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Sliding Mode Control With Limits on the Control Signal and Linear State Combinations

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ABSTRACT In this paper the sliding mode controller is designed using the reaching law approach. The system model considers the time-varying uncertainties and the unknown, external disturbances. The main idea is to guarantee that the state limitations (in form of linear combinations) hold for the whole regulation process. Moreover, the control signal restriction is taken into account and the chattering on the boundary of the admissible set is fully eliminated. Furthermore, the controller ensures fast, monotonic and finite-time convergence of the representative point to the switching hyperplane.

INDEX TERMS Continuous-time system, control signal restriction, state variables limitation, sliding mode control, reaching law technique.

I. INTRODUCTION

Currently, an increasing number of variable structure control strategies appear in practical applications [1]–[8]. Across many different types of these regulation methods the sliding mode control was demonstrated to be a fine solution to deal with systems perturbed by unknown disturbances and model uncertainties. The effectiveness and computational efficiency of this approach encourage researchers to enhance theory and find new fields where the sliding mode control can be implemented [9]–[17].

The origins of this strategy can be found in the work of Emelyanov [18], where the author made an unexpected observation that switching between two unstable systems, can result in a stable one. Next results were presented by Drazenowić in [19], where she proved that the sliding mode control is not only robust, but also insensitive to perturbations. Chronologically the sliding mode control was firstly applied into the continuous-time systems [20] and then theory was expanded for discrete-time objects [21].

The idea of the sliding mode control is to drive the representative point to the predefined switching hyperplane and after that establish the stable sliding motion along the mentioned manifold until the desired state is reached. Hence, it is reasonable to distinguish two phases of the regulation process: the reaching phase and the sliding phase. It is also important to notice that controller design can be made in two different ways. We can propose the control input and examine the stability in the sliding phase or use the reaching law (called also approach law). The second method consist of predetermining the dynamics of the sliding variable and was firstly introduced by Gao and Hung in [22]. Then, this approach was studied by many researchers. As a consequence over the years plenty of different reaching laws were presented, offering a wide range of system properties.

An interesting approach to manipulator control was presented in [23]. In this work the discontinuous part of the input signal is adaptively selected to minimize chattering while still ensuring the existence of the sliding mode. Unfortunately, the results are not general and can be applied only to a link robotic manipulator dynamics. Moreover, there are no strict limits imposed on the state variables. Nevertheless, the problem of constraining the state variables and the control signal in sliding mode control has been undertaken only by few researchers [24]–[32]. Unfortunately, all of the works exhibit some drawbacks. In paper [29] only the third order plant was taken in account subject to the velocity and acceleration restrictions. In [28] the nonlinear n-th order dynamical system was analyzed, however the approach enables constraining only one of the state variables and the control signal. In the work [27] a system of arbitrary order is controlled allowing to limit all of the state variables and control signal. Nevertheless, the disturbances are assumed to be sufficiently smooth functions, which is not common in practice. Moreover, in [26] the disturbances and systems uncertainties were assumed to be zero, which limits the practical applicability. In [30] the linear servomechanism was controlled, however only the input constraint was guaranteed. Similarly in [31] the control signal

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was limited for Euler-Lagrange plants. In the paper [32] authors considered control of the PMSM drive system. The trajectory was constrained, unfortunately it is not clear how to generalize the results for any dynamical system. An adaptive neural network nonsingular sliding mode controller was used in [33] to tackle the problem of permanent magnet linear synchronous motor control. The main difficulty in such a control plant are the unknown parameters of the motor, which can moreover vary with position. The sliding mode control paradigm therefore seems the best choice for this task. The neural network is used to estimate on-line the model parameters, and the adaptive part "fine-tunes" the discontinuous part of the control signal to minimize chattering. The approach is then tested on an experimental stand. In [34] a sliding mode controller for uncertain non-linear time-varying systems is proposed. A small drawback of the proposed solution is that the time-varying sliding hyperplane converges asymptotically to the desired position, which means that the exact system dynamics specified by choosing its final orientation will not be achieved in finite time. Despite of this, the simulation results confirm some advantages over the previous control methods, such as slightly faster convergence rate. The paper [35] proposes a terminal sliding mode controller, used in tandem with a sliding mode observer, to control second order systems, such as robot manipulators. Unfortunately, the constraints of states are not taken into account – the authors assume that the states are bounded, but do not demonstrate that their approach in fact ensures this to be true. However, computer simulations of a PUMA560 manipulator show the application potential of the method. A survey of sliding mode control in the presence of constraints is presented in [36]. The drawback of a large part of the described algorithms is that they take into account either the control signal limit or the state constraints, and not both of them. Moreover, some of the proposed solutions allow only to limit the part of the states, that form a "chain" of integrators, while the others evolve freely. In [41] the control of a second order integrator subjected to disturbances is considered. Such a system can model e.g. mechanical systems, such as joint dynamics in manipulators. The admissible region (in which the constraints are not violated) is divided into two regions. If the state is in one of them, satisfying the constraints in the future evolution of the trajectory is ensured, from the other one it is not. The proposed solution allows to limit both the control signal as well as the two state variables, however it is not entirely clear, how it could be extended to higher order systems. The authors propose one such extension, however it allows only to limit two state variables, irrespective of the order of the system. An interesting approach for controlling constrained nonlinear systems is presented in [42]. It not only ensures satisfying state and input constraints, but it also guarantees, that each state variable will not change its sign (it is assumed that the desired state is the origin). The method relies on making the nonlinear system follow a trajectory of a linear, stable one, with eigenvalues chosen by the designer. Unfortunately, the above-mentioned advantageous properties are only achieved, assuming that the number of states and inputs is identical, which is very seldom the case.

Therefore, in this paper we examine this issue upon the perturbed continuous time system with time-varying model uncertainties and external disturbances (which do not need to satisfy the matching conditions). We propose a reaching law that ensures the fastest, monotonic and finite-time convergence of the representative point to the predefined switching hyperplane, simultaneously limiting state constraints and control input, extending results obtained in [37]. The novelty of this paper consists of considering the state constraints in the forms of linear combinations, fully eliminating the chattering effect on the boundaries of the admissible set and increasing the convergence speed by taking into account maximal impact of model uncertainties at every time instant. It is also worth to notice that the presented control strategy does not require any additional computer tuning in comparison to some advanced methods utilizing e.g. neural networks [38], [39].

This paper is organized as follows. Section 2 contains the description of the system dynamics and the general controller design, based on the proposed reaching law. Further, in Section 3, the problem of constraining the state variables and the control signal is studied. Sliding variable convergence rates are calculated in such a way to keep the certain restrictions from being violated. Section 4 focuses on the control strategy and the sufficient condition for monotonic and finite time convergence of the representative point to the switching hyperplane. Section 5 consists of the simulation example and lastly Section 6 presents the conclusions.

II. SYSTEM MODEL

Let us take into consideration the continuous-time, linear system perturbed by external disturbances and subjected to time-varying model uncertainties. Thus, the system dynamics is given by the following equation:

$$\dot{x}(t) = \left[A + \widetilde{A}(t)\right] x(t) + bu(t) + d(t), \tag{1}$$

where A is a $n \times n$ dimensional state matrix, $\widetilde{A}(t)$ describes time-varying model uncertainties, $x(t)$ is a $n \times 1$ dimensional state vector, $b = [b_1, \ldots, b_n]^T$ is a vector multiplying scalar control input $u(t)$ and $d(t)$ is a vector that represents the external disturbances. Although the perturbations and uncertainties are unknown, we consider that they are bounded for the whole regulation process: $|d_i(t)| < D_i$, $|\widetilde{a}_{ij}| < v_{ij}$ for $i, j \in \{1, \ldots, n\}$. In order to group these bounds we introduce $D = [D_1, \ldots, D_n]^T$, $V = \left[v_{ij}\right]$. Next, we define the sliding variable as $s(t) = c^T x(t)$, where $c^T = [c_1, \ldots, c_n]$. Therefore, the switching hyperplane is given by

$$c^T x(t) = 0. \tag{2}$$

Vector c needs to be chosen in such a manner to ensure $c^T b \neq 0$. Otherwise it would be impossible to design the sliding mode controller as the control signal would have no impact on the value of the sliding variable. In order to predetermine the evolution of the sliding variable we establish the reaching law

$$\dot{s}(t) = -K \operatorname{sgn}[s(t)] + c^T d(t) + c^T \widetilde{A}(t)\, x(t). \tag{3}$$

were K is a sliding variable convergence rate. In the remainder of the paper we will find the maximum value of K to ensure the fastest convergence of the sliding variable to zero. Even thought the above approach law contains the external disturbances and model uncertainties, the controller based on this law will be free from these unknown influences. It is also worth noticing that monotonic and finite-time convergence of the representative point to the switching hyperplane can be only obtained when a convergence rate K is large enough to overcome perturbations and uncertainties.

Using the reaching law approach and equations (1), (2) we obtain the formula for the control signal:

$$u(t) = -\left(c^T b\right)^{-1} \left\{c^T A x(t) + K \operatorname{sgn}[s(t)]\right\}. \tag{4}$$

This control signal will ensure that (3) is satisfied. As it was mentioned before, the controller does not depend on unknown terms that are present in the approach law. Furthermore, let us observe that if condition

$$-\operatorname{sgn}[s(t)]\, \dot{s}(t) \geq \lambda, \tag{5}$$

where $\lambda > 0$, is met in the reaching phase, then the monotonic and finite-time convergence of the representative point to the switching hyperplane is obtained. Taking into account the form of the reaching law we substitute (3) into (5) getting

$$K - c^T d(t) \operatorname{sgn}[s(t)] - c^T \widetilde{A} x(t) \operatorname{sgn}[s(t)] \geq \lambda > 0. \tag{6}$$

The maximal possible impacts of the external disturbances and model uncertainties on the sliding variable are

$$D_{\max} = |c_1| D_1 + \ldots |c_{n-1}| D_{n-1} + |c_n| D_n, \tag{7}$$
$$V_{\max} = [|c_1|, \ldots, |c_n|]\, V\, [|x_1(t)|, \ldots, |x_n(t)|]^T. \tag{8}$$

Therefore, (7), (8) are the worst case scenario values for $c^T d(t) \operatorname{sgn}[s(t)]$ and $c^T \widetilde{A} x(t) \operatorname{sgn}[s(t)]$, correspondingly. Thus, the convergence rate needs to guarantee

$$K - D_{\max} - V_{\max} \geq \lambda. \tag{9}$$

during the whole reaching phase. Nevertheless, the exact moment when the sliding phase begins remains unknown, due to the presence of perturbation and uncertainties. However, the finite settling time is not considered in this paper. To obtain such a system property, it is necessary to select a nonlinear switching surface.

III. STATE VARIABLES AND CONTROL SIGNAL CONSTRAINTS

This section will focus on designing the convergence rates K_{α_i}, $i \in \{1, \ldots, m\}$, where m is the number of state constraints and α_i represents the certain linear combination, i.e. $\alpha_i(x) = \alpha_{i1} x_1 + \cdots + \alpha_{in} x_n$, $\alpha_{ij} \in \Re$. In order to simplify further calculations we specify $\alpha_{ie} = [\alpha_{i1}, \ldots, \alpha_{in}]$. Therefore, the admissible set is given as follows

$$X = \{x \in \Re^n : \forall_{i \in \{1,\ldots,m\}} \alpha_i(x) \leq r_i, |c^T x| \leq |c^T x_0|\}, \tag{10}$$

where $x_0 = x(0)$. Let us notice that satisfying (9) results in decreasing the sliding variable, hence it will never exceed its initial value. However, if the initial state is unknown, then we define $X = \{x \in \Re^n : \forall_{i \in \{1,\ldots,m\}} \alpha_i(x) \leq r_i\}$. We assume that the state constraints result in a compact admissible set.

Now, we move to the problem of keeping the state from exceeding the constraint given by $\alpha_i(x) \leq r_i$. When the representative point reaches this constraint i.e. $\alpha_i(x) = r_i$, then we need to guarantee $\dot{\alpha}_i(x) \leq 0$, which is equivalent to $\alpha_i(\dot{x}) \leq 0$. Using the properties of the linear combination and equation (1) we get

$$\alpha_i(\dot{x}) = \alpha_{ie}\left[A + \widetilde{A}(t)\right] x(t) + \alpha_i(d) - \alpha_i(b)\left(c^T b\right)^{-1} \{c^T A x(t) + K \operatorname{sgn}[s(t)]\}. \tag{11}$$

Next, we calculate the biggest possible influence of model uncertainties on the value of $\alpha_i(x)$, getting

$$V_{\alpha_i} = [|\alpha_{i1}|, \ldots, |\alpha_{in}|]\, V\, [|x_1(t)|, \ldots, |x_n(t)|]^T. \tag{12}$$

Let us also introduce a similar denotation for disturbances

$$D_{\alpha_i} = [|\alpha_{i1}|, \ldots, |\alpha_{in}|]\, D. \tag{13}$$

After equating the right-hand side of equation (11) to zero, deriving K and taking into account the unknown influences we get the convergence rate corresponding to the motion along the i-th constraint

$$K_{\alpha_i} = \operatorname{sgn}[s(t)]\, [\alpha_i(b)]^{-1} c^T b \left[\alpha_{ie} A x(t) + D_{\alpha_i} + V_{\alpha_i}\right] - \operatorname{sgn}[s(t)]\, c^T A x(t). \tag{14}$$

Therefore, it is important to assume that $\alpha_i(b)$ differs from zero. Otherwise, it would be impossible to guarantee the monotonic convergence of the representative point to the switching hyperplane both with maintaining the considered constraint. Moreover, from (11) we have that if

$$\operatorname{sgn}\left[\alpha_i(b)\, c^T b s(t)\right] = -1 \tag{15}$$

is true, then decreasing K below (14) will not result in violating the constraint. However, increasing K above (14) can result in exceeding the limitation, depending on the influence of external disturbances and model uncertainties. On the other hand, when (15) is not true, then we can implement the biggest convergence rate within the range of the control signal. Therefore, we denote $K_{\alpha_i} = K_{\alpha_i}^+$ when (15) is false and $K_{\alpha_i} = K_{\alpha_i}^-$ when (15) is true. Let us also notice that sign of $\alpha_i(b)\, c^T b s(t)$ is constant and not zero for the whole reaching phase.

Moreover, we will present the convergence rate K_u, responsible for maintaining the control signal limitation $|u(t)| \leq r_u$. From (4) we have

$$-r_u \leq -\left(c^T b\right)^{-1} \left\{c^T A x(t) + K \operatorname{sgn}[s(t)]\right\} \leq r_u. \tag{16}$$

Let us observe, that during the reaching phase the sign of the sliding variable is constant. Thus, the biggest convergence rate that guarantees the control signal limitation is

$$K_u = \left|c^T b\right| r_u - \mathrm{sgn}\left[s\left(t\right)\right] c^T Ax\left(t\right). \tag{17}$$

Let us observe that substituting (17) for K in (16) will result in one of the boundary values of the control signal limitation. However, the lowest convergence rate in the range of the control input is

$$K_{\min_u} = -\left|c^T b\right| r_u - \mathrm{sgn}\left[s\left(t\right)\right] c^T Ax\left(t\right). \tag{18}$$

Similarly, substituting (18) for K in (16) will result in achieving the other constraint. Therefore, K must be between (17) and (18) in order to satisfy the control input restriction.

IV. SUFFICIENT CONDITION AND CONTROL STRATEGY

Previously we calculated convergence rates connected with all of the constraints, taken separately. In this section we will demonstrate the and theorems that give the sufficient condition for the monotonic and finite-time convergence of the representative point to the switching hyperplane, simultaneously satisfying state and control signal limitations.

At first, we need to ensure that the control signal range is wide enough to enable the monotonic convergence of the sliding variable to zero, even in the presence of the external disturbances and model uncertainties. Therefore, we require

$$K_u - D_{\max} - V_{\max} \geq \lambda > 0 \tag{19}$$

for the whole reaching phase.

Theorem 1: To satisfy (19) it is sufficient that

$$\left|c^T b\right| r_u \geq sgn\left[s\left(t\right)\right] c^T Ax\left(t\right) + D_{\max} + V_{\max} + \lambda \tag{20}$$

is true for $x\left(t\right)$ in X.

Proof: Using the form of the convergence rate K_u and the condition (19) we obtain (20). Let us notice that the initial point is in the the admissible set. Thus, it is sufficient to take into account only points from X. □

Let us notice that condition (20) will also result in $K_{\min_u} \leq 0$.

Secondly, we have to guarantee analogous property also for the convergence rates connected with the state constraints:

$$K^-_{\alpha_i} - D_{\max} - V_{\max} \geq \lambda > 0, \tag{21}$$

$i \in \{1, \ldots, m\}$.

Theorem 2: To satisfy (21) it is sufficient that

$$\begin{aligned}\left|\left[\alpha_i\left(b\right)\right]^{-1} c^T b\right| &\left(\alpha_{ie}Ax\left(t\right) + D_{\alpha_i} + V_{\alpha_i}\right)\\ &\geq sgn\left[s\left(t\right)\right] c^T Ax\left(t\right) + D_{\max} + V_{\max} + \lambda\end{aligned} \tag{22}$$

is true for $i \in \{1, \ldots, m\}$, when $K_{\alpha_i} = K^-_{\alpha_i}$, $\alpha_i\left(x\right) = r_i$ and $x\left(t\right)$ is in X.

Proof: Using (15) we can rewrite the above condition as

$$\begin{aligned}\mathrm{sgn}\left[s\left(t\right)\right]&\left[\alpha_i\left(b\right)\right]^{-1} c^T b\left(\alpha_{ie}Ax\left(t\right) + D_{\alpha_i} + V_{\alpha_i}\right)\\ &\geq \mathrm{sgn}\left[s\left(t\right)\right] c^T Ax\left(t\right) + D_{\max} + V_{\max} + \lambda.\end{aligned} \tag{23}$$

After moving all terms of this inequality to the left-hand side and using the definition for K_{α_i} we obtain (21). □

The similar condition for $K^+_{\alpha_i}$ is not required, because then we can increase the convergence rate up to convergence rate K_u. However, this implies the need of

$$K_u \geq K^+_{\alpha_i}, \tag{24}$$

otherwise there would not exist any value K which satisfies both control input and i-th state variable input.

Theorem 3: To satisfy (24) it is sufficient that

$$\left|\alpha_i\left(b\right)\right| r_u + \alpha_{ie}Ax\left(t\right) + D_{\alpha_i} + V_{\alpha_i} \geq 0 \tag{25}$$

is true for $i \in \{1, \ldots, m\}$, when $K_{\alpha_i} = K^+_{\alpha_i}$, $\alpha_i\left(x\right) = r_i$ and $x\left(t\right)$ is in X.

Proof: At first, we subtract from both sides of above inequality the term $\alpha_{ie}Ax\left(t\right) + D_{\alpha_i} + V_{\alpha_i}$ and then we multiply by $\left|\alpha_i\left(b\right)\right|^{-1}\left|c^T b\right|$ obtaining

$$\left|c^T b\right| r_u \geq -\left|\left[\alpha_i\left(b\right)\right]^{-1} c^T b\right|\left(\alpha_{ie}Ax\left(t\right) + D_{\alpha_i} + V_{\alpha_i}\right). \tag{26}$$

Subsequently, using the fact that (15) is false we have

$$\begin{aligned}&\left|c^T b\right| r_u\\ &\geq \mathrm{sgn}\left[s\left(t\right)\right]\left[\alpha_i\left(b\right)\right]^{-1} c^T b\left(\alpha_{ie}Ax\left(t\right) + D_{\alpha_i} + V_{\alpha_i}\right).\end{aligned} \tag{27}$$

Taking into acocunt the forms of convergence rates K_u and K_{α_i} we subtract from both sides the expression $\mathrm{sgn}\left[s\left(t\right)\right] c^T Ax\left(t\right)$ getting (24). □

Lastly, we need to ensure that it is possible to maintain multiple state constraints simultaneously. Therefore, on the intersection of i-th and j-th state constraints it is necessary to guarantee

$$K^-_{\alpha_i} \geq K^+_{\alpha_j} \tag{28}$$

Otherwise, at least one restriction would be violated.

Theorem 4: To satisfy (28) it is sufficient that

$$\begin{aligned}&\left|\alpha_i\left(b\right)\right|^{-1}\left[\alpha_{ie}Ax\left(t\right) + D_{\alpha_i} + V_{\alpha_i}\right]\\ &\quad + \left|\alpha_j\left(b\right)\right|^{-1}\left[\alpha_{je}Ax\left(t\right) + D_{\alpha_j} + V_{\alpha_j}\right] \geq 0\end{aligned} \tag{29}$$

is true for $i, j \in \{1, \ldots, m\}$, $i \neq j$, when $\alpha_i\left(x\right) = r_i$, $\alpha_j\left(x\right) = r_j$ and $x\left(t\right)$ is in X.

Proof: At first, we multiply both sides of above inequality by term $\left|c^T b\right|$ and then subtract the element connected with j-th index, obtaining

$$\begin{aligned}&\left|\left[\alpha_i\left(b\right)\right]^{-1} c^T b\right|\left[\alpha_{ie}Ax\left(t\right) + D_{\alpha_i} + V_{\alpha_i}\right] \geq\\ &\quad -\left|\left[\alpha_j\left(b\right)\right]^{-1} c^T b\right|\left[\alpha_{je}Ax\left(t\right) + D_{\alpha_j} + V_{\alpha_j}\right]\end{aligned} \tag{30}$$

If $K_{\alpha_i} = K^-_{\alpha_i}$ and $K_{\alpha_j} = K^+_{\alpha_j}$, then we have that $\mathrm{sgn}\left[\alpha_i\left(b\right) c^T b\right] = \mathrm{sgn}\left[s\left(t\right)\right]$, $\mathrm{sgn}\left[\alpha_j\left(b\right) c^T b\right] = -\mathrm{sgn}\left[s\left(t\right)\right]$. Therefore,

$$\begin{aligned}&\mathrm{sgn}\left[s\left(t\right)\right]\left[\alpha_i\left(b\right)\right]^{-1} c^T b\left[\alpha_{ie}Ax\left(t\right) + D_{\alpha_i} + V_{\alpha_i}\right]\\ &\geq \mathrm{sgn}\left[s\left(t\right)\right]\left[\alpha_j\left(b\right)\right]^{-1} c^T b\left[\alpha_{je}Ax\left(t\right) + D_{\alpha_j} + V_{\alpha_j}\right].\end{aligned} \tag{31}$$

Lastly, subtracting from both sides $\operatorname{sgn}[s(t)]\,c^T Ax(t)$ and using the forms of the the convergence rates $K_{\alpha_i}, K_{\alpha_j}$ we get condition (28). □

Further step is to demonstrate the control strategy, i.e. the methodology of selecting the convergence rate, depending on the current state. Our aim is to achieve the fastest, monotonic and finite-time convergence of the representative point to the switching hyperplane simultaneously ensuring state and control signal constraints.

Theorem 5: If conditions (20), (22), (25), (29) hold, the control strategy:

1) *when the state is inside the admissible set or on the boundary/boundaries for which (15) is false, select convergence rate K_u,*
2) *otherwise, set $K = \min\{K_u, K^-_{\alpha_{i_1}}, K^-_{\alpha_{i_2}}, \ldots, K^-_{\alpha_{i_l}}\}$, for those $K^-_{\alpha_j}$ for which $\alpha_j(x) = r_j$,*

ensures the fastest, monotonic, finite time convergence of the representative point to the sliding hyperplane, simultaneously satisfying state and control input constraints $\alpha_i(x) \le r_i$, $|u(t)| \le r_u$.

When the sliding motion begins the convergence rate must be selected large enough to match the maximal possible influences of external disturbances and model uncertainties on the sliding variable i.e. $K \ge D_{\max} + V_{\max}$.

It is worth noticing that the presented control strategy has an advantage over a common regulation method to just react in emergency when the boundary is reached. By verifying the sufficient condition the designer will know in advance whether any limitation will be violated.

In the end, we will present the technique to fully eliminate the chattering effect that can occur on the boundary of the admissible set [41], [42]. The mentioned chattering is a result of switching between K_u and certain K_{α_i}, caused by the fact that convergence rate K_{α_i} is designed with a safety margin. This margin is necessary, because we cannot predict the influence of the external disturbances, so we need to consider the worst possible case. Nevertheless, this will usually result in pushing the representative point back into the interior of the admissible set. As a consequence the convergence rate K_u will be selected, bringing the state back onto the constraint. Therefore, the chattering will occur. To avoid this problem we can implement a smooth transition between K_u and $K^-_{\alpha_i}$ near the constraint, by using for example the following convex combination: $\frac{\alpha_i(x)+\varepsilon-r_i}{\varepsilon}K^-_{\alpha_i} + \frac{r_i-\alpha_i(x)}{\varepsilon}K_u$, when $\alpha_i(x) \ge r_i - \varepsilon$. The selection of ε is arbitrary, however it should be a relatively small number compared to r_i. When $\alpha_i(x) \ge r_i - \varepsilon$ holds for multiple constraints we take into account the one in which $K^-_{\alpha_i}$ is minimal.

To be clear, let us point out that the above chattering elimination approach is used to deal with the chattering only on the boundary of the admissibly set, and not during the sliding phase. In order to reduce the chattering on the switching hyperplane, one of the known approaches can be applied. We have not done this, to focus on the main results.

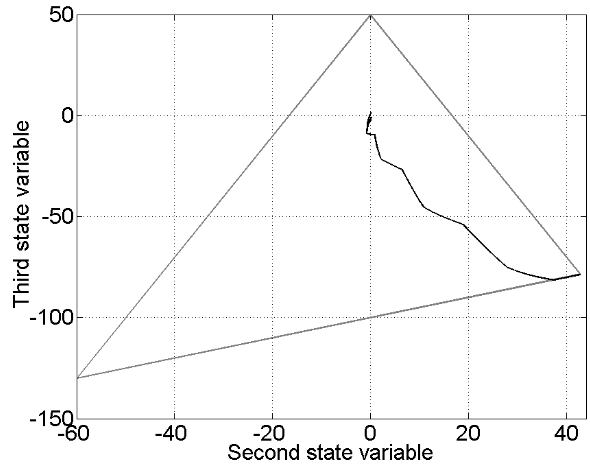

FIGURE 1. State trajectory.

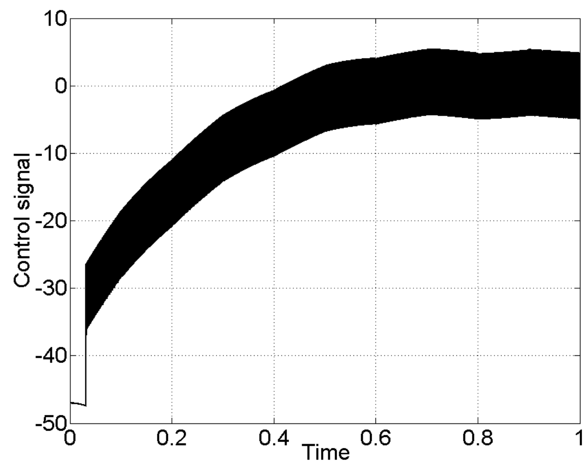

FIGURE 2. Control signal.

V. SIMULATION EXAMPLE

In this section the theoretical considerations will be verified by the numerical example. The model parameters, describing the system dynamics, are:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 0 & -6 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 12 \end{bmatrix}, \quad c = \begin{bmatrix} 10 \\ 0.5 \\ 0.1 \end{bmatrix},$$

$$\widetilde{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.1\sin(10t) & 0.2\sin(20t) & 0.3\sin(30t) \end{bmatrix},$$

$$d(t) = \begin{bmatrix} 5(-1)^{\lfloor 10t \rfloor} & 5\sin(10t) & 5\sin(20t) \end{bmatrix}. \qquad (32)$$

Calculating the greatest possible influence of the external disturbances on the sliding variable we get: $D_{\max} = 53$.

The switching hyperplane parameters were chosen is such a way to ensure $c^T b \neq 0$ and guarantee the stable sliding motion. What is more, we require that the following constraints hold for the whole regulation process: $0.5x_2(t) - x_3(t) \le 100$, $3x_2(t) + x_3(t) \le 50$, $-3x_2(t) + x_3(t) \le 50$, $|x_1(t)| \le 1.2$, $|u(t)| \le 50$. Therefore, $\alpha_{1e} = [0, 0.5, -1]$,

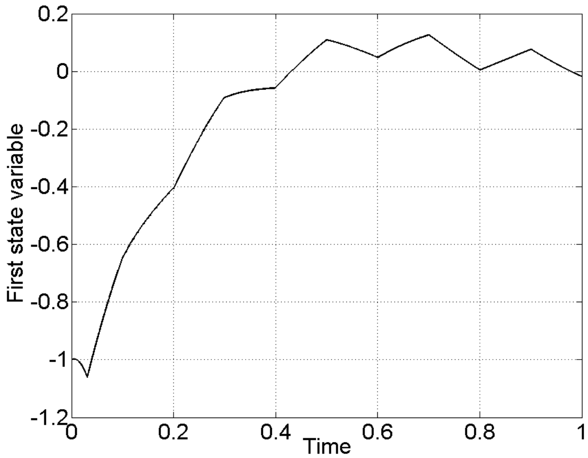

FIGURE 3. First state variable.

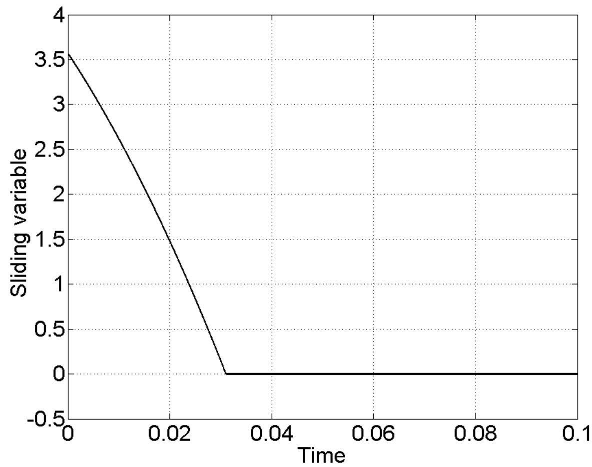

FIGURE 4. Sliding variable.

$\alpha_{2e} = [0, 3, 1]$, $\alpha_{3e} = [0, -3, 1]$, $\alpha_{4e} = [1, 0, 0]$, $\alpha_{5e} = [-1, 0, 0]$ and $r_1 = 100, r_2 = r_3 = 50, r_4 = r_5 = 1.2$, $r_u = 50$.

Our purpose is to drive the state to the sliding hyperplane monotonically and in finite-time. Moreover, we will ensure that all previously mentioned constraints hold for the whole regulation process. The initial point has been selected on the boundary of X in order to present the properties of our controller: $x_1(0) = -1, x_2(0) = 42.86, x_3(0) = -78.57$. First of all, let us observe that taking into account the initial state all conditions in theorems from the previous section are met for this model.

Fig. 1 depicts the evolution of the second and the third state variables. The triangle presented in this figure represents the linear combinations of respective state constraints. We can observe that at the beginning of the control process the representative point glides along the surface represented by $-3x_2(t) + x_3(t) \leq 50$. This is a result of applying the convergence rate close to the $K^-_{\alpha_3}$. The real implemented convergence is given by the following combination:

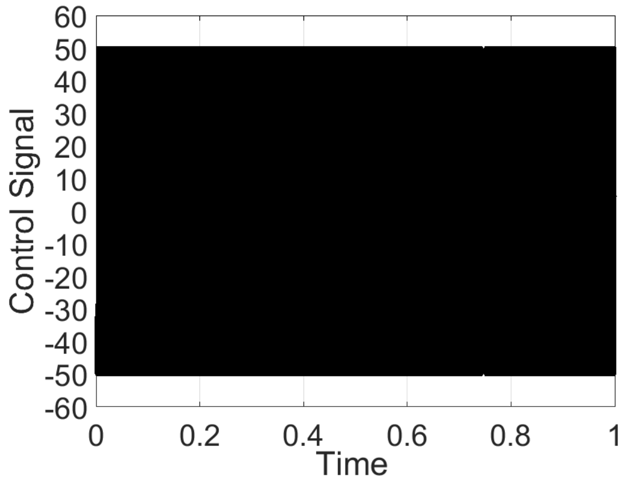

FIGURE 5. Control signal [41].

$\frac{\alpha_i(x)+\varepsilon-50}{\varepsilon}K^-_{\alpha_3} + \frac{50-\alpha_3(x)}{\varepsilon}K_u$, where threshold ε was select as 0.001. As a consequence the chattering was eliminated in the reaching phase, which can be seen on Fig 2. The control signal value at the beginning of the regulation process is the result of maintaining the state on one of the constraints. The observable chattering phenomena of the control signal is the effect of sliding motion and it can be reduced for example by replacing the signum function by saturation. The first state variable evolution was shown at the Fig. 3. We can notice that for the whole regulation process the constraint $|x_1(t)| \leq 1.2$ holds. Due to the fact that the matching conditions are not assumed to be true, the state error would not converge to zero, which can be noticed from figure 3. Moreover, from Fig 4. we have that that monotonic, finite time convergence of the representative point to the sliding hyperplane was achieved and the consecutive stable sliding motion was obtained, which demonstrates the robustness to unknown terms (external disturbances and parameters uncertainties) during the sliding phase.

Let us compare our results with the results obtained in [41]. The strategy proposed in [41] was presented for the double integrator case, however the idea can be extended to match the system considered in the simulation example. The controller, that corresponds to the one proposed in [41], behaves as follows: switch between the minimal admissible value and maximal admissible value, i.e. $-r_u, r_u$, if the state is on the limit or when the sliding hyperplane is reached. Therefore, in the considered case, when the initial point is placed at the constraint, the chattering effect will occur for the whole regulation process, which can be verified at Fig. 5. The other figures are practically the same as for the proposed solution. As we mentioned before, our approach eliminates the chattering on the state constraints. The chattering that is the result of applying the control strategy presented in [41] can be omitted only if the state would reach the sliding hyperplane before any constraint is reached. However, this is not ensured.

VI. CONCLUSION

This paper considered the problem of keeping state constraints (in forms of linear combinations of state variables) both with control signal limitation in the sliding mode control. The continuous-time model took in consideration the unknown, external disturbances and time-varying uncertainties. We analyzed the impact of these unknown terms on the dynamics of the obtained control system. The approach law was applied to design the sliding mode controller, which results in much greater control over the state dynamics in the reaching phase. Then, the regulation strategy was presented to establish the manner of selecting the convergence rate in every situation. As a consequence the fastest, monotonic and finite time convergence of the representative point to the predefined switching hyperplane in the presence of mentioned constraints was obtained. The sufficient condition ensuring these properties was stated and formally proved. Moreover, the technique to fully reduce the chattering on the boundary of the admissible set was demonstrated. Our next goal is to extrapolate this methodology also to nonlinear systems.

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