s_1+1)}^{\mathcal{U}}\otimes\mathbb{I}_{jt}^{\mathcal{V}}\right)\\
&=\left(\begin{array}{c}
\left(\begin{array}{c}
\left[\begin{array}{c}
\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{R-\ell(s_1+1)}}^{-\ \ \mathcal{U}}\mu_{\mathbb{I}_{R-jt}}^{-\ \ \mathcal{V}}\right)\right)\right),\\
\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{R-\ell(s_1+1)}}^{+\ \ \mathcal{U}}\mu_{\mathbb{I}_{R-jt}}^{+\ \ \mathcal{V}}\right)\right)\right)
\end{array}\right]
e^{i2\pi\left[\begin{array}{c}
\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{I-\ell(s_1+1)}}^{-\ \ \mathcal{U}}\mu_{\mathbb{I}_{I-jt}}^{-\ \ \mathcal{V}}\right)\right)\right),\\
\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{I-\ell(s_1+1)}}^{+\ \ \mathcal{U}}\mu_{\mathbb{I}_{I-jt}}^{+\ \ \mathcal{V}}\right)\right)\right)
\end{array}\right]},\\
\left[\begin{array}{c}
\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell(s_1+1)}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-jt}}^{-}\right)^{\mathcal{V}}\right),\\
\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell(s_1+1)}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-jt}}^{+}\right)^{\mathcal{V}}\right)
\end{array}\right]
e^{i2\pi\left[\begin{array}{c}
\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell(s_1+1)}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-jt}}^{-}\right)^{\mathcal{V}}\right)\\
,\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell(s_1+1)}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-jt}}^{+}\right)^{\mathcal{V}}\right)
\end{array}\right]}
\end{array}\right),\\
\left(\begin{array}{c}
\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{R-\ell(s_1+1)}}^{\ \ \mathcal{U}}\mu_{\mathbb{I}_{R-jt}}^{\ \ \mathcal{V}}\right)\right)\right)
e^{i2\pi\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{I-\ell(s_1+1)}}^{\ \ \mathcal{U}}\mu_{\mathbb{I}_{I-jt}}^{\ \ \mathcal{V}}\right)\right)\right)},\\
\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell(s_1+1)}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-jt}}\right)^{\mathcal{V}}\right)
e^{i2\pi\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell(s_1+1)}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-jt}}\right)^{\mathcal{V}}\right)\right)}
\end{array}\right)
\end{array}\right)
\end{aligned}
\tag{IX}
$$

$$\oplus_{\substack{k,j=1\\k\neq j}}^{s_1+1}\oplus_{\substack{\ell,t=1\\\ell\neq t}}^{s_2+1}\left(\mathbb{I}_{\ell k}^{\mathcal{U}}\otimes\mathbb{I}_{jt}^{\mathcal{V}}\right)=\left(\begin{array}{c}\left(\begin{array}{c}\left[\begin{array}{c}\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-jt}}^{-}{}^{\mathcal{V}}\right)\right)\right),\\\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-jt}}^{+}{}^{\mathcal{V}}\right)\right)\right)\end{array}\right]e^{i2\pi\left[\begin{array}{c}\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-jt}}^{-}{}^{\mathcal{V}}\right)\right)\right),\\\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-jt}}^{+}{}^{\mathcal{V}}\right)\right)\right)\end{array}\right]},\\\left[\begin{array}{c}\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-jt}}^{-}\right)^{\mathcal{V}}\right),\\\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-jt}}^{+}\right)^{\mathcal{V}}\right)\end{array}\right]e^{i2\pi\left[\begin{array}{c}\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-jt}}^{-}\right)^{\mathcal{V}}\right),\\\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-jt}}^{+}\right)^{\mathcal{V}}\right)\end{array}\right]}\end{array}\right),\\\left(\begin{array}{c}\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-jt}}{}^{\mathcal{V}}\right)\right)\right)e^{i2\pi\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-jt}}{}^{\mathcal{V}}\right)\right)\right)},\\\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-jt}}\right)^{\mathcal{V}}\right)e^{i2\pi\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-jt}}\right)^{\mathcal{V}}\right)\right)}\end{array}\right)\end{array}\right)$$

$$
\oplus\left(\begin{array}{c}
\left(\begin{array}{c}
\left[\begin{array}{c}
\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{-}{}^{\mathcal{V}}\right)\right)\right),\\
\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}^{+}{}^{\mathcal{V}}\right)\right)\right)
\end{array}\right]\\
e^{i2\pi\left[\begin{array}{c}
\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{-}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{-}{}^{\mathcal{V}}\right)\right)\right),\\
\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}^{+}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}^{+}{}^{\mathcal{V}}\right)\right)\right)
\end{array}\right]},\\
\left[\begin{array}{c}
\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}^{-}\right)^{\mathcal{V}}\right),\\
\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}^{+}\right)^{\mathcal{V}}\right)
\end{array}\right]\\
e^{i2\pi\left[\begin{array}{c}
\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{-}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}^{-}\right)^{\mathcal{V}}\right)\\
,\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{+}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}^{+}\right)^{\mathcal{V}}\right)
\end{array}\right]}
\end{array}\right),\\
\left(\begin{array}{c}
\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{R-\ell k}}{}^{\mathcal{U}}\mu_{\mathbb{I}_{R-j(s_1+1)}}{}^{\mathcal{V}}\right)\right)\right)\\
e^{i2\pi\left(1-\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\mu_{\mathbb{I}_{I-\ell k}}{}^{\mathcal{U}}\mu_{\mathbb{I}_{I-j(s_1+1)}}{}^{\mathcal{V}}\right)\right)\right)},\\
\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-j(s_1+1)}}\right)^{\mathcal{V}}\right)\\
e^{i2\pi\left(\prod_{k=1}^{s_1}\prod_{\substack{\ell,t=1\\ \ell\neq t}}^{s_2+1}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-j(s_1+1)}}\right)^{\mathcal{V}}\right)\right)}
\end{array}\right)
\end{array}\right) \qquad \text{(X)}
$$

$$\begin{aligned}
\iff\ & 1-\min_k\min_\ell \eta^+_{\mathbb{I}_{R-\ell k}} \\
\le\ & \left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^+_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^+_{\mathbb{I}_{R-jt}}\right)^{\mathcal{V}}\right)^{\sigma}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} \\
\le\ & 1-\max_k\max_\ell \mu^+_{\mathbb{I}_{R-\ell k}} \\
\iff\ & \min_k\min_\ell \eta^+_{\mathbb{I}_{R-\ell k}} \\
\le\ & 1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^+_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^+_{\mathbb{I}_{R-jt}}\right)^{\mathcal{V}}\right)^{\sigma}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} \\
\le\ & \max_k\max_\ell \mu^+_{\mathbb{I}_{R-\ell k}}
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
& \min_k\min_\ell \eta^-_{\mathbb{I}_{R-\ell k}} \\
&\quad\le 1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^-_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^-_{\mathbb{I}_{R-jt}}\right)^{\mathcal{V}}\right)^{\sigma}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} \\
&\quad\le \max_k\max_\ell \mu^-_{\mathbb{I}_{R-\ell k}} \\
& \min_k\min_\ell \eta^+_{\mathbb{I}_{I-\ell k}} \\
&\quad\le 1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^+_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^+_{\mathbb{I}_{I-jt}}\right)^{\mathcal{V}}\right)^{\sigma}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} \\
&\quad\le \max_k\max_\ell \mu^+_{\mathbb{I}_{I-\ell k}}
\end{aligned}$$

$$CICFSBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11},\mathbb{I}_{12},\ldots,\mathbb{I}_{\mathbb{n}\mathbb{m}}\right)$$

$$= \left(\begin{array}{c}
\left(\begin{array}{c}
\left[\begin{array}{c}
\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\mu^{-\ \mathcal{U}}_{\mathbb{I}_R}\mu^{-\ \mathcal{V}}_{\mathbb{I}_R}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}, \\
\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\mu^{+\ \mathcal{U}}_{\mathbb{I}_R}\mu^{+\ \mathcal{V}}_{\mathbb{I}_R}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}
\end{array}\right] \\
e^{i2\pi\left[\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\mu^{-\ \mathcal{U}}_{\mathbb{I}_I}\mu^{-\ \mathcal{V}}_{\mathbb{I}_I}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\mu^{+\ \mathcal{U}}_{\mathbb{I}_I}\mu^{+\ \mathcal{V}}_{\mathbb{I}_I}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right]}, \\
\left[\begin{array}{c}
1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^-_{\mathbb{I}_R}\right)^{\mathcal{U}}\left(1-\eta^-_{\mathbb{I}_R}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}, \\
1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^+_{\mathbb{I}_R}\right)^{\mathcal{U}}\left(1-\eta^+_{\mathbb{I}_R}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}
\end{array}\right] \\
e^{i2\pi\left[1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(\left(1-\eta^-_{\mathbb{I}_I}\right)^{\substack{1-\\\mathcal{U}}}\left(1-\eta^-_{\mathbb{I}_I}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(\left(1-\eta^+_{\mathbb{I}_I}\right)^{\substack{1-\\\mathcal{U}}}\left(1-\eta^+_{\mathbb{I}_I}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right]}
\end{array}\right), \\
\left(\begin{array}{c}
\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\mu^{\mathcal{U}}_{\mathbb{I}_R}\mu^{\mathcal{V}}_{\mathbb{I}_R}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} e^{i2\pi\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\mu^{\mathcal{U}}_{\mathbb{I}_I}\mu^{\mathcal{V}}_{\mathbb{I}_I}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}}, \\
\left(1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta_{\mathbb{I}_R}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_R}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right) \\
e^{i2\pi\left(1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta_{\mathbb{I}_I}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_I}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right)}
\end{array}\right)
\end{array}\right)$$

$$\min_{k}\min_{\ell}\eta^{-}_{\mathbb{I}_{I-\ell k}} \leq 1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\left(1-\eta^{-}_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\left(1-\eta^{-}_{\mathbb{I}_{I-jt}}\right)^{\mathcal{V}}\right)^{\sigma}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} \leq \max_{k}\max_{\ell}\mu^{-}_{\mathbb{I}_{I-\ell k}}$$

Similarly, we have

$$\min_{k}\min_{\ell}\mu_{\mathbb{I}_{R-\ell k}}\mu_{\mathbb{I}_{R-jt}} \leq \left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\mu^{\mathcal{U}}_{\mathbb{I}_{R-\ell k}}\mu^{\mathcal{V}}_{\mathbb{I}_{R-jt}}\right)^{\sigma}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} \leq \max_{k}\max_{\ell}\mu_{\mathbb{I}_{R-\ell k}}\mu_{\mathbb{I}_{R-jt}}$$

$$\min_{k}\min_{\ell}\mu_{\mathbb{I}_{I-\ell k}}\mu_{\mathbb{I}_{I-jt}} \leq \left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\mu^{\mathcal{U}}_{\mathbb{I}_{I-\ell k}}\mu^{\mathcal{V}}_{\mathbb{I}_{I-jt}}\right)^{\sigma}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} \leq \max_{k}\max_{\ell}\mu_{\mathbb{I}_{I-\ell k}}\mu_{\mathbb{I}_{I-jt}}$$

and

$$\min_{k}\min_{\ell}\eta_{\mathbb{I}_{R-\ell k}} \leq 1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-jt}}\right)^{\mathcal{V}}\right)^{\sigma}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} \leq \max_{k}\max_{\ell}\eta_{\mathbb{I}_{R-\ell k}}$$

$$\min_{k}\min_{\ell}\eta_{\mathbb{I}_{I-\ell k}}$$

$$= \left(\begin{array}{c}\left(\begin{array}{c}\left[\begin{array}{c}\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\mu^{-\;\mathcal{U}+\mathcal{V}}_{\mathbb{I}_R}\right)^{\frac{1}{\mathfrak{m}\mathfrak{n}(\mathfrak{m}-1)(\mathfrak{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\ \left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\mu^{+\;\mathcal{U}+\mathcal{V}}_{\mathbb{I}_R}\right)^{\frac{1}{\mathfrak{m}\mathfrak{n}(\mathfrak{m}-1)(\mathfrak{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\end{array}\right]\\ e^{i2\pi\left[\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\mu^{-\;\mathcal{U}+\mathcal{V}}_{\mathbb{I}_I}\right)^{\frac{1}{\mathfrak{m}\mathfrak{n}(\mathfrak{m}-1)(\mathfrak{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\mu^{+\;\mathcal{U}+\mathcal{V}}_{\mathbb{I}_I}\right)^{\frac{1}{\mathfrak{m}\mathfrak{n}(\mathfrak{m}-1)(\mathfrak{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right]},\\ \left[\begin{array}{c}1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\left(1-\eta^{-}_{\mathbb{I}_R}\right)^{\mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathfrak{m}\mathfrak{n}(\mathfrak{m}-1)(\mathfrak{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\\ 1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\left(1-\eta^{+}_{\mathbb{I}_R}\right)^{\mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathfrak{m}\mathfrak{n}(\mathfrak{m}-1)(\mathfrak{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\end{array}\right]\\ e^{i2\pi\left[1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\left(1-\eta^{-}_{\mathbb{I}_I}\right)^{\mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathfrak{m}\mathfrak{n}(\mathfrak{m}-1)(\mathfrak{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\left(1-\eta^{+}_{\mathbb{I}_I}\right)^{\mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathfrak{m}\mathfrak{n}(\mathfrak{m}-1)(\mathfrak{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right]}\end{array}\right),\\ \left(\begin{array}{c}\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\mu^{\mathcal{U}+\mathcal{V}}_{\mathbb{I}_R}\right)^{\frac{1}{\mathfrak{m}\mathfrak{n}(\mathfrak{m}-1)(\mathfrak{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} e^{i2\pi\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\mu^{\mathcal{U}+\mathcal{V}}_{\mathbb{I}_I}\right)^{\frac{1}{\mathfrak{m}\mathfrak{n}(\mathfrak{m}-1)(\mathfrak{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}},\\ \left(1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\left(1-\eta_{\mathbb{I}_R}\right)^{\mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathfrak{m}\mathfrak{n}(\mathfrak{m}-1)(\mathfrak{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right)\\ e^{i2\pi\left(1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathfrak{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathfrak{n}}\left(1-\left(1-\eta_{\mathbb{I}_I}\right)^{\mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathfrak{m}\mathfrak{n}(\mathfrak{m}-1)(\mathfrak{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right)}\end{array}\right)\end{array}\right)$$

$$
\begin{aligned}
&= \left(\begin{array}{c}
\left(\begin{array}{c}
\left[\begin{array}{c} \left(1-\left(1-\mu_{\mathbb{I}_R}^{-\ \mathcal{U}+\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}, \\ \left(1-\left(1-\mu_{\mathbb{I}_R}^{+\ \mathcal{U}+\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} \end{array} \right] e^{i2\pi\left[\left(1-\left(1-\mu_{\mathbb{I}_I}^{-\ \mathcal{U}+\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\left(1-\left(1-\mu_{\mathbb{I}_I}^{+\ \mathcal{U}+\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right]}, \\
\left[\begin{array}{c} 1-\left(1-\left(1-\left(1-\eta_{\mathbb{I}_R}^{-}\right)^{\mathcal{U}+\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}, \\ 1-\left(1-\left(1-\left(1-\eta_{\mathbb{I}_R}^{+}\right)^{\mathcal{U}+\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} \end{array} \right] \\
e^{i2\pi\left[\begin{array}{c} 1-\left(1-\left(1-\left(1-\eta_{\mathbb{I}_I}^{-}\right)^{\mathcal{U}+\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}, \\ 1-\left(1-\left(1-\left(1-\eta_{\mathbb{I}_I}^{+}\right)^{\mathcal{U}+\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} \end{array} \right]}
\end{array} \right), \\
\left(\begin{array}{c}
\left(1-\left(1-\mu_{\mathbb{I}_R}^{\mathcal{U}+\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} e^{i2\pi\left(1-\left(1-\mu_{\mathbb{I}_I}^{\mathcal{U}+\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}}, \\
\left(1-\left(1-\left(1-\left(1-\eta_{\mathbb{I}_R}\right)^{\mathcal{U}+\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right) \\
e^{i2\pi\left(1-\left(1-\left(1-\left(1-\eta_{\mathbb{I}_I}\right)^{\mathcal{U}+\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right)}
\end{array} \right)
\end{array} \right) \\
&= \left(\begin{array}{c}
\left(\begin{array}{c}
\left[\left(\mu_{\mathbb{I}_R}^{-\ \mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\left(\mu_{\mathbb{I}_R}^{+\ \mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right] e^{i2\pi\left[\left(\mu_{\mathbb{I}_I}^{-\ \mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}},\left(\mu_{\mathbb{I}_I}^{+\ \mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right]}, \\
\left[\begin{array}{c} 1-\left(\left(1-\eta_{\mathbb{I}_R}^{-}\right)^{\mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}, \\ 1-\left(\left(1-\eta_{\mathbb{I}_R}^{+}\right)^{\mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} \end{array} \right] \\
e^{i2\pi\left[\begin{array}{c} 1-\left(\left(1-\eta_{\mathbb{I}_I}^{-}\right)^{\mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}, \\ 1-\left(\left(1-\eta_{\mathbb{I}_I}^{+}\right)^{\mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} \end{array} \right]}
\end{array} \right), \\
\left(\begin{array}{c}
\left(\mu_{\mathbb{I}_R}^{\mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} e^{i2\pi\left(\mu_{\mathbb{I}_I}^{\mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}}, \\
\left(1-\left(\left(1-\eta_{\mathbb{I}_R}\right)^{\mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right) \\
e^{i2\pi\left(1-\left(\left(1-\eta_{\mathbb{I}_I}\right)^{\mathcal{U}+\mathcal{V}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right)}
\end{array} \right)
\end{array} \right) \\
&= \left(\begin{array}{c}
\left(\begin{array}{c}
\left[\mu_{\mathbb{I}_R}^{-},\mu_{\mathbb{I}_R}^{+}\right] e^{i2\pi\left(\left[\mu_{\mathbb{I}_I}^{-},\mu_{\mathbb{I}_I}^{+}\right]\right)}, \\
\left[\eta_{\mathbb{I}_R}^{-},\eta_{\mathbb{I}_R}^{+}\right] e^{i2\pi\left(\left[\eta_{\mathbb{I}_I}^{-},\eta_{\mathbb{I}_I}^{+}\right]\right)}
\end{array} \right), \\
\left(\mu_{\mathbb{I}_R} e^{i2\pi\left(\mu_{\mathbb{I}_I}\right)},\ \eta_{\mathbb{I}_R} e^{i2\pi\left(\eta_{\mathbb{I}_I}\right)}\right)
\end{array} \right) \\
&= \mathbb{I} \qquad \text{(XI)}
\end{aligned}
$$

$$\leq 1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathrm{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathrm{n}}\left(1-\begin{pmatrix}\left(1-\eta_{\mathbb{I}_{I-\ell k}}\right)^{\mathcal{U}}\\\left(1-\eta_{\mathbb{I}_{I-jt}}\right)^{\mathcal{V}}\end{pmatrix}\right)^{\sigma}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}$$
$$\leq \max_k \max_\ell \eta_{\mathbb{I}_{I-\ell k}}$$

Then, by using the information in Eq. (20) and Eq. (21), we get

$$\mathbb{I}^{+} \leq CICFSBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11},\mathbb{I}_{12},\ldots,\mathbb{I}_{\mathrm{nm}}\right) \leq \mathbb{I}^{-}.$$

Property 3 (Monotonicity): If $\mathbb{I}_{\ell k} \leq \mathbb{I}'_{\ell k}$, then

$$CICFSBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11},\mathbb{I}_{12},\ldots,\mathbb{I}_{\mathrm{nm}}\right) \leq CICFSBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}'_{11},\mathbb{I}'_{12},\ldots,\mathbb{I}'_{\mathrm{nm}}\right) \quad (26)$$

Proof: Similar to Property 2.

Property 4 (Commutativity): If $\mathbb{I}'_{\ell k}$ is the permutation of $\mathbb{I}_{\ell k}$, then

$$CICFSBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11},\mathbb{I}_{12},\ldots,\mathbb{I}_{\mathrm{nm}}\right) = CICFSBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}'_{11},\mathbb{I}'_{12},\ldots,\mathbb{I}'_{\mathrm{nm}}\right) \quad (27)$$

Proof: Using the information in Eq. (22), we have

$$CICFSBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11},\mathbb{I}_{12},\ldots,\mathbb{I}_{\mathrm{nm}}\right)$$
$$=\left(\frac{1}{\mathrm{mn}\left(\mathrm{m}-1\right)\left(\mathrm{n}-1\right)}\oplus_{\substack{k,j=1\\k\neq j}}^{\mathrm{m}}\oplus_{\substack{\ell,t=1\\\ell\neq t}}^{\mathrm{n}}\left(\mathbb{I}_{\ell k}^{\mathcal{U}}\otimes\mathbb{I}_{jt}^{\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}$$
$$=\left(\frac{1}{\mathrm{mn}\left(\mathrm{m}-1\right)\left(\mathrm{n}-1\right)}\oplus_{\substack{k,j=1\\k\neq j}}^{\mathrm{m}}\oplus_{\substack{\ell,t=1\\\ell\neq t}}^{\mathrm{n}}\left(\mathbb{I'}_{\ell k}^{\mathcal{U}}\otimes\mathbb{I'}_{jt}^{\mathcal{V}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}$$
$$= CICFSBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}'_{11},\mathbb{I}'_{12},\ldots,\mathbb{I}'_{\mathrm{nm}}\right).$$

Definition 12: The theory of CICFSWBM operator based on the collection of CICFSNs

$$\mathbb{I}_{\ell k}=\left(\begin{pmatrix}\left[\mu^{-}_{\mathbb{I}_{R-\ell k}},\mu^{+}_{\mathbb{I}_{R-\ell k}}\right]e^{i2\pi\left(\left[\mu^{-}_{\mathbb{I}_{I-\ell k}},\mu^{+}_{\mathbb{I}_{I-\ell k}}\right]\right)},\\\left[\eta^{-}_{\mathbb{I}_{R-\ell k}},\eta^{+}_{\mathbb{I}_{R-\ell k}}\right]e^{i2\pi\left(\left[\eta^{-}_{\mathbb{I}_{I-\ell k}},\eta^{+}_{\mathbb{I}_{I-\ell k}}\right]\right)}\end{pmatrix},\left(\mu_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\mu_{\mathbb{I}_{I-\ell k}}\right)},\eta_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\eta_{\mathbb{I}_{I-\ell k}}\right)}\right)\right),$$ deliberated by:

$$CICFSWBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11},\mathbb{I}_{12},\ldots,\mathbb{I}_{\mathrm{nm}}\right)=\left(\frac{1}{\mathrm{mn}\left(\mathrm{m}-1\right)\left(\mathrm{n}-1\right)}\oplus_{\substack{k,j=1\\k\neq j}}^{\mathrm{m}}\oplus_{\substack{\ell,t=1\\\ell\neq t}}^{\mathrm{n}}\left(\left(\mathbb{W}_k\left(w_\ell\mathbb{I}_{\ell k}\right)\right)\otimes\left(\mathbb{W}_t\left(w_j\mathbb{I}_{jt}\right)\right)\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}} \quad (28)$$

where $\mathcal{U},\mathcal{V}\geq 0$ with $\sum_{k=1}^{\mathrm{m}}\mathbb{W}_k=1$ and $\sum_{\ell=1}^{\mathrm{n}}w_\ell=1$.

Theorem 2: The main theory of the CICFSBM operator is initiated in the shape of Eq. (29), as shown at the bottom of the next page, by using Eq. (28) and Def. (9), we Eq. (29), as shown at the bottom of the next page, and Eq. (XII), as shown at the bottom of page 27.

Proof: Straightforward.

Additionally, using the information in Eq. (29), we illustrated various properties.

Property 5 (Idempotency): If $\mathbb{I}_{\ell k}=\mathbb{I}=$

$$\left(\left(\left[\mu^{-}_{\mathbb{I}_R},\mu^{+}_{\mathbb{I}_R}\right]e^{i2\pi\left(\left[\mu^{-}_{\mathbb{I}_I},\mu^{+}_{\mathbb{I}_I}\right]\right)},\left[\eta^{-}_{\mathbb{I}_R},\eta^{+}_{\mathbb{I}_R}\right]e^{i2\pi\left(\left[\eta^{-}_{\mathbb{I}_I},\eta^{+}_{\mathbb{I}_I}\right]\right)}\right),\left(\mu_{\mathbb{I}_R}e^{i2\pi\left(\mu_{\mathbb{I}_I}\right)},\eta_{\mathbb{I}_R}e^{i2\pi\left(\eta_{\mathbb{I}_I}\right)}\right)\right),$$

deliberated by:

$$CICFSWBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11},\mathbb{I}_{12},\ldots,\mathbb{I}_{\mathrm{nm}}\right)=\mathbb{I} \quad (30)$$

Proof: Straightforward.

Property 6 (Boundedness): If

$$\mathbb{I}^{+}=\left(\begin{pmatrix}\left[\max_k\max_\ell\mu^{-}_{\mathbb{I}_{R-\ell k}},\max_k\max_\ell\mu^{+}_{\mathbb{I}_{R-\ell k}}\right]e^{i2\pi\left(\left[\max_k\max_\ell\mu^{-}_{\mathbb{I}_{I-\ell k}},\max_k\max_\ell\mu^{+}_{\mathbb{I}_{I-\ell k}}\right]\right)},\\\left[\min_k\min_\ell\eta^{-}_{\mathbb{I}_{R-\ell k}},\min_k\min_\ell\eta^{+}_{\mathbb{I}_{R-\ell k}}\right]e^{i2\pi\left(\left[\min_k\min_\ell\eta^{-}_{\mathbb{I}_{I-\ell k}},\min_k\min_\ell\eta^{+}_{\mathbb{I}_{I-\ell k}}\right]\right)}\end{pmatrix},\begin{pmatrix}\max_k\max_\ell\mu_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\max_k\max_\ell\mu_{\mathbb{I}_{I-\ell k}}\right)},\\\min_k\min_\ell\eta_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\min_k\min_\ell\eta_{\mathbb{I}_{I-\ell k}}\right)}\end{pmatrix}\right)$$ and

$$\mathbb{I}^{-}=\left(\begin{pmatrix}\left[\min_k\min_\ell\mu^{-}_{\mathbb{I}_{R-\ell k}},\min_k\min_\ell\mu^{+}_{\mathbb{I}_{R-\ell k}}\right]e^{i2\pi\left(\left[\min_k\min_\ell\mu^{-}_{\mathbb{I}_{I-\ell k}},\min_k\min_\ell\mu^{+}_{\mathbb{I}_{I-\ell k}}\right]\right)},\\\left[\max_k\max_\ell\eta^{-}_{\mathbb{I}_{R-\ell k}},\max_k\max_\ell\eta^{+}_{\mathbb{I}_{R-\ell k}}\right]e^{i2\pi\left(\left[\max_k\max_\ell\eta^{-}_{\mathbb{I}_{I-\ell k}},\max_k\max_\ell\eta^{+}_{\mathbb{I}_{I-\ell k}}\right]\right)}\end{pmatrix},\begin{pmatrix}\min_k\min_\ell\mu_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\min_k\min_\ell\mu_{\mathbb{I}_{I-\ell k}}\right)},\\\max_k\max_\ell\eta_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\max_k\max_\ell\eta_{\mathbb{I}_{I-\ell k}}\right)}\end{pmatrix}\right),$$ deliberated by:

$$\mathbb{I}^{+}\leq CICFSWBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11},\mathbb{I}_{12},\ldots,\mathbb{I}_{\mathrm{nm}}\right)\leq\mathbb{I}^{-} \quad (31)$$

Proof: Straightforward.

Property 7 (Monotonicity): If $\mathbb{I}_{\ell k}\leq\mathbb{I}'_{\ell k}$, then

$$CICFSWBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11},\mathbb{I}_{12},\ldots,\mathbb{I}_{\mathrm{nm}}\right)\leq CICFSWBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}'_{11},\mathbb{I}'_{12},\ldots,\mathbb{I}'_{\mathrm{nm}}\right) \quad (32)$$

Proof: Straightforward.

Property 8 (Commutativity): If $\mathbb{I}'_{\ell k}$ is the permutation of $\mathbb{I}_{\ell k}$, then

$$CICFSWBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11},\mathbb{I}_{12},\ldots,\mathbb{I}_{\mathrm{nm}}\right)=CICFSWBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}'_{11},\mathbb{I}'_{12},\ldots,\mathbb{I}'_{\mathrm{nm}}\right) \quad (33)$$

Proof: Straightforward.

V. APPLICATION ("MULTI-ATTRIBUTE DECISION-MAKING TECHNIQUE")

Clustering analysis is a feasible technique data-mining tool for any institution that needs to discover discrete collections of intellectuals, sales transfers, or other sorts of behaviors and things. Similarly, pattern recognition, medical diagnosis, image segmentation, networking systems, decision-making, and MADM techniques are also played a very essential role in the environment of classical set theory. A lot of scholars have utilized the above-discussed applications in the field of fuzzy set (FS) theory. The main theme of this theory is to evaluate deficiencies with the help of invented operators, we established a MADM tool under the availability of CICFSBM, CICFSWBM operators. Finally, we described the supremacy and reliability of the diagnosed work with the help of comparative analysis and also explained their graphical representation is to enhance the worth of the established approaches.

A. DECISION-MAKING SKILL/PROCEDURE

In every day of our loves, we choose but with the space array of decisions expressed to us, we're frequently afflicted with indecision and remorse over prospect cost sustained from "wrong" decisions. For this, we suggest that $\{\mathbb{I}_1, \mathbb{I}_2, \ldots, \mathbb{I}_t\}$ and $\mathbb{E}_P = \{e_1, e_2, \ldots, e_m\}$ represented the collection of alternatives and set of parameters for the distinct value of weight vectors, such that $\sum_{k=1}^{\mathbb{m}} \mathbb{W}_k = 1$ and $\sum_{\ell=1}^{\mathbb{n}} \mathcal{w}_\ell = 1$. For

$$
\begin{aligned}
&CICFSWBM^{\mathcal{U},\mathcal{V}}\left(\mathbb{I}_{11}, \mathbb{I}_{12}, \ldots, \mathbb{I}_{\mathbb{n}\mathbb{m}}\right) \\
&\quad = (A, B)
\end{aligned}
$$

$$
A = \left(\begin{array}{l}
\left[\begin{array}{l}
\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\left(1-\mu^{-}_{\mathbb{I}_{R-\ell k}}\right)^{\mathbb{W}_k\mathcal{w}_\ell}\right)^{\mathcal{U}}\left(1-\left(1-\mu^{-}_{\mathbb{I}_{R-jt}}\right)^{\mathbb{W}_t\mathcal{w}_j}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}, \\
\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\left(1-\mu^{+}_{\mathbb{I}_{R-\ell k}}\right)^{\mathbb{W}_k\mathcal{w}_\ell}\right)^{\mathcal{U}}\left(1-\left(1-\mu^{+}_{\mathbb{I}_{R-jt}}\right)^{\mathbb{W}_t\mathcal{w}_j}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}
\end{array}\right] \\
e^{i2\pi\left[\begin{array}{l}
\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\left(1-\mu^{-}_{\mathbb{I}_{I-\ell k}}\right)^{\mathbb{W}_k\mathcal{w}_\ell}\right)^{\mathcal{U}}\left(1-\left(1-\mu^{-}_{\mathbb{I}_{I-jt}}\right)^{\mathbb{W}_t\mathcal{w}_j}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}, \\
\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\left(1-\mu^{+}_{\mathbb{I}_{I-\ell k}}\right)^{\mathbb{W}_k\mathcal{w}_\ell}\right)^{\mathcal{U}}\left(1-\left(1-\mu^{+}_{\mathbb{I}_{I-jt}}\right)^{\mathbb{W}_t\mathcal{w}_j}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}
\end{array}\right]}, \\
\left[\begin{array}{l}
1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^{-}_{\mathbb{I}_{R-\ell k}}{}^{\mathbb{W}_k\mathcal{w}_\ell}\right)^{\mathcal{U}}\left(1-\eta^{-}_{\mathbb{I}_{R-jt}}{}^{\mathbb{W}_t\mathcal{w}_j}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}, \\
1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^{+}_{\mathbb{I}_{R-\ell k}}{}^{\mathbb{W}_k\mathcal{w}_\ell}\right)^{\mathcal{U}}\left(1-\eta^{+}_{\mathbb{I}_{R-jt}}{}^{\mathbb{W}_t\mathcal{w}_j}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}
\end{array}\right] \\
e^{i2\pi\left[\begin{array}{l}
1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^{-}_{\mathbb{I}_{I-\ell k}}{}^{\mathbb{W}_k\mathcal{w}_\ell}\right)^{\mathcal{U}}\left(1-\eta^{-}_{\mathbb{I}_{I-jt}}{}^{\mathbb{W}_t\mathcal{w}_j}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}, \\
1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta^{+}_{\mathbb{I}_{I-\ell k}}{}^{\mathbb{W}_k\mathcal{w}_\ell}\right)^{\mathcal{U}}\left(1-\eta^{+}_{\mathbb{I}_{I-jt}}{}^{\mathbb{W}_t\mathcal{w}_j}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{m}\mathbb{n}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}
\end{array}\right]}
\end{array}\right) \tag{29}
$$

this, we computed a matrix, whose every term is in the shape of CICFS information such that

$$\mathbb{I}_{\ell k}=\left(\begin{array}{c}\left(\begin{array}{c}\left[\mu_{\mathbb{I}_{R-\ell k}}^{-},\mu_{\mathbb{I}_{R-\ell k}}^{+}\right]e^{i2\pi\left(\left[\mu_{\mathbb{I}_{I-\ell k}}^{-},\mu_{\mathbb{I}_{I-\ell k}}^{+}\right]\right)},\\ \left[\eta_{\mathbb{I}_{R-\ell k}}^{-},\eta_{\mathbb{I}_{R-\ell k}}^{+}\right]e^{i2\pi\left(\left[\eta_{\mathbb{I}_{I-\ell k}}^{-},\eta_{\mathbb{I}_{I-\ell k}}^{+}\right]\right)}\end{array}\right),\\ \left(\mu_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\mu_{\mathbb{I}_{I-\ell k}}\right)},\eta_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\eta_{\mathbb{I}_{I-\ell k}}\right)}\right)\end{array}\right),$$

$\ell=1,2,\ldots,\mathbb{n}$, where the term

$$\begin{aligned}\mu_{\mathbb{I}_{CIF-k}}\left(x_{\ell}\right)&=\left[\mu_{\mathbb{I}_{R-k}}^{-}\left(x_{\ell}\right),\mu_{\mathbb{I}_{R-k}}^{+}\left(x_{\ell}\right)\right]\\&\quad\times e^{i2\pi\left(\left[\mu_{\mathbb{I}_{I-k}}^{-}\left(x_{\ell}\right),\mu_{\mathbb{I}_{I-k}}^{+}\left(x_{\ell}\right)\right]\right)},\\ \mu_{\mathbb{I}_{IF-k}}\left(x_{\ell}\right)&=\mu_{\mathbb{I}_{R-k}}\left(x_{\ell}\right)e^{i2\pi\left(\mu_{\mathbb{I}_{I-k}}\left(x_{\ell}\right)\right)}\end{aligned}$$

and

$$\begin{aligned}\eta_{\mathbb{I}_{CIF-k}}\left(x_{\ell}\right)&=\left[\eta_{\mathbb{I}_{R-k}}^{-}\left(x_{\ell}\right),\eta_{\mathbb{I}_{R-k}}^{+}\left(x_{\ell}\right)\right]\\&\quad\times e^{i2\pi\left(\left[\eta_{\mathbb{I}_{I-k}}^{-}\left(x_{\ell}\right),\eta_{\mathbb{I}_{I-k}}^{+}\left(x_{\ell}\right)\right]\right)},\\ \eta_{\mathbb{I}_{IF-k}}\left(x_{\ell}\right)&=\eta_{\mathbb{I}_{R-k}}\left(x_{\ell}\right)e^{i2\pi\left(\eta_{\mathbb{I}_{I-k}}\left(x_{\ell}\right)\right)}\end{aligned}$$

are represented the well-known truth and falsity grades in the shape of complex numbers with techniques: $0\leq\mu_{\mathbb{I}_{R-k}}^{+}\left(x_{\ell}\right)+\eta_{\mathbb{I}_{R-k}}^{+}\left(x_{\ell}\right)\leq 1,0\leq\mu_{\mathbb{I}_{R-k}}\left(x_{\ell}\right)+\eta_{\mathbb{I}_{R-k}}\left(x_{\ell}\right)\leq 1$ and $0\leq\mu_{\mathbb{I}_{I-k}}^{+}\left(x_{\ell}\right)+\eta_{\mathbb{I}_{I-k}}^{+}\left(x_{\ell}\right)\leq 1,0\leq\mu_{\mathbb{I}_{I-k}}\left(x_{\ell}\right)+\eta_{\mathbb{I}_{I-k}}\left(x_{\ell}\right)\leq 1$. Then, to evaluate the above dilemma, the following steps have been explained for evaluating the problematic decision-making technique using CICFS information based on invented operators.

Step 1: Arrange the data for each alternative based on the distinct value of parameters in the term of the CICFS information matrix

$$D=\left(\begin{array}{c}\left(\begin{array}{c}\left[\mu_{\mathbb{I}_{R-\ell k}}^{-},\mu_{\mathbb{I}_{R-\ell k}}^{+}\right]e^{i2\pi\left(\left[\mu_{\mathbb{I}_{I-\ell k}}^{-},\mu_{\mathbb{I}_{I-\ell k}}^{+}\right]\right)},\\ \left[\eta_{\mathbb{I}_{R-\ell k}}^{-},\eta_{\mathbb{I}_{R-\ell k}}^{+}\right]e^{i2\pi\left(\left[\eta_{\mathbb{I}_{I-\ell k}}^{-},\eta_{\mathbb{I}_{I-\ell k}}^{+}\right]\right)}\end{array}\right),\\ \left(\mu_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\mu_{\mathbb{I}_{I-\ell k}}\right)},\eta_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\eta_{\mathbb{I}_{I-\ell k}}\right)}\right)\end{array}\right)_{\mathbb{n}\times\mathbb{m}}.$$

Step 2: Further, if the data given in the matrix is benefit type, then ignore it, but if the data given in the matrix is cost type, then it's needed to be normalized, using the below theory, such that

$$N=\begin{cases}\left(\begin{array}{c}\left(\begin{array}{c}\left[\mu_{\mathbb{I}_{R-\ell k}}^{-},\mu_{\mathbb{I}_{R-\ell k}}^{+}\right]e^{i2\pi\left(\left[\mu_{\mathbb{I}_{I-\ell k}}^{-},\mu_{\mathbb{I}_{I-\ell k}}^{+}\right]\right)},\\ \left[\eta_{\mathbb{I}_{R-\ell k}}^{-},\eta_{\mathbb{I}_{R-\ell k}}^{+}\right]e^{i2\pi\left(\left[\eta_{\mathbb{I}_{I-\ell k}}^{-},\eta_{\mathbb{I}_{I-\ell k}}^{+}\right]\right)}\end{array}\right),\\ \left(\mu_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\mu_{\mathbb{I}_{I-\ell k}}\right)},\eta_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\eta_{\mathbb{I}_{I-\ell k}}\right)}\right)\end{array}\right)\\ \textit{for benefit}\\ \left(\begin{array}{c}\left(\begin{array}{c}\left[\eta_{\mathbb{I}_{R-\ell k}}^{-},\eta_{\mathbb{I}_{R-\ell k}}^{+}\right]e^{i2\pi\left(\left[\eta_{\mathbb{I}_{I-\ell k}}^{-},\eta_{\mathbb{I}_{I-\ell k}}^{+}\right]\right)},\\ \left[\mu_{\mathbb{I}_{R-\ell k}}^{-},\mu_{\mathbb{I}_{R-\ell k}}^{+}\right]e^{i2\pi\left(\left[\mu_{\mathbb{I}_{I-\ell k}}^{-},\mu_{\mathbb{I}_{I-\ell k}}^{+}\right]\right)}\end{array}\right),\\ \left(\eta_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\eta_{\mathbb{I}_{I-\ell k}}\right)},\mu_{\mathbb{I}_{R-\ell k}}e^{i2\pi\left(\mu_{\mathbb{I}_{I-\ell k}}\right)}\right)\end{array}\right)\\ \textit{for cost}\end{cases}$$

Step 3: In the presence of the normalized information, we collect the information by using the information of CICFSBM operators.

Step 4: Compute the SVs of the collective information.

Step 5: Rank the alternatives using the information of SV and determine the best decision.

$$B=\left(\begin{array}{c}\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\left(1-\mu_{\mathbb{I}_{R-\ell k}}\right)^{\mathbb{W}_{k}w_{\ell}}\right)^{\mathcal{U}}\left(1-\left(1-\mu_{\mathbb{I}_{R-jt}}\right)^{\mathbb{W}_{t}w_{j}}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{mn}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\\ e^{i2\pi\left(1-\left(\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\left(1-\mu_{\mathbb{I}_{I-\ell k}}\right)^{\mathbb{W}_{k}w_{\ell}}\right)^{\mathcal{U}}\left(1-\left(1-\mu_{\mathbb{I}_{I-jt}}\right)^{\mathbb{W}_{t}w_{j}}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{mn}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}},\\ \left(1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta_{\mathbb{I}_{R-\ell k}}^{\mathbb{W}_{k}w_{\ell}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{R-jt}}^{\mathbb{W}_{t}w_{j}}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{mn}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right)\\ e^{i2\pi\left(1-\left(1-\prod_{\substack{k,j=1\\k\neq j}}^{\mathbb{m}}\prod_{\substack{\ell,t=1\\\ell\neq t}}^{\mathbb{n}}\left(1-\left(1-\eta_{\mathbb{I}_{I-\ell k}}^{\mathbb{W}_{k}w_{\ell}}\right)^{\mathcal{U}}\left(1-\eta_{\mathbb{I}_{I-jt}}^{\mathbb{W}_{t}w_{j}}\right)^{\mathcal{V}}\right)^{\frac{1}{\mathbb{mn}(\mathbb{m}-1)(\mathbb{n}-1)}}\right)^{\frac{1}{\mathcal{U}+\mathcal{V}}}\right)}\end{array}\right) \tag{XII}$$

TABLE 1. Cubic intuitionistic complex fuzzy soft matrix $\mathbb{I}_1$.

	e_1	e_2
$\mathbb{I}_{A-1}$	$\left(\begin{pmatrix}[0.3,0.4]e^{i2\pi([0.2,0.3])},\\ [0.1,0.2]e^{i2\pi([0.2,0.4])}\end{pmatrix}, \left(0.7e^{i2\pi(0.6)}, 0.1e^{i2\pi(0.2)}\right)\right)$	$\left(\begin{pmatrix}[0.31,0.41]e^{i2\pi([0.21,0.31])},\\ [0.11,0.21]e^{i2\pi([0.21,0.41])}\end{pmatrix}, \left(0.71e^{i2\pi(0.61)}, 0.11e^{i2\pi(0.21)}\right)\right)$
$\mathbb{I}_{A-2}$	$\left(\begin{pmatrix}[0.2,0.5]e^{i2\pi([0.1,0.5])},\\ [0.2,0.3]e^{i2\pi([0.3,0.4])}\end{pmatrix}, \left(0.4e^{i2\pi(0.3)}, 0.2e^{i2\pi(0.1)}\right)\right)$	$\left(\begin{pmatrix}[0.21,0.51]e^{i2\pi([0.11,0.51])},\\ [0.21,0.31]e^{i2\pi([0.31,0.41])}\end{pmatrix}, \left(0.41e^{i2\pi(0.31)}, 0.21e^{i2\pi(0.11)}\right)\right)$
$\mathbb{I}_{A-3}$	$\left(\begin{pmatrix}[0.5,0.7]e^{i2\pi([0.5,0.6])},\\ [0.1,0.2]e^{i2\pi([0.1,0.2])}\end{pmatrix}, \left(0.5e^{i2\pi(0.3)}, 0.2e^{i2\pi(0.2)}\right)\right)$	$\left(\begin{pmatrix}[0.51,0.71]e^{i2\pi([0.51,0.61])},\\ [0.11,0.21]e^{i2\pi([0.11,0.21])}\end{pmatrix}, \left(0.51e^{i2\pi(0.31)}, 0.21e^{i2\pi(0.21)}\right)\right)$
$\mathbb{I}_{A-4}$	$\left(\begin{pmatrix}[0.1,0.2]e^{i2\pi([0.3,0.4])},\\ [0.2,0.3]e^{i2\pi([0.3,0.4])}\end{pmatrix}, \left(0.5e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.1)}\right)\right)$	$\left(\begin{pmatrix}[0.11,0.21]e^{i2\pi([0.31,0.41])},\\ [0.21,0.31]e^{i2\pi([0.31,0.41])}\end{pmatrix}, \left(0.51e^{i2\pi(0.41)}, 0.31e^{i2\pi(0.11)}\right)\right)$
	e_3	e_4
$\mathbb{I}_{A-1}$	$\left(\begin{pmatrix}[0.32,0.42]e^{i2\pi([0.22,0.32])},\\ [0.12,0.22]e^{i2\pi([0.22,0.42])}\end{pmatrix}, \left(0.72e^{i2\pi(0.62)}, 0.12e^{i2\pi(0.22)}\right)\right)$	$\left(\begin{pmatrix}[0.33,0.43]e^{i2\pi([0.23,0.33])},\\ [0.13,0.23]e^{i2\pi([0.23,0.43])}\end{pmatrix}, \left(0.73e^{i2\pi(0.63)}, 0.13e^{i2\pi(0.23)}\right)\right)$
$\mathbb{I}_{A-2}$	$\left(\begin{pmatrix}[0.22,0.52]e^{i2\pi([0.12,0.52])},\\ [0.22,0.32]e^{i2\pi([0.32,0.42])}\end{pmatrix}, \left(0.42e^{i2\pi(0.32)}, 0.22e^{i2\pi(0.12)}\right)\right)$	$\left(\begin{pmatrix}[0.23,0.53]e^{i2\pi([0.13,0.53])},\\ [0.23,0.33]e^{i2\pi([0.33,0.43])}\end{pmatrix}, \left(0.43e^{i2\pi(0.33)}, 0.23e^{i2\pi(0.13)}\right)\right)$
$\mathbb{I}_{A-3}$	$\left(\begin{pmatrix}[0.52,0.72]e^{i2\pi([0.52,0.62])},\\ [0.12,0.22]e^{i2\pi([0.12,0.22])}\end{pmatrix}, \left(0.52e^{i2\pi(0.32)}, 0.22e^{i2\pi(0.22)}\right)\right)$	$\left(\begin{pmatrix}[0.53,0.73]e^{i2\pi([0.53,0.63])},\\ [0.13,0.23]e^{i2\pi([0.13,0.23])}\end{pmatrix}, \left(0.53e^{i2\pi(0.33)}, 0.23e^{i2\pi(0.23)}\right)\right)$
$\mathbb{I}_{A-4}$	$\left(\begin{pmatrix}[0.12,0.22]e^{i2\pi([0.32,0.42])},\\ [0.22,0.32]e^{i2\pi([0.32,0.42])}\end{pmatrix}, \left(0.52e^{i2\pi(0.42)}, 0.32e^{i2\pi(0.12)}\right)\right)$	$\left(\begin{pmatrix}[0.13,0.23]e^{i2\pi([0.33,0.43])},\\ [0.23,0.33]e^{i2\pi([0.33,0.43])}\end{pmatrix}, \left(0.53e^{i2\pi(0.43)}, 0.33e^{i2\pi(0.13)}\right)\right)$

B. NUMERICAL EXAMPLE

The information cited above has been explained with the help of numerical/genuine life using invented work. For this, we use the idea of CICFSBM and CICFSWBM operator to improve the worth of the invented work.

Monsoon rains in river catchments have caused flooding in the Nowshera main city, Pakistan in 2010. The Kabul River and its related or near places overflow as a result of monsoon rains which affected over ten lack people and their homes and more than ten thousand helicopters are working for giving facilitation to the suffering people. The main area affected by monsoon rains is represented by:

$\mathbb{I}_1$: Kheshgi Bala.
$\mathbb{I}_2$: Kheshgi Payan.
$\mathbb{I}_3$: Ahmad Nager.
$\mathbb{I}_4$: Machine Korona.

A panel of four experts represented by:
$\mathbb{E}_P = \{e_1 = Roman\ Khan, e_2 = Shahan, e_3 = Mehran\ Ali, e_4 = Luqman\}$ whose meanings are of the shape: crop loss, livestock, loss of life, and damage of houses. For this, the expert gives their opinion on alternative and their attributes, representing the weight vector such that 0.4, 0.3, 0.2, 0.1, and 0.1, 0.2, 0.3, 0.4. Then, to evaluate the above dilemma, the following steps have been explained for evaluating the problematic decision-making technique using CICFS information based on invented operators.

Step 1: Arrange the data for each alternative based on the distinct value of parameters in the term of the CICFS information matrix, available in Table 1, Table 2, Table 3, and Table 4.

Step 2: Further, if the data given in the matrix is benefit type, then ignore it, but if the data given in the matrix is cost

TABLE 2. Cubic intuitionistic complex fuzzy soft matrix $\mathbb{I}_2$.

	e_1	e_2
$\mathbb{I}_{A-1}$	$\left(\begin{pmatrix}[0.5,0.7]e^{i2\pi([0.5,0.6])},\\ [0.1,0.2]e^{i2\pi([0.1,0.2])}\end{pmatrix}, \\ \left(0.5e^{i2\pi(0.3)}, 0.2e^{i2\pi(0.2)}\right)\right)$	$\left(\begin{pmatrix}[0.51,0.71]e^{i2\pi([0.51,0.61])},\\ [0.11,0.21]e^{i2\pi([0.11,0.21])}\end{pmatrix}, \\ \left(0.51e^{i2\pi(0.31)}, 0.21e^{i2\pi(0.21)}\right)\right)$
$\mathbb{I}_{A-2}$	$\left(\begin{pmatrix}[0.1,0.2]e^{i2\pi([0.3,0.4])},\\ [0.2,0.3]e^{i2\pi([0.3,0.4])}\end{pmatrix}, \\ \left(0.5e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.1)}\right)\right)$	$\left(\begin{pmatrix}[0.11,0.21]e^{i2\pi([0.31,0.41])},\\ [0.21,0.31]e^{i2\pi([0.31,0.41])}\end{pmatrix}, \\ \left(0.51e^{i2\pi(0.41)}, 0.31e^{i2\pi(0.11)}\right)\right)$
$\mathbb{I}_{A-3}$	$\left(\begin{pmatrix}[0.3,0.4]e^{i2\pi([0.2,0.3])},\\ [0.1,0.2]e^{i2\pi([0.2,0.4])}\end{pmatrix}, \\ \left(0.7e^{i2\pi(0.6)}, 0.1e^{i2\pi(0.2)}\right)\right)$	$\left(\begin{pmatrix}[0.31,0.41]e^{i2\pi([0.21,0.31])},\\ [0.11,0.21]e^{i2\pi([0.21,0.41])}\end{pmatrix}, \\ \left(0.71e^{i2\pi(0.61)}, 0.11e^{i2\pi(0.21)}\right)\right)$
$\mathbb{I}_{A-4}$	$\left(\begin{pmatrix}[0.2,0.5]e^{i2\pi([0.1,0.5])},\\ [0.2,0.3]e^{i2\pi([0.3,0.4])}\end{pmatrix}, \\ \left(0.4e^{i2\pi(0.3)}, 0.2e^{i2\pi(0.1)}\right)\right)$	$\left(\begin{pmatrix}[0.21,0.51]e^{i2\pi([0.11,0.51])},\\ [0.21,0.31]e^{i2\pi([0.31,0.41])}\end{pmatrix}, \\ \left(0.41e^{i2\pi(0.31)}, 0.21e^{i2\pi(0.11)}\right)\right)$
	e_3	e_4
$\mathbb{I}_{A-1}$	$\left(\begin{pmatrix}[0.52,0.72]e^{i2\pi([0.52,0.62])},\\ [0.12,0.22]e^{i2\pi([0.12,0.22])}\end{pmatrix}, \\ \left(0.52e^{i2\pi(0.32)}, 0.22e^{i2\pi(0.22)}\right)\right)$	$\left(\begin{pmatrix}[0.53,0.73]e^{i2\pi([0.53,0.63])},\\ [0.13,0.23]e^{i2\pi([0.13,0.23])}\end{pmatrix}, \\ \left(0.53e^{i2\pi(0.33)}, 0.23e^{i2\pi(0.23)}\right)\right)$
$\mathbb{I}_{A-2}$	$\left(\begin{pmatrix}[0.12,0.22]e^{i2\pi([0.32,0.42])},\\ [0.22,0.32]e^{i2\pi([0.32,0.42])}\end{pmatrix}, \\ \left(0.52e^{i2\pi(0.42)}, 0.32e^{i2\pi(0.12)}\right)\right)$	$\left(\begin{pmatrix}[0.13,0.23]e^{i2\pi([0.33,0.43])},\\ [0.23,0.33]e^{i2\pi([0.33,0.43])}\end{pmatrix}, \\ \left(0.53e^{i2\pi(0.43)}, 0.33e^{i2\pi(0.13)}\right)\right)$
$\mathbb{I}_{A-3}$	$\left(\begin{pmatrix}[0.32,0.42]e^{i2\pi([0.22,0.32])},\\ [0.12,0.22]e^{i2\pi([0.22,0.42])}\end{pmatrix}, \\ \left(0.72e^{i2\pi(0.62)}, 0.12e^{i2\pi(0.22)}\right)\right)$	$\left(\begin{pmatrix}[0.33,0.43]e^{i2\pi([0.23,0.33])},\\ [0.13,0.23]e^{i2\pi([0.23,0.43])}\end{pmatrix}, \\ \left(0.73e^{i2\pi(0.63)}, 0.13e^{i2\pi(0.23)}\right)\right)$
$\mathbb{I}_{A-4}$	$\left(\begin{pmatrix}[0.22,0.52]e^{i2\pi([0.12,0.52])},\\ [0.22,0.32]e^{i2\pi([0.32,0.42])}\end{pmatrix}, \\ \left(0.42e^{i2\pi(0.32)}, 0.22e^{i2\pi(0.12)}\right)\right)$	$\left(\begin{pmatrix}[0.23,0.53]e^{i2\pi([0.13,0.53])},\\ [0.23,0.33]e^{i2\pi([0.33,0.43])}\end{pmatrix}, \\ \left(0.43e^{i2\pi(0.33)}, 0.23e^{i2\pi(0.13)}\right)\right)$

type, then it's needed to be normalized. From the information in Table 1 to Table 4, it is clear that they cannot be needed to be stabilized.

Step 3: In the presence of the normalized information, we collect the information by using the information of the CICFSBM operator, described in Table 5.

Step 4: Compute the SVs of the collective information, such that

$$\mathbb{S}^{SV}(\mathbb{I}_1) = 0.8686, \quad \mathbb{S}^{SV}(\mathbb{I}_2) = 0.8698,$$
$$\mathbb{S}^{SV}(\mathbb{I}_3) = 0.8686, \quad \mathbb{S}^{SV}(\mathbb{I}_4) = 0.8642$$

Step 5: Rank the alternatives using the information of SV and determine the best decision, such that

$$\mathbb{I}_2 > \mathbb{I}_1 > \mathbb{I}_3 > \mathbb{I}_4$$

The best option is $\mathbb{I}_2$. Further, for the different values of parameters $\mathcal{U}$ and $\mathcal{V}$, we checked the feasibility and stability of the proposed work in the shape of Table 6.

Form Table 6, we noticed that the best decision is still $\mathbb{I}_2$, represented the kheshgi Payan is more affected area due to monsoon rains in 2010.

C. COMPARATIVE ANALYSIS

One of the important and dominant parts of the qualitative manuscript is called comparative analysis which is a technique of comparing two or more ideas, theories, and mathematical structures. The main theme of this analysis is to classify the supremacy advantages and disadvantages of the diagnosed theory. Therefore, to evaluate deficiencies with the help of invented operators, we established a MADM tool under the availability of CICFSBM and CICFSWBM operators in section 5.2. Here, we described the supremacy and reliability of the diagnosed work with the help of comparative analysis and also explained their graphical representation is to enhance the worth of the established approaches. For this, we choose some existing theories based on prevailing ideas, for instance, prioritized aggregation operators for complex intuitionistic fuzzy soft (CIFS) sets were diagnosed by Ali *et al.* [21], generalized BM operators for complex intuitionistic fuzzy information were utilized by Garg and Rani [33], BM operators for cubic IFS was invented by Kaur and Garg [35] and BM operators based on intuitionistic fuzzy soft sets were utilized by Garg and Arora [36].

TABLE 3. Cubic intuitionistic complex fuzzy soft matrix $\mathbb{I}_3$.

	e_1	e_2
$\mathbb{I}_{A-1}$	$\left(\left([0.3,0.4]e^{i2\pi([0.2,0.3])},[0.1,0.2]e^{i2\pi([0.2,0.4])}\right),\left(0.7e^{i2\pi(0.6)},0.1e^{i2\pi(0.2)}\right)\right)$	$\left(\left([0.31,0.41]e^{i2\pi([0.21,0.31])},[0.11,0.21]e^{i2\pi([0.21,0.41])}\right),\left(0.71e^{i2\pi(0.61)},0.11e^{i2\pi(0.21)}\right)\right)$
$\mathbb{I}_{A-2}$	$\left(\left([0.2,0.5]e^{i2\pi([0.1,0.5])},[0.2,0.3]e^{i2\pi([0.3,0.4])}\right),\left(0.4e^{i2\pi(0.3)},0.2e^{i2\pi(0.1)}\right)\right)$	$\left(\left([0.21,0.51]e^{i2\pi([0.11,0.51])},[0.21,0.31]e^{i2\pi([0.31,0.41])}\right),\left(0.41e^{i2\pi(0.31)},0.21e^{i2\pi(0.11)}\right)\right)$
$\mathbb{I}_{A-3}$	$\left(\left([0.5,0.7]e^{i2\pi([0.5,0.6])},[0.1,0.2]e^{i2\pi([0.1,0.2])}\right),\left(0.5e^{i2\pi(0.3)},0.2e^{i2\pi(0.2)}\right)\right)$	$\left(\left([0.51,0.71]e^{i2\pi([0.51,0.61])},[0.11,0.21]e^{i2\pi([0.11,0.21])}\right),\left(0.51e^{i2\pi(0.31)},0.21e^{i2\pi(0.21)}\right)\right)$
$\mathbb{I}_{A-4}$	$\left(\left([0.1,0.2]e^{i2\pi([0.3,0.4])},[0.2,0.3]e^{i2\pi([0.3,0.4])}\right),\left(0.5e^{i2\pi(0.4)},0.3e^{i2\pi(0.1)}\right)\right)$	$\left(\left([0.11,0.21]e^{i2\pi([0.31,0.41])},[0.21,0.31]e^{i2\pi([0.31,0.41])}\right),\left(0.51e^{i2\pi(0.41)},0.31e^{i2\pi(0.11)}\right)\right)$
	e_3	e_4
$\mathbb{I}_{A-1}$	$\left(\left([0.32,0.42]e^{i2\pi([0.22,0.32])},[0.12,0.22]e^{i2\pi([0.22,0.42])}\right),\left(0.72e^{i2\pi(0.62)},0.12e^{i2\pi(0.22)}\right)\right)$	$\left(\left([0.33,0.43]e^{i2\pi([0.23,0.33])},[0.13,0.23]e^{i2\pi([0.23,0.43])}\right),\left(0.73e^{i2\pi(0.63)},0.13e^{i2\pi(0.23)}\right)\right)$
$\mathbb{I}_{A-2}$	$\left(\left([0.22,0.52]e^{i2\pi([0.12,0.52])},[0.22,0.32]e^{i2\pi([0.32,0.42])}\right),\left(0.42e^{i2\pi(0.32)},0.22e^{i2\pi(0.12)}\right)\right)$	$\left(\left([0.23,0.53]e^{i2\pi([0.13,0.53])},[0.23,0.33]e^{i2\pi([0.33,0.43])}\right),\left(0.43e^{i2\pi(0.33)},0.23e^{i2\pi(0.13)}\right)\right)$
$\mathbb{I}_{A-3}$	$\left(\left([0.52,0.72]e^{i2\pi([0.52,0.62])},[0.12,0.22]e^{i2\pi([0.12,0.22])}\right),\left(0.52e^{i2\pi(0.32)},0.22e^{i2\pi(0.22)}\right)\right)$	$\left(\left([0.53,0.73]e^{i2\pi([0.53,0.63])},[0.13,0.23]e^{i2\pi([0.13,0.23])}\right),\left(0.53e^{i2\pi(0.33)},0.23e^{i2\pi(0.23)}\right)\right)$
$\mathbb{I}_{A-4}$	$\left(\left([0.12,0.22]e^{i2\pi([0.32,0.42])},[0.22,0.32]e^{i2\pi([0.32,0.42])}\right),\left(0.52e^{i2\pi(0.42)},0.32e^{i2\pi(0.12)}\right)\right)$	$\left(\left([0.13,0.23]e^{i2\pi([0.33,0.43])},[0.23,0.33]e^{i2\pi([0.33,0.43])}\right),\left(0.53e^{i2\pi(0.43)},0.33e^{i2\pi(0.13)}\right)\right)$

The comparative analysis of the invented theory with the above-cited theories is illustrated below.

1. Ali *et al.* [21] diagnosed the theory of prioritized aggregation operators for CIFS information which depends on the value of the parameter which is working behind the truth and falsity grades. The data given in truth and falsity grade is in the shape of a singleton set which copes only with one-dimension information at a time. The invented theory of CICFS information can easily be able to cope with it. But the theory given in [21] is not able to handle our invented types of information because the information contained in the invented idea is in the shape of truth and falsity grade with a parameter that deals with two-dimension information at a time. Therefore, the invented theory is more powerful than the theory given in [21].
2. Garg and Rani [33] diagnosed the theory of generalized BM operators for CIFS which depends on the value truth and falsity grades. The data given in truth and falsity grade is in the shape of complex numbers which cope only with two-dimension information at a time without parameters. The invented theory of CICFS information can easily be able to cope with it. But the theory given in [33] is not able to handle our invented types of information because the information contained in the invented idea is in the shape of truth and falsity grade with a parameter that deals with two-dimension information at a time. Therefore, the invented theory is more powerful than the theory given in [33] due to parameters.
3. Kaur and Garg [35] diagnosed the theory of BM operators for cubic IFS which depends on the value of truth and falsity grades. The data given in truth and falsity grade is in the shape of a singleton set and also in the form of interval-valued sets which cope only with one-dimension information at a time. The invented theory of CICFS information can easily be able to cope with it. But the theory given in [35] is not able to handle our invented types of information because the information contained in the invented idea is in the shape of truth and falsity grade with a parameter that deals with two-dimension information at a time.

TABLE 4. Cubic intuitionistic complex fuzzy soft matrix $\mathbb{I}_1$.

	e_1	e_2
$\mathbb{I}_{A-1}$	$\left(\begin{pmatrix}[0.2,0.5]e^{i2\pi([0.1,0.5])},\\ [0.2,0.3]e^{i2\pi([0.3,0.4])}\end{pmatrix}, \left(0.4e^{i2\pi(0.3)}, 0.2e^{i2\pi(0.1)}\right)\right)$	$\left(\begin{pmatrix}[0.21,0.51]e^{i2\pi([0.11,0.51])},\\ [0.21,0.31]e^{i2\pi([0.31,0.41])}\end{pmatrix}, \left(0.41e^{i2\pi(0.31)}, 0.21e^{i2\pi(0.11)}\right)\right)$
$\mathbb{I}_{A-2}$	$\left(\begin{pmatrix}[0.3,0.4]e^{i2\pi([0.2,0.3])},\\ [0.1,0.2]e^{i2\pi([0.2,0.4])}\end{pmatrix}, \left(0.7e^{i2\pi(0.6)}, 0.1e^{i2\pi(0.2)}\right)\right)$	$\left(\begin{pmatrix}[0.31,0.41]e^{i2\pi([0.21,0.31])},\\ [0.11,0.21]e^{i2\pi([0.21,0.41])}\end{pmatrix}, \left(0.71e^{i2\pi(0.61)}, 0.11e^{i2\pi(0.21)}\right)\right)$
$\mathbb{I}_{A-3}$	$\left(\begin{pmatrix}[0.2,0.5]e^{i2\pi([0.1,0.5])},\\ [0.2,0.3]e^{i2\pi([0.3,0.4])}\end{pmatrix}, \left(0.4e^{i2\pi(0.3)}, 0.2e^{i2\pi(0.1)}\right)\right)$	$\left(\begin{pmatrix}[0.21,0.51]e^{i2\pi([0.11,0.51])},\\ [0.21,0.31]e^{i2\pi([0.31,0.41])}\end{pmatrix}, \left(0.41e^{i2\pi(0.31)}, 0.21e^{i2\pi(0.11)}\right)\right)$
$\mathbb{I}_{A-4}$	$\left(\begin{pmatrix}[0.5,0.7]e^{i2\pi([0.5,0.6])},\\ [0.1,0.2]e^{i2\pi([0.1,0.2])}\end{pmatrix}, \left(0.5e^{i2\pi(0.3)}, 0.2e^{i2\pi(0.2)}\right)\right)$	$\left(\begin{pmatrix}[0.51,0.71]e^{i2\pi([0.51,0.61])},\\ [0.11,0.21]e^{i2\pi([0.11,0.21])}\end{pmatrix}, \left(0.51e^{i2\pi(0.31)}, 0.21e^{i2\pi(0.21)}\right)\right)$
	e_3	e_4
$\mathbb{I}_{A-1}$	$\left(\begin{pmatrix}[0.22,0.52]e^{i2\pi([0.12,0.52])},\\ [0.22,0.32]e^{i2\pi([0.32,0.42])}\end{pmatrix}, \left(0.42e^{i2\pi(0.32)}, 0.22e^{i2\pi(0.12)}\right)\right)$	$\left(\begin{pmatrix}[0.23,0.53]e^{i2\pi([0.13,0.53])},\\ [0.23,0.33]e^{i2\pi([0.33,0.43])}\end{pmatrix}, \left(0.43e^{i2\pi(0.33)}, 0.23e^{i2\pi(0.13)}\right)\right)$
$\mathbb{I}_{A-2}$	$\left(\begin{pmatrix}[0.32,0.42]e^{i2\pi([0.22,0.32])},\\ [0.12,0.22]e^{i2\pi([0.22,0.42])}\end{pmatrix}, \left(0.72e^{i2\pi(0.62)}, 0.12e^{i2\pi(0.22)}\right)\right)$	$\left(\begin{pmatrix}[0.33,0.43]e^{i2\pi([0.23,0.33])},\\ [0.13,0.23]e^{i2\pi([0.23,0.43])}\end{pmatrix}, \left(0.73e^{i2\pi(0.63)}, 0.13e^{i2\pi(0.23)}\right)\right)$
$\mathbb{I}_{A-3}$	$\left(\begin{pmatrix}[0.22,0.52]e^{i2\pi([0.12,0.52])},\\ [0.22,0.32]e^{i2\pi([0.32,0.42])}\end{pmatrix}, \left(0.42e^{i2\pi(0.32)}, 0.22e^{i2\pi(0.12)}\right)\right)$	$\left(\begin{pmatrix}[0.23,0.53]e^{i2\pi([0.13,0.53])},\\ [0.23,0.33]e^{i2\pi([0.33,0.43])}\end{pmatrix}, \left(0.43e^{i2\pi(0.33)}, 0.23e^{i2\pi(0.13)}\right)\right)$
$\mathbb{I}_{A-4}$	$\left(\begin{pmatrix}[0.52,0.72]e^{i2\pi([0.52,0.62])},\\ [0.12,0.22]e^{i2\pi([0.12,0.22])}\end{pmatrix}, \left(0.52e^{i2\pi(0.32)}, 0.22e^{i2\pi(0.22)}\right)\right)$	$\left(\begin{pmatrix}[0.53,0.73]e^{i2\pi([0.53,0.63])},\\ [0.13,0.23]e^{i2\pi([0.13,0.23])}\end{pmatrix}, \left(0.53e^{i2\pi(0.33)}, 0.23e^{i2\pi(0.23)}\right)\right)$

TABLE 5. Represented the aggregated values.

	CICFSBM Operator
$\mathbb{I}_{A-1}$	$\left(\begin{pmatrix}[0.8644,0.8983]e^{i2\pi([0.8743,0.9017])},\\ [0.0217,0.0302]e^{i2\pi([0.0299,0.0413])}\end{pmatrix}, \left(0.9055e^{i2\pi(0.9169)}, 0.0222e^{i2\pi(0.0335)}\right)\right)$
$\mathbb{I}_{A-2}$	$\left(\begin{pmatrix}[0.8604,0.8929]e^{i2\pi([0.8631,0.8934])},\\ [0.0217,0.0302]e^{i2\pi([0.0209,0.0284])}\end{pmatrix}, \left(0.9096e^{i2\pi(0.9288)}, 0.0296e^{i2\pi(0.0335)}\right)\right)$
$\mathbb{I}_{A-3}$	$\left(\begin{pmatrix}[0.8644,0.8983]e^{i2\pi([0.8743,0.9017])},\\ [0.0217,0.0302]e^{i2\pi([0.0299,0.0413])}\end{pmatrix}, \left(0.9055e^{i2\pi(0.9169)}, 0.0222e^{i2\pi(0.0335)}\right)\right)$
$\mathbb{I}_{A-4}$	$\left(\begin{pmatrix}[0.8713,0.9093]e^{i2\pi([0.8501,0.9028])},\\ [0.0335,0.0407]e^{i2\pi([0.0404,0.0414])}\end{pmatrix}, \left(0.9036e^{i2\pi(0.92)}, 0.0339e^{i2\pi(0.0217)}\right)\right)$

TABLE 6. Represented the stability of the parameters.

Parameter	Score Value	Best Decision
$\mathcal{U} = 1, \mathcal{V} = 2$	0.7896,0.7911,0.7896,0.7796	$\mathbb{I}_2$
$\mathcal{U} = 3, \mathcal{V} = 4$	0.4373,0.4454,0.4373,0.4154	$\mathbb{I}_2$
$\mathcal{U} = 5, \mathcal{V} = 6$	0.2384,0.2516,0.2384,0.2157	$\mathbb{I}_2$
$\mathcal{U} = 7, \mathcal{V} = 8$	0.1188,0.1352,0.1188,0.0962	$\mathbb{I}_2$
$\mathcal{U} = 9, \mathcal{V} = 10$	0.0382,0.0565,0.0382,0.0153	$\mathbb{I}_2$

Therefore, the invented theory is more powerful than the theory given in [35].

4. Garg and Arora [36] diagnosed the theory of BM operators for intuitionistic fuzzy soft information which depends on the value of the parameter which is working behind the truth and falsity grades. The data given in truth and falsity grade is in the shape of a singleton set which copes only with one-dimension information at a time. The invented theory of CICFS information can easily be able to cope with it. But the theory

given in [36] is not able to handle our invented types of information because the information contained in the invented idea is in the shape of truth and falsity grade with a parameter that deals with two-dimension information at a time. Therefore, the invented theory is more powerful than the theory given in [36].

The diagnosed theory in this manuscript is more powerful because the mathematical shape of CICFS information includes truth grade and falsity grade in the shape of simple and interval-valued whose value is in the shape of polar coordinates. Therefore, based on the above analysis, we obtained that the invented theory is more feasible for demonstrating the best decision.

VI. CONCLUSION

The key diagnosis of this analysis is illustrated below:

1. We diagnosed the fundamental theory of CICFS information which is a very informative and effective tool for handling ambiguity and complications in reality.
2. In the consideration of CICFS information, we utilized the important algebraic laws, score value, and accuracy value and try to determine some rules for finding the relation between any CICFS numbers.
3. Using the presented information, we diagnosed the CICFSBM operator, CICFSWBM operator and evaluated their important results, and described their important properties (Idempotency, Monotonicity, and Boundedness).
4. We evaluated deficiencies with the help of invented operators, for this, we established a MADM tool under the availability of CICFSBM, CICFSWBM operators.
5. We described the supremacy and reliability of the diagnosed work with the help of comparative analysis and also explained their graphical representation is to enhance the worth of the established approaches.

In upcoming years, we try to revise the theory of complex spherical fuzzy sets [37], complex T-spherical fuzzy sets [38], T-spherical fuzzy sets [39], linear Diophantine fuzzy sets [40], fuzzy N-soft sets [41], intuitionistic fuzzy N-soft sets [42], Maclaurin symmetric mean operators [43] and decision-making [44]. In the above analysis, we will try to utilize the theory of soft sets or try to mix it with complex fuzzy sets to enhance the worth of the research gap.

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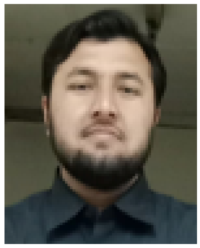

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